Higher dimensional conformal metrics from PDEs and null-surface formulation of GR

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Abstract
We analyze the relationship between n-dimensional conformal metrics and a certain class of partial differential equations (PDEs) that are in duality with the eikonal equation. In particular, we extend the null-surface formulation of general relativity (GR) to higher dimensions and give explicit expressions for the components of the metric and the generalized Wünschmann-like metricity conditions. We also compute the equation that the conformal factor must satisfy in order the metric to be a solution of the Einstein equations.

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(Some figures may appear in colour only in the online journal)

1. Introduction

About 30 years ago, Kozameh and Newman presented an unorthodox point of view of general relativity (GR) called null-surface formulation (NSF) [1] where the dynamics shifted from a metric tensor on a four-dimensional manifold to null surfaces and partial differential equations (PDEs) in two dimensions. In this version of GR, the conformal spacetime, i.e. a four-dimensional manifold equipped with a conformal structure, arises as the solution space of a particular pair of PDEs in a two-dimensional space representing the sphere of null directions. For the existence of this conformal structure, the pair of PDEs must satisfy the so-called metricity conditions, or Wünschmann conditions whose geometric meaning was unknown until recently [10, 18].

In order to carry out this approach, these authors started with a four-dimensional Lorentzian manifold, already containing a metric g_{ab} and a complete integral to the eikonal equation

$$g^{ab}(x^a)\nabla_a Z\nabla_b Z = 0. \tag{1}$$

The complete integral, expressed as,

$$u = Z(x^a, s, s^*) \tag{2}$$

contains the spacetime coordinates, x^a, and two parameters or constants of integration (s, s^*). By defining the four functions

$$\theta^A \equiv (u, \omega, \omega^*, R) \equiv (Z, \partial_s Z, \partial_{s^*} Z, \partial_{s^* s} Z), \tag{3}$$

from equation (2) and its derivatives, and by eliminating x^a, via the algebraic inversion

$$x^a = X^a(s, s^*, \theta^A), \tag{4}$$

they found that $u = Z(x^a, s, s^*)$ satisfies in addition to equation (1) the pair of second-order PDEs in (s, s^*), of the form

$$\partial_{ss} Z = S(Z, \partial_s Z, \partial_{s^*} Z, \partial_{ss^*} Z, s, s^*), \tag{5}$$

$$\partial_{s^* s^*} Z = S^*(Z, \partial_s Z, \partial_{s^*} Z, \partial_{ss^*} Z, s, s^*). \tag{6}$$

The x^a, in the solution of equations (5) and (6), appear now as constants of integration. Therefore, the roles of x^a and (s, s^*) are exchanged, and it is said that this system of PDEs and the eikonal equation are in duality. Furthermore, in this system of PDEs, the metric is not present anymore. In fact, as was shown by Frittelli, Kozameh and Newman, a conformal metric can be reconstructed from a pair of PDEs of the form (5)–(6) if (S, S^*) satisfy an integrability condition, a weak inequality and a certain set of differential conditions $\mathfrak{m}[S, S^*] = 0$, $\mathfrak{m}^*[S, S^*] = 0$ (the metricity or generalized Wünschmann conditions), i.e. the conformal structure is codified in Z or alternatively in the functions S and S^*.

The solutions to the pair determine a conformal four-dimensional Lorentzian metric and, in fact, all conformal Lorentzian metrics can be obtained from equivalence classes of equations of the form (5)–(6). When certain specific conditions [9], in addition to the Wünschmann condition, are imposed on the (S, S^*), the metrics, determined by the solutions, are in the conformal Einstein vacuum class.

The purpose of this work is to show that a similar program to the NSF of GR can be developed in dimensions higher than 3 and 4 [1–5]. Analogous results were presented in [21] for the Hamilton–Jacobi formulation of GR from Montiel-Piña, Newman and Silva-Ortigoza [22]. In particular, we show that from a special system of PDEs, (conformal) Lorentzian metrics can always be constructed , which are in duality with the eikonal equation.

In order to achieve this objective, we will use the same techniques as were developed by Frittelli, Kozameh and Newman [1–3] in order to obtain NSF in its four-dimensional version.

We will present the essential results, and we will see how the imposition of the Einstein equations to the system determines a unique n-dimensional metric.

2. The duality between the eikonal equation and a system of second-order PDEs

Let $\mathcal{M}$ be an n-dimensional manifold with local coordinates $x^a = (x^0, \ldots, x^{(n-1)})$ and let us assume that we are given an $(n-2)$-parameter set of functions $u = Z(x^a, s, s^*, \gamma^m)$, with $m = 1, \ldots, (n-4)$.

The parameters s, s^* and γ^m can take values on an open neighborhood of a manifold $\mathcal{N}$ of dimension $(n-2)$. It will also be assumed that for fixed values of the parameters s, s^* and γ^m, the level surfaces

$$u = \text{constant} = Z(x^a, s, s^*, \gamma^m) \tag{7}$$

locally foliate the manifold $\mathcal{M}$ and that $Z(x^a, s, s^*, \gamma^m)$ satisfies the eikonal equation

$$g^{ab}(x^a) \nabla_a Z(x^a, s, s^*, \gamma^m) \nabla_b Z(x^a, s, s^*, \gamma^m) = 0, \tag{8}$$

for some Lorentzian metric $g_{ab}(x^a)$.

Therefore, for each fixed value of $\{s, s^*, \gamma^m\}$, the level surfaces $Z(x^a, s, s^*, \gamma^m) = $ constant are null surfaces of $(\mathcal{M}, g_{ab})$.

We want to find now a system of PDEs in duality with the eikonal equation, i.e. a system that admits the same solutions, but where the role of integration constants and parameters is exchanged.

From the assumed existence of $Z(x^a, s, s^*, \gamma^m)$, we define n parameterized scalars θ^A, with $(A = 0, +, -, m, R)$, in the following way:

$$\theta^0 = u = Z, \tag{9}$$

$$\theta^+ = w^+ = \partial_s Z, \tag{10}$$

$$\theta^- = w^- = \partial_{s^*} Z, \tag{11}$$

$$\theta^m = w^m = \partial_m Z, \tag{12}$$

$$\theta^R = R = \partial_{ss^*} Z, \tag{13}$$

where the derivatives with respect to the parameters s, s^* and γ^m are denoted by ∂_s, ∂_{s^*} and $\partial_{\gamma^m} = \partial_m$. In a similar way, differentiation with respect to the local coordinates x^a will be denoted as ∇_a or 'comma a,', and for an arbitrary function $F(\theta^A, s, s^*, \gamma^m)$, F_{θ^A} will be the partial derivative of F with respect to θ^A.

We will assume that $Z(x^a, s, s^*, \gamma^m)$ is such that equations (9)–(13) can be solved for the x^as for all values of $\{s, s^*, \gamma^m\}$ in an open neighborhood $\mathcal{O} \in \mathcal{N}$; that is, we require

$$\det \theta^A{}_{,b} \neq 0, \tag{14}$$

and therefore

$$x^a = X^a(u, w^+, w^-, w^m, R, s, s^*, \gamma^m). \tag{15}$$

It can be shown that in the case of flat Lorentzian spacetimes, there exist families of null surfaces where equation (14) is satisfied (see appendix A for a discussion). Therefore, it follows the existence on arbitrary spacetimes of (local) families of null surfaces where equation (14) is also satisfied.

Assuming this, let us note that for each fixed value of s, s^* and γ^m, equations (9)–(13) can be thought as a coordinate transformation between the x^as and the θ^As.

Defining the following $n(n-3)/2$ scalars:

$$\tilde{S}(x^a, s, s^*, \gamma^m) = \partial_{ss} Z(x^a, s, s^*, \gamma^m), \tag{16}$$

$$\tilde{S}^*(x^a, s, s^*, \gamma^m) = \partial_{s^*s^*} Z(x^a, s, s^*, \gamma^m), \tag{17}$$

$$\tilde{\Phi}_m(x^a, s, s^*, \gamma^m) = \partial_{sm} Z(x^a, s, s^*, \gamma^m), \tag{18}$$

$$\tilde{\Phi}_m^*(x^a, s, s^*, \gamma^m) = \partial_{s^*m} Z(x^a, s, s^*, \gamma^m), \tag{19}$$

$$\tilde{\Upsilon}_{lm}(x^a, s, s^*, \gamma^m) = \partial_{lm} Z(x^a, s, s^*, \gamma^m), \tag{20}$$

and taking into account equation(15), we obtain a system of PDEs dual to the eikonal equation given by

$$\partial_{ss} Z = S(u, w^+, w^-, w^m, R, s, s^*, \gamma^m), \tag{21}$$

$$\partial_{s^*s^*} Z = S^*(u, w^+, w^-, w^m, R, s, s^*, \gamma^m), \tag{22}$$

$$\partial_{sm} Z = \Phi_m(u, w^+, w^-, w^m, R, s, s^*, \gamma^m), \tag{23}$$

$$\partial_{s^*m}Z = \Phi_m^*(u, w^+, w^-, w^m, R, s, s^*, \gamma^m), \tag{24}$$

$$\partial_{lm}Z = \Upsilon_{lm}(u, w^+, w^-, w^m, R, s, s^*, \gamma^m), \tag{25}$$

where

$$S(u, w^+, w^-, w^m, R, s, s^*, \gamma^m) = \tilde{S}(x^a(u, w^+, w^-, w^m, R, s, s^*, \gamma^m), s, s^*, \gamma^m), \tag{26}$$

and so on.

It means that the $(n-2)$-parametric family of level surfaces (equation (7)) can be obtained as solutions of the $n(n-3)/2$ system of second-order PDEs (21)–(25). In this case, $(S, S^*, \Phi_m, \Phi_m^*, \Upsilon_{lm})$ satisfy the following integrability conditions:

$$D_k S = D_s \Phi_k, \tag{27}$$

$$D_k S^* = D_{s^*} \Phi_k^*, \tag{28}$$

$$D_k \Phi_m = D_m \Phi_k = D_s \Upsilon_{mk}, \tag{29}$$

$$D_k \Phi_m^* = D_m \Phi_k^* = D_{s^*} \Upsilon_{mk}, \tag{30}$$

$$D_i \Upsilon_{mk} = D_k \Upsilon_{mi}, \tag{31}$$

$$D_s T^* = D_{s^*} T, \tag{32}$$

$$D_m Q_k = D_k Q_m, \tag{33}$$

$$D_m T = D_s Q_m, \tag{34}$$

$$D_m T^* = D_{s^*} Q_m, \tag{35}$$

where

Definition 1: *The total s, s^* and γ^m derivatives of a function $F = F(\theta^A, s, s^*, \gamma^n)$ are defined by*

$$D_s F \equiv \partial_s F + F_u w^+ + F_{w^+} S + F_{w^-} R + F_R T + F_{w^m} \Phi_m, \tag{36}$$

$$D_{s^*} F \equiv \partial_{s^*} F + F_u w^- + F_{w^-} S^* + F_{w^+} R + F_R T^* + F_{w^m} \Phi_m^*, \tag{37}$$

$$D_m F \equiv \partial_m F + F_u w^m + F_{w^+} \Phi_m + F_{w^-} \Phi_m^* + F_R Q_m + F_{w^k} \Upsilon_{km}, \tag{38}$$

respectively, with

$$T = \frac{1}{1 - S_R S_R^*} [S_{s^*} + S_u w^- + S_{w^-} S^* + S_{w^+} R + S_{w^m} \Phi_m^* \\ + S_R (S_s^* + S_u^* w^+ + S_{w^+}^* S + S_{w^-}^* R + S_{w^m}^* \Phi_m)], \tag{39}$$

$$T^* = \frac{1}{1 - S_R S_R^*} [S_s^* + S_u^* w^+ + S_{w^+}^* S + S_{w^-}^* R + S_{w^m}^* \Phi_m \\ + S_R^* (S_{s^*} + S_u w^- + S_{w^-} S^* + S_{w^+} R + S_{w^m} \Phi_m^*)], \tag{40}$$

$$Q_m = \Phi_{m,s^*} + \Phi_{m,u} w^- + \Phi_{m,w^-} S^* + \Phi_{m,w^+} R + \Phi_{m,R} T^* + \Phi_{m,w^k} \Phi_k^*. \tag{41}$$

Note that

$$T \equiv D_s R = D_{s^*} S, \tag{42}$$

$$T^* \equiv D_{s^*} R = D_s S^*, \tag{43}$$

$$Q_m \equiv D_m R = D_{s^*} \Phi_m = D_s \Phi_m^*. \tag{44}$$

It is easy to show that if the functions $(S, S^*, \Phi_m, \Phi_m^*, \Upsilon_{lm})$ satisfy the integrability conditions, the solution space of equations (21)–(25) is n dimensional, as we show next.

The system of PDEs (21)–(25) is equivalent to the Pfaffian system generated by the n one-forms $\beta^A = (\beta^0, \beta^+, \beta^-, \beta^m, \beta^R)$,

$$\beta^0 = \mathrm{d}u - w^+ \, \mathrm{d}s - w^- \, \mathrm{d}s^* - w^m \, \mathrm{d}\gamma^m, \tag{45}$$

$$\beta^+ = \mathrm{d}w^+ - S \, \mathrm{d}s - R \, \mathrm{d}s^* - \Phi_m \, \mathrm{d}\gamma^m, \tag{46}$$

$$\beta^- = \mathrm{d}w^- - R \, \mathrm{d}s - S^* \, \mathrm{d}s^* - \Phi_m^* \, \mathrm{d}\gamma^m, \tag{47}$$

$$\beta^m = \mathrm{d}w^m - \Phi_m \, \mathrm{d}s - \Phi_m^* \, \mathrm{d}s^* - \Upsilon_{mk} \, \mathrm{d}\gamma^k, \tag{48}$$

$$\beta^R = \mathrm{d}R - T \, \mathrm{d}s - T^* \, \mathrm{d}s^* - Q_k \, \mathrm{d}\gamma^k. \tag{49}$$

A direct computation shows that

$$\mathrm{d}\beta^0 = \mathrm{d}s \wedge \beta^+ + \mathrm{d}s^* \wedge \beta^- + \mathrm{d}\gamma^k \wedge \beta^k, \tag{50}$$

$$\begin{aligned}
\mathrm{d}\beta^+ = &[S_u \, \mathrm{d}s + \Phi_{k,u} \, \mathrm{d}\gamma^k] \wedge \beta^0 + [S_{w^+} \, \mathrm{d}s + \Phi_{k,w^+} \, \mathrm{d}\gamma^k] \wedge \beta^+ \\
&+ [S_{w^-} \, \mathrm{d}s + \Phi_{k,w^-} \, \mathrm{d}\gamma^k] \wedge \beta^- + [S_{w^k} \, \mathrm{d}s + \Phi_{m,w^k} \, \mathrm{d}\gamma^m] \wedge \beta^k \\
&+ [S_R \, \mathrm{d}s + \Phi_{k,R} \, \mathrm{d}\gamma^k + \mathrm{d}s^*] \wedge \beta^R \\
&+ [D_k S - D_s \Phi_k] \, \mathrm{d}s \wedge \mathrm{d}\gamma^k - D_k \Phi_m \, \mathrm{d}\gamma^k \wedge \mathrm{d}\gamma^m,
\end{aligned} \tag{51}$$

$$\begin{aligned}
\mathrm{d}\beta^- = &[S_u^* \, \mathrm{d}s^* + \Phi_{k,u}^* \, \mathrm{d}\gamma^k] \wedge \beta^0 + [S_{w^-}^* \, \mathrm{d}s^* + \Phi_{k,w^-}^* \, \mathrm{d}\gamma^k] \wedge \beta^- \\
&+ [S_{w^+}^* \, \mathrm{d}s^* + \Phi_{k,w^+}^* \, \mathrm{d}\gamma^k] \wedge \beta^+ + [S_{w^k}^* \, \mathrm{d}s^* + \Phi_{m,w^k}^* \, \mathrm{d}\gamma^m] \wedge \beta^k \\
&+ [\mathrm{d}s + S_R^* \, \mathrm{d}s^* + \Phi_{k,R}^* \, \mathrm{d}\gamma^k] \wedge \beta^R \\
&+ [D_k S^* - D_{s^*} \Phi_k^*] \, \mathrm{d}s^* \wedge \mathrm{d}\gamma^k - D_m \Phi_k^* \, \mathrm{d}\gamma^k \wedge \mathrm{d}\gamma^m,
\end{aligned} \tag{52}$$

$$\begin{aligned}
\mathrm{d}\beta^m = &[\Phi_{m,u} \, \mathrm{d}s + \Phi_{m,u}^* \, \mathrm{d}s^* + \Upsilon_{mk,u} \, \mathrm{d}\gamma^k] \wedge \beta^0 \\
&+ [\Phi_{m,w^+} \, \mathrm{d}s + \Phi_{m,w^+}^* \, \mathrm{d}s^* + \Upsilon_{mk,w^+} \, \mathrm{d}\gamma^k] \wedge \beta^+ \\
&+ [\Phi_{m,w^-} \, \mathrm{d}s + \Phi_{m,w^-}^* \, \mathrm{d}s^* + \Upsilon_{mk,w^-} \, \mathrm{d}\gamma^k] \wedge \beta^- \\
&+ [\Phi_{m,w^j} \, \mathrm{d}s + \Phi_{m,w^j}^* \, \mathrm{d}s^* + \Upsilon_{mk,w^j} \, \mathrm{d}\gamma^k] \wedge \beta^j \\
&+ [\Phi_{m,R} \, \mathrm{d}s + \Phi_{m,R}^* \, \mathrm{d}s^* + \Upsilon_{mk,R} \, \mathrm{d}\gamma^k] \wedge \beta^R \\
&+ [D_i \Phi_m^* - D_{s^*} \Upsilon_{mi}] \, \mathrm{d}s^* \wedge \mathrm{d}\gamma^i \\
&+ [D_i \Phi_m - D_s \Upsilon_{mi}] \, \mathrm{d}s \wedge \mathrm{d}\gamma^i - D_i \Upsilon_{mk} \, \mathrm{d}\gamma^i \wedge \mathrm{d}\gamma^k,
\end{aligned} \tag{53}$$

$$\begin{aligned}
\mathrm{d}\beta^R = &[T_u \, \mathrm{d}s + T_u^* \, \mathrm{d}s^* + Q_{k,u} \, \mathrm{d}\gamma^k] \wedge \beta^0 + [T_{w^+} \, \mathrm{d}s + T_{w^+}^* \, \mathrm{d}s^* + Q_{k,w^+} \, \mathrm{d}\gamma^k] \wedge \beta^+ \\
&+ [T_{w^-} \, \mathrm{d}s + T_{w^-}^* \, \mathrm{d}s^* + Q_{k,w^-} \, \mathrm{d}\gamma^k] \wedge \beta^- \\
&+ [T_{w^j} \, \mathrm{d}s + T_{w^j}^* \, \mathrm{d}s^* + Q_{k,w^j} \, \mathrm{d}\gamma^j] \wedge \beta^k \\
&+ [T_R \, \mathrm{d}s + T_R^* \, \mathrm{d}s^* + Q_{k,R} \, \mathrm{d}\gamma^k] \wedge \beta^R \\
&+ [D_{s^*} T - D_s T^*] \, \mathrm{d}s \wedge \mathrm{d}s^* + [D_i T^* - D_{s^*} Q_i] \, \mathrm{d}s^* \wedge \mathrm{d}\gamma^i \\
&+ [D_i T - D_s Q_i] \, \mathrm{d}s \wedge \mathrm{d}\gamma^i - D_i Q_k \, \mathrm{d}\gamma^i \wedge \mathrm{d}\gamma^k,
\end{aligned} \tag{54}$$

where in order to perform these computations, we have used the fact that for an arbitrary function

$$\Lambda = \Lambda(u, w^+, w^-, R, s, s^*, \gamma^m),$$

$$d\Lambda = \Lambda_u \beta^0 + \Lambda_{w^+} \beta^+ + \Lambda_{w^-} \beta^- + \Lambda_{w^m} \beta^m + \Lambda_R \beta^R + D_s\Lambda \; ds + D_{s^*}\Lambda \; ds^* + D_m\Lambda \; d\gamma^m.$$

(55)

Therefore, using integrability conditions, $d\beta^A = 0$ (modulo β^A). From this result and the Frobenius theorem, we conclude that the solution space of equations (21)–(25) is n dimensional.

In this way, we have obtained a system of differential equations in duality with the eikonal equation, with the metric disappeared from these equations. The natural question is as follows: Could one start with this system of PDEs, and then find the eikonal equation, with a metric $g^{ab}(x^a)$? As in three and four dimensions, we will show that when the functions $(S, S^*, \Phi_m, \Phi_m^*, \Upsilon_{lm})$ satisfy the integrability conditions, and a set of differential conditions (the metricity or generalized Wünschmann conditions), the procedure can be reversed. The solutions of the system determine a conformal n-dimensional metric.

In fact, there exist several geometrical ways to study these equations. One could, for example, study the equivalence problem associated with these equations, asking for a class of PDEs that can be obtained from the original one from the so-called point or contact transformations [14, 17–20], or by direct construction of a conformal connection with vanishing torsion tensor. However, there exists a more straightforward method to reconstruct the metric from Z (or $(S, S^*, \Phi_m, \Phi_m^*, \Upsilon_{lm})$), and it is the method that was developed by Frittelli, Kozameh and Newman in order to obtain the so-called null surface formulation of GR. Due to the simplicity of this technique, we extend its use to higher dimensions.

3. *n*-dimensional conformal metrics

The basic idea now is to solve equation (8) for the components of the metric in terms of $Z(x^a, s, s^*, \gamma^m)$. To do so, we will consider a number of parameter derivatives of condition (8), and then by manipulation of these derivatives obtain both the n-dimensional metric and the conditions that the $n(n - 3)/2$ PDEs defining the surfaces must satisfy. From the n scalars, θ^A, we have their associated gradient basis $\theta^A{}_a$ given by

$$\theta^A{}_a = \nabla_a \theta^A = \{Z_a, D_s Z_a, D_{s^*} Z_a, D_m Z_a, D_{ss^*} Z_a\}$$

(56)

and its dual vector basis $\theta_B{}^a$, so that

$$\theta_A{}^a \theta^B{}_a = \delta_A{}^B, \quad \theta_A{}^a \theta^A{}_b = \delta_b{}^a.$$

(57)

As was shown in the original works on NSF, it is easier to search for the components of the n-dimensional metric in the gradient basis rather than in the original coordinate basis. Furthermore, it is preferable to use the contravariant components rather than the covariant components of the metric; that is, we want to determine

$$g^{AB}(x^a, s, s^*, \gamma^m) = g^{ab}(x^a)\theta^A{}_a \theta^B{}_b.$$

(58)

The metric components and the Wünschmann-like conditions are obtained by repeatedly operating with D_s, D_{s^*} and D_m on equation (8), that is, by definition, on

$$g^{00} = g^{ab} Z_a Z_b = 0.$$

(59)

Applying D_s to equation (59) yields $D_s g^{00} = 2g^{ab}\partial_s Z_{,a} Z_{,b} = 0$, i.e.

$$g^{+0} = 0.$$

(60)

In the same way, we obtain from $D_{s^*}g^{00} = 0$ and $D_m g^{00} = 0$ that

$$g^{-0} = g^{m0} = 0. \tag{61}$$

Computing the second derivative $D_{ss}(g^{00}/2) = 0$, we obtain

$$\begin{aligned}
D_{ss}(g^{00}/2) &= g^{ab}(x^a)D_{ss}Z_{,a}Z_{,b} + g^{ab}D_sZ_{,a}D_sZ_{,b} = 0 \\
&= S_R g^{0R} + g^{++} = 0, \tag{62}
\end{aligned}$$

and therefore

$$g^{++} = -S_R g^{0R}. \tag{63}$$

Similarly applying the second derivatives $D_{s^*s^*}$, D_{ss^*}, D_{ms}, D_{ms^*} and D_{nm} to g^{00} yields

$$g^{--} = -S_R^* g^{0R}, \tag{64}$$

$$g^{+-} = -g^{0R}, \tag{65}$$

$$g^{+m} = -\Phi_{m,R} g^{0R}, \tag{66}$$

$$g^{-m} = -\Phi_{m,R}^* g^{0R}, \tag{67}$$

$$g^{nm} = -\Upsilon_{nm,R} g^{0R}, \tag{68}$$

where in all these computations, we used the fact that for an arbitrary function $F(\theta^A, s, s^*, \gamma^m)$, one has $F_{,a} = F_{\theta^A}\theta^A{}_{,a}$.

From the third derivatives

$$D_{sss^*}(g^{00}/2) = g^{ab}(x^a)T^*{}_{,a}u_{,b} + g^{ab}(x^a)S^*{}_{,a}w^+{}_{,b} + 2g^{-R} = 0, \tag{69}$$

$$D_{s^*ss^*}(g^{00}/2) = g^{ab}(x^a)T_{,a}u_{,b} + g^{ab}(x^a)S_{,a}w^-{}_{,b} + 2g^{+R} = 0, \tag{70}$$

we obtain a linear system for g^{+R} and g^{-R} that can be solved if $S_R S_R^* \neq 4$; then we find

$$g^{+R} = -\frac{g^{0R}}{4 - S_R S_R^*}[2(T_R - S_{w^+} - S_{w^-}S_R^* - S_{w^m}\Phi_{m,R}^*) - S_R(T_R^* - S_{w^-}^* - S_{w^+}^*S_R - S_{w^m}^*\Phi_{m,R})], \tag{71}$$

and

$$g^{-R} = -\frac{g^{0R}}{4 - S_R S_R^*}[2(T_R^* - S_{w^-}^* - S_{w^+}^*S_R - S_{w^m}^*\Phi_{m,R}) - S_R^*(T_R - S_{w^+} - S_{w^-}S_R^* - S_{w^m}\Phi_{m,R}^*)]. \tag{72}$$

By computing

$$D_{mss^*}(g^{00}/2) = g^{ab}(x^a)Q_{m,a}u_{,b} + g^{ab}(x^a)R_{,a}w^m{}_{,b} + g^{ab}(x^a)\Phi_{m,a}^*w^+{}_{,b} + g^{ab}(x^a)\Phi_{m,a}w^-{}_{,b} = 0, \tag{73}$$

we obtain

$$\begin{aligned}
g^{mR} &= -\Phi_{m,R}^* g^{+R} - \Phi_{m,R}g^{-R} - g^{0R}(Q_{m,R} - \Phi_{m,w^+} - \Phi_{m,w^-}S_R^* \\
&\quad - \Phi_{m,w^n}\Phi_{n,R}^* - \Phi_{m,w^+}^*S_R - \Phi_{m,w^-}^* - \Phi_{m,w^n}^*\Phi_{n,R}). \tag{74}
\end{aligned}$$

Finally, from

$$\begin{aligned}
D_{ss^*ss^*}(g^{00}/2) &= g^{ab}(x^a)U_{,a}u_{,b} + 2g^{ab}(x^a)T^*{}_{,a}w^+{}_{,b} + 2g^{ab}(x^a)T_{,a}w^-{}_{,b} \\
&\quad + 2g^{RR} + g^{ab}(x^a)S_{,a}S^*{}_{,b} = 0, \tag{75}
\end{aligned}$$

with $U = D_{s^*}T$, we obtain the last metric component g^{RR} (if $S_R S_R^* \neq -2$),

$$
\begin{aligned}
g^{RR} = -\frac{1}{2 + S_R S_R^*} \Big\{ & S_{w^m}^* S_R g^{mR} + [U_R - 2T_{w^+} - 2T_{w^-} S_R^* - 2T_{w^m} \Phi_{m,R}^* - 2T_{w^+}^* S_R \\
& - 2T_{w^-}^* - 2T_{w^m}^* \Phi_{m,R} + S_u^* S_R + S_R^* S_u - S_{w^+}^* (S_{w^+} S_R + S_{w^-} + S_{w^m} \Phi_{m,R}) \\
& - S_{w^-}^* (S_{w^+} + S_{w^-} S_R^* + S_{w^m} \Phi_{m,R}^*) - S_{w^m}^* (S_{w^+} \Phi_{m,R} + S_{w^-} \Phi_{m,R}^* + S_{w^n} \Upsilon_{mn,R})] g^{0R} \\
& + (2T_R^* + S_{w^+}^* S_R + S_R^* S_{w^+}) g^{+R} + (2T_R + S_{w^-} S_R^* + S_R S_{w^-}^*) g^{-R} \Big\}.
\end{aligned}
\tag{76}
$$

The metric then is expressed as

$$
g^{AB} = \Omega^2 \mathfrak{g}^{AB} = \Omega^2
\begin{pmatrix}
0 & 0 & 0 & \dots & 0 & 1 \\
0 & -S_R & -1 & \dots & -\Phi_{m,R} & \mathfrak{g}^{+R} \\
0 & -1 & -S_R^* & \dots & -\Phi_{m,R}^* & \mathfrak{g}^{-R} \\
\vdots & \vdots & \vdots & [-\Upsilon_{nm,R}] & \vdots & \vdots \\
0 & -\Phi_{m,R} & -\Phi_{m,R}^* & \dots & -\Upsilon_{mm,R} & \mathfrak{g}^{mR} \\
1 & \mathfrak{g}^{+R} & \mathfrak{g}^{-R} & \dots & \mathfrak{g}^{mR} & \mathfrak{g}^{RR}
\end{pmatrix}.
\tag{77}
$$

Therefore, we have found that all contravariant components of the metric can be expressed in term of derivatives of the functions $(S, S^*, \Phi_m, \Phi_m^*, \Upsilon_{ml})$ and a conformal factor $g^{0R} = \Omega^2$.

It is worthwhile to mention that by construction, the metric obtained in equation (77) has Lorentzian signature. However, if one starts from a system of PDEs that satisfy the Wünschmann conditions that we will present in the next section, and wishes to restrict the type of metrics to those with Lorentzian signature, one must impose extra conditions on $(S, S^*, \Phi_m, \Phi_m^*, \Upsilon_{lm})$. For example in four dimensions, such a condition reads $1 - S_R S_R^* > 0$, which follows from requiring that $\det(g^{ab}) < 0$, but unfortunately, in higher dimensions, this last condition is not sufficient. In these cases, for each particular set of PDEs, one must study the eigenvalue problem associated with the metric, and count the number of positive and negative signs [6, 7].

4. The generalized Wünschmann or metricity conditions

If we compute the remaining third derivatives, we obtain relations that automatically satisfy Z, but if we start with the point of view that we want to construct a conformal metric from the system of PDEs, then these relations are converted in Wünschmann-like conditions that must satisfy our system to assure the existence of the conformal metric in the solution space. These conditions read (following by applying $D_{sss}, D_{s^*s^*s^*}, D_{kmn}, D_{s^*s^*m}, D_{ssm}, D_{smn}, D_{s^*nm}$ to g^{00})

$$
\mathfrak{m} = D_s[S \cdot u] + 2[S \cdot w^+] = 0,
\tag{78}
$$

$$
\mathfrak{m}^* = D_{s^*}[S^* \cdot u] + 2[S^* \cdot w^-] = 0,
\tag{79}
$$

$$
\mathfrak{m}_{kmn} = D_k[\Upsilon_{mn} \cdot u] + [\Upsilon_{km} \cdot w^n] + [\Upsilon_{kn} \cdot w^m] = 0,
\tag{80}
$$

$$
\mathfrak{m}_m = D_m[S \cdot u] + 2[\Phi_m \cdot w^+] = 0,
\tag{81}
$$

$$
\mathfrak{m}_m^* = D_m[S^* \cdot u] + 2[\Phi_m^* \cdot w^-] = 0,
\tag{82}
$$

$$
\mathfrak{m}_{mn} = D_s[\Upsilon_{mn} \cdot u] + [\Phi_m \cdot w^n] + [\Phi_n \cdot w^m] = 0,
\tag{83}
$$

$$
\mathfrak{m}_{mn}^* = D_{s^*}[\Upsilon_{mn} \cdot u] + [\Phi_m^* \cdot w^n] + [\Phi_n^* \cdot w^m] = 0,
\tag{84}
$$

where we use the notation $F \cdot G = g^{ab} F_{,a} G_{,b}$, for the arbitrary functions F and G.

Explicit expressions for these metricity conditions are found in appendix B.

Remark 2: Due to the fact that $\mathfrak{m}_{mn} = \mathfrak{m}_{nm}$ and $\mathfrak{m}_{kmn} = \mathfrak{m}_{(knm)}$, there are in total $\frac{1}{6}(n^2 - 4)(n - 3)$ independent conditions in n dimensions. If we continue applying higher order derivatives to g^{00} as it happens in the four-dimensional version of NSF, we do not obtain new information. From these derivatives, we obtain only identities from the previous relations.

On the other hand, if we apply the derivatives D_s, D_{s^*}, D_m to the component $g^{0R} = \Omega^2$ of the metric, i.e. to the conformal factor, we obtain the equations

$$D_s\Omega = \tfrac{1}{2}(\mathfrak{g}^{+R} + T_R)\Omega, \tag{85}$$

$$D_{s^*}\Omega = \tfrac{1}{2}(\mathfrak{g}^{-R} + T_R^*)\Omega, \tag{86}$$

$$D_m\Omega = \tfrac{1}{2}(\mathfrak{g}^{mR} + Q_{m,R})\Omega. \tag{87}$$

Again, if we start with the point of view that we want to construct a conformal metric from a system of PDEs, these relations must be satisfied to assure the existence of conformal metrics. Note that these equations do not determine completely the conformal factor, but they are necessary for the conformal equivalence between the $(n - 2)$-parameter family of metrics.

In this way, we have proved that, in particular, all n-dimensional spacetime can be considered as originated as the solution space of an $n(n - 3)/2$ PDE. Note that in four dimensions, the system is relatively simpler than its nearby five-dimensional system, where one must consider five PDEs.

In a similar way as in four dimensions, one could construct the Cartan normal conformal connections associated with these PDEs, and from this reduce the system to one which is compatible with Einstein spaces [8–11]. However, although in principle this generalization is direct, in practice the algebraical manipulation of these equations appears as a tedious work, even using algebraic manipulators as Maple or Mathematica.

5. The Einstein equations

If we want to fix completely the conformal metric, an extra condition must be imposed. As we wish to extend NSF to higher dimensions, we will impose the Einstein equations to the system.

As in four dimensions, the vacuum Einstein equations can be obtained by requiring $R^{00} = 0$. This equation determines the conformal factor, which is necessary to convert the different conformal metrics into physical ones. We now adopt a global point of view toward geometry on an n-dimensional manifold. Instead of a metric $g^{ab}(x^a)$ on $\mathcal{M}$, as the fundamental variable, we consider as the basic variables a family of surfaces on $\mathcal{M}$ given by $u = \text{constant} = Z(x^a, s, s^*, \gamma^i)$ or preferably its second derivatives with respect to (s, s^*, γ^i). From this new point of view, these surfaces are basic and the metric is a derived concept. Now we will find the conditions on $u = Z(x^a, s, s^*, \gamma^i)$ or more accurately on the second-order system such that the n-dimensional metric be a solution to the Einstein equations.

We start with the Einstein equations in n dimensions, which are given by (see for example [12, 13])

$$R_{ab} = 8\pi G_{(n)} \left(T_{ab} - \frac{1}{n - 2} g_{ab} T \right), \tag{88}$$

with $G_{(n)}$ the gravitational constant in higher dimensions.

The Ricci tensor is given by

$$R_{ab} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^c} \left(\Gamma_{ab}^c \sqrt{-g} \right) - \frac{\partial^2}{\partial x^a \partial x^b} \ln \sqrt{-g} - \Gamma_{ad}^c \Gamma_{bc}^d, \tag{89}$$

with $g = \det(g_{ab})$ and

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd} \left(\frac{\partial g_{da}}{\partial x^b} + \frac{\partial g_{db}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d} \right), \tag{90}$$

the Christoffel symbols.

As happens in four dimensions, the Einstein equations are given by

$$R^{ab} Z_{,a} Z_{,b} = 8\pi G_{(n)} \left(T^{ab} Z_{,a} Z_{,b} \right). \tag{91}$$

That is, to obtain the Einstein equation in this case, we need to compute $R^{00} \equiv R^{ab} Z_{,a} Z_{,b}$, which is *one* of the components of $R^{AB} \equiv R^{ab} \theta^A_a \theta^B_b$. From the form of the metric (77), we have that $R^{00} = \Omega^4 R_{RR}$.

Before the computation of the equation for the conformal factor Ω^2, we will take the following definitions.

(1) The contravariant components of the metric g^{AB} will be written as $g^{AB} = \Omega^2 \mathfrak{g}^{AB}$; in a similar way, we will write $g_{AB} = \Omega^{-2} \mathfrak{g}_{AB}$.
(2) Latin indices $A, B, \ldots, K$, etc belong to the set $\{0, +, -, m, R\}$, while the indices $\{+, -, m\}$ will be denoted with Greek letters $\{\alpha, \beta, \kappa\}$.
(3) The determinant of g^{AB} and g_{AB} will be decomposed as

$$\det(g^{AB}) = \Omega^{2n} q, \tag{92}$$

$$\det(g_{AB}) = \Omega^{-2n} \frac{1}{q} = \Omega^{-2n} \Delta. \tag{93}$$

(4) The derivative $\frac{\partial}{\partial R}$ will be denoted as $\mathfrak{D}$.

Let us compute now R_{RR}:

$$R_{RR} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^C} \left(\Gamma^C_{RR} \sqrt{-g} \right) - \mathfrak{D}^2 \left[\ln \sqrt{-g} \right] - \Gamma^C_{RD} \Gamma^D_{RD}. \tag{94}$$

Note that the only not vanishing term of Γ^C_{RR} is $\Gamma^R_{RR} = -2 \frac{\mathfrak{D}\Omega}{\Omega}$. Therefore, the first term of R_{RR} reads

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^C} \left(\Gamma^C_{RR} \sqrt{-g} \right) = -2 \frac{\mathfrak{D}^2 \Omega}{\Omega} - \frac{\mathfrak{D}\Omega}{\Omega} \frac{\mathfrak{D}\Delta}{\Delta} + 2(n+1) \frac{(\mathfrak{D}\Omega)^2}{\Omega^2}. \tag{95}$$

For the second term, we obtain

$$\mathfrak{D}^2 \left[\ln \sqrt{-g} \right] = -n \frac{\mathfrak{D}^2 \Omega}{\Omega} + n \frac{(\mathfrak{D}\Omega)^2}{\Omega^2} + \frac{\mathfrak{D}^2 \Delta}{2\Delta} - \frac{(\mathfrak{D}\Delta)^2}{2\Delta^2}. \tag{96}$$

Finally, let us compute the remaining terms of R_{RR}, i.e.

$$\Gamma^C_{RD} \Gamma^D_{RD} = \Gamma^0_{RD} \Gamma^D_{R0} + \Gamma^\alpha_{RD} \Gamma^D_{R\alpha} + \Gamma^R_{RD} \Gamma^D_{RR}. \tag{97}$$

Due to the fact that $\Gamma^0_{RD} = \Gamma^\alpha_{R0} = 0$ and that the only not vanishing term of Γ^D_{RR} is Γ^R_{RR}, such an expression is reduced to

$$\Gamma^C_{RD} \Gamma^D_{RC} = \Gamma^\alpha_{R\beta} \Gamma^\beta_{R\alpha} + \left(\Gamma^R_{RR} \right)^2. \tag{98}$$

Using the decomposition $g^{AB} = \Omega^2 \mathfrak{g}^{AB}$, $g_{AB} = \Omega^{-2} \mathfrak{g}_{AB}$, one finds

$$\Gamma^\alpha_{R\beta} = -\frac{\mathfrak{D}\Omega}{\Omega} \delta^\alpha_\beta - \frac{1}{2} \mathfrak{D}[\mathfrak{g}^{\alpha\kappa}] \mathfrak{g}_{\beta\kappa}. \tag{99}$$

Therefore,

$$\Gamma^\alpha_{R\beta} \Gamma^\beta_{R\alpha} = (n-2) \frac{(\mathfrak{D}\Omega)^2}{\Omega^2} + \frac{\mathfrak{D}\Omega}{\Omega} \mathfrak{D}[\mathfrak{g}^{\alpha\beta}] \mathfrak{g}_{\alpha\beta} + \frac{1}{4} \mathfrak{D}[\mathfrak{g}^{\alpha\epsilon}] \mathfrak{D}[\mathfrak{g}^{\beta\kappa}] \mathfrak{g}_{\beta\epsilon} \mathfrak{g}_{\alpha\kappa}. \tag{100}$$

Adding all these terms, we obtain

$$R_{RR} = (n-2)\frac{\mathfrak{D}^2\Omega}{\Omega} - \frac{\mathfrak{D}\Omega}{\Omega}\left(\frac{\mathfrak{D}\Delta}{\Delta} + \mathfrak{D}[\mathfrak{g}^{\alpha\beta}]\mathfrak{g}_{\alpha\beta}\right)$$
$$- \frac{\mathfrak{D}^2\Delta}{2\Delta} + \frac{(\mathfrak{D}\Delta)^2}{2\Delta^2} + \frac{1}{4}\mathfrak{D}[\mathfrak{g}^{\alpha\epsilon}]\mathfrak{D}[\mathfrak{g}^{\beta\kappa}]\mathfrak{g}_{\beta\epsilon}\mathfrak{g}_{\alpha\kappa}, \tag{101}$$

but the second term is vanishing, since as is well known from the properties of determinants (see for example [23]),

$$\frac{\mathfrak{D}\Delta}{\Delta} + \mathfrak{D}[\mathfrak{g}^{\alpha\beta}]\mathfrak{g}_{\alpha\beta} = 0. \tag{102}$$

Finally, from (101), we obtain the equation that determines Ω, *the Einstein equations*, where in order to compare with the four-dimensional case discussed in the literature, we have replaced Δ by $1/q$,

$$\mathfrak{D}^2\Omega = \frac{8\pi G_{(n)}}{n-2}T^{00}\Omega^{-3} + \frac{1}{n-2}\left[\frac{1}{2}\frac{(\mathfrak{D}q)^2}{q^2} - \frac{\mathfrak{D}^2q}{2q} + \frac{1}{4}\mathfrak{D}[\mathfrak{g}^{\alpha\epsilon}]\mathfrak{D}[\mathfrak{g}^{\beta\kappa}]\mathfrak{g}_{\beta\epsilon}\mathfrak{g}_{\alpha\kappa}\right]\Omega. \tag{103}$$

This equation in addition to the metricity conditions and the other three relations equations (140)–(142) that Ω must satisfy constitute a system equivalent to the ten Einstein equations for the metric g_{ab}.

As a particular case, we cite the well-known case of NSF in four dimensions, where one has a system of two PDEs, namely

$$\partial_{ss}Z = S(u, w^+, w^-, R, s, s^*), \tag{104}$$

$$\partial_{s^*s^*}Z = S^*(u, w^+, w^-, R, s, s^*). \tag{105}$$

The contravariant components of the metric are

$$g^{AB} = \Omega^2\mathfrak{g}^{AB} = \Omega^2\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & a & -1 & b \\ 0 & -1 & a^* & b^* \\ 1 & b & b^* & c \end{pmatrix}, \tag{106}$$

with $a = -S_R$ y $a^* = -S_R^*$. (The expressions for b, b^* and c are not necessary to write the Einstein equations.)

From this expression for the metric, we have that

$$\frac{1}{4}\mathfrak{D}[\mathfrak{g}^{\alpha\epsilon}]\mathfrak{D}[\mathfrak{g}^{\beta\kappa}]\mathfrak{g}_{\beta\epsilon}\mathfrak{g}_{\alpha\kappa} = \frac{1}{4}\frac{(\mathfrak{D}q)^2}{q^2} + \frac{1}{2}\frac{\mathfrak{D}a\mathfrak{D}a^*}{q}, \tag{107}$$

with $q = 1 - S_R S_R^*$.

In this way by replacing in equation (103), we obtain

$$\mathfrak{D}^2\Omega = 4\pi GT^{00}\Omega^{-3} + \left[\frac{3}{8}\frac{(\mathfrak{D}q)^2}{q^2} - \frac{1}{4}\frac{\mathfrak{D}^2q}{q} + \frac{1}{4}\frac{\mathfrak{D}^2[S]\mathfrak{D}^2[S^*]}{q}\right]\Omega, \tag{108}$$

and therefore, we recover the well-known equation for the conformal factor obtained first (with an error of a factor -1) in [3] (see the correction given in [4], in equation (5)).

6. NSF of GR in five dimensions

As an explicit case, we will consider now the NSF of GR in five dimensions. The motivation is twofold: on one hand, the Einstein equations in five dimensions can be used as an attempt of a geometrical unification between the Maxwell electrodynamics (more precisely null

electromagnetic fields) and gravitational fields in the manner of *Kaluza–Klein*. Although it is not so clear if this program can be studied in terms of NSF, it would be very interesting to be able to put both fields in terms of the function Z. On the other hand, this is the most simple example of the formalism that one can study in dimensions higher than 4. We also present some simple known metrics expressed in this formalism.

6.1. The equations

The null surfaces can be associated with general solutions from the following system of five PDEs:

$$\partial_{ss}Z = S(Z, \partial_s Z, \partial_{s^*} Z, \partial_\gamma Z, \partial_{ss^*} Z, s, s^*, \gamma), \tag{109}$$

$$\partial_{s^*s^*}Z = S^*(Z, \partial_s Z, \partial_{s^*} Z, \partial_\gamma Z, \partial_{ss^*} Z, s, s^*, \gamma), \tag{110}$$

$$\partial_{s\gamma}Z = \Phi(Z, \partial_s Z, \partial_{s^*} Z, \partial_\gamma Z, \partial_{ss^*} Z, s, s^*, \gamma), \tag{111}$$

$$\partial_{s^*\gamma}Z = \Phi^*(Z, \partial_s Z, \partial_{s^*} Z, \partial_\gamma Z, \partial_{ss^*} Z, s, s^*, \gamma), \tag{112}$$

$$\partial_{\gamma\gamma}Z = \Upsilon(Z, \partial_s Z, \partial_{s^*} Z, \partial_\gamma Z, \partial_{ss^*} Z, s, s^*, \gamma). \tag{113}$$

These functions must satisfy the Wünschmann conditions (78)–(84),

$$\mathfrak{m} = (D_s S)_R - 3(S_{w^+} S_R + S_{w^-} + S_{w^1} \Phi_R - S_R \mathfrak{g}^{+R}) = 0, \tag{114}$$

$$\mathfrak{m}^* = (D_{s^*} S^*)_R - 3(S^*_{w^-} S^*_R + S^*_{w^+} + S^*_{w^1} \Phi^*_R - S^*_R \mathfrak{g}^{-R}) = 0, \tag{115}$$

$$\mathfrak{m}_{111} = (D_1 \Upsilon)_R - 3(\Upsilon_{w^+} \Phi_R + \Upsilon_{w^-} \Phi^*_R + \Upsilon_{w^1} \Upsilon_R - \Upsilon_R \mathfrak{g}^{1R}) = 0, \tag{116}$$

$$\begin{aligned}\mathfrak{m}_1 = (D_1 S)_R &- (S_{w^-} \Phi^*_R + S_{w^+} \Phi_R + S_{w^1} \Upsilon_R - S_R \mathfrak{g}^{1R}) \\ &- 2(\Phi_{w^-} + \Phi_{w^+} S_R + \Phi_{w^1} \Phi_R - \Phi_R \mathfrak{g}^{+R}) = 0,\end{aligned} \tag{117}$$

$$\begin{aligned}\mathfrak{m}^*_1 = (D_1 S^*)_R &- (S^*_{w^+} \Phi_R + S^*_{w^-} \Phi^*_R + S^*_{w^1} \Upsilon_R - S^*_R \mathfrak{g}^{1R}) \\ &- 2(\Phi^*_{w^+} + \Phi^*_{w^-} S^*_R + \Phi^*_{w^1} \Phi^*_R - \Phi^*_R \mathfrak{g}^{-R}) = 0,\end{aligned} \tag{118}$$

$$\begin{aligned}\mathfrak{m}_{11} = (D_s \Upsilon)_R &- (\Upsilon_{w^+} S_R + \Upsilon_{w^-} + \Upsilon_{w^1} \Phi_R) \\ &- 2(\Phi_{w^+} \Phi_R + \Phi_{w^-} \Phi^*_R + \Phi_{w^1} \Upsilon_R) + \Upsilon_R \mathfrak{g}^{+R} + 2\Phi_R \mathfrak{g}^{1R} = 0,\end{aligned} \tag{119}$$

$$\begin{aligned}\mathfrak{m}^*_{11} = (D_{s^*} \Upsilon)_R &- (\Upsilon_{w^-} S^*_R + \Upsilon_{w^+} + \Upsilon_{w^1} \Phi^*_R) \\ &- 2(\Phi_{w^-} \Phi_R + \Phi^*_{w^+} \Phi_R + \Phi^*_{w^1} \Upsilon_R) + \Upsilon_R \mathfrak{g}^{-R} + 2\Phi^*_R \mathfrak{g}^{1R} = 0.\end{aligned} \tag{120}$$

In these relations, $\mathfrak{g}^{+R}$, $\mathfrak{g}^{-R}$ and $\mathfrak{g}^{1R}$ are some of the components of the conformal metric

$$\mathfrak{g}^{+R} = -\frac{1}{4 - S_R S^*_R}[2(T_R - S_{w^+} - S_{w^-} S^*_R - S_{w^1} \Phi^*_R) - S_R(T^*_R - S^*_{w^-} - S^*_{w^+} S_R - S^*_{w^1} \Phi_R)], \tag{121}$$

$$\mathfrak{g}^{-R} = -\frac{1}{4 - S_R S^*_R}[2(T^*_R - S^*_{w^-} - S^*_{w^+} S_R - S^*_{w^1} \Phi_R) - S^*_R(T_R - S_{w^+} - S_{w^-} S^*_R - S_{w^1} \Phi^*_R)], \tag{122}$$

$$\mathfrak{g}^{1R} = -\Phi^*_R \mathfrak{g}^{+R} - \Phi_R \mathfrak{g}^{-R} - (Q_{1R} - \Phi_{w^+} - \Phi_{w^-} S^*_R - \Phi_{w^1} \Phi^*_R - \Phi^*_{w^+} S_R - \Phi^*_{w^-} - \Phi^*_{w^1} \Phi_R). \tag{123}$$

Remark 3: Note also that using the commutator relations (valid in n dimensions)

$$[\partial_y, D_s] = \delta_{w^+,y}\partial_u + S_{.y}\partial_{w^+} + \delta_{R,y}\partial_{w^-} + \Phi_{k,y}\partial_{w^k} + T_y\partial_R, \tag{124}$$

$$[\partial_y, D_{s^*}] = \delta_{w^-,y}\partial_u + \delta_{R,y}\partial_{w^+} + S_{.y}^*\partial_{w^-} + \Phi_{k,y}^*\partial_{w^k} + T_y^*\partial_R, \tag{125}$$

$$[\partial_y, D_m] = \delta_{w^m,y}\partial_u + \Phi_{m,y}\partial_{w^+} + \Phi_{m,y}^*\partial_{w^-} + \Upsilon_{km,y}\partial_{w^k} + Q_{m,y}\partial_R, \tag{126}$$

with $y \in \{u, w^+, w^-, w^m, R\}$ and $\delta_{y',y}$ the Kronecker symbol, we have

$$(D_s S)_R = D_s(S_R) + S_R S_{w^+} + S_{w^-} + \Phi_R S_{w^1} + T_R S_R, \tag{127}$$

$$(D_{s^*} S^*)_R = D_{s^*}(S_R^*) + S_R^* S_{w^-}^* + S_{w^+}^* + \Phi_R^* S_{w^1}^* + T_R^* S_R^*, \tag{128}$$

$$(D_1 S)_R = D_1(S_R) + \Phi_R S_{w^+} + \Phi_R^* S_{w^-} + \Upsilon_R S_{w^1} + Q_{1,R} S_R, \tag{129}$$

$$(D_1 S^*)_R = D_1(S_R^*) + \Phi_R^* S_{w^-}^* + \Phi_R S_{w^+}^* + \Upsilon_R S_{w^1}^* + Q_{1,R} S_R^*, \tag{130}$$

$$(D_s \Upsilon)_R = D_s(\Upsilon_R) + S_R \Upsilon_{w^+} + \Upsilon_{w^-} + \Phi_R \Upsilon_{w^1} + T_R \Upsilon_R, \tag{131}$$

$$(D_{s^*} \Upsilon)_R = D_{s^*}(\Upsilon_R) + S_R^* \Upsilon_{w^-} + \Upsilon_{w^+} + \Phi_R^* \Upsilon_{w^1} + T_R^* \Upsilon_R, \tag{132}$$

$$(D_1 \Upsilon)_R = D_1(\Upsilon_R) + \Phi_R \Upsilon_{w^+} + \Phi_R^* \Upsilon_{w^-} + \Upsilon_R \Upsilon_{w^1} + Q_{1,R} \Upsilon_R. \tag{133}$$

The other components of the conformal metric are

$$\mathfrak{g}^{00} = \mathfrak{g}^{0+} = \mathfrak{g}^{0-} = \mathfrak{g}^{01} = 0, \tag{134}$$

$$\mathfrak{g}^{0R} = -\mathfrak{g}^{+-} = 1, \tag{135}$$

$$\mathfrak{g}^{++} = -S_R, \quad \mathfrak{g}^{--} = -S_R^*, \tag{136}$$

$$\mathfrak{g}^{+1} = -\Phi_R, \quad \mathfrak{g}^{-1} = -\Phi_R^*, \tag{137}$$

$$\mathfrak{g}^{11} = -\Upsilon_R, \tag{138}$$

and

$$\begin{aligned}
\mathfrak{g}^{RR} = -\frac{1}{2 + S_R S_R^*}\{ & S_{w^1}^* S_R \mathfrak{g}^{1R} + [U_R - 2T_{w^+} - 2T_{w^-} S_R^* - 2T_{w^1}\Phi_R^* - 2T_{w^+}^* S_R \\
& - 2T_{w^-}^* - 2T_{w^1}^* \Phi_R + S_u^* S_R + S_R^* S_u - S_{w^+}^*(S_{w^+} S_R + S_{w^-} + S_{w^1}\Phi_R) \\
& - S_{w^-}^*(S_{w^+} + S_{w^-} S_R^* + S_{w^1}\Phi_R^*) - S_{w^1}^*(S_{w^+}\Phi_R + S_{w^-}\Phi_R^* + S_{w^1}\Upsilon_R)]\mathfrak{g}^{0R} \\
& + (2T_R^* + S_{w^+}^* S_R + S_R^* S_{w^+})\mathfrak{g}^{+R} + (2T_R + S_{w^-} S_R^* + S_R S_{w^-}^*)\mathfrak{g}^{-R} \}.
\end{aligned} \tag{139}$$

Finally the conformal factor must satisfy the relations

$$D_s \Omega = \tfrac{1}{2}(\mathfrak{g}^{+R} + T_R)\Omega, \tag{140}$$

$$D_{s^*} \Omega = \tfrac{1}{2}(\mathfrak{g}^{-R} + T_R^*)\Omega, \tag{141}$$

$$D_1 \Omega = \tfrac{1}{2}(\mathfrak{g}^{1R} + Q_{m,R})\Omega, \tag{142}$$

and the Einstein equation (103)

$$\begin{aligned}
\mathfrak{D}^2 \Omega = \frac{8\pi G_{(5)}}{3} T^{00}\Omega^{-3} + \frac{1}{6}\Big\{ & \frac{3}{2}\frac{(\mathfrak{D}q)^2}{q^2} - \frac{\mathfrak{D}^2 q}{q} + \frac{1}{q}[2(\Phi_R \mathfrak{D}\Phi_R \mathfrak{D}S_R^* \\
& + \Phi_R^* \mathfrak{D}\Phi_R^* \mathfrak{D}S_R - \mathfrak{D}\Phi_R \mathfrak{D}\Phi_R^*) - S_R \mathfrak{D}\Upsilon_R \mathfrak{D}S_R^* - S_R^* \mathfrak{D}\Upsilon_R \mathfrak{D}S_R \\
& - \Upsilon_R \mathfrak{D}S_R \mathfrak{D}S_R^* + S_R(\mathfrak{D}\Phi_R)^2 + S_R^*(\mathfrak{D}\Phi_R^*)^2]\Big\}\Omega,
\end{aligned} \tag{143}$$

with

$$q = 2\Phi_R \Phi_R^* - S_R \Phi_R^{*2} - S_R^* \Phi_R^2 - \Upsilon_R(1 - S_R S_R^*). \tag{144}$$

13

6.2. A simple example

Let there be the following family of null surfaces of the five-dimensional Minkowski spacetime:

$$u = Z(x^a, \zeta, \bar{\zeta}, \gamma) = x^a l_a(\zeta, \bar{\zeta}, \gamma), \tag{145}$$

with $x^a = \{t, x, y, z, v\}$ the Minkowskian coordinates and l_a the covariant components of the null vector l^a given by

$$l^a(\zeta, \bar{\zeta}, \gamma) = \frac{1}{\sqrt{2}(1 + \zeta\bar{\zeta})} \left((1 + \zeta\bar{\zeta}), (\zeta + \bar{\zeta}) \sin\gamma, i(\bar{\zeta} - \zeta) \sin\gamma, \right.$$
$$\left. (\zeta\bar{\zeta} - 1) \sin\gamma, (1 + \zeta\bar{\zeta}) \cos\gamma \right), \tag{146}$$

with $\{\zeta, \bar{\zeta}, \gamma\}$ the coordinates on a three-dimensional sphere S^3, and $\bar{\zeta}$ the complex conjugate of ζ.

We will make the following identification between parameters: $s \equiv \zeta$, $s^* \equiv \bar{\zeta}$ and $\gamma^1 \equiv \gamma$. Therefore, from these relations, we can construct the scalars θ^A,

$$u = \frac{\sqrt{2}}{2} \left\{ t - \frac{\sin\gamma}{1 + \zeta\bar{\zeta}} [(\zeta + \bar{\zeta})x + i(\bar{\zeta} - \zeta)y - (1 - \zeta\bar{\zeta})z] - \cos\gamma\, v \right\}, \tag{147}$$

$$w^+ = \frac{\sqrt{2}}{2} \frac{\sin\gamma}{(1 + \zeta\bar{\zeta})^2} [(\bar{\zeta}^2 - 1)x + i(\bar{\zeta}^2 + 1)y - 2\bar{\zeta}z], \tag{148}$$

$$w^- = \frac{\sqrt{2}}{2} \frac{\sin\gamma}{(1 + \zeta\bar{\zeta})^2} [(\zeta^2 - 1)x - i(\zeta^2 + 1)y - 2\zeta z], \tag{149}$$

$$w^1 = \frac{\sqrt{2}}{2} \left\{ -\frac{\cos\gamma}{1 + \zeta\bar{\zeta}} [(\zeta + \bar{\zeta})x + i(\bar{\zeta} - \zeta)y - (1 - \zeta\bar{\zeta})z] + \sin\gamma\, v \right\}, \tag{150}$$

$$R = \sqrt{2} \frac{\sin\gamma}{(1 + \zeta\bar{\zeta})^3} [(\zeta + \bar{\zeta})x + i(\bar{\zeta} - \zeta)y - (1 - \zeta\bar{\zeta})z]. \tag{151}$$

Solving these equations for the x^as, we obtain

$$t = \sqrt{2}[u + \cot\gamma\, w^1 + \frac{(1 + \zeta\bar{\zeta})^2}{2\sin^2\gamma} R], \tag{152}$$

$$x = \frac{\sqrt{2}}{2\sin\gamma} [(\bar{\zeta} - 1)w^- + (\zeta^2 - 1)w^+ + (\zeta + \bar{\zeta})(1 + \zeta\bar{\zeta})R], \tag{153}$$

$$y = \frac{\sqrt{2}i}{2\sin\gamma} [(\bar{\zeta} + 1)w^- - (\zeta^2 + 1)w^+ + (\zeta - \bar{\zeta})(1 + \zeta\bar{\zeta})R], \tag{154}$$

$$z = -\frac{\sqrt{2}}{2\sin\gamma} [2\bar{\zeta}w^- + 2\zeta w^+ + (1 - \zeta^2\bar{\zeta}^2)R], \tag{155}$$

$$v = \frac{\sqrt{2}}{2\sin\gamma} [w^1 + \cos\gamma(1 + \zeta\bar{\zeta})^2 R]. \tag{156}$$

Remember that these equations can be interpreted as a three-parameter family of coordinate transformations between x^a and $\{u, w^+, w^-, w^1, R\}$. In particular, for fixed values of ζ, $\bar{\zeta}$ and γ, we can write the Minkowskian metric in the new coordinates $\{u, w^+, w^-, w^1, R\}$,

$$ds^2 = \eta_{ab}dx^a dx^b = \frac{\partial x^a}{\partial \theta^A} \frac{\partial x^b}{\partial \theta^B} \eta_{ab} d\theta^A d\theta^B$$
$$= 2du^2 + 2\frac{(1 + \zeta\bar{\zeta})^2}{\sin^2\gamma} du\, dR + 2\cot\gamma\, dw^1 du - 2\frac{(1 + \zeta\bar{\zeta})^2}{\sin^2\gamma} dw^+ dw^- - 2(dw^1)^2.$$
$$\tag{157}$$

14

On the other hand, from equations (16)–(20), and transformations (152)–(156), we see that

$$S = -\frac{2\bar{\zeta}}{1 + \zeta\bar{\zeta}} w^+, \tag{158}$$

$$S^* = -\frac{2\zeta}{1 + \zeta\bar{\zeta}} w^-, \tag{159}$$

$$\Phi = \cot\gamma\, w^+, \tag{160}$$

$$\Phi^* = \cot\gamma\, w^-, \tag{161}$$

$$\Upsilon = \frac{1}{2\sin^2\gamma}[(1 + \zeta\bar{\zeta})^2 R + 2\sin\gamma\cos\gamma\, w^1]. \tag{162}$$

In a similar way, we obtain

$$T = D_{\bar{\zeta}}S = -\frac{2}{(1 + \zeta\bar{\zeta})^2}[w^+ + \bar{\zeta}(1 + \zeta\bar{\zeta})R], \tag{163}$$

$$T = D_{\zeta}S^* = -\frac{2}{(1 + \zeta\bar{\zeta})^2}[w^- + \zeta(1 + \zeta\bar{\zeta})R], \tag{164}$$

$$Q_1 = D_{\bar{\zeta}}\Phi = \cot\gamma\, R. \tag{165}$$

Now instead of starting with the flat metric (157), we want to consider the system of PDEs (109)–(113) and reconstruct from it the (conformal) metric.

It is an easy task to shown that, as it should be, the functions $(S, S^*, \Phi, \Phi^*, \Upsilon)$ satisfy the Wünshmann conditions (114)–(120).

Finally, from expressions (121)–(123) and (134)–(139) for the contravariant components of the (conformal) metric in terms of $(S, S^*, \Phi, \Phi^*, \Upsilon)$, we obtain that the only non-vanishing components are

$$g^{0R} = \Omega^2, \tag{166}$$

$$g^{+-} = -\Omega^2, \tag{167}$$

$$g^{RR} = -\frac{2}{(1 + \zeta\bar{\zeta})^2}\Omega^2, \tag{168}$$

$$g^{1R} = \cot\gamma\,\Omega^2, \tag{169}$$

$$g^{11} = -\frac{(1 + \zeta\bar{\zeta})^2}{\sin^2\gamma}\Omega^2, \tag{170}$$

or in its covariant version

$$g_{AB} = \Omega^{-2}\begin{pmatrix} \frac{2\sin^2\gamma}{(1+\zeta\bar{\zeta})^2} & 1 & 0 & 0 & \frac{2\sin\gamma\cos\gamma}{(1+\zeta\bar{\zeta})^2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \frac{2\sin\gamma\cos\gamma}{(1+\zeta\bar{\zeta})^2} & 0 & 0 & 0 & -\frac{2\sin^2\gamma}{(1+\zeta\bar{\zeta})^2} \end{pmatrix}. \tag{171}$$

On the other hand, the conformal factor must satisfy equations (140)–(142),

$$D_{\zeta}\Omega = \frac{T_R}{2}\Omega = -\frac{\bar{\zeta}}{1 + \zeta\bar{\zeta}}\Omega, \tag{172}$$

$$D_{\bar{\zeta}}\Omega = \frac{T_R^*}{2}\Omega = -\frac{\zeta}{1 + \zeta\bar{\zeta}}\Omega, \tag{173}$$

$$D_\gamma \Omega = \tfrac{1}{2}(\mathfrak{g}^{1R} + Q_{1,R})\Omega = \cot \gamma \; \Omega, \tag{174}$$

and Einstein's equation (143) (with $T^{00} = 0$)

$$\mathfrak{D}^2 \Omega = 0. \tag{175}$$

This system admits as a particular solution

$$\Omega_\eta = \frac{\sin \gamma}{1 + \zeta \bar\zeta}. \tag{176}$$

Therefore, it is a simple task to show that this choice makes the metric g_{AB} the Minkowski metric (equation (157)).

However, they admit other conformal flat solutions, as for example, the de Sitter solution

$$\Omega = \Omega_\eta + \Lambda \eta_{ab} x^a x^b \tag{177}$$

$$= \frac{\sin \gamma}{1 + \zeta \bar\zeta} + 2\Lambda \left[u^2 - (w^1)^2 + 2\cot \gamma \, w^1 u + \frac{(1 + \zeta \bar\zeta)^2}{\sin^2 \gamma}(uR - w^+ w^-) \right], \tag{178}$$

with $\Lambda = $ constant.

Remark 4: In the four-dimensional case, the system is given by the pair of PDEs (104) and (105), with the two functions S and S^* as in equations (158) and (159). On the other hand, in the literature [3], one can find that the conformal flat metrics can be obtained from the most simple system of equations

$$\eth^2 Z = 0, \tag{179}$$

$$\bar\eth^2 Z = 0, \tag{180}$$

where $\eth$ is the edth operator that acts on a function η of spin-weight s as $\eth\eta = 2P^{1-s}\partial_\zeta(P^s\eta)$, and $P = \tfrac{1}{2}(1 + \zeta \bar\zeta)$. There is not incompatibility between these two systems, in fact acting on a function Z ($s = 0$), we have

$$\eth^2 Z = 4\partial_\zeta(P^2 \partial_\zeta Z) = 4P^2 \partial_{\zeta\zeta} Z + 8P\partial_\zeta P \partial_\zeta Z, \tag{181}$$

and therefore from $\eth^2 Z = 0$, we obtain

$$\partial_{\zeta\zeta} Z = -2\partial_\zeta(\ln P)\partial_\zeta Z = -\frac{2\bar\zeta}{1 + \zeta \bar\zeta}\partial_\zeta Z, \tag{182}$$

which coincides with our S. It would be desirable to reformulate NSF in higher dimensions in terms of covariant operators on the sphere S^{n-2}.

7. Final comments

We have shown that the NSF can be extended to higher dimensions. In particular, all conformal n-dimensional metric can be constructed from a particular class of $n(n-3)/2$ PDEs. In order to assure the existence of the metric, this class must satisfy a set of metricity conditions. However, this system of PDEs becomes more involved when the considered dimension increases, and it is not so easy to write explicit conditions in order the metric to be Lorentzian. This in an important caveat. Yet in this case it is notable that all the information contained in the $n(n + 1)/2$ metric components (local point of view) can be globally codified in only two functions Z and Ω dependent on $n - 2$ parameters (global point of view).

On the other hand, it would be expected that the Wünschmann conditions have a similar geometrical meaning as in three and four dimensions. In the last two cases, one can show that these conditions can be understood as the requirement of a vanishing torsion tensor of a

canonical connection defined on the solution space associated with the differential equations. In fact, if there exists such a connection, then the Lie derivative of the metric can be put in terms of the torsion components, and if the torsion vanishes, the Lie derivative of the metric is proportional to the metric, i.e. they are all in the same conformal class. Although in higher dimensions the equations are more complicated, the program should in principle be possible. These results will be presented elsewhere.

Finally, as was mentioned before, it should be desirable to express this formalism in terms of covariant operators, and from then find a geometrical meaning to the functions $(S, S^*, \Phi_m, \Phi_m^*, \Upsilon_{nm})$.

Acknowledgments

Appendix A

Here we would like to give some examples of families of null surfaces of n-dimensional Lorentzian flat spacetimes, such that equation (14) is satisfied. Let us begin with the following $(n - 2)$-parametric family of functions:

$$u = Z(x^a, s_1, s_2, \ldots, s_{n-2}) = x_0 - \frac{2s_1}{1 + r^2}x_1 - \frac{2s_2}{1 + r^2}x_2 - \cdots - \frac{2s_{n-2}}{1 + r^2}x_{n-2} - \frac{r^2 - 1}{1 + r^2}x_{n-1},$$
(A.1)

with $r^2 = s_1^2 + s_2^2 + \cdots + s_{n-2}^2$ and $\{s_1, s_2, \ldots, s_{n-2}\} \in \mathbb{R}^{n-2}$. The parameters s_i can be thought as stereographic coordinates of the sphere S^{n-2}. This family of functions satisfies the eikonal equation

$$(Z_{x_0})^2 - (Z_{x_1})^2 - \cdots - (Z_{x_{n-1}})^2 = 0$$
(A.2)

and geometrically they represent the null plane waves $u = x^a l_a(s_1, \ldots, s_{n-2})$ spanned by the null vectors

$$l^a = \left(1, \frac{2s_1}{1 + r^2}, \ldots, \frac{2s_{n-2}}{1 + r^2}, \frac{r^2 - 1}{1 + r^2}\right).$$
(A.3)

Now, if we define the scalars $\theta^A{}_{,a}$ from this Z (with any identification between the parameters $\{s_1, s_2, \ldots, s_{n-2}\}$ and $\{s, s^*, \ldots, \gamma^m\}$), we get that det $\theta^A{}_{,a} = 0$, i.e. they are not independent scalars. However, by doing the following (complex) transformation:

$$s \equiv \zeta = s_1 + \mathrm{i}s_2,$$
(A.4)

$$s^* \equiv \bar{\zeta} = s_1 - \mathrm{i}s_2,$$
(A.5)

$$\gamma^m = s_m,$$
(A.6)

we obtain that the new $\theta^A{}_{,a}$ satisfy

$$\det \theta^A{}_{,a} = -\left(\frac{-2}{1 + r^{*2}}\right)^{n-2} \mathrm{i},$$
(A.7)

with $r^{*2} = 1 + \zeta\bar{\zeta} + \gamma_1^2 + \cdots + \gamma_{n-2}^2$. Therefore, equation (14) is satisfied (with the exception of the pole $r^* \to \infty$).

Let us consider now the family

$$Z = x_0 - \tilde{s}_1 x_1 - \tilde{s}_2 x_2 - \cdots - \tilde{s}_{n-2} x_{n-2} - \sqrt{1 - \tilde{r}^2}\, x_{n-1},$$
(A.8)

with $\tilde{r}^2 = \tilde{s}_1^2 + \tilde{s}_2^2 + \cdots + \tilde{s}_{n-2}^2$ and $s_i \in [-1, 1]$. Note that this family can be obtained from the previous one, equation (A.1), by the transformation

$$\tilde{s}_1 = \frac{2s_1}{1 + r^2}, \tag{A.9}$$

$$\tilde{s}_2 = \frac{2s_2}{1 + r^2}, \tag{A.10}$$

$$\vdots$$

$$\tilde{s}_{n-2} = \frac{2s_{n-2}}{1 + r^2}. \tag{A.11}$$

In this case, by doing the identification $s \equiv s_1$, $s^* \equiv s_2$ and $\gamma^m \equiv s_m$, we obtain

$$\det \theta^A{}_{,a} = \frac{(-1)^n s s^*}{(1 - \tilde{r}^2)^{3/2}}, \tag{A.12}$$

and therefore the scalars θ^A are independent in the regions where s and s^* are non-vanishing.

An alternative possibility was used in the example of five-dimensional Minkowski spacetime (equation (145)). In that case, $\det \theta^A{}_{,a} = \frac{4 \mathrm{i} \sin^4 \gamma_1}{(1 + \zeta \bar{\zeta})^4}$.

Note that the examples given by equations (A.1) and (A.8) geometrically represent the same family of null waves, but they give origin to distinct (although equivalent) PDEs. They are related to each other by a fiber-preserving transformation. In fact, in three and four dimensions, Frittelli, Kamran and Newman show the equivalence between PDEs with vanishing Wünschmann invariants under more general transformations than fiber preserving, known as contact transformations [14–16].

Appendix B

The explicit expressions for the generalized Wünschmann conditions (78)–(84) are

$$\mathfrak{m} = (D_s S)_R - 3(S_{w^+} S_R + S_{w^-} + S_{w^m} \Phi_{m,R} - S_R \mathfrak{g}^{+R}) = 0, \tag{B.1}$$

$$\mathfrak{m}^* = (D_{s^*} S^*)_R - 3(S^*_{w^-} S^*_R + S^*_{w^+} + S^*_{w^m} \Phi^*_{m,R} - S^*_R \mathfrak{g}^{-R}) = 0, \tag{B.2}$$

$$\begin{aligned}
\mathfrak{m}_{kmn} = {} & (D_k \Upsilon_{mn})_R - \Upsilon_{mn,w^+} \Phi_{k,R} - \Upsilon_{mk,w^+} \Phi_{n,R} - \Upsilon_{kn,w^+} \Phi_{m,R} \\
& - \Upsilon_{mn,w^-} \Phi^*_{k,R} - \Upsilon_{mk,w^-} \Phi^*_{n,R} - \Upsilon_{kn,w^-} \Phi^*_{m,R} \\
& - \Upsilon_{mn,w^l} \Upsilon_{lk,R} - \Upsilon_{mk,w^l} \Upsilon_{ln,R} - \Upsilon_{kn,w^l} \Upsilon_{lm,R} \\
& + \Upsilon_{mn,R} \mathfrak{g}^{kR} + \Upsilon_{mk,R} \mathfrak{g}^{nR} + \Upsilon_{kn,R} \mathfrak{g}^{mR} = 0,
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
\mathfrak{m}_m = {} & (D_m S)_R - (S_{w^-} \Phi^*_{m,R} + S_{w^+} \Phi_{m,R} + S_{w^n} \Upsilon_{nm,R} - S_R \mathfrak{g}^{mR}) \\
& - 2(\Phi_{m,w^-} + \Phi_{m,w^+} S_R + \Phi_{m,w^n} \Phi_{n,R} - \Phi_{m,R} \mathfrak{g}^{+R}) = 0,
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
\mathfrak{m}^*{}_m = {} & (D_m S^*)_R - (S_{w^+} \Phi_{m,R} + S^*_{w^-} \Phi^*_{m,R} + S^*_{w^n} \Upsilon_{nm,R} - S^*_R \mathfrak{g}^{mR}) \\
& - 2(\Phi^*_{m,w^+} + \Phi^*_{m,w^-} S^*_R + \Phi^*_{m,w^n} \Phi^*_{n,R} - \Phi^*_{m,R} \mathfrak{g}^{-R}) = 0,
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
\mathfrak{m}_{mn} = {} & (D_s \Upsilon)_{mn,R} - (\Upsilon_{mm.w^+} S_R + \Upsilon_{mn,w^-} + \Upsilon_{mn,w^k} \Phi_{k,R}) \\
& - (\Phi_{m,w^+} \Phi_{n,R} + \Phi_{m,w^-} \Phi^*_{n,R} + \Phi_{m,w^k} \Upsilon_{nk,R}) \\
& - (\Phi_{n,w^+} \Phi_{m,R} + \Phi_{n,w^-} \Phi^*_{m,R} + \Phi_{n,w^k} \Upsilon_{mk,R}) \\
& + \Upsilon_{mn,R} \mathfrak{g}^{+R} + \Phi_{m,R} \mathfrak{g}^{nR} + \Phi_{n,R} \mathfrak{g}^{mR} = 0,
\end{aligned} \tag{B.6}$$

$$\mathfrak{m}^*{}_{mn} = (D_{s^*}\Upsilon)_{mn,R} - (\Upsilon_{mm.w^-}S_R^* + \Upsilon_{mn,w^+} + \Upsilon_{mn,w^k}\Phi_{k,R}^*)$$
$$- (\Phi_{m,w^-}^* \Phi_{n,R}^* + \Phi_{m,w^+}^* \Phi_{n,R} + \Phi_{m,w^k}^* \Upsilon_{nk,R})$$
$$- (\Phi_{n,w^-}^* \Phi_{m,R}^* + \Phi_{n,w^+}^* \Phi_{m,R} + \Phi_{n,w^k}^* \Upsilon_{mk,R})$$
$$+ \Upsilon_{mn,R}\mathfrak{g}^{-R} + \Phi_{m,R}^*\mathfrak{g}^{nR} + \Phi_{n,R}^*\mathfrak{g}^{mR} = 0. \tag{B.7}$$

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