MINIMIZATION OF CONVEX FUNCTIONALS OVER FRAME OPERATORS

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Abstract. We present results about minimization of convex functionals defined over a finite set of vectors in a finite dimensional Hilbert space, that extend several known results for the Benedetto-Fickus frame potential. Our approach depends on majorization techniques. We also consider some perturbation problems, where a positive perturbation of the frame operator of a set of vectors is realized as the frame operator of a set of vectors which is close to the original one.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space. A set of vectors $\mathcal{F} = \{\phi_i\}_{i\in I}$ in $\mathcal{H}$ is a *frame* if there exist a pair of constants $a, b > 0$ such that, for every $x \in \mathcal{H}$,

$$a\ \|x\|^2 \leq \sum_{i\in I} |\ \langle x\ ,\ \phi_i\rangle\ |^2 \leq b\ \|x\|^2. \tag{1}$$

The optimal constants a, b in (1) are called the *frame bounds*. We say that the frame is *tight* if $a = b$. In general, if the inequality on the right hand side of (1) holds for $x \in \mathcal{H}$ we say that $\mathcal{F}$ is a Bessel sequence. Given a Bessel sequence $\mathcal{F}$ we consider its *synthesis operator* $T^{\mathcal{F}} : l_2(I) \to \mathcal{H}$ defined as $T^{\mathcal{F}}(e_i) = \phi_i$, where $\{e_i\}_{i\in I}$ is the canonical orthonormal basis of $l_2(I)$. We also consider its *frame operator* given by $S^{\mathcal{F}} = T^{\mathcal{F}}(T^{\mathcal{F}})^*$ and its *Grammian*, defined by $G^{\mathcal{F}} = (T^{\mathcal{F}})^* T^{\mathcal{F}}$.

Frames where introduced by Duffin and Schaeffer [8] in their work on nonharmonic Fourier series. These were later rediscovered by Daubechies, Grossmann and Meyer in the fundamental paper [7]. In recent years the study of frames has increased considerably due to the wide range of applications in which frames play an important role. In this note we shall focus on finite frames i.e. $\mathcal{H} = \mathbb{F}^d$ where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ and I is a finite set. Note that in this setting, a frame is just a set of generators for $\mathcal{H}$.

In [4] Benedetto and Fickus introduced the notions frame force (FF) and frame potential (FP) for a finite frame. More explicitly they defined, for $\mathcal{F} = \{\phi_i\}_{i=1}^m \subseteq \mathcal{H}$ a finite sequence of vectors

$$\mathrm{FP}(\mathcal{F}) = \sum_{i,j=1}^{m} |\langle \phi_i, \phi_j\rangle|^2 = \mathrm{tr}((S^{\mathcal{F}})^2) \tag{2}$$

It is shown in [4] that the finite unit norm tight frames are the minimizers of the frame potential among all unit norm frames with a fixed number of vectors. If we

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now impose restrictions on the lengths of the vectors, the structure of minimizers changes since tight frames with a prescribed set of norms may not exist. The complete characterization of global and local minimizers for the frame potential was done in [6].

The equality $\mathrm{FP}(\mathcal{F}) = \mathrm{tr}((S^{\mathcal{F}})^2)$ suggests that, more generally, we can consider functionals of the form $P_f(\mathcal{F}) = \mathrm{tr}(f(S^{\mathcal{F}}))$, where f is a non-negative, non-decreasing and convex function defined on $[0, \infty)$. In this context, the problem of describing the geometrical structure of minimizers of these convex functionals arises; surprisingly, this structure does not depend on f. In order to state the following results we introduce the sets $\mathcal{A}(c) = \{\{\phi_i\}_{i=1}^m \subset \mathbb{C}^d,\ \sum_{i=1}^m \|\phi_i\|^2 = c\}$ and $\mathcal{B}(\mathbf{a}) = \{\{\phi_i\}_{i=1}^m \subset \mathbb{C}^d,\ \|\phi_i\|^2 = a_i$ for every $i\}$, where $\mathbf{a} = (a_i)_{i=1}^m$ is a non-increasing finite sequence of positive real numbers.

Theorem (A). *Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a convex function and P_f the functional associated to f. Let $c > 0$ and $\mathbf{a} = (a_i)_{i=1}^m$ be a non-decreasing finite sequence of positive real numbers. Then,*

a) *If $\mathcal{F} \in \mathcal{A}(c)$ is a tight frame then it is a global minimizer of P_f in $\mathcal{A}(c)$. If we assume further that f is strictly convex then every global minimizer in $\mathcal{A}(c)$ is tight.*

b) *If $\mathcal{F} \in \mathcal{B}(\mathbf{a})$ is of the form*

$$\{\sqrt{a_i}\, e_i\}_{i=1}^r \cup \{\phi_i\}_{i=r+1}^m \tag{3}$$

where $\{e_i\}_{i=1}^d$ is an o.n.b. for $\mathbb{C}^d$, r is the d-irregularity *of* $\mathbf{a}$ *(see definition 2.2 below) and $\{\phi_i\}_{i=r+1}^m$ is a tight frame for span $\{e_i\}_{i=r+1}^d$ then, it is a global minimizer of P_f. If we assume further that f is strictly convex then every global minimizer in $\mathcal{B}(\mathbf{a})$ is as in* (3) *for some o.n.b. $\{e_i\}_{i=1}^d$.*

It is also interesting to study the structure of the *local* minimizers of P_f in the previous sets $\mathcal{A}(c)$ and $\mathcal{B}(\mathbf{a})$. A natural metric in this context is the vector-vector distance $d(\mathcal{F}, \mathcal{G}) = \max_{1 \le i \le m} \|\phi_i - \psi_i\|$ for sequences $\mathcal{F} = \{\phi_i\}_{i=1}^m$, $\mathcal{G} = \{\psi_i\}_{i=1}^m$. But this characterization problem turns out to be quite difficult for the local minimizers of P_f in $\mathcal{B}(\mathbf{a})$. Hence, we alternatively consider the description of the structure of local minimizers of P_f in $\mathcal{R}(\mathbf{a}) = \{S^{\mathcal{F}},\ \mathcal{F} \in \mathcal{B}(\mathbf{a})\}$ endowed with the norm topology. Notice that this last point of view is weaker. Indeed, $\|S^{\mathcal{F}} - S^{\mathcal{G}}\| \le 2\sqrt{m}\, \max(\|T^{\mathcal{F}}\|, \|T^{G}\|)\, d(\mathcal{F}, \mathcal{G})$, where $T^{\mathcal{F}}$ and $S^{\mathcal{F}}$ denote the synthesis and frame operator of $\mathcal{F}$ (see the beginning of section 3), while there are pairs of different sequences that share the frame operator.

Theorem (B). *Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a non decreasing strictly convex function and P_f the functional associated to f. Let $c > 0$ and $\mathbf{a} = (a_i)_{i=1}^m$ be a non-decreasing finite sequence of positive real numbers. Then,*

a) *Every local minimizer of P_f in $\mathcal{A}(c)$ with respect to $d(\cdot, \cdot)$ is a tight frame and hence, a global minimizer.*

b) *Every local minimizer of P_f in $\mathcal{R}(\mathbf{a})$ with respect to the operator norm is of the form* (3) *for some o.n.b. $\{e_i\}_{i=1}^d$ of $\mathbb{C}^d$ and hence, a global minimizer.*

The previous results show that the structure of the local minimizers of P_f (when P_f is considered as a function of the frame operators) does not depend on the strictly convex function chosen. Unfortunately, we get only partial results related with the local minimizers of P_f in $\mathcal{B}(\mathbf{a})$ with respect to the vector-vector distance, for a general convex function f.

Our approach depends on solving some perturbation problems concerning the frame operator for a generic case of frame.

More explicitly, if $\mathcal{F}$ is a frame in $\mathcal{B}(\mathbf{a})$ which can not be partitioned in two mutually orthogonal sets of vectors (i.e. its Grammian is not block-diagonal) and S_i is a sequence in $\mathcal{M}_d(\mathbb{C})^+$ which converges to $S^{\mathcal{F}}$, then for every $\varepsilon > 0$ there exists i_0 such that, for $i \geq i_0$ there is a frame $\mathcal{G} \in \mathcal{B}(\mathbf{a})$ such that $S^{\mathcal{G}} = S_i$ and $d(\mathcal{F}, \mathcal{G}) \leq \varepsilon$. Our approach to this problem depends on differential geometric tools that we describe in an appendix at the end of the paper. In the particular case of the Benedetto-Fickus frame potential, we recover a theorem by Casazza et al. [6] describing its local minimizers.

The paper is organized as follows: Section 2 contains preliminary facts together with some new results about majorization of vectors in $\mathbb{R}^d$ that we shall need in the sequel; Propositions 2.1 and 2.3 give a characterization of minimal points of certain sets of vectors with respect to majorization. Section 3 is devoted to the basic facts about frames in $\mathbb{C}^d$ together with some previous results from [3] about some design problems for frames. In Section 4, some properties of the convex functions P_f defined on frame operators are given. In this section we consider the sets of frame operators $\mathcal{R}(c)$ and $\mathcal{T}(\mathbf{a})$, consisting of frame operators of elements in $\mathcal{A}(c)$ and $\mathcal{B}(\mathbf{a})$ respectively. Theorems 4.6 and 4.7 deal with the characterization of global and local minimizers for every P_f (for a non decreasing strictly convex function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$) on $\mathcal{R}(c)$ and $\mathcal{T}(\mathbf{a})$. At the end of this section, some examples and applications are given. Finally, in Section 5 we focus on the structure of minimizers of the functions P_f when they are defined on frames instead of frame operators. This leads to some geometrical problems which are developed in the Appendix.

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2. Preliminares

In this section we present some basic aspects of majorization theory together with some new results that we shall need in what follows. For a more detailed treatment of majorization see [10]. Given $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{R}^d$, denote by $\mathbf{b}^{\downarrow} \in \mathbb{R}^d$ the vector obtained by rearranging the coordinates of $\mathbf{b}$ in non increasing order. If $\mathbf{b}, \mathbf{c} \in \mathbb{R}^d$ then we say that $\mathbf{b}$ is *majorized* by $\mathbf{c}$, and write $\mathbf{b} \prec \mathbf{c}$, if

$$\sum_{i=1}^{k} b_i^{\downarrow} \leq \sum_{i=1}^{k} c_i^{\downarrow} \quad k = 1, \ldots, d-1 \ \text{ and } \ \sum_{i=1}^{d} b_i^{\downarrow} = \sum_{i=1}^{d} c_i^{\downarrow}.$$

Majorization is a preorder relation in $\mathbb{R}^d$ that occurs naturally in matrix analysis and plays an important role in convex optimization problems.

Proposition 2.1. *Let $c > 0$ and consider the set*

$$\mathcal{K}(c) = \{\mathbf{b} \in (\mathbb{R}_{\geqslant 0})^d : \ \sum_{i=1}^{d} b_i = c\} \tag{4}$$

Then the vector $\mathbf{v} = (\frac{c}{d}, \ldots, \frac{c}{d})$ satisfies $\mathbf{b} \succ \mathbf{v}$ for every $\mathbf{b} \in \mathcal{K}(c)$. Moreover, if $\mathbf{b} \in \mathcal{K}(c)$ is such that $\mathbf{b}^{\downarrow} \neq \mathbf{v}$, then for every $0 < \varepsilon$ sufficiently small, there exists $\mathbf{b}_\varepsilon \in \mathcal{K}(c)$ such that $\mathbf{b}^{\downarrow} \neq \mathbf{b}_\varepsilon^{\downarrow}$, $\mathbf{b} \succ \mathbf{b}_\varepsilon$ and $\|\mathbf{b}^{\downarrow} - \mathbf{b}_\varepsilon^{\downarrow}\| \leq \varepsilon$.

Proof. The first part is a well known fact about majorization, and it is easy to check. For the proof of the moreover part, suppose that $\mathbf{b} \in \mathcal{K}(c)$ is such that $\mathbf{b} \neq \mathbf{v}$, then there exists a index j, $1 \leq j \leq d$ such that $b_j^\downarrow > b_{j+1}^\downarrow$ were we denote by $b_i^\downarrow$ the entries of $\mathbf{b}^\downarrow$.

Let $0 < \varepsilon$ such that $b_j^\downarrow - \sqrt{\frac{\varepsilon}{2}} \geq b_{j+1}^\downarrow + \sqrt{\frac{\varepsilon}{2}}$ and denote by $\mathbf{b}_\varepsilon$ the vector $\mathbf{b}_\varepsilon = \mathbf{b}^\downarrow - \sqrt{\frac{\varepsilon}{2}}\,\mathbf{e}_j + \sqrt{\frac{\varepsilon}{2}}\,\mathbf{e}_{j+1}$ were $\{\mathbf{e}_i\}_{i=1}^d$ is the canonical basis in $\mathbb{R}^d$. Clearly $\mathbf{b}_\varepsilon \in \mathcal{K}(c)$, $\mathbf{b} \succ \mathbf{b}_\varepsilon$, and by construction of $\mathbf{b}_\varepsilon$, $\|\mathbf{b}^\downarrow - \mathbf{b}_\varepsilon^\downarrow\|^2 = \varepsilon$. $\square$

Following [6] we consider the d-irregularity of a sequence as follows

Definition 2.2. Let $\mathbf{a} = (a_i)_{i=1}^m$ be a non increasing sequence of positive numbers and $d \in \mathbb{N}$ with $d \leq m$. The d-irregularity of $\mathbf{a}$, denoted $r_d(\mathbf{a}) \in \mathbb{N}$, is defined as

$$r_d(\mathbf{a}) = \max\Big\{1 \leq j \leq d-1 : (d-j)a_j > \sum_{i=j+1}^m a_i\Big\},$$

if the set on the right is non empty, and $r_d(\mathbf{a}) = 0$ otherwise.

Notice that in particular, with the notations of Definition 2.2, we have:

(1) $(d-j)a_j \leq \sum_{i=j+1}^m a_i$, for $r_d(\mathbf{a}) < j \leq d$ whenever $r_d(\mathbf{a}) > 0$,

(2) $(d-j)a_j > \sum_{i=j+1}^m a_i$, for every $1 \leq j \leq r_d(\mathbf{a})$.

Proposition 2.3. *Let $0 < d \leq m$ and let $\mathbf{a} = (a_i)_{i=1}^m$ be a non increasing sequence of positive numbers with d-irregularity $r = r_d(\mathbf{a})$. Consider the set*

$$\mathcal{P}(\mathbf{a}) = \{\mathbf{b} \in (\mathbb{R}_{\geqslant 0})^d : \ \sum_{i=1}^k b_i^\downarrow \geq \sum_{i=1}^k a_i \ \text{ for } \ 1 \leq k \leq d \ \text{ and } \ \sum_{i=1}^d b_i = \sum_{i=1}^m a_i\}.$$

Let $\mathbf{v} = (a_1, \ldots, a_r, \overbrace{c, \ldots, c}^{d-r \text{ times}})$, where $c = (d-r)^{-1}\sum_{j=r+1}^m a_j$. Then $\mathbf{v}$ belongs to $\mathcal{P}(\mathbf{a})$ and, for every $\mathbf{b} \in \mathcal{P}(\mathbf{a})$, $\mathbf{b} \succ \mathbf{v}$. Moreover, if $\mathbf{b} \in \mathcal{P}(\mathbf{a})$ and $\mathbf{b}^\downarrow \neq \mathbf{v}$, then for every $0 < \varepsilon$ sufficiently small, there exists $\mathbf{b}_\varepsilon$ in $\mathcal{P}(\mathbf{a})$ such that $\mathbf{b}_\varepsilon^\downarrow \neq \mathbf{b}^\downarrow$, $\mathbf{b} \succ \mathbf{b}_\varepsilon$ and $\|\mathbf{b}^\downarrow - \mathbf{b}_\varepsilon^\downarrow\| \leq \varepsilon$.

Proof. By the comments after Definition 2.2, $\mathbf{v} = \mathbf{v}^\downarrow$. First, we show that $\mathbf{v} \in \mathcal{P}(\mathbf{a})$. Note that $\sum_{j=1}^k a_j = \sum_{j=1}^k v_j$ for $1 \leq j \leq r$. On the other hand,

$$a_{r+1}(d-r) - a_{r+1} = a_{r+1}(d-(r+1)) \leq \sum_{j=r+2}^m a_j \ \Rightarrow \ a_{r+1} \leq (d-r)^{-1} \sum_{j=r+1}^m a_j.$$

Therefore $c \geq a_{r+1} \geq a_j$ for every $r+1 \leq j \leq m$. Then, for every $r+1 \leq k \leq d$ we have

$$\sum_{j=1}^k v_j = \sum_{j=1}^r a_j + \sum_{j=r+1}^k c_j \geq \sum_{j=1}^k a_j.$$

Since $\sum_{j=1}^d v_j = \sum_{j=1}^m a_j$ it follows that $\mathbf{v} \in \mathcal{P}(\mathbf{a})$. Let $\mathbf{b} = (b_i)_{i=1}^d \in \mathcal{P}(\mathbf{a})$ and, without loss of generality, assume that $\mathbf{b} = \mathbf{b}^\downarrow$. Then, it is clear that $\sum_{j=1}^k v_j \leq \sum_{j=1}^k b_j$ for every $1 \leq k \leq r$. Let $\alpha = \sum_{j=1}^r b_j - \sum_{j=1}^r a_j \geq 0$. Therefore

$$\left(\sum_{j=1}^r b_j - \sum_{j=1}^r a_j\right) + \sum_{j=r+1}^d b_j = \sum_{j=r+1}^m a_j \ \Rightarrow \ \sum_{j=r+1}^d (b_j + (d-r)^{-1}\alpha) = \sum_{j=r+1}^m a_j \tag{5}$$

which implies, by Proposition 2.1, that $(c)_{i=r+1}^{d} \prec ((d-r)^{-1}\alpha + b_i)_{i=r+1}^{d} \in \mathbb{R}^{d-r}$. Then, for every $r+1 \leq k \leq d$ we have

$$\begin{aligned}\sum_{j=1}^{k} b_j &= \sum_{j=1}^{r} b_j - \sum_{j=r+1}^{k} (d-r)^{-1}\alpha + \sum_{j=r+1}^{k} (b_j + (d-r)^{-1}\alpha) \\ &\geq \sum_{j=1}^{r} b_j - \alpha + \sum_{j=r+1}^{k} c = \sum_{j=1}^{r} a_j + \sum_{j=r+1}^{k} c = \sum_{j=1}^{k} v_j .\end{aligned}$$

On the other hand

$$\sum_{j=1}^{d} b_j = \sum_{j=1}^{m} a_j = \sum_{j=1}^{d} v_j$$

so we see that $\mathbf{v} \prec \mathbf{b}$. For the second part, let $\mathbf{b} \in \mathcal{P}(\mathbf{a})$, $\mathbf{b}^{\downarrow} \neq \mathbf{v}$. Again we assume that $\mathbf{b} = \mathbf{b}^{\downarrow}$.

Claim: There exists j, $1 \leq j \leq d-1$ such that $b_j > b_{j+1}$ and $\sum_{i=1}^{j} b_i > \sum_{i=1}^{j} a_i$.

It is clear that for some $1 \leq k \leq d-1$, $b_k > b_{k+1}$. Otherwise, $b_i = b_1$ for all i which would imply that $\mathbf{b} = \mathbf{v}$ (the d-irregularity of a would be 0). Denote by $b_{t_1} \geq b_{t_2} \geq \ldots \geq b_{t_m}$ all the entries of $\mathbf{b}$ which satisfy $b_{t_n} > b_{t_n+1}$.

Suppose that, for every t_n, $\sum_{i=1}^{t_n} b_i = \sum_{i=1}^{t_n} a_i$. Then, since by hypothesis $kb_1 = \sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} a_i$ for all $k \leq t_1$, we have that $a_i = b_1 = b_i$ for all $i \leq t_1$. By the same reasoning, $a_i = b_{t_1+1} = b_i$ for all $t_1 + 1 \leq i \leq t_2$. Finally, we get that $a_i = b_i$ for all $1 \leq i \leq t_m$ moreover, $b_k = (d-t_m)^{-1} \sum_{i=t_m+1}^{m} a_i$ for $t_m + 1 \leq k$. The definition of the irregularity of $\mathbf{a}$ implies that $t_m \leq r$ (otherwise, the decreasing order of $\mathbf{b}$ would be violated), but if $t_m \leq r-1$, then by the comments following Def. 2.2,

$$a_{t_m+1} > (d-(t_m+1))^{-1} \sum_{i=t_m+2}^{m} a_i ,$$

which in turn implies that $a_{t_m+1} > (d-t_m)^{-1} \sum_{i=t_m+1}^{m} a_i = b_{t_m+1}$, which contradicts $\mathbf{b} \in \mathcal{P}(\mathbf{a})$. The only possible case is $t_m = r$, but in this case, $\mathbf{b} = \mathbf{v}$, a contradiction.

Now, given $1 \leq j \leq d-1$ such that $b_j > b_{j+1}$ and $\sum_{i=1}^{j} \mathbf{b}_i > \sum_{i=1}^{j} a_i$, let ε such that $b_j - \varepsilon/\sqrt{2} \geq b_{j+1} + \varepsilon/\sqrt{2}$ and $\sum_{i=1}^{j} b_i - \varepsilon/\sqrt{2} \geq \sum_{i=1}^{j} a_i$. Now, denote by $\mathbf{b}_\varepsilon$ the vector $\mathbf{b} - \varepsilon/\sqrt{2}\,\mathbf{e}_j + \varepsilon/\sqrt{2}\,\mathbf{e}_{j+1}$. Then is easy to see that $\mathbf{b}_\varepsilon$ satisfy the desired properties.

□

Remark 2.4. Note that the proof of the previous claim shows that the only vector $\mathbf{b}$ in $\mathcal{P}(\mathbf{a})$ such that: $\mathbf{b}^{\downarrow} = (a_1, a_2, \ldots, a_k, c, \ldots, c)$ is $\mathbf{v}$.

Finally, we consider the following extension of majorization to self-adjoint operators due to Ando [2] which will be useful for the study of convex functions on frame operators: given self-adjoint matrices B, $C \in \mathcal{M}_d(\mathbb{C})$ we say that B is majorized by C, and write $B \prec C$ if and only if $\lambda(B) \prec \lambda(C)$, where $\lambda(A) \in \mathbb{R}^d$ denotes the d-tuple of eigenvalues of a selfadjoint matrix $A \in \mathcal{M}_d(\mathbb{C})$ counted with multiplicity and arranged in decreasing order.

3. Preliminaries on frames

Let $\mathcal{H} = \mathbb{F}^d$, ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$), and let $\mathcal{F} = \{\phi_i\}_{i=1}^m$ be a set of vectors in $\mathcal{H}$, we say that $\mathcal{F}$ is a *frame* if there exist $a, b > 0$ such that for every vector x in $\mathcal{H}$

$$a\|x\|^2 \leq \sum_{i=1}^m |\langle x, \phi_i\rangle|^2 \leq b\|x\|^2 \tag{6}$$

the optimal bounds a and b are the upper and lower *frame bounds* for $\mathcal{F}$.

We can define the following bounded linear operator

$$T^{\mathcal{F}} : \mathbb{F}^m \to \mathcal{H}, \quad T^{\mathcal{F}}(e_i) = \phi_i,\ 1 \leq i \leq m$$

The positive semidefinite operators

$$G^{\mathcal{F}} := (T^{\mathcal{F}})^* T^{\mathcal{F}} \quad \text{and} \quad S^{\mathcal{F}} := T^{\mathcal{F}} (T^{\mathcal{F}})^*$$

are called *Grammian* and the *frame operator* respectively, of the sequence $\mathcal{F} = \{\phi_i\}_{i=1}^m$. Throughout this note we shall consider the matrices of those operators with respect to the canonical bases of $\mathbb{F}^m$ and $\mathbb{F}^d$, maintaining the notation . Thus, $S^{\mathcal{F}} \in \mathcal{M}_d(\mathbb{F})^+$ and $G^{\mathcal{F}} \in \mathcal{M}_m(\mathbb{F})^+$.

In particular, it can be seen that the upper and lower frame bound for $\mathcal{F}$ are the greatest and smallest positive eigenvalues of $S^{\mathcal{F}}$, denoted by λ_1 and λ_d respectively.

Proposition 3.1 ([3]). *Let $\mathcal{F} = \{\phi_i\}_{i=1}^m \subseteq \mathcal{H}$ and let G and S be the Grammian and frame operators of $\mathcal{F}$. Then, there exists a Hilbert space $\mathcal{H}_0$ with dimension $m - d$ and an isometric isomorphism $U : \mathbb{F}^m \to \mathcal{H} \oplus \mathcal{H}_0$ such that*

$$UGU^* = \begin{array}{cc} \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} & \begin{array}{c} \mathcal{H} \\ \mathcal{H}_0 \end{array} \end{array} \tag{7}$$

Therefore, $(\|\phi_i\|^2)_{i=1}^m \prec (\sigma(S^{\mathcal{F}}), 0_\sim)$ where $0_\sim \in \mathbb{R}^{m-d}$.

As a consequence of Proposition 3.1 we see that, if $\sigma(G) \in \mathbb{F}^m$ (resp $\sigma(S) \in \mathbb{C}^d$) denote the eigenvalues of G counted with multiplicity then $\sigma(G) = (\sigma(S), 0_\sim)$ where $0_\sim \in \mathbb{F}^{m-d}$.

Theorem 3.2 ([3, 12]). *Let $S \in \mathcal{M}_d(\mathbb{F})^+$ and let $\mathbf{a} = (a_i)_{i=1}^m$ be a sequence of positive numbers. Then, there exists a sequence $\{\phi_i\}_{i=1}^m \subset \mathcal{H}$ with frame operator S and such that $\|\phi_i\| = a_i$ for every $1 \leq i \leq m$ if and only if*

$$\sum_{i=1}^k a_i^2 \leq \sum_{i=1}^k \lambda(S)_i, \quad \textit{for} \ \ 1 \leq i \leq d-1, \quad \textit{and} \quad \sum_{i=1}^m a_i^2 = \operatorname{tr}(S).$$

4. Convex functions defined on frame operators.

In this section we define a family functions P_f on the set of frame operators of sequences $\mathcal{F}$ in $\mathbb{C}^d$, starting from a convex function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$. As a particular case, we recover the frame potential, introduced by Benedetto and Fickus, in [4] with a specific convex function f.

When we restrict our attention to special sets of sequences, namely, those sequences with a prescribed set of norms, we are able to compute the minimum value taken by P_f on the corresponding set of frame operators and to characterize the spectrum of minimizers of P_f, for every f non decreasing and convex function which satisfies $f(0) = 0$.

Definition 4.1. *Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a non decreasing convex function. Then, the frame potential associated to f, denoted P_f, is the functional defined on the set of frame operators of sequences in $\mathbb{C}^d$ given by*

$$P_f(S^{\mathcal{F}}) = \operatorname{tr}(f(S^{\mathcal{F}})) \tag{8}$$

for every $\mathcal{F} = \{\phi_i\}_{i=1}^m \subset \mathbb{C}^d$. In detail, if we denote by $\lambda = (\lambda_i)_{i=1}^d$ the eigenvalues of $S^{\mathcal{F}}$ counted with multiplicity, then $P_f(S^{\mathcal{F}}) = \sum_{i=1}^d f(\lambda_i)$.

Remark 4.2. Using the relation between $G^{\mathcal{F}}$ and $S^{\mathcal{F}}$ shown in Proposition 3.1, we have

$$\operatorname{tr}(f(G^{\mathcal{F}})) = P_f(S^{\mathcal{F}}) + (m-d)f(0)$$

In particular, if $f(0) = 0$, $P_f(S^{\mathcal{F}})$ can be computed using the Grammian matrix.

Example 4.3 (Benedetto-Fickus's potential)**.** Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be the strictly convex function $f(x) = x^2$. Then, the frame potential associated to f is

$$P_f(S^{\mathcal{F}}) = \operatorname{tr}((S^{\mathcal{F}})^2) = \operatorname{tr}((G^{\mathcal{F}})^2) = \sum_{i,\, j=1}^{m} |\langle \phi_i, \phi_j\rangle|^2 \tag{9}$$

that is, the frame potential as defined by Benedetto and Fickus in [4]

In what follows, given $\mathcal{F} = \{\phi_i\}_{i=1}^m \subset \mathbb{C}^d$ and $\alpha \in \mathbb{C}$ we denote by $\alpha\mathcal{F} = \{\alpha\, \phi_i\}_{i=1}^m$. On the other hand, given $\mathcal{F}_1 = \{\phi_i\}_{i=1}^{M_1}$, $\mathcal{F}_2 = \{\psi_i\}_{i=1}^{M_2} \subset \mathbb{C}^d$ then $\mathcal{F}_1 \sqcup \mathcal{F}_2$ denotes the list of $M_1 + M_2$ vectors obtained by juxtaposition of $\mathcal{F}_1$ and $\mathcal{F}_2$. Note that, if $\mathcal{G} = \alpha\mathcal{F}_1 \sqcup \mathcal{F}_2$ then

$$S^{\mathcal{G}} = |\alpha|^2\, S^{\mathcal{F}_1} + S^{\mathcal{F}_2}.$$

Theorem 4.4. *Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ be a non decreasing convex function and $\mathcal{F}_1 = \{\phi_i\}_{i=1}^{M_1}$, $\mathcal{F}_2 = \{\psi_i\}_{i=1}^{M_2} \subset \mathbb{C}^d$.*

1. *If $S^{\mathcal{F}_1} \prec S^{\mathcal{F}_2}$ then*
$$P_f(S^{\mathcal{F}_1}) \leq P_f(S^{\mathcal{F}_2}).$$
2. *Assume further that f is a strictly convex function, $S^{\mathcal{F}_1} \prec S^{\mathcal{F}_2}$ and $P_f(S^{\mathcal{F}_1}) = P_f(S^{\mathcal{F}_2})$. Then, there exists a unitary operator $U \in \mathcal{M}_d(\mathbb{C})$ such that*
$$U S^{\mathcal{F}_1} U^* = S^{\mathcal{F}_2}.$$
3. *If $t \in [0,1]$ and $\mathcal{G} = t^{1/2}\mathcal{F}_1 \sqcup (1-t)^{1/2}\mathcal{F}_2$ then*
$$P_f(S^{\mathcal{G}}) \leq t P_f(S^{\mathcal{F}_1}) + (1-t) P_f(S^{\mathcal{F}_2}).$$
4. *If $\mathcal{G} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ then*
$$P_f(S^{\mathcal{G}}) \geq P_f(S^{\mathcal{F}_1}) + P_f(S^{\mathcal{F}_2}).$$

Proof. The first two items are well known (see [5, 10]). The last two inequalities above are also well known (see [1, Theorem 1-24]) for these functionals. □

Remark 4.5. For $g = -f$, $P_g(S) = \operatorname{tr}(g(S))$ for $S \in \mathcal{M}_d(\mathbb{C})^+$ are called "entropy-like" functionals in [1]. Notice that the minimization of the functions P_f corresponds to the maximization of the entropy-like functional P_g.

Let $c > 0$ and $\mathbf{a} = \{a_i\}_{i=1}^m$ be a sequence of positive elements arranged in decreasing order. In what follows we shall consider the following sets:

$$\mathcal{A}(c) = \{\mathcal{F} = \{\phi_i\}_{i=1}^m \subset \mathbb{C}^d,\ \sum_{i=1}^m \|\phi_i\|^2 = \operatorname{tr}(S^{\mathcal{F}}) = c\},$$

$$\mathcal{B}(\mathbf{a}) = \{\mathcal{F} = \{\phi_i\}_{i=1}^m \subset \mathbb{C}^d,\ \|\phi_i\|^2 = a_i \text{ for every } i\}.$$

Observe that, by Theorem 3.2, the sets of frame operators for sequences in $\mathcal{A}(c)$ and $\mathcal{B}(\mathbf{a})$ can be well characterized:

$$\mathcal{T}(c) = \{S^{\mathcal{F}},\ \mathcal{F} \in \mathcal{A}(c)\} = \{S \in \mathcal{M}_d(\mathbb{C})^+,\ \lambda(S) \in \mathcal{K}(c)\}. \tag{10}$$

$$\mathcal{R}(\mathbf{a}) = \{S^{\mathcal{F}},\ \mathcal{F} \in \mathcal{B}(\mathbf{a})\} = \{S \in \mathcal{M}_d(\mathbb{C})^+,\ \lambda(S) \in \mathcal{P}(\mathbf{a})\}. \tag{11}$$

Theorem 4.6. *Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a non decreasing convex function and P_f the functional associated to f and let $c > 0$. Then, if $\mathcal{F} \in \mathcal{A}(c)$ is a tight frame, then*

$$P_f(S^{\mathcal{F}}) \leq P_f(S^{\mathcal{G}}) \quad \forall S^{\mathcal{G}} \in \mathcal{T}(c).$$

Moreover, if in addition f is strictly convex and $S^{\mathcal{F}}$ is a local minimum of P_f considering the operator norm in $\mathcal{T}(c)$, then $S^{\mathcal{F}} = \frac{c}{d} I$ so $\mathcal{F}$ is a tight frame.

Proof. The proof follows immediately from Proposition 2.1. Indeed, $S^{\mathcal{F}} \in \mathcal{T}(c)$ is a global minimum for P_f if and only if $\lambda(S^{\mathcal{F}}) = \mathbf{v}$, i.e. $S^{\mathcal{F}} = \frac{c}{d} I$, which means that $\mathcal{F}$ is a tight frame in $\mathcal{A}(c)$. On the other side, if $\lambda = \lambda(S^{\mathcal{F}}) \neq \mathbf{v}$, then by Prop. 2.1 for every $\varepsilon > 0$ sufficiently small, there exist $\lambda_\varepsilon \in \mathcal{K}(c)$ such that $\lambda_\varepsilon \prec \lambda$, $\lambda^\downarrow \neq \lambda_\varepsilon^\downarrow$ and $\|\lambda - \lambda_\varepsilon\| < \varepsilon$. Thus, if $S^{\mathcal{F}} = U^* \operatorname{diag}(\lambda) U$ with U unitary, it is clear that $S_\varepsilon = U^* \operatorname{diag}(\lambda_\varepsilon) U \in \mathcal{T}(c)$ satisfies $\|S^{\mathcal{F}} - S_\varepsilon\| < \varepsilon$ and $P_f(S_\varepsilon) < P_f(S^{\mathcal{F}})$, by Thm. 4.4.

□

Theorem 4.7. *Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a non decreasing convex function and P_f the functional associated to f. Let $\mathbf{a} = (a_i)_{i=1}^m$ be a non increasing sequence of strictly positive numbers with $d \leq m$. Suppose that $\mathcal{F} \in \mathcal{B}(\mathbf{a})$ is of the form*

$$\{\sqrt{a_i}\, e_i\}_{i=1}^r \cup \{\phi_i\}_{i=r+1}^m \tag{12}$$

where $\{e_i\}_{i=1}^d$ is an o.n. basis for $\mathbb{C}^d$, r is the d-irregularity *of* $\mathbf{a}$ *and $\{\phi_i\}_{i=r+1}^m$ is a tight frame for span$\{e_i\}_{i=r+1}^d$ with frame constant $c = (d-r)^{-1} \sum_{i \geq r+1} a_i$.*

Then, $S^{\mathcal{F}}$ is a global minimum for P_f in $\mathcal{R}(\mathbf{a})$. Moreover, if f is strictly convex and $S^{\mathcal{F}}$ is a local minimum for P_f in $\mathcal{R}(\mathbf{a})$ (considering the operator norm), then $\mathcal{F}$ is as in (12).

Proof. Let $\mathcal{F} \in \mathcal{B}(\mathbf{a})$ be of the form given in (12). Therefore, the (ordered) spectrum of the frame operator $S^{\mathcal{F}}$ is $\mathbf{v} = (a_1, \ldots, a_r, c, \ldots, c)$ where $c = (d-r)^{-1} \sum_{i \geq r+1} a_i$ is an eigenvalue with multiplicity $d-r$. Then, by the Proposition 2.3 and Theorem 4.4, we can conclude that $\mathcal{F}$ is a global minimum in $\mathcal{B}(\mathbf{a})$.

Now, let $\mathcal{G} = \{\psi_i\}_{i=1}^m \in \mathcal{B}(\mathbf{a})$ be such that $\lambda(S^{\mathcal{G}}) = \mathbf{v}$. Then the (optimal) upper frame bound of $\mathcal{G}$ is a_1 and we have

$$\|\psi_1\|^4 + \sum_{j>1} |\langle \psi_j\,,\, \psi_1 \rangle|^2 \leq a_1 \|\psi_1\|^2 = \|\psi_1\|^4$$

Therefore, ψ_1 is orthogonal to ψ_j for $j \neq 1$. By restriction to $\operatorname{span}\{\psi_i\}_{i=2}^m$, we deduce that $\langle \psi_2 \, , \, \psi_i \rangle = 0$ for $i \neq 2$ in the same way. Therefore we can conclude that $\langle \psi_i \, , \psi_j \rangle = 0$ for every $1 \leq i \leq r$, $j \neq i$, in particular we define the orthonormal set $e_i = a_i^{-1/2} \psi_i$ for $1 \leq i \leq r$. We then complete it to an o.n.b. $\{e_i\}_{i=1}^d$.

Finally, since the frame operator restricted in the orthogonal complement of the space spanned by $\{e_i\}_{i=1}^r$ is a multiple of the identity, the rest of the frame is a tight frame in its span. Then, $\mathcal{G}$ can be described as in (12).

Let $S^{\mathcal{F}} \in \mathcal{R}(\mathbf{a})$ be such that $\lambda(S^{\mathcal{F}})$ is not $\mathbf{v} \in \mathcal{P}(\mathbf{a})$. Therefore, by the last statement of Prop. 2.3 and arguing as in Thm. 4.6, given $\varepsilon > 0$, we can find a positive definite operator $S_\varepsilon \in \mathcal{R}(\mathbf{a})$ such that $\|S_\varepsilon - S^{\mathcal{F}}\| < \varepsilon$, $\lambda(S_\varepsilon) \neq \lambda(S^{\mathcal{F}})$ and $S_\varepsilon \prec S^{\mathcal{F}}$. Then $P_f(S_\varepsilon) < P_f(\mathcal{F})$ for every strictly convex function f, by Theorem 4.4. In particular, by the previous paragraph, every local minimum for P_f in $\mathcal{R}(\mathbf{a})$ is a global minimum, so it is a frame operator of a frame given by (12).

□

Theorem (A) in the Introduction is now a consequence of the identities (10), (11) and Theorems 4.6, 4.7.

Corollary 4.8. *Let $\mathcal{F} = \{\phi_i\}_{i=1}^m \in \mathcal{A}(c)$ and let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a non decreasing convex function. We have the following inequalities:*

$$(d-1) \cdot f(0) + f(c) \geq P_f(S^{\mathcal{F}}) \geq d \cdot f(\frac{c}{d}), \tag{13}$$

And, for $\mathcal{F} \in \mathcal{B}(\mathbf{a})$ we have

$$(d-1) \cdot f(0) + f(\sum_{i=1}^m a_i) \geq P_f(S^{\mathcal{F}}) \geq \sum_{i=1}^r f(a_i) + (d-r) \cdot f(h) \tag{14}$$

with $h = (d-r)^{-1} \sum_{i \geq r+1} a_i$. Moreover, if in addition f is strictly convex and the lower bound is attained in (13) *(respectively in* (14)*) then $\mathcal{F}$ is a tight frame (respectively is as in* (12) *for some o.n.b. $\{e_i\}_{i=1}^d$ for $\mathbb{C}^d$).*

4.1. **Some applications of the previous results.** Let us begin with the following example in order to illustrate the content of our previous results.

Example 4.9 (continuation of example 4.3)**.** Let $f(x) = x^2$ and note that, by equation (9), if $\mathcal{F} = \{\phi_i\}_{i=1}^m \in \mathcal{A}(c)$ then

$$P_f(S^{\mathcal{F}}) = \sum_{i,j=1}^m |\langle \phi_i, \phi_j \rangle|^2 .$$

Note that f is a strictly convex function and that $f(\lambda \cdot x) = \lambda^2 \cdot f(x)$ for every $\lambda \geq 0$, so we can take $g(\lambda) = \lambda^2$. Then, equation (14) becomes

$$1 \geq \frac{\sum_{i=1}^m |\langle \phi_i, \phi_j \rangle|^2}{(\sum_{i=1}^m \|\phi_i\|^2)^2} \geq d \cdot \frac{1}{d^2} = \frac{1}{d} \tag{15}$$

which is the *generalized Welch inequality* of [13]. Moreover, by Theorem 4.8 we deduce that the lower bound (resp the upper bound) in equation (15) is attained if and only if $\mathcal{F}$ is a tight frame with frame bound $\frac{c}{d}$ (resp if and only if $\operatorname{span}(\mathcal{F})$ has dimension 1).

Of course, the function $f(x) = x^2$ is probably the most simple function that can be used to produce a reasonable frame potential. In the following examples we shall investigate other choices of convex functions.

Example 4.10 (n-th frame potential)**.** Let $n \geq 2$ and consider $f_n(x) = x^n$ for $x \geq 0$. Then, f is an increasing strictly convex function and produce the *n-th frame potential* given by

$$P_n(S^{\mathcal{F}}) = \operatorname{tr}((S^{\mathcal{F}})^n)$$

where $S^{\mathcal{F}}$ is the frame operator of the sequence $\mathcal{F} = \{\phi_i\}_{i \in m} \subset \mathbb{C}^d$. Since $f(0) = 0$ then we have

$$P_n(\{\phi_i\}_{i=1}^m) = \operatorname{tr}((G^{\mathcal{F}})^n) = \sum_{i_1, \ldots, i_n = 1}^{m} \prod_{j=1}^{n} \langle \phi_{i_j}, \phi_{i_{j+1}} \rangle \tag{16}$$

where we follow the convention $i_{n+1} = i_1$. Note that P_2 is the usual frame potential. Indeed, formula (16) is a consequence of the identity

$$\langle (G^{\mathcal{F}})^n e_k, e_k \rangle = \sum_{\substack{i_2, \ldots, i_n = 1 \\ i_1 = k}}^{m} \prod_{j=1}^{n} \langle \phi_{i_j}, \phi_{i_{j+1}} \rangle \geq 0 \tag{17}$$

In this case, using equation (16), equation (14) becomes

$$1 \geq \frac{\sum_{i_1, \ldots, i_n = 1}^{m} \prod_{j=1}^{n} \langle \phi_{i_j}, \phi_{i_{j+1}} \rangle}{(\sum_{i=1}^{m} \|\phi_i\|^2)^n} \geq \frac{1}{d^{n-1}} \tag{18}$$

while equation (17) implies

$$\max_{1 \leq k \leq m} \sum_{\substack{i_2, \ldots, i_n = 1 \\ i_1 = k}}^{m} \prod_{j=1}^{n} \langle \phi_{i_j}, \phi_{i_{j+1}} \rangle \geq \frac{(\sum_{i=1}^{m} \|\phi_i\|^2)^n}{m \cdot d} \tag{19}$$

As before, the lower bound in formula (18) is attained if and only if $\mathcal{F}$ is a tight frame with frame bound $\frac{c}{d}$. Analogously, the bound in equation (19) is attained if and only if $\mathcal{F}$ is a tight frame.

Example 4.11 (von Neumann Entropy)**.** If we consider the concave function $f(x) = -x \ln(x)$, then P_f restricted to density matrices is the well known von Neumann entropy in quantum information theory. Roughly speaking, it measures the lack of information about the state of a system. Theorems 4.6 and 4.7 show, as a particular case, the structure of maximizers of the entropy without restrictions in the first case and with the restriction: $\{S$ a density matrix with $(\lambda(S), 0_{m-d}) \succ \mathbf{a}\}$ for a fixed positive sequence $\mathbf{a}$ with $\sum_{i=1}^{m} a_i = 1$.

4.2. **Convex functions over CGU frames.** In this section we use the previous techniques to characterize the global minimizers of P_f when restricted to the compound geometrically uniform frames, with a prescribed list of norms.

Definition 4.12. *Let G be a finite abelian group of unitaries in $\mathcal{M}_d(\mathbb{C})$, and $\varphi \in \mathbb{C}^d$. If the set $\mathsf{G} \cdot \varphi = \{U\varphi : U \in \mathsf{G}\}$ is a frame the we say that $\mathsf{G} \cdot \varphi$ is a* **geometrically uniform frame** *(GU). When G acts on a larger set of functions, $\Phi = \{\varphi_i \in \mathbb{C}^d : 1 \leq i \leq m\}$ and $\mathsf{G} \cdot \Phi$ is a frame, we say that it is a* **compound geometrically uniform frame** *(CGU).*

From now on, in order to simplify the computations, we assume also that G is cyclic. Let suppose then that we have $\mathsf{G} = \{U^i : 0 \leq i \leq n-1\}$, where U is a unitary such that $U^n = I$. Thus, we shall consider frame sequences of the form $\mathcal{F} = \mathsf{G} \cdot \Phi = \{U^i \varphi_j : 0 \leq i \leq n-1\,, \quad 1 \leq j \leq m\}$.

We are interested in minimizing P_f when we restrict P_f to the set of frame operators of CGU frames:

$$\mathsf{G}\cdot\mathcal{B}(\mathbf{a}) = \{\mathsf{G}\cdot\mathcal{F} \,:\, \mathcal{F}\in\mathcal{B}(\mathbf{a})\},$$

where G is a fixed cyclic group of unitaries, $\mathbf{a}$ is fixed. Clearly $\mathsf{G}\cdot\mathcal{B}(\mathbf{a})\subset\mathcal{B}(\mathbf{b})$, where $\mathbf{b}=\{b_i\}_{i=1}^{nm}$ is the sequence $\mathbf{a}$ repeated n times. Then, by Corollary 4.8, if $\mathcal{F}\in\mathsf{G}\cdot\mathcal{B}(\mathbf{a})$,

$$P_f(S^{\mathcal{F}}) \geq \sum_{i=1}^{r} f(b_i) + (d-r)\cdot f(h), \tag{20}$$

where $h = (d-r)^{-1}\sum_{i=r+1}^{nm} b_i$ and r is the d-irregularity of $\mathbf{b}$. The previous inequality can be stated in terms of $\mathbf{a}$ if we characterize the d-irregularity of $\mathbf{b}$.

Proposition 4.13. *Let $\mathbf{a}=(a_i)_{i=1}^m$ be a non increasing sequence of positive numbers and let $\mathbf{b}=(b_i)_{i=1}^{nm}$ be a sequence given by:*

$$b_j = a_i \quad \text{for} \quad j=(i-1)n+s, \quad 1\leq s\leq n,\ 1\leq i\leq m.$$

Then, if r_0 is the d-irregularity of $\mathbf{b}$, $r_0 = nr$, where

$$r = \max\{j : (\frac{d}{n}-j)\,a_j > \sum_{k=j+1}^{m} a_k\}.$$

Proof. The result is clear if $r_0=0$. If $r_0\neq 0$, then it holds that n divides r_0. Indeed, by definition of r_0, $b_{r_0}\neq b_{r_0+1}$ which can only occur if $r_0 = nr$, $r\in\{1,\dots m\}$. Finally,

$$r_0 = \max\{nj:\ (d-nj)\,b_{nj} > \sum_{k=nj+1}^{nm} b_k\} = n\ \max\{j:\ (\frac{d}{n}-j)\,a_j > \sum_{k=j+1}^{m} a_k\}.$$

□

Theorem 4.14. *Let G, $\mathbf{a}$ and $\mathcal{B}(\mathbf{a})$ as before. Suppose that $n|d$ and that there exists an orthonormal family $\{e_i\}_{i=1}^N$, with $N=\frac{d}{n}$ such that the set $\{U^k e_j\ 1\leq k\leq n,\ 1\leq j\leq N\}$ is an orthonormal basis of $\mathbb{C}^d$. Let $\mathcal{F}$ in $\mathcal{B}(\mathbf{a})$ be of the form*

$$\mathcal{F}' = \{\sqrt{a_i}\, b_i\}_{i=1}^r \cup \mathcal{D} \tag{21}$$

where $\mathcal{E}=\{b_i\}_{i=1}^r$ is an orthonormal set such that $\mathsf{G}\cdot\mathcal{E}$ is orthonormal, r is the N-irregularity *of* $\mathbf{a}$ *and $\mathsf{G}\cdot\mathcal{D}$ is a tight frame for* span $(\mathsf{G}\cdot\mathcal{E})^{\perp}$ *with frame constant $h=(N-r)^{-1}\sum_{k=r+1}^m a_k$. Denote $\mathcal{F}=\mathsf{G}\cdot\mathcal{F}'\in\mathsf{G}\cdot\mathcal{B}(\mathbf{a})$.*

Then $S^{\mathcal{F}}$ is a global minimum for P_f in the set of frame operators of $\mathsf{G}\cdot\mathcal{B}(\mathbf{a})$. Conversely, if in addition f is strictly convex, and $S^{\mathcal{F}}$ is a global minimum for P_f, then $\mathcal{F}$ is of the form $\mathsf{G}\cdot\mathcal{V}$, with $\mathcal{V}$ as in (21).

Proof. By Thm. 4.7 and Prop. 4.13 it is clear that if such sequence exists, then $S^{\mathcal{F}}$ is a global minimum in $\mathcal{R}(\mathbf{b})$ (using the previous notation), so it is a global minimum when we restrict P_f to the frame operators of $\mathsf{G}\cdot\mathcal{B}(\mathbf{a})$. Moreover, if f is strictly convex, every global minimum must be of this form, by Thm. 4.7.

Then, in order to prove the statement we need to show that such sequence exists. Indeed let $\mathcal{F}'$ be the sequence given by

$$\{\sqrt{a_i}\, e_i\}_{i=1}^r \cup \{\phi_i\}_{i=r+1}^m$$

where $\{e_i\}_{i=1}^N$ is the orthonormal set existing by the hypotheses, r is the N-*irregularity* of $\mathbf{a}$ and $\{\phi_i\}_{i=r+1}^m$ is a tight frame for $\operatorname{span}\{e_k\}_{k=r+1}^N$, with frame constant $h=(N-r)^{-1}\sum_{k=r+1}^m a_k$. Such frame exists by Theorem 4.7.

Clearly, for every $1\le k\le n$, the set $\{U^k\phi_i\}_{i=r+1}^m$ is a tight frame (with the same constant $h=(N-r)^{-1}\sum_{k=r+1}^m a_k$) for $\operatorname{span}\{U^k e_i\}_{i=r+1}^N$, therefore, $\mathcal{D}=\mathsf{G}\cdot\{\phi\}_{i=r+1}^m$ is a tight frame of $\operatorname{span}(\mathsf{G}\cdot\mathcal{E})^\perp$ with frame constant $c=(N-r)^{-1}\sum_{k=r+1}^m a_k$, where $\mathcal{E}=\{e_i\}_{i=1}^r$.

□

Remark 4.15. If in addition we assume that the initial vectors $\mathcal{F}$ lie on the $\frac{d}{n}$-dimensional subspace $\mathcal{K}$ generated by $\{e_i\}_{i=1}^N$ ($\frac{d}{n}=N$) of $\mathbb{C}^d$ we can conclude that the global minimizers are of the form given in (21), where r is the $\frac{d}{n}$-irregularity of $\mathbf{a}$ and $\mathcal{D}$ forms a tight frame on $\mathcal{K}\cap(\operatorname{span}\{b_i\}_{i=1}^r)^\perp$. Indeed, in this case the Grammian matrix of $\mathsf{G}\cdot\mathcal{F}$ is block-diagonal.

A special case of this situation is given on convolutional frames studied in [9]. In particular, previous Theorem can be seen as a partial generalization to [9, Thm. 6].

Corollary 4.16. *Under the hypotheses of Theorem 4.14, for $\mathcal{F}\in\mathcal{B}(\mathbf{a})$ we have*

$$(d-1)\cdot f(0)+f(n\cdot\sum_{i=1}^m a_i)\ge P_f(S^{\mathsf{G}\cdot\mathcal{F}})\ge n\left\{\sum_{i=1}^r f(a_i)+(d-r)\cdot f(h)\right\}\tag{22}$$

with $h=(d-r)^{-1}\sum_{i\ge r+1}a_i$. Moreover, if in addition f is strictly convex and the lower bound is attained in (22) *then $\mathcal{F}$ is as in* (21).

5. From frame operators to frames.

In the previous section we have considered the function P_f associated to a convex function f as a function of the frame operators; we have described the structure of local minimizers of P_f when restricted to the sets $\mathcal{T}(c)$ and $\mathcal{R}(\mathbf{a})$ with respect to the norm topology.

We are now interested in considering P_f defined on frames

$$P_f(\mathcal{F}):=P_f(S^{\mathcal{F}})=\operatorname{tr}(f(S^{\mathcal{F}}))$$

for $\mathcal{F}=\{\phi_i\}_{i=1}^m\subset\mathbb{C}^d$, and studying the structure of global and local minimizers of these functions when restricted to the sets $\mathcal{A}(c)$ and $\mathcal{B}(\mathbf{a})$, with respect to the vector-vector distance

$$d(\mathcal{F},\mathcal{G})=\max_{1\le i\le m}\|\phi_i-\psi_i\|\tag{23}$$

for sequences $\mathcal{F}=\{\phi_i\}_{i=1}^m$, $\mathcal{G}=\{\psi_i\}_{i=1}^m$. It is worth noting that the norm distance between frame operators can not bound the vector-vector distance; indeed if σ is a permutation of order m and $\mathcal{G}=\{f_{\sigma(i)}\}_{i=1}^m$ then $S^{\mathcal{F}}=S^{\mathcal{G}}$ while $d(\mathcal{F},\mathcal{G})\neq 0$ possibly. This implies that the results in the previous section can not be used to obtain a complete characterization of the local minimizers in this new setting.

Our approach to this new point of view involves the study of the existence local cross sections of the map $\mathcal{F}\mapsto S^{\mathcal{F}}$ when it is restricted to $\mathcal{A}(c)$ and $\mathcal{B}(\mathbf{a})$ respectively (note that the restriction on the norms which defines $\mathcal{B}(\mathbf{a})$ is a condition on the main diagonal of $G^{\mathcal{F}}$).

To begin with, Theorem 4.6 implies that if a sequence $\mathcal{F} = \{\phi_i\}_{i=1}^m \in \mathcal{A}(c)$ does not have the structure of a local (global) minimizer of P_f on $\mathcal{T}(c)$, for a strictly convex function f, then for every $\varepsilon > 0$ there exists a $S \in \mathcal{T}(c)$ such that $\|S - S^{\mathcal{F}}\| \le \varepsilon$ and $P_f(S) < P_f(S^{\mathcal{F}})$. In order to show that $\mathcal{F}$ is not a local minimum of P_f on $\mathcal{A}(c)$ with respect to the vector-vector distance the following problem arises: given such S, is there any sequence $\mathcal{G} = \{\psi_i\}_{i=1}^m \in \mathcal{A}(c)$ such that $S^G = S$ and $d(\mathcal{F}, \mathcal{G}) \le \delta(\varepsilon)$, with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$? A positive answer to this question is given in the following

Proposition 5.1. *Let $\mathcal{F} = \{\phi_i\}_{i=1}^m \in \mathcal{A}(c)$ and let $S \in \mathcal{T}(c)$ be such that $\|S - S^{\mathcal{F}}\| < \varepsilon$. Then there exist $\mathcal{G} = \{\psi_i\}_{i=1}^m \in \mathcal{A}(c)$ such that $d(\mathcal{F}, \mathcal{G}) < \varepsilon^{1/2}$ and $S^{\mathcal{G}} = S$.*

Proof. Consider $T^{\mathcal{F}} = (S^{\mathcal{F}})^{1/2} W$ the polar decomposition of $T^{\mathcal{F}}$. Then, since $\|S - S^{\mathcal{F}}\| < \varepsilon\,,\ \|S^{1/2} - (S^{\mathcal{F}})^{1/2}\| < \varepsilon^{1/2}$ by [5, Thm. X.1.1]).

Now let $\mathcal{G} = \{\psi_i\}_{i=1}^m$, where $\psi_i = S^{1/2} W e_i$ for $1 \le i \le m$. Then $T^{\mathcal{G}} = S^{1/2} W$, $S^{\mathcal{G}} = T^{\mathcal{G}} (T^{\mathcal{G}})^* = S$ and for $1 \le i \le m$

$$\|\psi_i - \phi_i\| \le \|T^{\mathcal{G}} - T^{\mathcal{F}}\| \le \|S^{1/2} - (S^{\mathcal{F}})^{1/2}\| < \varepsilon^{1/2}.$$

□

The previous result combined with Theorem 4.6 provide a complete characterization of the local (global) minimizers of P_f on $\mathcal{A}(c)$ with respect to the vector-vector distance, for a strictly convex f.

Theorem 5.2. *Let $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a non decreasing convex function. If $\mathcal{F} \in \mathcal{A}(c)$ is a tight frame then it is a global minimizer of P_f on $\mathcal{A}(c)$. Moreover, if f is a strictly convex function then every local minimum of P_f on $\mathcal{A}(c)$ with respect to the vector-vector distance is a tight frame.*

Proof. The first part of the statement follows from Theorem 4.6 and (10). By the proof of 4.6, if $\mathcal{F} \in \mathcal{A}(c)$ is not tight, then for every $\varepsilon > 0$, there exists $S_\varepsilon \in \mathcal{T}(c)$ such that $\|S^{\mathcal{F}} - S_\varepsilon\| < \varepsilon^2$ and $P_f(S_\varepsilon) < P_f(S^{\mathcal{F}})$. Finally, by Proposition 5.1, there exist $\mathcal{G} = \{\psi_i\}_{i=1}^m \in \mathcal{A}(c)$ such that $S^{\mathcal{G}} = S_\varepsilon$ and $\|\phi_i - \psi_i\| < \varepsilon$. □

As before, in order to obtain a characterization of local minimizers of P_f on $\mathcal{B}(\mathbf{a})$ with respect to the vector-vector distance using Theorem 4.7 we are led to consider the following perturbation problem: given a sequence $\mathcal{F} = \{\phi_i\}_{i=1}^m \in \mathcal{B}(\mathbf{a})$ and $S \in \mathcal{R}(\mathbf{a})$ with $\|S^{\mathcal{F}} - S\| \le \varepsilon$, is there a sequence $\mathcal{G} = \{\psi_i\}_{i=1}^m \in \mathcal{B}(\mathbf{a})$ with $S^{\mathcal{G}} = S$ and $d(\mathcal{F}, \mathcal{G}) \le \delta(\varepsilon)$ with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$. The constrain $\mathcal{G} \in \mathcal{B}(\mathbf{a})$ seems to be hard to deal with. For example, notice that we have no control on the norms of the vectors in $\mathcal{G}$ constructed in Proposition 5.1. On the other hand, it is convenient to work with the Grammian since the restriction $\mathcal{G} \in \mathcal{B}(\mathbf{a})$ is equivalent to $\mathrm{d}(G^{\mathcal{G}}) = \mathbf{a}$, where $\mathrm{d}(X) \in \mathbb{C}^m$ denotes the main diagonal of the $m \times m$ complex matrix X.

We have only obtained partial results which are presented in the following Proposition. The proof depends strongly on geometrical aspects and it is developed in the appendix.

Proposition 5.3. *Let $\mathcal{F} = \{\phi_j\}_{j=1}^m \subseteq \mathbb{C}^d$ be a frame, let $S = S^{\mathcal{F}}$ be its frame operator and assume that $\mathcal{F}$ can not be partitioned in two sets of mutually orthogonal vectors. Let $\{S_i\}_i \subseteq \mathcal{M}_d(\mathbb{C})^+$ be a sequence converging to S. Then, for every $\eta > 0$ there exists $i_1 \in \mathbb{N}$ such that for each $i \ge i_1$ there exists a frame $\mathcal{G}(i) = \mathcal{G} = \{\psi_j\}_{j=1}^m$ such that:*

(1) $\|\psi_j\| = \|\phi_j\|$ *for* $1 \leq j \leq m$.
(2) $\|\psi_j - \phi_j\| \leq \eta$ *for* $1 \leq j \leq m$.
(3) $S^{\mathcal{G}} = S_i$.

Theorem 5.4. *Let* $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *be a non decreasing convex function. If* $\mathcal{F} \in \mathcal{B}(\mathbf{a})$ *has the structure as in* (12) *then it is a global minimizer of* P_f *on* $\mathcal{B}(\mathbf{a})$.

If in addition f *is a strictly convex function, then every global minimum of* P_f *on* $\mathcal{B}(\mathbf{a})$ *is as in* (12). *Moreover, for such* f *then every* $\mathcal{F} = \{\phi_i\}_{i=1}^m \in \mathcal{B}(a)$ *such that it can not be partitioned in two mutually orthogonal sets of vectors is a local minimum if and only if is a global minimum.*

Proof. The first part of the statement follows from Theorem 4.7 and (11).

Assume now that $\mathcal{F}$ is not a global minimum; by the proof of Thm. 4.7, there is a sequence of operators $\{S_n\}$ such that S_n converges to $S^{\mathcal{F}}$ and such that $P_f(S_n) < P_f(S^{\mathcal{F}}), \forall n$.

Let $\varepsilon > 0$, then, by Thm. 6.4, for a sufficient large $n_0 \in \mathbb{N}$, there exist a frame $\mathcal{G} = \{\psi_i\}_{i=1}^m \in \mathcal{B}(\mathbf{a})$ such that $\|\phi_i - \psi_i\| < \varepsilon$ and $S^{\mathcal{G}} = S_n$. In particular, $P_f(\mathcal{G}) < P_f(\mathcal{F})$. □

Theorem (B) in the Introduction follows immediately from Theorems 5.2 and 5.4.

It is clear that $P_f(\mathcal{F}) = P_f(\mathcal{F}_1) + P_f(\mathcal{F}_2)$ if $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ with the vectors in $\mathcal{F}_2$ being orthogonal to those in $\mathcal{F}_1$ (we shall denote this by $\mathcal{F}_1 \perp \mathcal{F}_2$). This simple observation and the previous result, allows a reduction of the set of possible local minimizers for P_f:

Corollary 5.5. *Let* $\mathcal{F} \in \mathcal{B}(\mathbf{a})$ *such that* $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ *with* $\mathcal{F}_1 = \{\phi_i\}_{i=1}^{M_1} \perp \mathcal{F}_2 = \{\phi_i'\}_{i=1}^{M_2}$, *and suppose that* $\mathcal{F}_1$ *can not be partitioned into two mutually orthogonal sequences and it is not a global minimizer for* P_f *restricted to the set*

$$\mathcal{B}(\mathbf{a}_1) = \{\{\psi_j\}_{j=1}^{M_1} : \psi_j \in \operatorname{span} \mathcal{F}_1, \ \|\psi_j\| = \|\phi_j\| \quad 1 \leq j \leq M_1\}.$$

Then, $\mathcal{F}$ *is not a local minimizer for* P_f.

Note that the general structure of local minimums of arbitrary function P_f can not be inferred from Theorem 5.4 and Corollary 5.5. Still, these results allow to a reduction of the general situation to a particular case (see Problem ($\star$) below). In order to exemplify the ideas involved, we recover [6, Theorem 10] about the structure of general minimizers in the particular caso of the Benedetto-Fickus potential.

Theorem 5.6. *Any local minimizer* $\mathcal{F} = \{\phi_i\}_{i=1}^m$ *of the Benedetto-Fickus potential in* $\mathcal{B}(\mathbf{a})$ *with respect to the distance* $d(\cdot,\cdot)$ *is a global minimizer of this potential and hence has the structure given in* (12)

Proof. Suppose that we have a frame $\mathcal{F} \in \mathcal{B}(\mathbf{a})$ which is not a global minimum for the Benedetto - Fickus potential FP. We must show that then it is not a local minimum.

Let $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$ its minimal decomposition in pairwise orthogonal subsets (minimal in the sense that no $\mathcal{F}_j$ can be partitioned in two mutually orthogonal subsets). By Corollary 5.5, if there exist $1 \leq i \leq k$ such that $\mathcal{F}_i$ is not a global minimum for FP (restricted to $\mathcal{B}(\mathbf{a}_i)$), then $\mathcal{F}$ is not a local minimizer.

So we can suppose that every $\mathcal{F}_i$ is a global minimum on $\mathcal{B}(\mathbf{a}_i)$. Then by Theorem 4.7, $\mathcal{F}_i$ is tight on its span (possibly with a single vector), with frame constant c_i, for $1 \leq i \leq k$. We claim that in this case, there is a pair i, j such that the frame $\mathcal{F}_i \cup \mathcal{F}_j$ is not a global minimum for FP on span $\mathcal{F}_1 \cup \mathcal{F}_2$ with the restriction given by the vector norms in $\mathcal{F}_i$, $i = 1, 2$.

Indeed, if there exists a pair $\mathcal{F}_i$ and $\mathcal{F}_j$, each with two or more vectors, and with constants $c_i \neq c_j$, then $\mathcal{F}_i \cup \mathcal{F}_j$ is not a global minimum for FP (in the adequate restriction), since by the structure given in Thm. 4.7, if a global minimum is a union of two mutually orthogonal tight subframes (on their spans), then one of them must be a single vector. On the other side, if every $\mathcal{F}_i$, consisting of more than one vector has the same frame constant c, then there must be a j such that $\mathcal{F}_j$ has only a single vector, with $c_j \neq c$ (since $\mathcal{F}$ can not be a tight frame). Moreover, by Remark 2.4 and Thm. 4.7, $c_j < c$ which implies that $\mathcal{F}_j \cup \mathcal{F}_i$ is not a global minimum, again by Thm. 4.7.

So, let $\mathcal{F}_i$, $\mathcal{F}_j$ be such pair of subsets. Notice that if $c_i > c_j$, then the vectors in $\mathcal{F}_j$ must be linear dependent, since it always have more than one vector (recall that the partition on orthogonal subsets of $\mathcal{F}$ is minimal). Then, from the proof of Claim 3 in the proof of [6, Thm. 10] we deduce that given $\varepsilon > 0$ there exist a set $\mathcal{F}(\varepsilon)$ such that $d(\mathcal{F}_i \cup \mathcal{F}_j, \mathcal{F}(\varepsilon)) \leq \varepsilon$ and FP($\mathcal{F}(\varepsilon)$) <FP($\mathcal{F}$). Hence, $\mathcal{F}$ is not a local minimizer of FP on $\mathcal{B}(\mathbf{a})$. □

By inspection of the previous proof, we see that the complete characterization of local minimum for every P_f on $\mathcal{B}(\mathbf{a})$ depends on the following problem:

Problem($\star$): let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \in \mathcal{B}(\mathbf{a})$ such that $\mathcal{F}_1 \perp \mathcal{F}_2$ and $\mathcal{F}_i$ is a tight frame on its span. Suppose that $\mathcal{F}$ is not a global minimum for P_f. Given $\varepsilon > 0$. Is there a frame $\mathcal{G} \in \mathcal{B}(\mathbf{a})$ such that $\|\psi_i - \phi_i\| < \varepsilon, \forall i$ and $P_f(\mathcal{G}) < P_f(\mathcal{F})$?

6. APPENDIX: A Geometrical approach to the frame perturbation problem

We now consider some well known facts from differential geometry that we shall need in the sequel. In what follows we consider the unitary group $\mathcal{U}(m)$ together with its natural differential geometric (Lie) structure. Given $U \in \mathcal{U}(m)$ we shall identify its tangent space

$$\mathcal{T}_U\,\mathcal{U}(m) = \{X \in \mathcal{M}_m(\mathbb{C}) : \; U^*X \in i \cdot \mathcal{M}_m(\mathbb{C})^{sa}\}$$

with the fixed space $\mathcal{T}_I\,\mathcal{U}(m) = i \cdot \mathcal{M}_m(\mathbb{C})^{sa}$ of $m \times m$ anti-hermitian matrices, via the isometric isomorphism $X \mapsto U^*X$. Given $G \in \mathcal{M}_m(\mathbb{C})^+$ we consider the smooth map $\Psi_G : \mathcal{U}(m) \to \mathcal{U}_m(G)$ given by $\Psi_G(U) = U^*GU$. Under the previous identification of the tangent spaces of $\mathcal{U}(m)$, the differential of Ψ_G at a point $U \in \mathcal{U}(m)$ in the direction given by $X \in i \cdot \mathcal{M}_m(\mathbb{C})^{sa}$ is given by

$$(D\Psi_G)_U(X) = [X, U^*GU]. \tag{24}$$

As it is well known, the differential $(D\Psi_G)_U$ is an epimorphism at every $U \in \mathcal{U}(m)$ and hence (24) gives us a description of the tangent space of the manifold $\mathcal{U}_m(G)$ at a point U^*GU.

Let $\Delta(G) = \{x \in \mathbb{R}^m : \; \sum_{i=1}^m x_i = \operatorname{tr}(G)\}$ and consider $\Phi_G : \mathcal{U}(m) \to \Delta(G)$ given by $\Phi_G(U) = \operatorname{d}(U^*GU)$, where $\operatorname{d}(A) \in \mathbb{R}^m$ is the main diagonal of the matrix $A \in \mathcal{M}_m(\mathbb{C})$. Notice that $\Delta(G)$ is a sub-manifold of $\mathbb{R}^m$ with tangent space at

$x \in \Delta(G)$

$$\mathcal{T}_x\Delta(G) = \{y \in \mathbb{R}^m : \sum_{i=1}^m y_i = 0\}.$$

Using (24), we get (identifying again the tangent spaces of $\mathcal{U}(m)$ as before) that the differential of Φ_G at a point $U \in \mathcal{U}(m)$ in the direction of $X \in i \cdot \mathcal{M}_m(\mathbb{C})^{sa}$ is

$$(D\Phi_G)_U(X) = \mathrm{d}([X, U^*GU]). \tag{25}$$

We shall be concerned with the existence of local cross sections of the map Φ_G around the identity $I \in \mathcal{U}(m)$. Since the map Φ_G is smooth, the existence of local cross sections of Φ_G is equivalent to the surjectivity of its differential $(D\Phi_G)_I$ around the identity.

Let us fix some notation first: we shall denote by $\mathbb{I}_m$ the (ordered) set $(1, 2, \ldots, m)$. Let $\{e_i\}_{i\in\mathbb{I}_m}$ be the canonical orthonormal basis in $\mathbb{C}^m$, for $I \subseteq \mathbb{I}_m$ we let P_I denote the (diagonal) projection onto the span$\{e_i : i \in I\}$. Finally, by $B_\delta(x)$ we mean a ball centered on x with radius δ, in the metric given by the context.

The following result is part of Step 1 in [11].

Lemma 6.1. *Let $G \in \mathcal{M}_m(\mathbb{C})^+$ with $d(G) = \mathbf{a}$ and consider Φ_G as before. Then, the differential $(D\Phi_G)_I : i \cdot \mathcal{M}_m(\mathbb{C})^{sa} \to \mathcal{T}_{\mathbf{a}}\Delta(G)$ is surjective, and hence Φ_G is open in $\Delta(G)$, if for $I \subseteq \mathbb{I}_m$ such that $P_IG = GP_I$ then $I = \mathbb{I}_m$ or $I = \emptyset$.*

Proof. Assume that $(D\Phi_G)_I$ is not surjective. Then, there exists $0 \neq x \in \mathcal{T}_{\mathbf{a}}\Delta(G)$ which is orthogonal to the image of $(D\Phi_G)_I$. Let D be the diagonal matrix with main diagonal $x \in \mathbb{R}^m$. Using (25) we get

$$0 = \langle \mathrm{d}([X, G]), x\rangle = \mathrm{tr}([X, G]D) = \mathrm{tr}(X[G, D]), \quad \forall X \in i \cdot \mathcal{M}_m(\mathbb{C})^{sa}. \tag{26}$$

Since $[G, D]$ is also anti-hermitian we get that $[G, D] = 0$ and hence G and D commute. If we let $I = \{i : x_i > 0\}$ we see, since P_I is a polynomial in D, that $[G, P_I] = 0$. Notice that $I \neq \emptyset$ and $I \neq \mathbb{I}_m$ since $\sum_{i=1}^m x_i = 0$. $\square$

Lemma 6.2. *Let us assume that the map $\Phi := \Phi_G$, defined as before for $G \in \mathcal{M}_m(\mathbb{C})^+$, has a local cross section around the identity. Let $\{G_i\}_i \subseteq \mathcal{M}_m(\mathbb{C})^+$ be a sequence converging to G and for $i \in \mathbb{N}$ let $\Phi_i := \Phi_{G_i}$ be defined as before. Then there exist $\delta > 0$ and $i_0 \in \mathbb{N}$ such that for $i \geq i_0$ then*

$$B_\delta(I) \cap \mathcal{U}(m) = \mathcal{S} + \mathcal{K}_i$$

where $\mathcal{S}$ and $\mathcal{K}_i$ are submanifolds with $I = (I_{\mathcal{S}}, I_{\mathcal{K}_i})$ and

$$\Phi_i|_{\mathcal{S}} : \mathcal{S} \to \Phi_i(\mathcal{S}), \quad \Phi|_{\mathcal{S}} : \mathcal{S} \to \Phi(\mathcal{S})$$

are diffeomorphisms.

Proof. First note that without loss of generality we can assume, as we shall, that $\mathrm{tr}(G_i) = \mathrm{tr}(G)$ for $i \in \mathbb{N}$. Also note that the maps Φ_i converges uniformly to Φ since

$$\Phi_i(U) - \Phi(U) = \mathrm{d}(U^*(G_i - G)U). \tag{27}$$

On the other hand, there is uniform convergence at the level of the differentials of these transformations. Indeed, under the previous identification of the tangent spaces of $\mathcal{U}(m)$ we can apply (25) and get

$$\|(D\Phi)_U(X) - (D\Phi_i)_U(X)\| = \|\mathrm{d}([X, U^*(G - G_i)U])\| \leq 2\sqrt{m}\, \|X\|\, \|G - G_i\|. \tag{28}$$

where $X \in i \cdot \mathcal{M}_m(\mathbb{C})^{sa}$ is arbitrary.

We now consider $\Gamma : W \to B_{\delta_1}(I) \cap \mathcal{U}(m)$ a diffeomorphic local chart, where $W \subseteq \mathbb{R}^p$ is an open set with $\Gamma(0) = I$. Let $\Phi \circ \Gamma : W \to \Delta(G)$ and notice that $(D(\Phi \circ \Gamma))_0 : \mathbb{R}^p \to \mathcal{T}_{\mathbf{a}}\Delta(G)$ is surjective. By continuity, we can assume that $(D(\Phi \circ \Gamma))_x$ is surjective for all $x \in W$. Hence, the orthogonal projection Q_x to $(\ker(D(\Phi \circ \Gamma))_x)^\perp$ is continuous on W. Indeed in this case we have that $Q_x = D_x^*(D_x D_x^*)^{-1} D_x$ since $D_x := (D(\Phi \circ \Gamma))_x$ is surjective on W. By continuity of the projections Q_x we can assume without loss of generality that $\|Q_0(1 - Q_x)\| \leq 1/4$ for all $x \in W$.

By taking $0 < \delta \leq \delta_1$ and using the uniform convergence of the differentials (28), we can assure that there exists a $i_1 \in \mathbb{N}$ such that for all $i \geq i_1$ then $(D(\Phi_i \circ \Gamma))_x$ is surjective for all $x \in W$. If $Q_{x,\,i}$ denotes the orthogonal projection onto $(\ker(D(\Phi_i \circ \Gamma))_x)^\perp$ then, using the previous description of $Q_{x,\,i}$ we see that for every $\epsilon > 0$ there exists $i(\epsilon)$ such that $\|Q_{x,\,i} - Q_x\| \leq \epsilon$ for $i \geq i(\epsilon)$ and for every $x \in W$. Let $i_2 = i(1/4) \in \mathbb{N}$, then if $i_0 = \max\{i_1, 1_2\}$, for every $x \in W$ and every $i \geq i_0$ we have

$$\|Q_0(1 - Q_{x,\,i})\| \leq \|Q_0(Q_x - Q_{x,\,i})\| + \|Q_0(1 - Q_x)\| \leq 1/2$$

and hence

$$(\ker(D(\Phi \circ \Gamma))_0)^\perp \cap \ker(D(\Phi_i \circ \Gamma))_x = \{0\}. \tag{29}$$

We now define $\mathcal{S} := \Gamma(\ker(D(\Phi \circ \Gamma))_0)^\perp \cap W)$ and $\mathcal{K} := \Gamma(\ker(D(\Phi \circ \Gamma))_0 \cap W)$. An straightforward argument using (29) now shows that $(D\Phi|_{\mathcal{S}})_x$ is injective and using a dimension argument we conclude that $(D\Phi|_{\mathcal{S}})_x$ is also surjective for all $x \in \mathcal{S}$; similarly with Φ_i for $i \geq i_0$. The lemma follows from these last facts.

□

Lemma 6.3. *Using the notations and assumptions of the previous lemma, let $\Psi : A\,(= A^0 \subseteq \mathbb{R}^t) \to \mathcal{S}$ be a local chart of $\mathcal{S}$ with $\Psi(0) = I_{\mathcal{S}}$ and let $V(r) := \Psi(B_r(0)) \subseteq \mathcal{S}$, where $\overline{B_r(0)} \subseteq A$. Then, for any such $r > 0$ there exists $\varepsilon > 0$ such that for $i \geq i_0$ then*

$$B_\varepsilon(d(G_i)) \subseteq \Phi_i(V(r)). \tag{30}$$

Proof. Fix r as above and let $V = V(r)$. Note that for $i \geq i_0$ then $\Phi_i(I_{\mathcal{S}}) = \mathrm{d}(G_i)$ is an interior point of $\Phi_i(V)$ and similarly $\Phi(I_{\mathcal{S}}) = \mathrm{d}(G)$ is an interior point of $\Phi(V)$. We show that there exists $\epsilon > 0$ such that for all $i \geq i_0$ then

$$\inf_{x \in \partial\Phi_i(V)} \|\mathrm{d}(G_i) - x\| = \min_{x \in \partial\Phi_i(V)} \|\mathrm{d}(G_i) - x\| \geq \epsilon \tag{31}$$

where $\partial\Phi_i(V)$ stands for boundary of the image $\Phi_i(V)$ in $\Delta(G)$. Observe that the lemma is a consequence of the condition given in (31).

Indeed, assume that (31) is not true. Then, there exists a (sub)-sequence (Φ_{i_k}) such that

$$\inf_{x \in \partial\Phi_{i_k}(V)} \|\mathrm{d}(G_{i_k}) - x\| = \|\ \mathrm{d}(G_{i_k}) - x_k\| \leq \frac{1}{k} \tag{32}$$

for some $x_k = \Phi_{i_k}(U_k)$ with $U_k \in \partial V \subseteq \mathcal{S}$ since $\Phi_{i_k}(\partial V) = \partial\Phi_{i_k}(V) \subseteq \Delta$. But then for every $k \in \mathbb{N}$ and $U_k \in \partial V$ then

$$\begin{aligned} \|\mathrm{d}(G) - \Phi(U_k)\| &\leq \|\mathrm{d}(G) - \mathrm{d}(G_{i_k})\| + \|\mathrm{d}(G_{i_k}) - \Phi_{i_k}(U_k)\| + \|\Phi_{i_k}(U_k) - \Phi(U_k)\| \\ &= \|\mathrm{d}(G) - \mathrm{d}(G_{i_k})\| + \|\mathrm{d}(G_{i_k}) - x_k\| + \|\Phi_{i_k}(U_k) - \Phi(U_k)\| \underset{k}{\to} 0 \end{aligned}$$

by (32) and the convergences $\mathrm{d}(G_{i_k}) \to \mathrm{d}(G)$ and $\Phi_{i_k}(U_k) \to \Phi(U_k)$. But this implies that $\mathrm{d}(G)$ is not an interior point of $\Phi(V)$ since in this case

$$\inf_{x\in\partial\Phi(V)} \|\mathrm{d}(G)-x\| = \inf_{z\in\partial V} \|\mathrm{d}(G)-\Phi(z)\| = 0$$

which contradicts the claims at the beginning of this proof. □

Theorem 6.4. *Let $\mathcal{F} = \{\phi_j\}_{j=1}^m \subseteq \mathbb{C}^d$ be a list of vectors, let $G = G^{\mathcal{F}}$ be its Grammian operator and assume that $\Phi := \Phi_G$ has a local cross section around the identity.*

Let $\{S_i\}_i \in \mathcal{M}_d(\mathbb{C})^+$ be a sequence converging to $S = S^{\mathcal{F}}$. Then, for every $\eta > 0$ there exists $i_1 \in \mathbb{N}$ such that for each $i \geq i_1$ there exists a frame $\mathcal{G}(i) = \mathcal{G} = \{\psi_j\}_{j=1}^m$ such that:

1. $\|\psi_j\| = \|\phi_j\|$ *for* $1 \leq j \leq m$.
2. $\|\psi_j - \phi_j\| \leq \eta$ *for* $1 \leq j \leq m$.
3. $S^{\mathcal{G}} = S_i$.

Proof. Let $T = T^{\mathcal{F}} : \mathbb{C}^m \to \mathbb{C}^d$ be the frame operator of the list $\mathcal{F}$ with polar decomposition $T = |T^*|\, W = S^{1/2}\, W$ for a co-isometry $W : \mathbb{C}^m \to \mathbb{C}^d$. Define $G_i = W^* S_i W$ and notice that, by our hypothesis, $\|G_i - G\| \xrightarrow{i} 0$.

Using the notation introduced in the previous lemmas, let $\Psi : A\,(= A^0 \subseteq \mathbb{R}^t) \to \mathcal{S}$ be a local chart and $r > 0$ be small enough so that $B_r(0) \subseteq A$ and for $U \in V(r) = \Psi(B_r(0))$ then

$$\|U - I\| \leq \frac{\eta}{2(\|S^{1/2}\| + \eta/2)}. \tag{33}$$

For this choice of $r > 0$ let $\varepsilon > 0$ be as in (30) for $i \geq i_0 \in \mathbb{N}$. Let $i_2 \in \mathbb{N}$ be such that, for $i \geq i_2$ then $\|S^{1/2} - S_i^{1/2}\| \leq \eta/2$ and $\|G_i - G\| \leq \frac{\varepsilon}{\sqrt{m}}$.

If we now define $i_1 = \max(i_0, i_2)$ then for $i \geq i_1$ we further have

$$\|\Phi(I) - \Phi_i(I)\| = \|\mathrm{d}(G - G_i)\| \leq \sqrt{m}\, \|G - G_i\| < \varepsilon \;\Rightarrow\; \mathrm{d}(G) \in \Phi_i(V(r)). \tag{34}$$

We fix $i \geq i_1$ and construct $\mathcal{G} = \mathcal{G}(i)$ with the desired properties. By Lemma 6.2 and (34) there exists $U \in V(r) \subseteq \mathcal{S}$ such that $\Phi_i(U) = \mathrm{d}(G)$.

Define $\tilde{T} := S_i^{1/2}\, WU$, and $\mathcal{G} = \{\psi_j\}_{j=1}^m = \{\tilde{T}(e_j)\}_{j=1}^m$ where $\{e_j\}_{j=1}^m$ denotes the canonical basis of $\mathbb{C}^m$. Since by construction $G^{\mathcal{G}} = U^* G_i U$ and $S^{\mathcal{G}} = S_i$, then items (1) and (3) hold true. Item (2) follows from the inequality

$$\|T - \tilde{T}\| = \|S^{1/2}W - S_i^{1/2}WU\| \leq \|S^{1/2} - S_i^{1/2}\| + \|S_i^{1/2}\|\, \|I - U\| \leq \eta.$$

□

Proof of Proposition 5.3. This is an immediate consequence of Lemma 6.1 and Theorem 6.4. □

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