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Quaternionic (super) twistors extensions and general superspaces

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In a attempt to treat a supergravity as a tensor representation, the four-dimensional N-extended quaternionic superspaces are constructed from the (diffeomorphyc) graded extension of the ordinary Penrose-twistor formulation, performed in a previous work of the authors [D. J . Cirilo-Lombardo and V. N. Pervushin, *Int. J. Geom. Methods Mod. Phys.*, doi: http://dx.doi.org/10.1142/S0219887816501139.], with $N = p + k$. These quaternionic superspaces have $4 + k(N - k)$ even-quaternionic coordinates and $4N$ odd-quaternionic coordinates, where each coordinate is a quaternion composed by four $\mathbb{C}$-fields (bosons and fermions respectively). The fields content as the dimensionality (even and odd sectors) of these superspaces are given and exemplified by selected physical cases. In this case, the number of fields of the supergravity is determined by the number of components of the tensor representation of the four-dimensional N-extended quaternionic superspaces. The role of tensorial central charges for any N even $USp(N) = Sp(N, \mathbb{H}_{\mathbb{C}}) \cap U(N, \mathbb{H}_{\mathbb{C}})$ is elucidated from this theoretical context.

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1. Introduction

In theoretical physics from long ago, there are an increment of the use of modern mathematical methods to treat several problems of diverse degrees of complexity [9]. From the hydrodynamics and the mechanics of the continuous media, passing for the quantum mechanics (QM), quantum field theory (QFT) and relativistic astrophysics, the correct description of the physical phenomena is based in the application of the geometry and group theory. Between these methods, the

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introduction of spinors constrained by conformal symmetries in order to map the spacetime, have a particular importance in classical and quantum field theories. For example, in classical mechanics, string theories and supersymmetrical extensions of the spacetime, this construction (map) has been successfully introduced for the description of different scenarios. Here, we introduce such "spinorial mapping" (supertwistors) to describe the diverse superspaces. According with Penrose's suggestion, the spacetime continuum can be considered as a derivative construction with respect to an underlying spinor structure. For instance, the spinor structure contains the pre-images of the fundamental properties of the classical spacetime: dimension, signature, connections, etc. These superspaces will be the basis of nonlinear realized unified theories containing the SM+GR that, with the help of a super biquaternionic extension of the coordinates, the correct number of fields will be reached.

2. Twistor Theory and Quaternionic Extension

As suggested by Penrose long ago [3], from the beginning in the twistor theory the starting point a complex space $\mathbb{C}M \sim \mathbb{C}_{2,4}(T)$ by mean conformal spinors $t_\alpha = (\omega^\alpha, \pi_\alpha)$ with $\alpha = 1, 2$ and $a = 1, 2, 3, 4$ as describing the prior geometry with the complex Minkowski space coordinates usually denoted

$$z^{\dot\alpha\beta} = \frac{1}{2}\sigma_\mu^{\dot\alpha\beta} z^\mu \tag{1}$$

related with the twistors coordinates $t_\alpha \subset T$ by the incidence equation

$$\omega^{\dot\alpha} = iz^{\dot\alpha\beta}\pi_\beta \tag{2}$$

that is in fact, a particular case of geometrical (in general harmonic) mapping. Notice that (1) is directly a biquaternion, namely

$$z^{\dot\alpha\beta} = \frac{1}{2}(\sigma_0^{\dot\alpha\beta} z^0 + \sigma_1^{\dot\alpha\beta} z^1 + \sigma_2^{\dot\alpha\beta} z^2 + \sigma_3^{\dot\alpha\beta} z^3), \quad z^\mu \in \mathbb{C} \tag{3}$$

that have eight real dimensions and can be extended even more to 16 real dimensional if each coordinate is quaternionic itself, namely

$$q^{\dot\alpha\beta} = \frac{1}{2}(\sigma_0^{\dot\alpha\beta} q^0 + \sigma_1^{\dot\alpha\beta} q^1 + \sigma_2^{\dot\alpha\beta} q^2 + \sigma_3^{\dot\alpha\beta} q^3), \quad q^\mu \in \mathbb{H}. \tag{4}$$

Now we can promote $t_\alpha = (\omega^{\dot\alpha}, \pi_\alpha)$ to fermionic quaternion, then a quaternionic twistor is a quaternion with conformal spinors as coefficients. From (2), a point in a CM space (in this case an element of $\mathbb{H}_1(\mathbb{C})$) defines a plane in T or a line in $CP(3)$. Consequently, the mapping (1) is a one-dimensional quaternionic one (minimal map in $\mathbb{H}$).

2.1. *Quaternionic conformal spinors*

A four component spinor over a field K can be realized following the scheme.

Remark 1.

$$\text{Majorana } K = \mathbb{R}\xi(x) \tag{5}$$

$$\text{Dirac } K = \mathbb{C}\psi(x) = \frac{1}{\sqrt{2}}(\xi_1(x) + i\xi_2(x)) \tag{6}$$

$$\text{Quaternionic } K = \mathbb{H}\Psi(x) = \frac{1}{\sqrt{2}}(\xi_0(x) + \widehat{i}_i\xi_i(x)) \quad (i = 1, 2, 3) \tag{7}$$

$$\text{Bi-Quaternionic } K = \mathbb{H}_\mathbb{C}\Psi(x) = \frac{1}{\sqrt{2}}(\psi_0(x) + \widehat{i}_i\psi_i(x)) \quad (i = 1, 2, 3). \tag{8}$$

Case 5 is an ordinary Majorana fermion realized over reals, Case 6 is the complex realized Dirac one, Case 7 is the quaternionic Dirac field (ordinary fermionic-quaternion) and Case 8 is a biquaternionic realized spinor where each coefficient is a Dirac field. Case 8 is what we are interested in.

2.2. *Quaternionic extension*

In the quaternionic-twistor theory our starting point a quaternionic space $\mathbb{H}M \sim \mathbb{H}_{1,4}(T)$ implemented as $\mathbb{C}M \times \mathbb{C}M$ by mean quaternionic spinors $t_\alpha = (\omega^{\dot\alpha}, \pi_\alpha)$ (with $\alpha = 1, 2$ and $a = 1, 2, 3, 4$ and $\omega^{\dot\alpha}, \pi_\alpha \in \mathbb{H}$) as describing the prior geometry with the quaternionic Minkowski space coordinates denoted

$$q^{\dot\alpha\beta} = \frac{1}{2}\sigma_\mu^{\dot\alpha\beta}q^\mu \tag{9}$$

related with the twistors coordinates $t_\alpha \subset T$ by the quaternionic incidence equation

$$\omega^{\dot\alpha} = iq^{\dot\alpha\beta}\pi_\beta \tag{10}$$

we have now 16 real dimensions being each coordinate quaternionic

$$q^{\dot\alpha\beta} = \frac{1}{2}(\sigma_0^{\dot\alpha\beta}q^0 + \sigma_1^{\dot\alpha\beta}q^1 + \sigma_2^{\dot\alpha\beta}q^2 + \sigma_3^{\dot\alpha\beta}q^3), \quad q^\mu \in \mathbb{H} \tag{11}$$

being $\omega^{\dot\alpha}, \pi_\alpha \in \mathbb{H}$ biquaternionic fermions of the form given by Case 8 (see [15] for the simplest case with reality condition), namely $\Psi(x) = \frac{1}{\sqrt{2}}(\psi_0(x) + \widehat{i}_i\psi_i(x))$ ($i = 1, 2, 3$), as we have pointed out before. As in the ordinary twistor case, we can introduce a pair of nonparallel quaternionic twistors to determine the corresponding "point" (really a $\mathbb{R}$-subspace) $q \in \mathbb{H}M$ by solving the matrix equation

$$Q = -i\Omega\Pi^{-1}$$

$$\rightarrow q^{\dot\alpha\beta} = -i\frac{(\omega_1^{\dot\alpha}\pi_2^\beta - \omega_2^{\dot\alpha}\pi_1^\beta)}{\pi_2^\alpha\pi_{\alpha 1}}, \tag{12}$$

where the matrices are $Q = (q^{\dot\alpha\beta}), \Omega = (\omega_1^{\dot\alpha}, \omega_2^{\dot\alpha}), \Pi = (\pi_{\alpha 1}, \pi_{\alpha 2})$.

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As is easily seen, the $\mathbb{H}M$ coordinates are invariant under the transformations in T as follows

$$\Omega' = X\Omega, \quad \Pi' = X\Pi \tag{13}$$

with $X \subset GL(2, \mathbb{H})$. Notice the important fact that the introduction of biquaternions (e.g. an almost complex structure) imply an underlying symplectic one.

3. Supergroups and Quaternionic Superframes

Some points to consider about complex graded vector spaces and quaternionic ones:

(i) For quaternions, the minimal dimension for its representation is 2 (e.g. $SU(2) \times \mathbb{R}_1$).

(ii) The corresponding complex graded vector space is consequently $\mathbb{C}^{2n;2m}$ equivalent to $\mathbb{H}^{n;m}$ due (i) with n quaternions with $4n$ complex commuting coefficients (bosons) and $4m$ complex anticommuting coefficients (fermions).

3.1. *Fundamental representations*

As was commented somewhere for the simplest complex case [8] (see also [13] for introduction to quaternionic structures in QFT), it is possible to introduce (in the biquaternionic case) the following two fundamental representations of $SU(2, 2; N \| \mathbb{H}_{\mathbb{C}})$:

(a) Quaternionic supertwistors: [10]

$$T_A^{(N)} = (t_1 \ldots t_4, \xi_1 \ldots \xi_N) \subset T_{\mathbb{H}_{\mathbb{C}}}^{(N)} \equiv \mathbb{H}_{\mathbb{C}}^{4;N}$$

$$(\mathbb{C}^{8;2N} \text{biquaternionic extension of the Ferber construction}). \tag{14}$$

(b) Quaternionic fermionic supertwistors:

$$\widetilde{T}_A^{(N)} = (\eta_1 \ldots \eta_4, u_1 \ldots u_N) \subset \widetilde{T}_{\mathbb{H}_{\mathbb{C}}}^{(N)} \equiv \mathbb{H}_{\mathbb{C}}^{N;4}$$

$$(\mathbb{C}^{2N;8} \text{biquaternionic extension of the}$$

$$\text{Litov–Pervushin construction}). \tag{15}$$

For these two biquaternionic representations the following $U(2, 2; N \| \mathbb{H}_{\mathbb{C}})$ scalar products can be introduced:

$$\langle T^{(N)}, T^{(N)\prime} \rangle = \overline{T}_A^{(N)} G^{AB} T_B^{(N)\prime} \quad (even)$$

$$\langle \widetilde{T}^{(N)}, \widetilde{T}^{(N)\prime} \rangle = \overline{\widetilde{T}}_A^{(N)} G^{AB} \widetilde{T}_B^{(N)\prime} \quad (even)$$

$$\langle \widetilde{T}^{(N)}, T^{(N)\prime} \rangle = \langle T^{(N)}, \widetilde{T}^{(N)\prime} \rangle = \overline{\widetilde{T}}_A^{(N)} G^{AB} T_B^{(N)\prime}$$

$$= \overline{T}_A^{(N)} G^{AB} \widetilde{T}_B^{(N)\prime} \quad (odd : \text{linear in fermionic coordinates}),$$

where $G_{AB}^{(N)}$ is the $U(2, 2; N \| \mathbb{H}_{\mathbb{C}})$ supermetric, schematically $G_{AB}^{(N)} = \begin{pmatrix} g & 0 \\ 0 & iI_N \end{pmatrix} g$ is symplectic and each entry is a quaternionic one (e.g a 2×2 block). Consequently,

the supergroup $U(2, 2; N \| \mathbb{H}_\mathbb{C})$ is defined as the set of graded $2(4 + N) \times 2(4 + N)$ matrices U ($(4+N)\times(4+N)$ biquaternionic matrices in the lowest $\mathbb{H}$-representation) satisfying the relation

$$^*U_{DA}G^{(N)}_{AB}U_{BC} = G^{(N)}_{DC}.$$

Let us to observe, in resume, the following points:

- (A) The graded quaternionic matrices U:

  (i) Are described by four *biquaternionic supertwistors* and N *biquaternionic fermionic supertwistors*.
  (ii) Are represented with $2(4 + N) \times 2(4 + N)$ matrices U ($(4 + N) \times (4 + N)$ biquaternionic matrices in the lowest $\mathbb{H}$-representation).

- (B) And the four *biquaternionic supertwistors* and N *biquaternionic fermionic supertwistors* satisfy:

  (i) $16 = 4 \times 4$ relations defining an arbitrary four frame in the biquaternionic supertwistor space $\mathbb{H}^{4;N}_\mathbb{C}$;
  (ii) $N \times N$ biquaternionic relations defining an arbitrary N-frame in the biquaternionic fermionic supertwistor space $\mathbb{H}^{N;N}_\mathbb{C}$;
  (iii) $4N$ biquaternionic relations defining graded structure (orthogonality in some cases) of the biquaternionic superframes in $\mathbb{H}^{4;N}_\mathbb{C}$ and $\mathbb{H}^{N;4}_\mathbb{C}$. Schematically these structures are:

  $$\begin{pmatrix} 4 \times 4 & 4 \times N(\mathbb{H}^{4,N}_\mathbb{C}) \\ N \times 4(\mathbb{H}^{N,4}_\mathbb{C}) & N \times N(\mathbb{H}^{N,N}_\mathbb{C}) \end{pmatrix}.$$

- (C) And the most important is that the biquaternionic fields t, ξ, η, u (indexes avoided) into the structures $T^{(N)}_A$ and $\widetilde{T}^{(N)}_A$ contain four spinors each one as coefficients: e.g.

$$t_a = e_0 t^0_a + e_i t^i_a, \tag{16}$$

$$u_N = e_0 u^0_N + e_i u^i_N, \quad (i = 1, 2, 3) \tag{17}$$

etc. In consequence, the supergroup $SU(2, 2; N \,|\, \mathbb{H}_\mathbb{C})$ describes all the superframes-(modulo a global phase factor) in $\mathcal{T}^{(N)}_{\mathbb{H}_\mathbb{C}} = T^{(N)}_{\mathbb{H}_\mathbb{C}} \oplus \widetilde{T}^{(N)}_{\mathbb{H}_\mathbb{C}} = \mathbb{H}^{4;N}_\mathbb{C} \oplus \mathbb{H}^{N;4}_\mathbb{C}$ (Penrose notation), being the corresponding parametric equations describing the respective coset written in $\mathcal{T}^{(N)}_{\mathbb{H}_\mathbb{C}}$.

4. Penrose Equations and Super-Quaternionic Extension: The General Case

The quaternionic superspace coordinates are ($N = p + k$)

$$\mathbb{H}_\mathbb{C}(N, N - k) = (q_\mu, \lambda^{\,r}_s, \theta^{\,\alpha}_s, \theta^{\dot\alpha r}), \tag{18}$$

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where in the general case, $k(N - k)$ bosonic quaternionic coordinates describe the internal symmetry coset $\frac{U(N \mid \mathbb{H}_{\mathbb{C}})}{U(N \mid \mathbb{H}_{\mathbb{C}})U(N-k \mid \mathbb{H}_{\mathbb{C}})}$ and shall be called *regular* or *Fueter-analytic* coordinates. The above quaternionic superspace coordinate can be determined in terms of two supertwistor and k fermionic supertwistors. This can be achieved introducing the following graded matrices:

(i) The $(2 + N - k) \times (2 + k)$ quaternionic matrix describing the quaternionic superspace coordinates

$$Q = \begin{pmatrix} q^{\dot{\alpha}\beta} & \theta^{\dot{\alpha}r} \\ \theta_s{}^{\beta} & \lambda_s{}^r \end{pmatrix} \tag{19}$$

also the graded $(2 + N - k) \times (2 + k)$ quaternionic matrix

$$\Omega = \begin{pmatrix} \omega^{\dot{\alpha}}_{\gamma} & \rho^{\dot{\alpha}}_r \\ \varsigma_{s\gamma} & u_{sr} \end{pmatrix} \tag{20}$$

and the $(2 + k) \times (2 + k)$ quaternionic matrix

$$\Pi = \begin{pmatrix} \pi_{\alpha\gamma} & \sigma_{\alpha r'} \\ \xi_{r\gamma} & v_{rr'} \end{pmatrix}, \tag{21}$$

where $\gamma = 1, 2$ is twistor index and $r = 1, \ldots, k, s = 1, \ldots, N - k(N - k = p$ in the Litov–Pervushin notation). Consequently, the superextension of the ordinary twistor equations becomes to

$$\begin{pmatrix} \omega^{\dot{\alpha}}_{\gamma} & \rho^{\dot{\alpha}}_{r'} \\ \varsigma_{s\gamma} & u_{sr'} \end{pmatrix} = \begin{pmatrix} q^{\dot{\alpha}\beta}\pi_{\beta\gamma} + \theta^{\dot{\alpha}r}\xi_{r\gamma} & q^{\dot{\alpha}\beta}\sigma_{\beta r'} + \theta^{\dot{\alpha}r}v_{rr'} \\ \theta_s{}^{\beta}\pi_{\beta\gamma} + \lambda_s{}^r\xi_{r\gamma} & \theta_s{}^{\beta}\sigma_{\beta r'} + \lambda_s{}^r v_{rr'} \end{pmatrix}. \tag{22}$$

Consequently, the reconstruction via twistors of the superspace namely

$$Q = -i\Omega\Pi^{-1} \tag{23}$$

using

$$\Pi^{-1} = \begin{pmatrix} A^{\gamma\beta} & B^{\gamma'r} \\ C^{\gamma\beta} & D^{r'r} \end{pmatrix}, \tag{24}$$

where

$$\begin{aligned}
A^{\gamma\beta} &= -(\pi^{-1})^{r''\gamma}\sigma_{r''r'}[v^{r'\beta} - \xi^{\beta}_{\gamma'}(\pi^{-1})^{\gamma''\gamma'}\sigma^{r'}_{\gamma''}]^{-1}, \\
B^{\gamma'r} &= -(\xi^{-1})^{r''\gamma'}v_{r''r'}[\sigma^{rr'} - \pi^{r\gamma}(\xi^{-1})_{\gamma r'''}v^{r'''r'}]^{-1}, \\
C^{r'\beta} &= [v^{r'\beta} - \xi^{\beta}_{\gamma'}(\pi^{-1})^{\gamma''\gamma'}\sigma^{r'}_{\gamma''}]^{-1}, \\
D^{r'r} &= [\sigma^{rr'} - \pi^r_{\gamma}(\xi^{-1})^{\gamma r'''}v^{r'}_{r'''}]^{-1}
\end{aligned} \tag{25}$$

carry explicitly to the following expressions in general form:

$$q^{\dot{\alpha}\beta} = (-\omega^{\dot{\alpha}}_{\gamma}(\pi^{-1})^{\gamma'\gamma}\sigma_{\gamma'r'} + \rho^{\dot{\alpha}}_{r'})[v^{r'\beta} - \xi^{\beta}_{\gamma'}(\pi^{-1})^{\gamma''\gamma'}\sigma^{r'}_{\gamma''}]^{-1}, \tag{26}$$

$$\theta^{\dot{\alpha}r} = (-\omega^{\dot{\alpha}}_{\gamma}(\xi^{-1})^{\gamma'\gamma}v_{\gamma'r'} + \rho^{\dot{\alpha}}_{r'})[\sigma^{rr'} - \pi^r_{\gamma}(\xi^{-1})^{\gamma r'''}v^{r'}_{r'''}]^{-1}, \tag{27}$$

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$$\theta_s^{\ \beta} = (-\varsigma_{s\gamma}(\pi^{-1})^{\gamma'\gamma}\sigma_{\gamma'r'} + u_{sr'})[v^{r'\beta} - \xi_{\gamma'}^{\beta}(\pi^{-1})^{\gamma''\gamma'}\sigma_{\gamma''}^{\ r'}]^{-1}, \tag{28}$$

$$\lambda_s^{\ r} = (-\varsigma_{s\gamma}(\xi^{-1})^{\gamma'\gamma}v_{\gamma'r'} + u_{sr'})[\sigma^{rr'} - \pi_{\ \gamma}^{r}(\xi^{-1})^{\gamma r'''}v_{r'''}^{\ r'}]^{-1}. \tag{29}$$

5. The $N = 4, K = 2$ Superspace

This is the corresponding to the case $p = q = 2(N = p + q)$ in the Litov–Pervushin notation [6]. Equations (23) and (24) describe for $N = 4, k = 2$ the supercoset

$$\frac{SU(2, 2; 4 \mid \mathbb{H}_{\mathbb{C}})}{SU\left(2; \underbrace{2}_{k} \mid \mathbb{H}_{\mathbb{C}}\right) \times SU\left(2; \underbrace{2}_{N-k=p} \mid \mathbb{H}_{\mathbb{C}}\right)}, \tag{30}$$

$$Q = \begin{pmatrix} q^{\dot{1}1} & q^{\dot{1}2} & \theta^{\dot{1}1} & \theta^{\dot{1}2} \\ q^{\dot{2}1} & q^{\dot{2}2} & \theta^{\dot{2}1} & \theta^{\dot{2}2} \\ \theta_1^1 & \theta_1^2 & \lambda_1^1 & \lambda_1^2 \\ \theta_2^1 & \theta_2^2 & \lambda_2^1 & \lambda_2^2 \end{pmatrix}, \tag{31}$$

$$\Pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \sigma_{11} & \sigma_{12} \\ \pi_{21} & \pi_{22} & \sigma_{21} & \sigma_{22} \\ \xi_{11} & \xi_{12} & v_{11} & v_{12} \\ \xi_{21} & \xi_{22} & v_{21} & v_{22} \end{pmatrix}, \tag{32}$$

$$\Omega = \begin{pmatrix} \omega_1^{\dot{1}} & \omega_2^{\dot{1}} & \rho_1^{\dot{1}} & \rho_2^{\dot{1}} \\ \omega_1^{\dot{2}} & \omega_2^{\dot{2}} & \rho_1^{\dot{2}} & \rho_2^{\dot{2}} \\ \varsigma_{11} & \varsigma_{12} & u_{11} & u_{12} \\ \varsigma_{12} & \varsigma_{22} & u_{12} & u_{22} \end{pmatrix}, \tag{33}$$

where $\gamma = 1, 2$ is twistor index $r = 1, \ldots, k, s = 1, \ldots, N - k$ $(N - k = p)$.

Explicitly

$$q^{\dot{\alpha}\beta} = (-\omega_1^{\dot{\alpha}}(\pi^{-1})^{21}\sigma_{2r'} + \omega_2^{\dot{\alpha}}(\pi^{-1})^{12}\sigma_{1r'} + \rho_{r'}^{\dot{\alpha}})$$
$$\times [v^{r'\beta} - \xi_1^{\beta}(\pi^{-1})^{21}\sigma_2^{\ r'} + \xi_2^{\beta}(\pi^{-1})^{12}\sigma_1^{\ r'}]^{-1}, \tag{34}$$

$$\theta^{\dot{\alpha}r} = (-\omega_1^{\dot{\alpha}}(\xi^{-1})^{21}v_{2r'} + \omega_2^{\dot{\alpha}}(\xi^{-1})^{12}v_{1r'} + \rho_{r'}^{\dot{\alpha}})$$
$$\times [\sigma^{rr'} - \pi_1^r(\xi^{-1})_2^{\ r'''}v_{r'''}^{\ r'} + \pi_2^r(\xi^{-1})_1^{\ r'''}v_{r'''}^{\ r'}]^{-1}, \tag{35}$$

$$\theta_s^{\ \beta} = (-\varsigma_{s1}(\pi^{-1})^{21}\sigma_{2r'} + \varsigma_{s2}(\pi^{-1})^{12}\sigma_{1r'} + u_{sr'})$$
$$\times [v^{r'\beta} - \xi_1^{\beta}(\pi^{-1})^{21}\sigma_2^{\ r'} + \xi_2^{\beta}(\pi^{-1})^{12}\sigma_1^{\ r'}]^{-1}, \tag{36}$$

$$\lambda_s{}^r = (-\varsigma_{s1}(\xi^{-1})^{21}v_{2r'} + \varsigma_{s2}(\xi^{-1})^{12}v_{1r'} + u_{sr'})$$
$$\times [\sigma^{rr'} - \pi^r{}_1(\xi^{-1})_2{}^{r'''}v_{r'''}^{r'} + \pi^r{}_2(\xi^{-1})_1{}^{r'''}v_{r'''}^{r'}]^{-1}. \tag{37}$$

Evidently, the above solutions are invariant under the arbitrary $R \in GL(2; 2 \,|\, \mathbb{H}_{\mathbb{C}})$ supertransformations (e.g.: superrotations) namely

$$\Omega' \to \Omega R, \quad \Pi' \to \Pi R \tag{38}$$

with Ω and Π for $N = 4, k = 2$. We easily see that two biquaternionic supertwistors and two biquaternionic fermionic supertwistors are described by 64 bosonic complex coordinates (given by eighth bosonic biquaternions) and by 64 fermionic complex coordinates (given by eighth fermionic biquaternions): one half of these coordinates describes the $N = 4, k = 2$ biquaternionic superspace and the second half, however, describes the $GL(2; 2 \,|\, \mathbb{H}_{\mathbb{C}})$ degrees of freedom. Schematically, from the matrix Q, each block describes faithfully the following supercoordinates

$$\begin{pmatrix} q^{11} & q^{12} \\ q^{\dot{2}1} & q^{\dot{2}2} \end{pmatrix} \to 2 \times 4 \times 4 = 32 \text{ bosonic fields}, \tag{39}$$

$$\begin{pmatrix} \lambda_1^1 & \lambda_1^2 \\ \lambda_2^1 & \lambda_2^2 \end{pmatrix} \to 2 \times 4 \times 4 = 32 \text{ bosonic fields}, \tag{40}$$

$$\begin{pmatrix} \theta_1^1 & \theta_1^2 \\ \theta_2^1 & \theta_2^2 \end{pmatrix} \to 2 \times 4 \times 4 = 32 \text{ fermionic fields}, \tag{41}$$

$$\begin{pmatrix} \theta^{\dot{1}1} & \theta^{\dot{1}2} \\ \theta^{\dot{2}1} & \theta^{\dot{2}2} \end{pmatrix} \to 2 \times 4 \times 4 = 32 \text{ fermionic fields}. \tag{42}$$

It is important to remark here that if the supermanifold described (spanned) by two biquaternionic supertwistors : $T_1^{(4)} \; T_2^{(4)}$ and two biquaternionic fermionic $\widetilde{T}_1^{(4)} \; \widetilde{T}_2^{(4)}$ supertwistors is totally null or supergeodesic with respect to the norm of $SU(2, 2; 4, 4\|\mathbb{H}_{\mathbb{C}})$, namely

$$\langle T_1^{(4)}, T_1^{(4)} \rangle = \langle T_2^{(4)}, T_2^{(4)} \rangle = \langle \widetilde{T}_1^{(4)}, \widetilde{T}_1^{(4)} \rangle = \langle \widetilde{T}_2^{(4)}, \widetilde{T}_2^{(4)} \rangle = 0 \tag{43}$$

$$\langle T_1^{(4)}, T_2^{(4)} \rangle = \langle T_2^{(4)}, \widetilde{T}_1^{(4)} \rangle = \langle T_1^{(4)}, \widetilde{T}_2^{(4)} \rangle$$
$$= \langle \widetilde{T}_1^{(4)}, T_1^{(4)} \rangle = \langle T_2^{(4)}, \widetilde{T}_2^{(4)} \rangle = \langle \widetilde{T}_1^{(4)}, \widetilde{T}_2^{(4)} \rangle = 0 \tag{44}$$

notice that this fact does not implies, in principle, a reality condition. However, these conditions enforce a Majorana condition over the antidiagonal fermionic sectors into the matrix Q.

Notice that the constraints (43) four bosonic complex variables can be eliminated, and the constraints (44) two complex bosonic variables and four fermionic ones can be eliminated. This counting evisently agrees with the restriction of

biquaternionic superspace degrees of freedom of by the graded pseudohermiticity condition given by construction.

6. The $N = 8, k = 4$ Superspace

This is the corresponding to the case $p = q = 2$ $(N = p + q)$ in the Litov–Pervushin notation. Equations (23) and (24) describe for $N = 8, k = 4$ the supercoset

$$\frac{SU(2,2;8 \mid \mathbb{H}_{\mathbb{C}})}{SU\left(2; \underbrace{4}_{k} \mid \mathbb{H}_{\mathbb{C}}\right) \times SU\left(2; \underbrace{4}_{N-k=p} \mid \mathbb{H}_{\mathbb{C}}\right)}, \tag{45}$$

$$Q = \begin{pmatrix} q^{\dot{1}1} & q^{\dot{1}2} & \theta^{\dot{1}1} & \theta^{\dot{1}2} & \theta^{\dot{1}3} & \theta^{\dot{1}4} \\ q^{\dot{2}1} & q^{\dot{2}2} & \theta^{\dot{2}1} & \theta^{\dot{2}2} & \theta^{\dot{2}3} & \theta^{\dot{2}4} \\ \theta_1^1 & \theta_1^2 & \lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 \\ \theta_2^1 & \theta_2^2 & \lambda_2^1 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 \\ \theta_3^1 & \theta_3^2 & \lambda_3^1 & \lambda_3^2 & \lambda_3^3 & \lambda_3^4 \\ \theta_4^1 & \theta_4^2 & \lambda_4^1 & \lambda_4^2 & \lambda_4^3 & \lambda_4^4 \end{pmatrix}, \tag{46}$$

$$\Pi = \begin{pmatrix} \pi_{11} & \pi_{12} & \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \pi_{21} & \pi_{22} & \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \xi_{11} & \xi_{12} & v_{11} & v_{12} & v_{13} & v_{14} \\ \xi_{21} & \xi_{22} & v_{21} & v_{22} & v_{23} & v_{24} \\ \xi_{31} & \xi_{32} & v_{31} & v_{32} & v_{33} & v_{34} \\ \xi_{41} & \xi_{42} & v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix}, \tag{47}$$

$$\Omega = \begin{pmatrix} \omega_1^{\dot{1}} & \omega_2^{\dot{1}} & \rho_1^{\dot{1}} & \rho_2^{\dot{1}} & \rho_3^{\dot{1}} & \rho_4^{\dot{1}} \\ \omega_1^{\dot{2}} & \omega_2^{\dot{2}} & \rho_1^{\dot{2}} & \rho_2^{\dot{2}} & \rho_3^{\dot{2}} & \rho_4^{\dot{2}} \\ \varsigma_{11} & \varsigma_{12} & u_{11} & u_{12} & u_{13} & u_{14} \\ \varsigma_{21} & \varsigma_{22} & u_{21} & u_{22} & u_{23} & u_{24} \\ \varsigma_{31} & \varsigma_{32} & u_{31} & u_{32} & u_{33} & u_{34} \\ \varsigma_{41} & \varsigma_{42} & u_{41} & u_{42} & u_{43} & u_{44} \end{pmatrix}, \tag{48}$$

where $\gamma = 1, 2$ is twistor index $r = 1, \ldots, k, s = 1, \ldots, N - k$ $(N - k = p)$. This case is very important to have the 64* dimensional sector to reproduce $SU(3)$.

Evidently, the above solutions are invariant under the arbitrary $R \in GL(2; 2 \mid \mathbb{H}_{\mathbb{C}})$ supertransformations (e.g.: superrotations) namely

$$\Omega' \to \Omega R, \quad \Pi' \to \Pi R \tag{49}$$

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with Ω and Π given by (47, 48) with $N = 8, k = 4$. We easily see that two biquaternionic supertwistors and two biquaternionic fermionic supertwistors are described by 64 bosonic complex coordinates (given by eighth bosonic biquaternions) and by 64 fermionic complex coordinates (given by eighth fermionic biquaternions): one half of these coordinates describes the $N = 8, k = 4$ biquaternionic superspace and the second half, however, describes the $GL(2; 2 \,|\, \mathbb{H}_{\mathbb{C}})$ degrees of freedom. Schematically, from the matrix (46), each block describes faithfully the following supercoordinates:

$$\begin{pmatrix} q^{\dot{1}1} & q^{\dot{1}2} \\ q^{\dot{2}1} & q^{\dot{2}2} \end{pmatrix} \rightarrow 2 \times 4 \times 4 = 32 \text{ bosonic sector}, \tag{50}$$

$$\begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^3 \\ \lambda_2^1 & \lambda_2^2 & \lambda_2^3 & \lambda_2^3 \\ \lambda_3^1 & \lambda_3^2 & \lambda_3^3 & \lambda_3^3 \\ \lambda_4^1 & \lambda_4^2 & \lambda_4^3 & \lambda_4^4 \end{pmatrix} \rightarrow 2 \times 4 \times 16 = 128 \text{ bosonic sector(internal "soul")},$$

$$\tag{51}$$

$$\begin{pmatrix} \theta^{\dot{1}1} & \theta^{\dot{1}2} & \theta^{\dot{1}3} & \theta^{\dot{1}4} \\ \theta^{\dot{2}1} & \theta^{\dot{2}2} & \theta^{\dot{2}3} & \theta^{\dot{2}4} \end{pmatrix} \rightarrow 2 \times 4 \times 8 = 64 \text{ fermionic sector}, \tag{52}$$

$$\begin{pmatrix} \theta_1^1 & \theta_1^2 \\ \theta_2^1 & \theta_2^2 \\ \theta_3^1 & \theta_3^2 \\ \theta_4^1 & \theta_4^2 \end{pmatrix} \rightarrow 2 \times 4 \times 8 = 64 \text{ fermionic sector}. \tag{53}$$

7. Discussion

As is well known, the knowledgement of the Casimir operators of any group is important: they are needed for the classification of the irreducible representations of the (extended supersymmetry) algebra. From the supersymmetic viewpoint, they also can be used to find covariant equations of motion for superfields [7]. If there are not central charges, the maximal possible internal symmetry group is $U(N)$: in this case, a complete set of Casimirs operators are well know: P^2, the square of the superspin vector and the corresponding super-Casimir extensions for $U(N)$.

In the general cases, was claimed that the central charges are needed being the maximal possible internal group for any N *even* $USp(N) = Sp(N, \mathbb{H}_{\mathbb{C}}) \cap U(N, \mathbb{H}_{\mathbb{C}})$, however as we have shown, that if $N = p + k$ with $p, k \neq 0$ these central charges are nothing more that the genereators associated to the tensorial representation (not chiral or antichiral) given by the $\lambda_s^{\,r}$ parameters into the superspace matrix Q.

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8. Superfields and Coherent States

We know that the equation relating the B_0 and B_1 parts of the superspace in the case of supertwistors can be in terms of quaternions into the form

(i) The fundamental representation can be decomposed, in principle, as in the case of [6] as

$$\mathcal{U} = t \cdot h,$$

where h is an element of the maximal compact subgroup $S(U(2m) \times U(2n))$ and t of the corresponding coset space $\frac{SU(2,2;k,N-k)}{S(U(2,k) \times U(2,N-k))}$. Explicitly (see [5])

$$h = \exp\left[i \begin{pmatrix} \chi & 0 \\ 0 & \varepsilon \end{pmatrix}\right] = \begin{pmatrix} \mu & 0 \\ 0 & \upsilon \end{pmatrix} \tag{56}$$

and

$$t = \exp\left[i \begin{pmatrix} 0 & \nu \\ \overline{\nu} & 0 \end{pmatrix}\right] = \frac{1}{\sqrt{1 - QQ^\dagger}} \begin{pmatrix} \mathbb{I} & Q \\ Q^\dagger & \mathbb{I} \end{pmatrix}, \tag{57}$$

where the parametrization is given by

$$Q^A_B = \left[\frac{\tanh\sqrt{\overline{\nu}\nu}}{\sqrt{\overline{\nu}\nu}} \nu\right]^A_B \tag{58}$$

because the quaternionic supervariable transforms nonlinearly under $SU(2,2; k,N-k)$

$$Q \to \frac{\alpha Q + \beta}{\gamma Q + \delta}; \quad \alpha, \beta, \gamma, \delta \in \mathbb{H}_\mathbb{C} \tag{59}$$

the superdisplacement operator is precisely

$$D = e^{\eta^\dagger Q^\dagger \xi^\dagger} e^{[\eta^\dagger \ln(1 - QQ^\dagger)^{1/2}\eta - \xi \ln(1 - QQ^\dagger)^{1/2}\xi^\dagger]} e^{-\xi Q\eta}, \tag{60}$$

where η and ξ are the quaternionic oscillator-like supertwistors of [6] (e.g.; super quaternionic analog of the standard a and $a^\dagger$ operators), namely

$$\xi^A = (a^c, -\xi^i); \quad \overline{\xi}_A = \left.\begin{pmatrix} a^\dagger_c \\ \xi^\dagger_i \end{pmatrix}\right\} 2n + 2q, \tag{61}$$

$$\overline{\eta}^M = \overbrace{(b^{\dagger m}, \quad \eta^{\dagger l})}^{2m+2p}; \quad \eta_M = \begin{pmatrix} b_m \\ \eta_l \end{pmatrix}, \tag{62}$$

where we have defined

$$a^\dagger_c = \frac{1}{\sqrt{2}}(\lambda_\alpha + \overline{\mu}^{\dot{\alpha}}), \tag{63}$$

$$b_m = -\frac{1}{\sqrt{2}}(\lambda_\alpha - \overline{\mu}^{\dot{\alpha}}). \tag{64}$$

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9. Concluding Remarks

In this paper, we construct the four-dimensional N-extended quaternionic superspaces from the supersymmetric extension of the ordinary Penrose-twistor formulation, with $N = p + k \cdot (p = N - k)$.

These quaternionic superspaces have $4 + k(N - k)$ bosonic quaternionic coordinates and $4N$ fermionic quaternionic coordinates, where each coordinate is a quaternion composed by four fields (bosons and fermions respectively).

The superspace coordinates are determined in terms of *two* quaternionic supertwistors corresponding to the fundamental Ferber's representation and k quaternionic fermionic supertwistors corresponding to the Litov–Pervushin representation as we show in a previous section.

The biquaternionic construction for $N = 8$, it is the more convenient to represent the SM with $N = 2k$, being also possible the nonlinear realization of the symmetries. This fact is achieved due the (super) symplectic (almost complex) structure of this construction conveniently extended to an even-orthogonal group $O(2N)$. The reason is fundamented by Ambrose–Singer theorem and extended Rothstein theorem (R-C-L theorem see [11] and therein) that clearly relates the graded structure of the tangent space and the field content of the realized physical theory e.g: GUT containing the standard model.

Having into account the developments made here, in the next work [12], we will perform the nonlinear realization for the $N = 8$ case to obtain GR+SM: [1, 2, 4].

Acknowledgments

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