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Metric geometry of partial isometries in a finite von Neumann algebra

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Abstract

We study the geometry of the set

$$\mathcal{I}_p = \{v \in M:\ v^*v = p\}$$

of partial isometries of a finite von Neumann algebra M, with initial space p (p is a projection of the algebra). This set is a C^∞ submanifold of M in the norm topology of M. However, we study it in the strong operator topology, in which it does not have a smooth structure. This topology allows for the introduction of inner products on the tangent spaces by means of a fixed trace τ in M. The quadratic norms do not define a Hilbert–Riemann metric, for they are not complete. Nevertheless certain facts can be established: a restricted result on minimality of geodesics of the Levi-Civita connection, and uniqueness of these as the only possible minimal curves. We prove also that $(\mathcal{I}_p, d_g)$ is a complete metric space, where d_g is the geodesic distance of the manifold (or the metric given by the infima of lengths of piecewise smooth curves).

Keywords: Partial isometries; Projections; Geodesics; Finite von Neumann algebras

1. Introduction

Let M be a finite von Neumann algebra with fixed trace τ, and let $p \in M$ be a projection. Consider the set

$$\mathcal{I}_p = \{v \in M:\ v^*v = p\}$$

of partial isometries in M with initial space p. This set is a C^∞ differentiable submanifold of M in the norm topology, and carries a natural action of the unitary group U_M of M:

$$u \cdot v = uv, \quad u \in U_M,\ v \in \mathcal{I}_p.$$

Therefore the set $\mathcal{I}_p$ is a homogeneous space of the group U_M, i.e. the quotient of U_M by a subgroup (the isotropy group), which is itself the unitary group of a subalgebra of M, the reduced algebra $(1-p)M(1-p)$. Our interest in the properties of unitary groups and homogeneous spaces of von Neumann algebras in the weak topologies, originates

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in a paper by Popa and Takesaki [17]. In this paper it was remarked that though much is known on the topology of the unitary groups in the norm topology (for instance, the homotopy type of the unitary group in terms of the type decomposition of the algebra, starting from Kuiper's theorem [12], and its generalizations), little is known about the relevant topologies in von Neumann algebras, which are the weak topologies. Our point of view, which is not new (see for instance [9]), consists in using tools from differential geometry, more precisely Finsler and Riemann metrics, to treat these homogeneous spaces (in the strong topologies) as geodesic metric spaces, as defined by Gromov [10]. On the other hand, we claim that these homogeneous spaces provide interesting (infinite dimensional) examples to the theory of geodesic metric spaces.

The action of U_M on $\mathcal{I}_p$ is locally transitive: if $v, v_0 \in \mathcal{I}_p$ with $\|v - v_0\| < 1/2$, then there exists a unitary valued C^∞ map $\omega = \omega(v, v_0)$ such that $\omega v_0 = v$ (see [14,3] and [2] for an account of these facts). In the case of a general von Neumann algebra the action is not transitive. For instance, in the space of isometries (initial space equal to 1) in the algebra $\mathcal{B}(H)$ of all bounded operators in an infinite dimensional Hilbert space H, the action preserves the Fredholm index, so it cannot be transitive. However, when M is finite, it is transitive. Indeed, considering the right pMp-Hilbert C*-module pM with the inner product $\langle x, y\rangle = x^*y$, the space $\mathcal{I}_p$ is what in [4] was called the unit sphere of the module (i.e. the set of elements x such that $\langle x, x\rangle = p$, the unit of pMp). In that paper it was proved that the unit sphere is connected when the algebra is finite, and therefore the action is transitive.

We introduce a metric in the tangent spaces of $\mathcal{I}_p$, the inner products defined are computed by means of the action of U_M, and in terms of the trace τ of M. The quadratic norms in the tangent spaces (which are real linear subspaces of M) are equivalent to the 2-norm of τ. Therefore the Riemannian metric is not complete, the tangent spaces are only pre-Hilbert spaces, and this setting differs from the classical Riemann–Hilbert theory of manifolds. In fact, the set $\mathcal{I}_p$ is not even a manifold in the strong operator topology. Nevertheless we carry on the geometric study of $\mathcal{I}_p$, with ad-hoc as well as classical tools. We consider the metric space $(\mathcal{I}_p, d_g)$, where d_g is the metric given by the infima of the lengths of curves joining two given points in $\mathcal{I}_p$, measured with the Riemannian metric.

In Section 2 we introduce the metric and compute its Levi-Civita connection. This connection was previously studied in the general setting of homogenous reductive spaces [15], where it was labeled the *classifying connection* of the homogeneous space. For instance the geodesics of this connection can be computed. Also in Section 2 we transcribe previous results [1,5,6] on the geometric structure of the unitary group in the strong operator topology. Our results on minimality of geodesics of the connection rely on these results, particularly on the convexity properties obtained in [6]. We prove that there exists a radius $R > 0$ such that for any $v \in \mathcal{I}_p$, the ball (in the usual norm $\|\ \|$ of M) $\mathcal{E}_v(R)$ centered at v has the following property: for any $v' \in \mathcal{E}_v(R)$ there exists a geodesic joining v and v', which is shorter than any other piecewise smooth curve $\nu(t)$ *inside* $\mathcal{E}_v(R)$ joining the same endpoints. Note that this result is not a Hopf–Rinow theorem, we were not able to prove that this geodesic is shorter than any other smooth curve (not contained in $\mathcal{E}_v(R)$) joining v and v'.

In Section 3 we prove that $(\mathcal{I}_p, d_g)$ is a complete metric space.

In $\mathcal{I}_p$, the strong operator topology is metrizable by the 2-norm given by the trace. This fact is certainly known, let us finish this introduction by giving a proof of it. Recall that

$$\|x\|_2 = \tau\big(x^*x\big)^{1/2}.$$

The usual norm of M will be denoted by $\|\ \|$.

Lemma 1.1. *The 2-norm $\|\ \|_2$ metrizes the strong operator topology in $\mathcal{I}_p$. The metric space $(\mathcal{I}_p, \|\ \|_2)$ is complete.*

Proof. Let $H = L^2(M, \tau)$ be the Hilbert space obtained by completion of $(M, \|\ \|_2)$, and suppose M represented in H (by left multiplication). Only for the purpose of this proof, the elements $x \in M$, when regarded as vectors in H, will be denoted by ξ_x, and when regarded as operators in H they will be denoted by l_x. Later on in the paper, these distinctions will not be needed. Note that the commutant of M in this representation consists of right multiplication operators r_x, for $x \in M$: r_x is the closure of $r_x\xi_y = \xi_{yx}$ $(y \in M)$. Suppose that $v_n \to v$ strongly in $\mathcal{I}_p$. Note that

$$\|v_n - v\|_2^2 = \tau\big((v_n - v)^*(v_n - v)\big) = 2\tau(p) - \tau\big(v_n^*v\big) - \tau\big(v^*v_n\big).$$

Then $\tau(v_n^*v) = \langle \xi_v, l_{v_n}\xi_1\rangle \to \langle \xi_v, l_v\xi_1\rangle = \tau(v^*v) = \tau(p)$ and $\tau(v^*v_n) = \overline{\tau(v_n^*v)} \to \tau(p)$, so that $\|v_n - v\|_2 \to 0$. Conversely, suppose that $\|v_n - v\|_2 \to 0$. Clearly this can be read as $\|l_{v_n}\xi_1 - l_v\xi_1\|_H \to 0$.

$$l_{v_n} r_x\xi_1 = r_x l_{v_n}\xi_1 \to r_x l_v\xi_1 = l_v r_x\xi_1.$$

The linear subspace $\{r_x\xi_1\colon x \in M\}$ is dense in H (ξ_1 is cyclic a separating for M), therefore $l_{v_n}\xi \to l_v\xi$ for all $\xi \in H$, because the sequence v_n is bounded in norm. Let us show now that $(\mathcal{I}_p, \| \|_2)$ is complete. Let v_n be a Cauchy sequence in $\mathcal{I}_p$ for the 2-norm. Then there exists $\xi \in H$ such that $\xi_{v_n} \to \xi$. Every element $\xi \in H$ defines a possibly unbounded operator l_ξ on H, whose domain includes $\{\xi_x\colon x \in M\}$: $l_\xi\xi_x = r_x\xi$. Then for all $x \in M$,

$$l_{v_n}\xi_x = r_x\xi_{v_n} \to r_x\xi_v = l_\xi\xi_x. \tag{1.1}$$

Since v_n are partial isometries with initial space p, this implies that

$$\|l_p\xi_x\| = \|l_{v_n}\xi_{px}\| \to \|l_\xi\xi_{px}\| = \|l_\xi l_p\xi_x\|.$$

It follows that l_ξ acts isometrically in $\{\xi_{px}\colon x \in M\}$ which is a dense linear subspace in the range of l_p. Analogously it can be proved that it acts trivially in $\{\xi_{(1-p)x}\colon x \in M\}$, which is dense in the range of $I - l_p$. Thus $l_\xi = l_v$ is a partial isometry with initial space p, and (1.1) implies that $v_n \to v$ strongly. □

2. The metric induced by the reductive structure

Denote by V_v the isotropy group of the action, i.e. the subgroup of unitaries which fix v:

$$V_v = \{w \in U_M\colon wv = v\},$$

and denote by $\mathcal{V}_v$ its Banach–Lie algebra,

$$\mathcal{V}_v = \{x \in M_{ah}\colon xv = 0\}.$$

Here M_{ah} denotes the real Banach space of anti-hermitian elements of M, which identifies with the Lie algebra of U_M. Denote by

$$L_v : U_M \to \mathcal{I}_p, \quad L_v(u) = uv$$

which is a submersion, and let

$$\ell_v = d(L_v)_1 : M_{ah} \to (T\mathcal{I}_p)_v, \quad \ell_v(x) = xv.$$

Note that the range of ℓ_v is $(T\mathcal{I}_p)_v = \{xv\colon x^* = -x\}$ and its kernel is $\mathcal{V}_v$. Also note that $y \in (T\mathcal{I}_p)_v$ satisfies $y^*v + v^*y = 0$, which is obtained by differentiating $v^*v = p$.

There is a natural reductive structure for this homogeneous space [11,18]: a smooth distribution of horizontal spaces $\{\mathcal{H}_v\colon v \in \mathcal{I}_p\}$ which are supplements for $\mathcal{V}_v$, $v \in \mathcal{I}_p$:

$$\mathcal{H}_v \oplus \mathcal{V}_v = M_{ah},$$

and are invariant under the inner action of V_v,

$$w\mathcal{H}_v w^* = \mathcal{H}_v, \quad w \in V_v.$$

Let us choose these supplements $\mathcal{H}_v$. Denote by $p^v = vv^*$ the final projection of v. An anti-hermitian element $x \in \mathcal{V}_v$ satisfies $xv = 0$ or equivalently $xp^v = 0$, and therefore if one represents it as a 2×2 matrix (in terms of p^v), it is of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & x_0 \end{pmatrix}$$

with x_0 anti-hermitian. A natural supplement for $\mathcal{V}_v$, which is clearly τ-orthogonal to $\mathcal{V}_v$, is the space $\mathcal{H}_v$ which consists of anti-hermitian elements of M whose matrices, in terms of p^v, are of the form

$$\begin{pmatrix} z_{11} & z_{12} \\ -z_{12}^* & 0 \end{pmatrix}.$$

That is

$$\mathcal{H}_v = \{z \in M\colon z^* = -z,\ (1 - p^v)z(1 - p^v) = 0\}.$$

We introduce an incomplete Riemannian metric in $\mathcal{I}_p$ by means of the trace τ, and compute its Levi-Civita connection. The relevant data which encode all the information of the reductive structure are the coordinate maps

$$\kappa_v : (T\mathcal{I}_p)_v \to \mathcal{H}_v,$$

which are the inverses of the isomorphisms

$$\ell_v|_{\mathcal{H}_v} : \mathcal{H}_v \to (T\mathcal{I}_p)_v.$$

Explicitly

$$\kappa_v(x) = xv^* - vx^* - p^v xv^*.$$

Consider in $(T\mathcal{I}_p)_v$ the inner product which makes the map κ_v an isometry, when $\mathcal{H}_v \subset M_{ah}$ is considered with the trace inner product

$$\langle x, y\rangle_v = \tau\big(\kappa_v(y)^*\kappa_v(x)\big), \quad x, y \in (T\mathcal{I}_p)_v. \tag{2.1}$$

Therefore

$$\|x\|_v = \big\|\kappa_v(x)\big\|_2.$$

These norms can be computed explicitly, and they equal (after routine calculations, which involve the identities $xp = x$ and $x^*v + v^*x = 0$, valid for $x \in (T\mathcal{I}_p)_v$):

$$\|x\|_v^2 = 2\tau\big(x^*x\big) - \tau\big(p^v xx^* p^v\big). \tag{2.2}$$

Note that this metric in $(T\mathcal{I}_p)_v$ is equivalent to the trace inner product metric, though we claim that it is geometricaly more relevant for the homogeneous structure. Indeed,

$$\|x\|_v^2 = \tau\big(x^*x\big) + \big(\tau\big(xx^*\big) - \tau\big(p^v xx^* p^v\big)\big) = \tau\big(x^*x\big) + \tau\big(x^*\big(1 - p^v\big)x\big).$$

Therefore $\|x\|_v = \|(2 - p^v)^{1/2}x\|_2$, and thus

$$\|x\|_2 \leqslant \|x\|_v \leqslant \sqrt{2}\|x\|_2. \tag{2.3}$$

In [15], Mata-Lorenzo and Recht introduced the classifying connection of a homogeneous reductive space. One of the main properties of this connection is that it has trivial torsion. Denote by P_v the map

$$P_v : M_{ah} \to M_{ah}, \quad P_v = \kappa_v \circ \ell_v.$$

Note that P_v takes values in $\mathcal{H}_v$, and since $\ell_v \circ \kappa_v$ restricted to $(T\mathcal{I}_p)_v$ is the identity map, P_v is an idempotent. Explicitly

$$P_v(x) = xp^v - p^v x^* - p^v xp^v = 2xp^v - p^v xp^v.$$

This idempotent P_v is in fact the orthogonal projection onto $\mathcal{H}_v$, with respect to the trace inner product: if $x, y \in M_{ah}$,

$$\tau\big(y^* P_v(x)\big) = \tau\big(y^* 2xp^v - y^* p^v xp^v\big) = -\tau\big(\big(2p^v y - p^v yp^v\big)x\big) = \tau\big(P_v(y)^* x\big).$$

The classifying connection ∇^c is given as follows. Suppose that x, y are tangent vector fields in $\mathcal{I}_p$, then $\nabla^c_x(y)$ is characterized by the value $\kappa_v(\nabla^c_x(y))$ at each point v,

$$\kappa_v\big(\nabla^c_x(y)\big) = P_v\big(x(y)\big), \tag{2.4}$$

where $x(y)$ denotes the derivative of y along x.

We shall prove next that this connection is the Levi-Civita connection of the metric (2.1) introduced above. By this we mean that it is symmetric (torsion free) and compatible with the metric.

Lemma 2.1. *The classifying connection ∇^c is the Levi-Civita connection of the metric $\langle\, ,\rangle_v$ in $\mathcal{I}_p$.*

Proof. It was proven in [15] that it is symmetric, and we now show that it is compatible with the metric. Let $x(t)$, $y(t)$ be two tangent fields along the curve $v(t)$ in $\mathcal{I}_p$. Then

$$\left\langle \frac{D^c x}{dt}, y\right\rangle_v = \tau\big(\kappa_v(y)^* P_v\big(\dot{\kappa_v}(x)\big)\big).$$

Note that since P_v is orthogonal with respect to the trace inner product, and projects onto $\mathcal{H}_v$, it follows that

$$\tau\big(\kappa_\nu(y)^* P_\nu\big(\dot{\kappa_\nu}(x)\big)\big) = \tau\big(\kappa_\nu(y)^* \dot{\kappa_\nu}(x)\big),$$

and similarly for the term $\langle x, \frac{D^c y}{dt}\rangle_\nu$. Then

$$\begin{aligned}\left\langle \frac{D^c x}{dt}, y\right\rangle_\nu + \left\langle x, \frac{D^c y}{dt}\right\rangle_\nu &= \tau\big(\kappa_\nu(y)^*\dot{\kappa_\nu}(x)\big) + \tau\big(\dot{\kappa_\nu}(y)^*\kappa_\nu(x)\big) \\ &= \frac{d}{dt}\big(\tau\big(\kappa_\nu(y)^*\kappa_\nu(x)\big)\big) = \frac{d}{dt}\big(\langle x, y\rangle_\nu\big). \qquad \square\end{aligned}$$

The geodesics of this connection are computed explicitly in [15]. The unique curve δ in $\mathcal{I}_p$ with $\delta(0) = v_0$ and $\dot{\delta}(0) = xv$ is given by

$$\delta(t) = e^{t\kappa_v(x)} v.$$

3. Minimality of geodesics

If ν is a smooth curve in $\mathcal{I}_p$ with $\nu(0) = v$, there is a unique smooth curve γ in U_M with the following properties

1. The curve γ lifts ν: $\gamma(t)v = \nu(t)$.
2. $\gamma(0) = 1$.
3. $\gamma^*\dot{\gamma} \in \mathcal{H}_\nu$.

This curve γ is called the *horizontal lifting* of ν, and is also characterized as the unique solution of the following linear differential equation

$$\begin{cases}\dot{\gamma} = \kappa_\nu(\dot{\nu})\gamma, \\ \gamma(0) = 1.\end{cases} \tag{3.1}$$

These are standard facts from the theory of homogeneous reductive spaces [15].

We shall need the following facts concerning the geometric structure of the unitary group U_M, which we take from [1] and [6]. The first fact states that if we measure the length of a smooth curve $\gamma(t)$ of unitaries, $t \in [a, b]$, using the norm $\|\ \|_2$, i.e.

$$L_2(\gamma) = \int_a^b \big\|\dot{\gamma}(t)\big\|_2\, dt,$$

then the curves of the form $\delta(t) = ue^{tx}$, with $x \in M_{ah}$, have minimal length along their paths for $t \in [0, 1]$, provided that $\|x\| \leqslant \pi$. If $\|x\| < \pi$, the curve δ is unique having this property. Note the fact that the condition is given on the *operator* norm of x, but the length is measured in the 2-norm. These norms are not equivalent, so this result is a weak form of a Hopf–Rinow theorem. We remark that U_M is not a Hilbert–Riemann manifold with the trace inner product. See [1] for the details.

A straightforward consequence of the definition of the metric in $\mathcal{I}_p$ is that the length of a curve ν in $\mathcal{I}_p$ coincides with the length of its horizontal lifting γ. Indeed,

$$L_2(\nu) = \int_0^1 \|\dot{\nu}\|_\nu\, dt = \int_0^1 \big\|\kappa_\nu(\dot{\nu})\big\|_2\, dt = \int_0^1 \big\|\dot{\gamma}\gamma^*\big\|_2\, dt = \int_0^1 \|\dot{\gamma}\|_2\, dt = L_2(\gamma).$$

Thus we shall use the metric structure of the unitary group with the trace metric.

The next results on the metric geometry of U_M concern a variation formula for the energy functional and the local convexity property of the geodesic distance (i.e. the distance given by the minima of lengths of curves joining two given unitaries, which by the above cited result, are achieved by one parameter groups of unitaries). These facts were proved in [6].

In Theorem 2.1 of [6] it was shown that if F_2 denotes the energy functional

$$F_2(\gamma) = \int_0^1 \|\dot\gamma\|_2^2 \, dt = \int_0^1 \tau\big(\dot\gamma^*\dot\gamma\big)\, dt,$$

for γ a piecewise smooth curve in U_M, and $\gamma_s(t)$ is a smooth variation of γ, i.e.

$$\gamma_s(t) \in U_M, \quad s \in (-r, r),\ t \in [0,1], \qquad \gamma_0 = \gamma,$$

then the *first variation of the energy functional* is

$$\frac{1}{2}\frac{d}{ds}F_2(\gamma_s)\Big|_{s=0} = \tau(x_0 y_0)\big|_{t=0}^{t=1} - \int_0^1 \tau\left(\frac{d}{dt}[x_0]y_0\right) dt, \tag{3.2}$$

where

$$x_s(t) = \gamma_s(t)^* \frac{d}{dt}\gamma_s(t) \quad \text{and} \quad y_s(t) = \gamma_s(t)^* \frac{d}{ds}\gamma_s(t).$$

The other result in [6] needed here is the following lemma. Here d_2 denotes the geodesic distance induced by the 2-norm.

Lemma 3.1. *(See [5, Theorem 4.5].) Let $u_0, u_1, u_2 \in U_M$, such that $\|u_i - u_j\| < \sqrt{2-\sqrt{2}} = r$. Let $\delta(t) = u_1 e^{tz}$ be the minimal geodesic joining u_1 and u_2. Then $f(s) = d_2(u_0, \delta(s))^2$ is a convex function ($s \in [0,1]$).*

Let us return now to $\mathcal{I}_p$.

Remark 3.2. By virtue of the inverse function theorem, the exponential map

$$\exp_v : \mathcal{H}_v \to \mathcal{I}_p$$

is a local diffeomorphism near the origin. For $r > 0$, denote by $B_r(v)$ the ball

$$B_r(v) = \big\{v' \in \mathcal{I}_p\colon\ \|v' - v\| < r\big\}.$$

Then there exists a number $R_0 > 0$, and an open set $\mathcal{E}_v(R_0)$ (in the norm topology), $0 \in \mathcal{E}_v(R_0) \subset \mathcal{H}_v$, such that

$$\exp_v : \mathcal{E}_v(R_0) \to B_{R_0}(v)$$

is a diffeomorphism. Moreover, one may adjust R_0, such that for any element $v_1 \in B_{R_0}(v)$, there exists a unique geodesic δ of $\mathcal{I}_p$ inside $B_{R_0}(v)$ with $\delta(0) = v$ and $\delta(1) = v_1$. Namely, $v_1 = \exp_v(z)$, for a unique $z \in \mathcal{E}_v(R_0)$, and $\delta(t) = \exp_v(tz) = e^{tz}v$.

Note that R_0 does not depend on v, due to the fact that the action of U_M on $\mathcal{I}_p$ is isometric.

Denote by L_∞ the length of curves, either of unitaries or partial isometries, measured with the operator norm $\|\ \|$: if $\alpha(t)$, $t \in [a,b]$, is a curve (in U_M or $\mathcal{I}_p$)

$$L_\infty(\alpha) = \int_a^b \|\dot\alpha\|\, dt.$$

Lemma 3.3. *Let $\nu(t)$ be a smooth curve in $\mathcal{I}_p$, $t \in [0,1]$, and let γ be its horizontal lifting. Then*

$$L_\infty(\gamma) \leqslant \sqrt{2}L_\infty(\nu).$$

Proof. Note that

$$L_\infty(\gamma) = \int_0^1 \|\dot\gamma\|\, dt = \int_0^1 \big\|\kappa_\nu(\dot\nu)\big\|\, dt.$$

Thus it suffices to show that for any tangent vector $x \in (T\mathcal{I}_p)_v$, $\|\kappa_v(x)\| \leqslant \sqrt{2}\|x\|$. Recall that $\kappa_v(x) = xv^* - vx^* - p^v x v^* = (1-p^v)xv^* - vx^*$. Since p^v is the range projection of v, it follows that the operators $(1-p^v)xv^*$ and vx^* have orthogonal ranges. Then, if ξ is a unit vector in any representation of M,

$$\|\kappa_v(x)\xi\|^2 = \|(1-p^v)xv^*\xi\|^2 + \|vx^*\xi\|^2$$

so that

$$\|\kappa_v(x)\|^2 \leqslant \|(1-p^v)xv^*\|^2 + \|vx^*\|^2 \leqslant 2\|x\|^2. \qquad \square$$

Lemma 3.4. *Let $\nu(t)$ be a smooth curve in $\mathcal{I}_p$, $t \in [0,1]$, joining v and $e^z v$, $z^* = -z$. Suppose that $\|z\| < \pi/4$ and $L_\infty(\nu) < \pi/4\sqrt{2}$. Denote by γ the horizontal lifting of ν. Then 1, e^z and $\gamma(1)$ lie at (norm) distance less than $\sqrt{2-\sqrt{2}}$.*

Proof. First note that

$$\|1 - e^z\| = \sqrt{2 - 2\cos(\|z\|)} < \sqrt{2-\sqrt{2}}.$$

Also it is clear that $L_\infty(\gamma) \leqslant \sqrt{2}L_\infty(\nu) < \sqrt{2-\sqrt{2}}/2$. Let d_∞ denote the geodesic distance in U_M given by the usual norm. It follows that $d_\infty(1, \gamma(1)) < \sqrt{2-\sqrt{2}}/2$. Therefore $\gamma(1) = e^y$, with $y^* = -y$ and $\|y\| < \sqrt{2-\sqrt{2}}/2 < \pi/4$. Then, similarly as above,

$$\|1 - \gamma(1)\| = \|1 - e^y\| = \sqrt{2 - 2\cos(\|y\|)} < \sqrt{2-\sqrt{2}}.$$

Finally

$$d_\infty(\gamma(1), e^z) \leqslant d_\infty(\gamma(1), 1) + d_\infty(1, e^z) \leqslant \|y\| + \|z\| < \sqrt{2-\sqrt{2}} < \pi/4.$$

Then there exists $w \in M_{ah}$, $\|w\| < \sqrt{2-\sqrt{2}} < \pi/4$ such that $\gamma(1) = e^z e^w$, and then again

$$\|\gamma(1) - e^z\| = \|1 - e^w\| < \sqrt{2-\sqrt{2}}. \qquad \square$$

The next result states that among curves in $\mathcal{I}_p$ which are sufficiently short for the L_∞ functional, the geodesics have minimal L_2 length. Let us refine our choice of R_0. Namely, if necessary we adjust it in order that

1. $R_0 < \pi/4\sqrt{2}$.
2. If $\|v - v'\| < R_0$, then the unique $z \in \mathcal{H}_v$ such that $e^z v = v'$ verifies that $\|z\| < \sqrt{2-\sqrt{2}}/2$.

Again, these further restricions on R_0 are uniform in $\mathcal{I}_p$ (independent of v).

Lemma 3.5. *Let ν be a smooth curve in $\mathcal{I}_p$ such that $L_\infty(\nu) < R_0$. Then there exists a geodesic δ with the same endpoints as ν, such that*

$$L_2(\delta) \leqslant L_2(\nu).$$

Proof. Put $v = \nu(0)$. Note that $L_\infty(\nu) < R_0$ implies in particular that $\nu(1) \in B_{R_0}(v)$, and therefore there exists $z \in \mathcal{H}_v$ with $\|z\| < \sqrt{2-\sqrt{2}}/2$ such that $\exp_v(z) = e^z v = \nu(1)$. Let γ be the horizontal lifting of ν. By the lemma above, the unitaries $\gamma(1)$, e^z and 1 differ in norm by less than $\sqrt{2-\sqrt{2}}$. Let $\mu(s) = e^z e^{sy}$ be the minimal geodesic (for the $\|\ \|_2$ metric) of U_M joining $\mu(0) = e^z$ and $\mu(1) = e^z e^y = \gamma(1)$ (note that indeed $\|y\| < \pi$).

Consider the map

$$f(s) = d_2(1, \mu(s))^2, \quad s \in [0,1].$$

By (3.1), $f(s)$ is convex. We claim that $f'(0) = 0$, so that f has an absolute minimum at $s = 0$.

Indeed, note that

$$\|\mu(s)-1\| = \|1-e^{sy}\| \leqslant \|1-e^{y}\| = \|e^{z}-\gamma(1)\| < \sqrt{2-\sqrt{2}} < 2.$$

Therefore the anti-hermitian logarithm

$$\log : \{u \in U_M : \|u-1\| < 2\} \to \{x \in M_{ah} : \|x\| < \pi\}$$

is well defined and smooth. Let $\epsilon_s(t) = e^{t\log(\mu(s))}$. Note that $\epsilon_s(t)$ is a smooth variation of $\epsilon_0 = e^{tz}$. Also note that for each fixed $s \in [0,1]$, $\epsilon_s(t)$ consists of minimizing geodesics, because $\|\log(\mu(s))\| < \pi$. Then

$$f(s) = L_2(\gamma_s)^2 = \|\log(\mu(s))\|_2^2 = F_2(\epsilon_s).$$

Then $f'(0)$ can be computed using the first variation formula (3.2). In our case

$$x_s = \epsilon_s^* \frac{d}{dt}\epsilon_s(t) = \log(\mu(s))$$

is independent of t, and therefore (3.2) reduces to

$$f'(0) = 2\{\tau(zy_0(1)) - \tau(zy_0(0))\}.$$

Note that $\gamma(1)$ and e^z verify that $\gamma(1)v = e^z v = \nu(1)$, i.e. $e^{-z}\gamma(1) \in V_v$. Also note that $V_v \subset U_M$ is geodesically convex: the minimal geodesic joining 1 and $e^{-z}\gamma(1)$, namely e^{sy}, lies in V_v. Therefore $y \in \mathcal{V}_v$.

At $t=0$, $\epsilon_s(0) = 1$ for all s, therefore $y_s(0) = 0$, and then $\tau(zy_0(0)) = 0$. At $t=1$, $\epsilon_s(1) = \mu(s)$, so that $y_s(1) = \mu^*(s)\dot{\mu}(s)$ and $y_0(1) = y$. Then

$$\tau(zy_0(1)) = \tau(zy) = 0$$

because $z \in \mathcal{H}_v$ and $y \in \mathcal{V}_v$ are τ-orthogonal subspaces.

Thus $f(0) \leqslant f(1)$, and therefore

$$L_2(\epsilon_0) = d_2(1, e^z) = f(0)^{1/2} \leqslant f(1)^{1/2} \leqslant d_2(1, \gamma(1)) = L_2(\gamma).$$

Since ϵ_0 and γ are the horizontal liftings of δ and ν, respectively, we have that

$$L_2(\delta) \leqslant L_2(\nu). \qquad \square$$

We call a piecewise smooth curve a geodesic polygon if it is a continuous path in $\mathcal{I}_p$, whose segments are geodesic paths. The next result states that given any piecewise smooth curve, there is a geodesic polygon joining the same endpoints which is shorter than the original curve.

Proposition 3.6. *Let $\nu(t)$, $t \in [0,1]$ be a piecewise smooth curve, then there is a geodesic polygon ρ such that $\rho(0) = \nu(0)$, $\rho(1) = \nu(1)$ and*

$$L_2(\rho) \leqslant L_2(\nu).$$

Proof. Clearly it suffices to consider the case when ν is smooth. Then there exists a partition $t_0 = 0 < t_1 < \cdots < t_n + 1$ of the unit interval such that for all $i = 1, \ldots, n$, $\|\nu(t_i) - \nu(t_{i-1})\| < R_0$. Thus each curve $\nu|_{[t_{i-1},t_i]}$ satisfies the hypothesis of the previous lemma, ans therefore there exists a geodesic ρ_i with the same endpoints as $\nu|_{[t_{i-1},t_i]}$ with

$$L_2(\rho_i) \leqslant L_2(\nu|_{[t_{i-1},t_i]}).$$

Clearly the curve ρ which consists of adjoining the paths ρ_i, is a continuous polygon which is shorter than ν. $\square$

To prove our main result on minimality of geodesics, we need to the following result, which establishes the local equivalence between the operator norm $\| \ \|$ and the geodesic distance d_∞ induced by this norm, in the space $\mathcal{I}_p$.

Proposition 3.7. *One may further adjust R_0 in order that if $v, v' \in \mathcal{I}_p$ with $\|v - v'\| < R_0$, then*

$$\|v - v'\| \leqslant d_\infty(v, v') \leqslant 2\|v - v'\|.$$

Proof. The first inequality is obvious, if one regards $\mathcal{I}_p$ as a subset of M. Let $z \in \mathcal{H}_v$ such that $\epsilon(t) = e^{tz}v$, $t \in [0, 1]$ is the geodesic joining v and v'. Clearly

$$d_\infty(v, v') \leqslant L_\infty(\epsilon) = \|zv\|.$$

On the other hand,

$$\begin{aligned}\|v - v'\| &= \big\|\big(e^z - 1\big)v\big\| \geqslant \|zv\| - \left\|\frac{1}{2}z^2v + \frac{1}{6}z^3v + \cdots\right\| \\ &= \|zv\| - \left\|\left(\frac{1}{2}z + \frac{1}{6}z^2 + \cdots\right)zv\right\| \geqslant \|zv\|\left(1 - \left\|\frac{1}{2}z + \frac{1}{6}z^2 + \cdots\right\|\right).\end{aligned}$$

Clearly $\|\frac{1}{2}z + \frac{1}{6}z^2 + \cdots\| \leqslant \frac{1}{2}\|z\| + \frac{1}{6}\|z\|^2 + \cdots$, which can be made smaller than $\frac{1}{2}$ if $\|z\|$ is small. Also it is clear that this can be achieved adjusting R_0. Thus

$$\|v - v'\| \geqslant \frac{1}{2}\|zv\| \geqslant \frac{1}{2}d_\infty(v, v'). \qquad \square$$

In what follows we choose R_0 so that, as in the proof above, if $\|v - v'\| < R_0$, then

$$L_\infty(\epsilon) \leqslant 2\|v - v'\|, \tag{3.3}$$

where $\epsilon(t) = ve^{tz}$ is the unique geodesic with $\epsilon(1) = v'$.

Theorem 3.8. *Given any point $v' \in B_{R_0/2}(v)$, there exists a unique geodesic $\delta(t) = e^{tz}v$ with $\delta(1) = e^z v = v'$ which is shorter than any other piecewise smooth curve $\nu \subset B_{R_0/2}(v)$ joining v and v'.*

Proof. Clearly, by Remark 3.2, there exists $z \in \mathcal{H}_v$ such that $e^z v = v'$. As in the proof of the previous Proposition 3.6, we may obtain a geodesic polygon with vertices $v_0 = v, v_1, \dots, v_n = v'$ in $B_{R_0/2}(v)$, with segments of L_∞-length less than R_0, which is shorter than ν for the L_2 functional. Let us denote by ϵ_i the segments of this polygon (joining v_{i-1} and v_i), and by $\epsilon_i \# \epsilon_{i+1}$ the path formed by adjoining two consecutive segments. Note that

$$L_\infty(\epsilon_1 \# \epsilon_2) \leqslant L_\infty(\epsilon_1) + L_\infty(\epsilon_2) < R_0.$$

Therefore, by Lemma 3.5, the geodesic polygon which results from replacing $\epsilon_1 \# \epsilon_2$ by the minimal geodesic ϵ_1' joining v_0 and v_2 in the original polygon, is shorter for the L_2 functional. Note that the new segment ϵ_1' lies inside $B_{R_0/2}(v)$. Moreover, by (3.3) above,

$$L_\infty\big(\epsilon_1'\big) \leqslant 2\|v - v_2\| < 2\frac{R_0}{2} = R_0.$$

Inductively, one arrives at the desired result. $\square$

We say that a curve $\nu(t)$, $t \in [0, 1]$, is minimal along its path if for any $0 \leqslant t_0 < t_1 \leqslant 1$, the curve $\nu|_{[t_0,t_1]}$ has minimal length among all curves in $\mathcal{I}_p$ joining $\nu(t_0)$ and $\nu(t_1)$. Note that we have proved that the geodesics of the connection that are minimal, are in fact minimal along their paths.

Next let us show that the geodesics of the connection are the only possible curves which have minimal length.

Theorem 3.9. *Suppose that ν is a piecewise smooth curve which has minimal L_2-length among smooth curves in $\mathcal{I}_p$ joining the same endpoints. Then ν is a geodesic of the linear connection.*

Proof. By the previous proposition, there exists a geodesic polygon ϵ with segments ϵ_i, $i = 1, \dots, n$, which is not longer than ν. Then ϵ is also minimal. Moreover, by construction, the elements $t_0, t_1, \dots, t_n$ of the partition provide points $\nu(t_i)$ which lie both in ν and ϵ. We claim that ϵ is smooth, i.e. a geodesic. This would prove our result, since the partition can be arbitrarily refined to contain as many (finite) points in common between ν and ϵ, and the smoothness of ϵ proves that all these polygons are in fact the same geodesic, which coincides with ν.

We assume that ϵ is not smooth to arrive to a contradiction. Namely, suppose that there are points v_i occurring at instants t_i of the polygon ϵ where $\dot\epsilon(t_i^-) \neq \dot\epsilon(t_i^+)$. The segments ϵ_i and ϵ_{i+1} of ϵ at the vertex v_i can be parametrized $\epsilon_i(t) = e^{tz_i^+} v_i$ and $\epsilon_{i+1}(t) = e^{tz_i^-} v_i$, with $z_i^+, z_i^- \in \mathcal{H}_{v_i}$. Then the jumps of the derivative of ϵ are

$$0 \neq \Delta\dot\epsilon_i = \dot\epsilon_{i+1}\big(0^+\big) - \dot\epsilon_i\big(1^-\big) = z_i^+ v_i - z_i^- v_i.$$

We may choose a variation γ_s of $\epsilon = \gamma_0$ which is constantly identical to ϵ except in small neighbourhoods of v_i (so that they do not overlap), and such that the variation field $V(t) = \frac{\partial}{\partial s}|_{s=0}\gamma_s$ equals these jumps (which are tangent vectors) at $t = t_i$, namely $V(t_i) = \Delta\dot\epsilon_i$.

According to the classic first variation formula in a Riemannian manifold (cf. [13], for example),

$$\frac{d}{ds}\Big|_{s=0} L_2(\gamma_s) = -\int_0^1 \big\langle V, D_t^c\dot\gamma\big\rangle dt - \sum \big\langle V(t_i), \Delta_i\dot\gamma\big\rangle.$$

Since ϵ consists of piecewise geodesics, and is also a critical point of the length distance (it is minimizing for the 2-metric) we obtain

$$0 = \frac{d}{ds}\Big|_{s=0} L_2(\gamma_s) = \sum_i \big\langle z_i^+ v_i - z_i^- v_i, z_i^+ v_i - z_i^- v_i\big\rangle_{v_i}^{1/2} = \sum_i \big\|z_i^+ - z_i^-\big\|_2,$$

which is a contradiction. □

4. Completeness of the Riemannian metric

In this section we shall prove that the geodesic distance d_g induced by the Riemannian metric is complete. This is an interesting fact, given that the tangent spaces of $\mathcal{I}_p$ are themselves not complete. This fact is related to the completness of the space

$$\mathcal{P} = \big\{q \in M\colon\ q^2 = q^* = q\big\}$$

of selfadjoint projections of M, also called the Grassmannian of M. The Grasmannian of a C*-algebra is a well-studied space, and there are several papers considering the geometric aspects of this set: [8,7,16]. In [5] we considered the Grassmannian of a finite von Neumann algebra, and endowed their tangent spaces with the incomplete Riemannian metric given by the 2-norm, in the same spirit as in the present paper. Let us list a few properties obtained there:

1. The Levi-Civita connection of the norm $\| \ \|_2$ at every tangent space of $\mathcal{P}$ is the reductive connection introduced in [8]. The geodesics of this connection are the curves of the form

   $$\rho(t) = e^{tx} q e^{-tx},$$

   where $x^* = -x$ is co-diagonal with respect to q: $x = xq + qx$.
2. Given two projections q_1, q_2, there is always a geodesic joining them.
3. The geodesic above can be chosen to be minimal. A minimal geodesic satisfies $\|x\| \leqslant \pi/2$. If $\|q_1 - q_2\| < 1$ (in general one has $\|q_1 - q_2\| \leqslant 1$), then the minimal geodesic is unique.
4. The geodesic distance (i.e. the metric given by the infima of lengths of curves in $\mathcal{P}$ joining the giving endpoints) is complete, and equivalent to the norm $\| \ \|_2$.

We shall use these facts to prove completeness of $(\mathcal{I}_p, d_g)$. There is one more property of $\mathcal{P}$ which shall be needed, and is not contained in [5]. It is the following sequential lifting property for the strong topology.

Proposition 4.1. *Let q_n be a sequence in $\mathcal{P}$ which converges strongly to q. Then there exist unitaries $u_n \in U_M$ such that*

$$u_n q u_n^* = q_n \quad \text{and} \quad u_n \to 1 \text{ strongly}.$$

Proof. Since the norm $\| \|_2$ metrizes the strong topology in $\mathcal{P}$, and is equivalent to the geodesic distance, there exist minimal geodesics $\rho_n(t) = e^{tx_n} q e^{-tx_n}$ joining q to q_n, such that

$$L_2(\rho_n) = \|x_n q - q x_n\|_2 \to 0.$$

Note that since $x_n = x_n q + q x_n$, then $\|x_n q - q x_n\|_2 = \|x_n\|_2$. Moreover, the norms $\|x_n\|$ are bounded by $\pi/2$. In [1], it was proven that if $\|x_n\| \leqslant \pi$, then $\|x_n\|_2 \to 0$ if and only if $e^{x_n} \to 1$ in U_M strongly. □

The link between these facts and the topology of $\mathcal{I}_p$ is the map

$$\Phi : \mathcal{I}_p \to \mathcal{P}, \quad \Phi(v) = vv^* = p^v \tag{4.1}$$

which assigns to each partial isometry its final projection. Since the algebra is finite, it is clearly strongly continuous. Also, when regarded as a map between the smooth structures (given by the usual norm) it is C^∞. For our purposes, the relevant property of Φ is the following:

Proposition 4.2. *The differential $d\Phi_v$ is contractive, when the tangent space of $\mathcal{I}_p$ is endowed with the Riemannian metric $\| \|_v$, and the tangent space of $\mathcal{P}$ is endowed with the norm $\| \|_2$.*

Proof. Note that

$$d\Phi_v(x) = xv^* + vx^*, \quad x \in (T\mathcal{I}_p)_v.$$

Then

$$\|d\Phi_v(x)\|_2^2 = \|xv^* + vx^*\|_2^2 = \tau(xpx^*) + \tau(xv^*xv^*) + \tau(vx^*vx^*) + \tau(vx^*xv^*).$$

Tangent vectors x at v satisfy $x^*v + v^*x = 0$ and also $xp = x$ (because $\mathcal{I}_p \subset Mp$). Therefore

$$\|d\Phi_v(x)\|_2^2 = 2\tau(xx^*) - 2\tau(p^v xx^* p^v) \leqslant 2\tau(xx^*) - \tau(p^v xx^* p^v) = \|x\|_v^2. \qquad \square$$

Our main result of this section follows:

Theorem 4.3. *The metric space $(\mathcal{I}_p, d_g)$ is complete.*

Proof. Let v_n be a Cauchy sequence in $\mathcal{I}_p$ for the geodesic distance. Then, by the above proposition, $q_n = v_n v_n^*$ is a Cauchy sequence for the geodesic distance in $\mathcal{P}$. Therefore there exists $q \in \mathcal{P}$ such that $q_n \to q$ strongly. By the sequential lifting property there exist $x_n^* = -x_n \in M$ with $\|x_n\| \leqslant \pi$ such that $e^{x_n} q e^{-x_n} = q_n$ and $\|x_n\|_2 \to 0$.

We claim that $e^{-x_n} v_n$ is also a Cauchy sequence in $\mathcal{I}_p$ for the geodesic distance. Indeed,

$$d_g(e^{-x_n} v_n, e^{-x_m} v_m) \leqslant d_g(e^{-x_n} v_n, v_n) + d_g(v_n, v_m) + d_g(v_m, e^{-x_m} v_m).$$

Note that $e^{-x_n} v_n$ can be joined to v_n by means of the curve $\nu_n(t) = e^{-tx_n} v_n$, whose length is

$$L_2(\nu_n) = \int_0^1 \|\dot{\nu}_n\|_{\nu_n} \, dt = \sqrt{2} \int_0^1 \|\dot{\nu}_n\|_2 \, dt = \sqrt{2}\|x_n v_n\|_2 \leqslant \sqrt{2}\|x_n\|_2.$$

It follows that $d_g(e^{-x_n} v_n, v_v) \leqslant L_2(\nu_n) \to 0$. Analogously, $d_g(e^{-x_m} v_m, v_m) \to 0$.

Note that $e^{-x_n} v_n$ are partial isometries with (initial space p and) final space equal to q:

$$e^{-x_n} v_n (e^{-x_n} v_n)^* = e^{-x_n} v_n v_n^* e^{x_n} = e^{-x_n} q_n e^{x_n} = q.$$

Since $q = upu^*$ for some unitary u, this subset of $\mathcal{I}_p$ is homeomorphic to the set $\mathcal{I}_p^p \subset \mathcal{I}_p$ of partial isometries with final and initial spaces equal to p, by means of

$$v \mapsto u^* v.$$

Note that in $\mathcal{I}_p^p$, the Riemannian metric of $\mathcal{I}_p$ coicides with the 2-norm: if $w \in \mathcal{I}_p^p$ and $x \in (T\mathcal{I}_p^p)_w$, then $p^w = p$ and $px = xp = x$. Thus

$$\|x\|_w^2 = 2\tau(x^*x) - \tau(p_w xx^* p^w) = \tau(x^*x) = \|x\|_2^2.$$

This set $\mathcal{I}_p^p$, clearly identifies isometrically with the unitary group of pMp. In [1] it was shown that the unitary group of a finite algebra is complete with the geodesic distance. Completeness of $\mathcal{I}_p$ follows. $\square$

A similar type of argument shows that the mapping

$$L_v : U_M \to \mathcal{I}_p, \quad L_v(u) = uv$$

has also the sequential lifting property for the strong topology. Note that in the norm topology it is a submersion, and therefore a fibration.

Proposition 4.4. *Let v_n be a sequence in $\mathcal{I}_p$ converging strongly to v. Then there exist unitaries $u_n \in U_M$ such that $u_n v = v_n$ and $u_n \to 1$ strongly. In particular, $\mathcal{I}_p$ is homeomorphic to U_M/V_v, when all the spaces involved are considered with the strong operator topology.*

Proof. The action of U_M on $\mathcal{I}_p$ is transitive, and for a fixed $u \in U_M$, the maps $\mathcal{I}_p \ni v \mapsto uv \in \mathcal{I}_p$ are homeomorphisms for the strong topology. Therefore we may suppose without loss of generality that $v = p$. Since M is finite and v_n are uniformly bounded, $q_n = v_n v_n^* \to p$ strongly. By the sequential lifting property in $\mathcal{P}$, there exist unitaries w_n in M which converge strongly to 1, such that $q_n = w_n p w_n^*$. One can choose $u_n = v_n + w_n(1-p)$. Note that $u_n \to 1$ strongly, and $u_n p = v_n p = v_n$. It remains to show that they are unitary elements:

$$u_n u_n^* = v_n v_n^* + w_n(1-p)w_n^* + v_n(1-p)w_n^* + w_n(1-p)v_n^* = q_n + 1 - q_n = 1,$$

because $v_n(1-p)w_n^* = 0 = (v_n(1-p)w_n^*)^* = w_n(1-p)v_n^*$.

The map $L_v : U_M \to \mathcal{I}_p$, which is continuous in the strong operator topology, induces a continuous bijection

$$\bar{L}_v : U_M/V_v \to \mathcal{I}_p,$$

when $\mathcal{I}_p$ is considered with the strong operator topology. The sequential lifting property implies that $\bar{L}_v$ is an open mapping, i.e. a homeomorphism. Indeed, if $v_n \to v$ strongly, then there exist unitaries u_n such that $u_n v = v_n$ and $u_n \to 1$ strongly. Therefore $[u_n] \to [1]$ in U_M/V_v. $\square$

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