Discrete Applied Mathematics 160 (2012) 2627–2635

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On minimal vertex separators of dually chordal graphs: Properties and characterizations

Pablo De Caria *, Marisa Gutierrez

CONICET/ Departamento de Matemática, Universidad Nacional de La Plata, Calle 50 y 115, C.C. 172, (1900), La Plata, Argentina

ARTICLE INFO

Article history:
Received 30 April 2010
Received in revised form 13 February 2012
Accepted 22 February 2012
Available online 10 March 2012

Keywords:
Chordal
Dually chordal
Separator
Tree
Clique
Neighborhood

ABSTRACT

Many works related to dually chordal graphs, their cliques and neighborhoods were published by Brandstädt et al. (1998) [1] and Gutierrez (1996) [6]. We will undertake a similar study by considering minimal vertex separators and their properties instead. We find a necessary and sufficient condition for every minimal vertex separator to be contained in the closed neighborhood of a vertex and two major characterizations of dually chordal graphs are proved. The first states that a graph is dually chordal if and only if it possesses a spanning tree such that every minimal vertex separator induces a subtree. The second says that a graph is dually chordal if and only if the family of minimal vertex separators is Helly, its intersection graph is chordal and each of its members induces a connected subgraph. We also found adaptations for them, requiring just $O(|E(G)|)$ minimal vertex separators if they are conveniently chosen. We obtain at the end a proof of a known characterization of the class of hereditary dually chordal graphs that relies on the properties of minimal vertex separators.

1. Introduction

The class of chordal graphs has been widely investigated and several of its characteristics are very useful in the resolution of many problems. One example of them are phylogenetic trees [7,8].

Clique graphs of chordal graphs form a class considered in many senses as dual to chordal graphs, hence the name dually chordal graphs. Several studies about them were done, and as a result, many characterizations of dually chordal graphs were discovered, mainly involving cliques and neighborhoods. However, not much has been revealed about their minimal vertex separators. For that reason, one of the purposes of this paper is to study minimal vertex separators of dually chordal graphs to determine if the properties known about cliques and neighborhoods have their counterparts dealing with minimal vertex separators.

After reviewing some terminology and previous results in Sections 2 and 3, we describe in Section 4 all the results we could prove about minimal vertex separators of dually chordal graphs.

In 4.1, we show the first results of the approach described above. To the known fact that dually chordal graphs are endowed with spanning trees such that any clique or neighborhood induces a subtree, we add that the same is true for minimal vertex separators.

In 4.2, we study minimal vertex separators contained in neighborhoods. We see that many of them could be found with the help of the trees mentioned in the previous paragraph and we discover, among other results, that every minimal vertex separator of a dually chordal graph is contained in the neighborhood of a vertex if and only if every chordless cycle of length greater than or equal to four is contained in the neighborhood of a vertex.

* Corresponding author. Tel.: +54 0221 4245875; fax: +54 0221 4229850.
E-mail addresses: pdecaria@mate.unlp.edu.ar, pablodc72@hotmail.com (P. De Caria), marisa@mate.unlp.edu.ar (M. Gutierrez).

0166-218X/$ – see front matter
doi:10.1016/j.dam.2012.02.022

In 4.3, we can see how the results of 4.1 lead to new characterizations of dually chordal graphs. We prove that a graph is dually chordal if and only if there is a spanning tree of the graph such that any minimal vertex separator induces a subtree. Another necessary and sufficient condition is that every minimal vertex separator induces a connected subgraph and that all the minimal vertex separators form a Helly family whose intersection graph is chordal. We also look for weaker conditions and we find that these characterizations not always require considering all the minimal vertex separators, but a subfamily whose number of members is of the order of the number of edges of the graph.

Finally, in 4.4, we show how the characterizations appearing in 4.3 can be used to find the family of minimal forbidden induced subgraphs for the class of hereditary dually chordal graphs. This family was already known, but minimal vertex separators had never been used as a tool to find it.

2. Some graph terminology

This paper deals just with finite simple (without loops or multiple edges) graphs. For a graph G, $V(G)$ denotes the set of its vertices and $E(G)$ that of its edges. A *complete* is a subset of pairwise adjacent vertices of $V(G)$. A *clique* is a maximal complete and the family of cliques of G will be denoted by $C(G)$. The subgraph *induced* by $A \subseteq V(G)$, $G[A]$, has A as vertex set and two vertices are adjacent in $G[A]$ if and only if they are adjacent in G.

Given two vertices v and w of G, the *distance* between v and w, or $d(v, w)$, is the length of any shortest path connecting v and w in G. The *open neighborhood* of v, or $N(v)$, is the set of all the vertices adjacent to v. The *closed neighborhood* of v, or $N[v]$, is defined by the equality $N[v] = N(v) \cup \{v\}$. The *disk* centered at v with radius k is the set $N^k[v] := \{w \in V(G),\ d(v, w) \leq k\}$.

Given two vertices u and v in the same connected component of G, a *uv-separator* is a set $S \subseteq V(G)$ such that u and v are in different connected components of $G - S := G[V(G) - S]$. It is *minimal* if no proper subset of S has the same property. We will just say *minimal vertex separator* to refer to a minimal set separating a pair of nonadjacent vertices. The family of minimal vertex separators of G is denoted by $\mathcal{S}(G)$.

Let G be a connected graph and let T be a spanning tree of G; for all $v, w \in V(G)$, $T[v, w]$ will denote the path in T from v to w or the vertices of this path, depending on the context. In the latter case, it is used to define $T(v, w)$ as the set $T[v, w] - \{v, w\}$.

Let $\mathcal{F}$ be a family of nonempty sets. $\mathcal{F}$ is *Helly* if the intersection of all the members of any subfamily of pairwise intersecting sets is not empty. If $C(G)$ is a Helly family, we say that G is a *clique-Helly graph*. The *intersection graph* of $\mathcal{F}$, $L(\mathcal{F})$, has the members of $\mathcal{F}$ as vertices, two of them being adjacent if and only if they are not disjoint. The *clique graph* $K(G)$ of G is the intersection graph of $C(G)$.

3. Basic notions and properties

A *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle. *Chordal* graphs are those without chordless cycles of length at least four.

A vertex w is a *maximum neighbor* of v if $N^2[v] \subseteq N[w]$. A linear ordering $v_1, \ldots, v_n$ of the vertices of G is a *maximum neighborhood ordering* of G if, for $i = 1, \ldots, n$, v_i has a maximum neighbor in $G[\{v_i, \ldots, v_n\}]$. *Dually chordal graphs* can be defined as those possessing a maximum neighborhood ordering. However, they were studied independently and simultaneously with different definitions and names, such as *HT-graphs*, *tree-clique graphs* and *expanded trees* [6,10]. Later, it became clear that the term *dually chordal* was the most fitting denomination for them [1].

The characterizations of dually chordal graphs are many. In fact, given a connected graph G, the following are equivalent [1]:

1. *G is dually chordal.*
2. *There is a spanning tree T of G such that any clique of G induces a subtree in T.*
3. *There is a spanning tree T of G such that any closed neighborhood of G induces a subtree in T.*
4. *There is a spanning tree T of G with any disk inducing a subtree in T.*
5. *G is clique-Helly and $K(G)$ is chordal.*

It is even true that any spanning tree fulfilling 2., 3. or 4. automatically fulfills the other two. Such a tree will be said to be *compatible* with G.

The characterization of a connected dually chordal graphs given by 3. can be rephrased as follows:

Theorem 1 (*[6]*)**.** *A connected graph G is dually chordal if and only if there is a spanning tree T of G such that, for all $x, y, z \in V(G)$, $xy \in E(G)$ and $z \in T(x, y)$ implies that $xz \in E(G)$ and $yz \in E(G)$.*

4. On minimal vertex separators of dually chordal graphs

From now on, all the graphs considered will be assumed to be connected for a better handling of the proofs of the properties we are going to discuss. It is not difficult to extend them to disconnected graphs by applying the proofs to each connected component.

4.1. Minimal vertex separators and compatible trees

We have seen that compatible trees have many interesting properties, which help to explain why they can be so helpful in modeling dually chordal graphs. The more we know about them, the more evident their importance is.

In this section, we see that they are related to minimal vertex separators in a way very similar to the known relationships with cliques and neighborhoods. More precisely, our main goal is to show that every minimal vertex separator induces a subtree of any compatible tree.

In order to start, we need the following lemma:

Lemma 1. *Let G be a dually chordal graph, T a tree compatible with G, u and v two nonadjacent vertices and S a minimal uv-separator of G. If $w \in S - T[u, v]$, then the vertices of the path in T from w to the vertex of $T[u, v]$ closest to w (with respect to T) are contained in S.*

Proof. Let x be a vertex in the above mentioned path, $x \neq w$, and suppose that $x \notin S$. As S is minimal and $w \in S$, there is a path P joining u and v such that w is the only vertex of S in P. Let A be the connected component of $T - x$ containing w. Suppose that y_1 and y_2 are two vertices adjacent in P and such that $y_1 \in A$ and $y_2 \notin A$. Then, $x \in T[y_1, y_2]$ and, since T is compatible with G, $y_1, y_2 \in N[x]$. This ensures that replacing any vertex of P that is also a vertex of A by x will yield a sequence connecting u and v and such that (removing identical consecutive vertices if necessary) consecutive vertices are adjacent. As w was replaced by x, no vertex of S is in the sequence, contradicting that S is a separator. Therefore, $x \in S$. □

Theorem 2. *Let G be a dually chordal graph and T a tree compatible with G. Then, every minimal vertex separator of G induces a subtree of T.*

Proof. Let u and v be two nonadjacent vertices and S a minimal uv-separator. We will prove that S induces a subtree of T. Let $T[A]$ and $T[B]$ be the connected components of $T - T(u, v)$ containing u and v, respectively. We first note that no vertex of S is in A. Otherwise, Lemma 1 would imply that $u \in S$. Likewise, no vertex of S is in B.

Assume that $x, y \in S$ and that x and y are not adjacent in T. In order to conclude that S induces a subtree, it will suffice to prove that $T(x, y) \cap S \neq \emptyset$.

Suppose on the contrary that $T(x, y) \cap S = \emptyset$. Let C be the connected component of $G - S$ containing u and C' the connected component of $G - S$ containing $T(x, y)$. We can prove that $C = C'$ by considering two cases:

If there are two adjacent vertices $w_1, w_2 \in C$ in different connected components of $T - T(x, y)$, then there is a vertex $w' \in T[w_1, w_2] \cap T(x, y)$, which is not in S by the assumption in the previous paragraph. Since T is compatible with G, w' is adjacent to both w_1 and w_2 and hence the three are in the same connected component of $G - S$, making C and C' equal.

Otherwise, let $T[D]$ be the only connected component of $T - T(x, y)$ intersecting C. Without loss of generality, assume that $y \notin D$. As S is a minimal uv-separator, y is adjacent to at least one vertex $w \in C$. If $w \in T(x, y)$, then $C \cap C' \neq \emptyset$ and hence $C = C'$. If $w \notin T(x, y)$, then $w \in D$ and we can find $w' \in T[w, y] \cap T(x, y)$. It holds that w is adjacent to w' because T is a compatible tree, so w and w' are in the same connected component of $G - S$. Since we knew that $w \in C$ and $w' \in C'$, we again conclude that $C = C'$.

By a similar argument, we can conclude that the vertices of $T(x, y)$ are in the same connected component of $G - S$ as v, contradicting that u and v are separated by S.

This contradiction is avoided if $T(x, y) \cap S \neq \emptyset$. Therefore, S induces a subtree of T. □

An immediate consequence of the previous theorem is that every minimal vertex separator of a dually chordal graph induces a connected subgraph. It is also known that a family of subtrees of a tree is Helly [5] and its intersection graph is chordal [11], which we use to get the following corollary:

Corollary 1. *Let G be a dually chordal graph. Then, $\mathcal{S}(G)$ is Helly, its intersection graph is chordal and every member of $\mathcal{S}(G)$ induces a connected subgraph of G.*

It will be proved later that the converse is also true.

4.2. Minimal vertex separators and neighborhoods

A typical characterization of chordal graphs is given by the fact that every minimal uv-separator induces a complete subgraph [2], implying that there is another vertex w whose neighborhood (excepting u or v if any of them is a neighbor of w) separates u and v. In the following, we see that dually chordal graphs satisfy a similar property and more connections between minimal vertex separators and neighborhoods will be established.

Theorem 3. *Let u and v be two nonadjacent vertices of a dually chordal graph G. Then, there is a vertex w, $w \neq u$ and $w \neq v$, such that $N[w] - \{u, v\}$ is a uv-separator.*

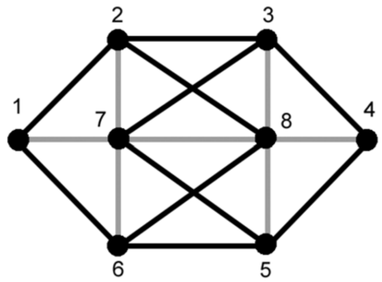

Fig. 1. A dually chordal graph with the edges of a compatible tree in gray. $\{1, 4, 7, 8\}$ is a minimal 26-separator but it is not contained in the neighborhood of any vertex.

Proof. Let T be a tree compatible with G. Note that u and v are not adjacent in T because they are not adjacent in G and thus $T(u, v)$ is not empty. Let w be any vertex of $T(u, v)$. If P is a path in G joining u and v, then it has vertices in different connected components of $T - w$. This implies that there are two vertices x_1 and x_2 consecutive in P such that $w \in T[x_1, x_2]$. Since T is compatible with G, it follows that $\{x_1, x_2\} \subseteq N[w]$. As u and v are nonadjacent, x_1 and/or x_2 belong to $N[w]-\{u, v\}$. Consequently, $N[w] - \{u, v\}$ separates u and v. □

Any vertex separator obviously contains a minimal vertex separator, which combined with the previous proof gives the following corollary:

Corollary 2. *Let G be a dually chordal graph, T a tree compatible with G and u and v two nonadjacent vertices of G. Then, the closed neighborhood of every vertex in $T(u, v)$ contains a minimal uv-separator.*

Despite this corollary, not every minimal vertex separator of a dually chordal graph is contained in a neighborhood (see Fig. 1).

Notwithstanding, that property becomes true under additional conditions. In fact, we get a necessary and sufficient condition for every minimal vertex separator of a dually chordal graph to be contained in a neighborhood.

We need the following previous result:

Lemma 2. *Let G be a dually chordal graph and $A \subseteq V(G)$ such that $d(x, y) \leq 2$ for all $x, y \in A$. Then, there exists a vertex w such that $A \subseteq N[w]$.*

Proof. Consider the family of all the closed neighborhoods centered at vertices of A. As there exists one tree T such that the closed neighborhood of every vertex of G induces a subtree of T, and subtrees of a tree always form a Helly family [5], $\{N[v]\}_{v\in A}$ is Helly. Furthermore, the fact that $d(x, y) \leq 2$ for all $x, y \in A$ implies that the members of the family are pairwise intersecting. Therefore, there is at least one vertex w such that $w \in \bigcap_{v\in A} N[v]$, that is, $A \subseteq N[w]$. □

Theorem 4. *Let G be a dually chordal graph. Then, every minimal vertex separator of G is contained in a closed neighborhood if and only if each chordless cycle of length at least four is contained in a closed neighborhood.*

Proof. Suppose that G is a dually chordal graph with every minimal vertex separator contained in a closed neighborhood. Let C be a chordless cycle of G of length at least four and let x, y be two nonconsecutive vertices of C. As there are two paths in C of length at least two from x to y, we can take u and v inner vertices of each of those paths. For every path in G from u to v not contained in $G[C]$, pick one vertex $z \notin C$ to form a set R. Set $S = R \cup \{x, y\}$.

Now we prove that S is a uv-separator. We just need to prove that each path P from u to v has a vertex in S. If P is not contained in $G[C]$, then it has a vertex in S because of the way R was constructed. Moreover, since C is chordless, there are only two paths in $G[C]$ from u to v, with x and y in each of them.

Let S' be a minimal uv-separator contained in S and let P_1 and P_2 be the paths in $G[C]$ containing x and y, respectively. As x is the only vertex in $S \cap V(P_1)$ and y is the only vertex in $S \cap V(P_2)$, we conclude that x and y are in S'. By the hypothesis, S' is contained in the neighborhood of a vertex and thus $N[x] \cap N[y] \neq \emptyset$. Therefore, $d(x, y) = 2$. We conclude that the vertices of C are either adjacent or are at distance two. By Lemma 2, there exists a vertex w such that $C \subseteq N[w]$.

Conversely, suppose that every chordless cycle is contained in the closed neighborhood of a vertex and let S be any minimal separator of two nonadjacent vertices u and v. The proof that S is contained in a closed neighborhood can be done by adapting the demonstration that a graph is chordal if and only if each minimal vertex separator is a complete as given in [5].

Let x and y be two vertices in S. Consider a cycle C containing x and y constructed as in [5]. If C has a chord, we conclude that x and y are adjacent. If C is chordless, then it is contained in the closed neighborhood of a vertex w and xwy is a path from x to y of length two. Therefore, again by Lemma 2, S is contained in the closed neighborhood of a vertex. □

If C is a cycle of length four or five, then any two vertices of it are at distance not greater than two and hence, by Lemma 2, C is contained in the closed neighborhood of a vertex. Therefore, dually chordal graphs whose chordless cycles, if any, have length not greater than five satisfy that every minimal vertex separator is contained in a closed neighborhood. Thus, any dually chordal graph not satisfying this property must have a chordless cycle of length at least six not contained in a neighborhood. In Fig. 1, that cycle is induced by $\{1, 2, 3, 4, 5, 6\}$.

Thus far, three Helly families of sets of vertices of a dually chordal graph were identified, namely, the family of its cliques, the family of its closed neighborhoods (and its disks) and the family of its minimal vertex separators. But the fact that every set of these families induces a subtree of a fixed compatible tree implies that the union of the three families is itself Helly. This has implications for some of the issues discussed before.

When we considered a minimal vertex separator S such that there was a vertex w satisfying that $S \subseteq N[w]$, it could happen that $w \notin S$. Now we can prove that, in dually chordal graphs, w can always be chosen in such a way that it is an element of S.

Proposition 1. *Let S be a minimal vertex separator of a dually chordal graph such that S is contained in the closed neighborhood of a vertex. Then, there exists a vertex w such that $w \in S$ and $S \subseteq N[w]$.*

Proof. Consider the family composed of S and the closed neighborhoods of all the vertices of S. Since S is contained in the neighborhood of a vertex, this family is intersecting. Therefore, there is a vertex which is an element of each member of the family, that is, a vertex in S whose neighborhood contains S. □

The following result is true for all graphs:

Lemma 3. *Let u and v be two nonadjacent vertices of a graph G, with $d(u, v) = k$. Then, for each $1 \leq i < k$, there exists a minimal uv-separator S_i contained in $\{w \in V(G) : d(v, w) = i\}$.*

Proof. For every path P from u to v, take a vertex w of it at distance i from v to form a set S. S is clearly a uv-separator and thus it contains a minimal uv-separator with the required characteristics. □

In the particular case of dually chordal graphs, we can see that this type of minimal vertex separators are contained in a neighborhood.

Proposition 2. *Let G be a dually chordal graph, $v \in V(G)$, T a tree compatible with G and T' a subtree of T. If all the vertices of T' are at distance i from v, then there exists a vertex v_i such that $d(v, v_i) = i - 1$ and $V(T') \subseteq N[v_i]$.*

Proof. As T' is a subtree of T, there exists a vertex x in $V(T')$ such that $x \in T[v, w]$ for every $w \in V(T')$, that is, a vertex in $V(T')$ which is the ancestor of all the other vertices of $V(T')$ when we consider T to be rooted at v. If $V(T') = \{x\}$, then $V(T') \subseteq N[x]$. Otherwise, take a vertex $w \in V(T')$, $w \neq x$, and consider a path P in G of length i from v to w. Let y be the vertex preceding w in P, which implies that $d(v, y) = i - 1$. We claim that w and y are in different connected components of $T - x$. Otherwise, since $x \in T[v, w]$, $x \in T[v, y]$ as well. But this contradicts that $v, y \in N^{i-1}[v]$ and that $N^{i-1}[v]$ induces a subtree in T.

We can infer from the last part of the previous paragraph that $x \in T[w, y]$. As T is compatible with G, we conclude that w is adjacent to x. This way, the inclusion $V(T') \subseteq N[x]$ follows. This implies that the family composed of $N^{i-1}[v]$ and the closed neighborhood of each vertex of T' is intersecting. As it is Helly, there exists one vertex v_i that is in every member of the family, which means that $V(T') \subseteq N[v_i]$ and $d(v, v_i) = i - 1$. □

Corollary 3. *Let G be a dually chordal graph and u and v two nonadjacent vertices with $d(u, v) = k$. If S_i is a minimal vertex separator contained in $\{w \in V(G) : d(v, w) = i\}$, $1 \leq i < k$, then there exists a vertex v_i such that $d(v, v_i) = i - 1$ and $S_i \subseteq N[v_i]$.*

Proof. Since G is dually chordal, we can take a tree T compatible with G. By Theorem 2, $T[S_i]$ is a subtree of T. Then, Proposition 2 can be applied. □

As a consequence, we can prove the following interesting property:

Proposition 3. *Let G be a dually chordal graph and u, v two nonadjacent vertices of G with $d(u, v) = k$. Then, there exists a path $v = v_1, \ldots, v_k u$ and $k - 1$ disjoint minimal uv-separators $S_1, \ldots, S_{k-1}$ such that for each i, $1 \leq i \leq k - 1$, $S_i \subseteq N[v_i]$.*

Proof. For each i, $1 \leq i \leq k - 1$, let S_i be a minimal uv-separator contained in $\{w \in V(G) : d(v, w) = i\}$. We can apply Proposition 2 to pick a vertex v_i such that $S_i \subseteq N[v_i]$ and $d(v, v_i) = i - 1$.

Furthermore, let v_k be a vertex adjacent to u and at distance $k - 1$ from v. We claim that $v_1, \ldots, v_k u$ is the desired path. By the construction, $d(v, v_1) = 0$, so $v_1 = v$. Therefore, it remains to prove that v_i and v_{i+1} are adjacent, $1 \leq i \leq k - 1$. We prove it first for $1 \leq i \leq k - 2$. Let Q be a shortest path from u to v and let z be the vertex of Q at distance $i + 1$ from v, so it is at distance $k - i - 1$ from u. Then, z is necessarily an element of S_{i+1}, making v_{i+1} adjacent to z and thus $d(v_{i+1}, u) \leq k - i$. We also knew that $d(v_{i+1}, v) = i$, so v_{i+1} can be found in a shortest path from u to v. Consequently, v_{i+1} is necessarily in S_i. As v_i is adjacent to all the vertices of S_i, we conclude that v_i is adjacent to v_{i+1}. For $i = k - 1$, it is an immediate consequence of our choice of v_k that $v_k \in S_{k-1}$, so v_{k-1} is adjacent to v_k. □

4.3. Characterizations

Recall that we have already proved that each minimal vertex separator of a dually chordal graph G induces a subtree of any tree T compatible with G and hence induces a connected subgraph. We also saw that the family of minimal vertex separators is Helly and its intersection graph is chordal. When we consider cliques, we have properties that can be stated in similar terms: every clique induces a subtree of any compatible tree on one side and $C(G)$ is Helly and $K(G)$ is chordal on the other. Given the fact that these two properties can be used to characterize dually chordal graphs, it is natural to wonder if it is also the case for minimal vertex separators. Fortunately, we can answer in the affirmative and this section is devoted to prove the characterizations of dually chordal graphs that can be obtained.

Theorem 5. *A graph G is dually chordal if and only if there is a spanning tree T of G such that every minimal vertex separator induces a subtree in T.*

Proof. By Theorem 2, it remains to prove one of the implications.

Assume that T is a spanning tree like the above described. Let $x, y \in V(G)$ be two adjacent vertices and $z \in T(x, y)$. If x and z were not adjacent, let S be a minimal xz-separator. As S must induce a subtree of T and must contain a vertex in $T[x, z]$, $T[y, z]$ is a path joining y and z in $G - S$ and we conclude that y and z are in the same connected component of this graph. But x is also in that component because it is adjacent to y, contradicting that x and z are separated by S. Therefore, x and z are necessarily adjacent. Similarly, we can prove that y is adjacent to z and consequently, by Theorem 1, T is compatible with G and the graph is dually chordal. □

By analyzing the steps of this proof, it is clear that we might not need to know that every minimal vertex separator of a graph G induces a subtree of a spanning tree T to conclude that G is dually chordal. Actually, the proof requires that, given two nonadjacent vertices u and v, there is at least one minimal uv-separator inducing a subtree of T. This leads to a slightly different characterization of dually chordal graphs.

Theorem 6. *Let G be a graph and $\mathcal{F}$ a family of minimal vertex separators of G such that, for any two nonadjacent vertices u and v, $\mathcal{F}$ contains a uv-separator. Then, G is dually chordal if and only if there is a spanning tree T of G such that every member of $\mathcal{F}$ induces a subtree in T.*

The proof of Theorem 6 is almost identical to that of Theorem 5. The only difference is that we ask S to be in $\mathcal{F}$.

Now we present the second major characterization:

Theorem 7. *A graph G is dually chordal if and only if each minimal vertex separator induces a connected subgraph, $\mathcal{S}(G)$ is Helly and $L(\mathcal{S}(G))$ is chordal.*

Proof. By Corollary 1, it remains to prove one of the implications.

Assume that each minimal vertex separator induces a connected subgraph, the family of minimal vertex separators is Helly and its intersection graph is chordal. The last two conditions imply that there is at least one tree whose vertex set is $V(G)$ with each minimal vertex separator inducing a subtree [9]. Of all those trees, choose T such that $s(T) := \sum_{vw \in E(T)} d(v, w)$ is minimum. We claim that T is a spanning tree of G. Otherwise, let v and w be two vertices adjacent in T but not in G, with $d(v, w) = k$. Consider the family $\mathcal{F}$ formed by all the minimal vertex separators containing v and w, if any, one minimal vw-separator S_1 contained in $N[v]$ and one minimal vw-separator S_2 contained in $\{u \in V(G),\ d(u, w) = k-1\}$, whose existence is ensured by Lemma 3. We now prove that the members of $\mathcal{F}$ are pairwise intersecting.

Let S be a minimal vertex separator containing v and w. As S induces a connected subgraph, it contains a path P from v to w. Since S_i, $i = 1, 2$, separates v and w, it must contain a vertex of P. Hence, $S \cap S_i \neq \emptyset$.

Now consider a shortest path P from v to w. Then, P contains a vertex x such that $d(v, x) = 1$ and $d(w, x) = k - 1$, which is unique in this respect. Consequently, $x \in S_1 \cap S_2$.

As $\mathcal{F}$ is intersecting and $\mathcal{S}(G)$ is Helly, there is a vertex u belonging to every member of $\mathcal{F}$. Let A and B be the set of vertices of the connected components of $T - vw$ containing v and w, respectively. If $u \in A$, let T' be a tree obtained from T by removing vw and adding uw. Then, every minimal vertex separator in G induces a subtree in T'. In order to prove it, let S be a minimal vertex separator of G. If $S \subseteq A$ or $S \subseteq B$, then S induces the same subtree in T and T'.

Otherwise we have two vertices $y, z \in S$ such that $y \in A$ and $z \in B$. As S induces a subtree in T and $v, w \in T[y, z]$, we conclude that $v, w \in S$ and thus $u \in S$ as well. Then, v and w are connected in T' by the path formed by merging $T[v, u] = T'[v, u]$ and uw, whose vertices are contained in S because $u, v \in S$ and $T[S]$ is connected. Moreover, any other two vertices of S adjacent in T are still adjacent in T'. Therefore, vertices of S adjacent in T are connected in T' by paths within S. This is enough to conclude that S induces a subtree in T'.

Furthermore, $s(T') < s(T)$ because $d(u, w) = k - 1$ and $d(v, w) = k$, contradicting our choice of T. If $u \in B$, we can remove vw and add uv to T and a similar contradiction arises. Thus, T is necessarily a spanning tree of G and, by Theorem 5, G is dually chordal. □

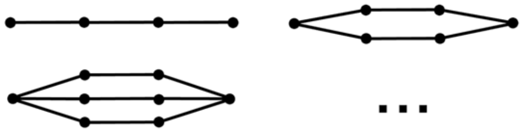

Fig. 2. A family of graphs for which the number of minimal vertex separators increases exponentially. These graphs consist of two vertices being connected by an increasing number of internally disjoint paths of length three.

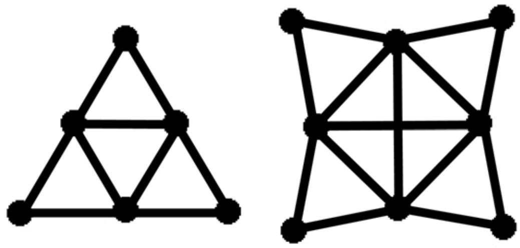

Fig. 3. A 3-sun and a 4-sun.

We note that this characterization does not lead to an efficient way to decide whether a graph is dually chordal or not. In order to test if the three conditions of the characterization are satisfied, we need a list of all the minimal vertex separators of the graph, but there could be an exponential number of minimal vertex separators. In fact, it is known that there are many graphs with an exponential number of minimal vertex separators (see Fig. 2). If we take any of them, we add a vertex to it and we make it universal, the new graph is dually chordal and still has an exponential number of minimal vertex separators.

However, we can get a condition characterizing dually chordal graphs depending on a number of minimal vertex separators not exceeding $2|E(\overline{G})|$, where $\overline{G}$ is the complement of G. In view of Theorem 6 and the steps of the proof of Theorem 7, the following can be proved:

Theorem 8. *Let G be a graph and $\mathcal{F}$ a family of minimal vertex separators of G such that, for any two nonadjacent vertices v and w, there are minimal vw-separators S_1 and S_2 in $\mathcal{F}$ (they could be equal) such that $S_1 \cap S_2 \neq \emptyset$ and $S_1 \cap S_2 \subseteq N^{k-1}[v] \cap N^{k-1}[w]$, where k is the distance from v to w. Then, G is dually chordal if and only if each member of $\mathcal{F}$ induces a connected subgraph of G, $\mathcal{F}$ is Helly and the intersection graph of $\mathcal{F}$ is chordal.*

Observe that, in order to build a family like $\mathcal{F}$, one of the possible ways to construct S_1 and S_2 is as in the previous proof. Theorem 6 allows us to consider just the members of $\mathcal{F}$ instead of all the minimal vertex separators (if there were more). $S_1 \cap S_2 \neq \emptyset$ ensures that, if we define $\mathcal{F}$ in a way similar to that of Theorem 7, then it will be intersecting, and the condition $S_1 \cap S_2 \subseteq N^{k-1}[u] \cap N^{k-1}[v]$ allows us again to conclude that $s(T') < s(T)$.

4.4. Minimal vertex separators and strongly chordal graphs

We present this subsection as an application of the previous one.

An analysis of the conditions stated in Theorem 7 for a graph to be dually chordal yields that none of them can be omitted in order to simplify the characterization. Some of the simplest graphs illustrating this are shown below.

C_4, the cycle of four vertices, satisfies that $\mathcal{S}(C_4)$ is Helly and $L(\mathcal{S}(C_4))$ is chordal, but the minimal vertex separators of C_4 do not induce connected subgraphs.

A *k-sun*, $k \geq 3$, is defined as a graph with vertex set $\{v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k\}$ such that $\{v_1, v_2, \ldots, v_k\}$ is a complete, $N[w_i] = \{v_i, v_{i+1}, w_i\}$, $i = 1, 2, \ldots, k-1$, and $N[w_k] = \{v_1, v_k, w_k\}$.

It is not difficult to see that the suns are not dually chordal graphs. Each one of their minimal vertex separators induces a connected subgraph (see Fig. 3). $L(\mathcal{S}(3-\text{sun}))$ is chordal, but $\mathcal{S}(3-\text{sun})$ is not Helly. On the other hand, for $k \geq 4$, $\mathcal{S}(k-\text{sun})$ is Helly, but $L(\mathcal{S}(k-\text{sun}))$ is not chordal.

A graph is *strongly chordal* if it is chordal and every cycle of length at least six has a chord joining two vertices at an odd distance in the cycle. This definition implies that the class of strongly chordal is hereditary. What is more, strongly chordal graphs were proved to be the hereditary dually chordal graphs [1], i.e., dually chordal graphs such that all their induced subgraphs are also dually chordal.

Every hereditary class of graphs admits a characterization by a family of minimal forbidden induced subgraphs. For hereditary dually chordal graphs, such a family is just formed by cycles of length at least four and the suns [3]. The current section is devoted to giving a new proof of this by using minimal vertex separators and the characterizations of dually chordal graphs that were found.

The proof will be divided in two propositions and the main theorem at the end.

Lemma 4. *Let S be a minimal vertex separator of a chordal graph G and $v \notin S$. Then, there exists a vertex w such that $S \subseteq N[w]$ and v and w are in different connected components of $G - S$.*

Proof. Since S is a minimal vertex separator of a chordal graph, there exist cliques C_1 and C_2 in G such that $C_1 \cap C_2 = S$ and $C_1 \setminus C_2$ and $C_2 \setminus C_1$ are contained in different connected components of $G - S$ [4]. Without loss of generality, we can assume that $C_1 \setminus C_2$ and v are not in the same connected component of $G - S$. Then, any vertex $w \in C_1 \setminus C_2$ will satisfy the required condition. □

Proposition 4. *Let G be a chordal graph such $\mathcal{S}(G)$ is not Helly. Then, the 3-sun is an induced subgraph of G.*

Proof. Since $\mathcal{S}(G)$ is not Helly, we can take $v_1, v_2, v_3 \in V(G)$ such that the members of $\mathcal{S}(G)$ containing at least two elements of $\{v_1, v_2, v_3\}$ have no common element. Then, there exist $S_1, S_2, S_3 \in \mathcal{S}(G)$ such that $S_1 \cap \{v_1, v_2, v_3\} = \{v_2, v_3\}, S_2 \cap \{v_1, v_2, v_3\} = \{v_1, v_3\}$ and $S_3 \cap \{v_1, v_2, v_3\} = \{v_1, v_2\}$.

By Lemma 4, we can take a vertex w_i such that $S_i \subseteq N[w_i]$ and v_i and w_i are in different connected components of $G - S_i$, $i = 1, 2, 3$.

As the vertices of minimal vertex separators of chordal graphs are pairwise adjacent, it is clear that $\{v_1, v_2, v_3\}$ induces a cycle of three vertices. Also, by our choice, $N[w_1] \cap \{v_1, v_2, v_3\} = \{v_2, v_3\}, N[w_2] \cap \{v_1, v_2, v_3\} = \{v_1, v_3\}$ and $N[w_3] \cap \{v_1, v_2, v_3\} = \{v_1, v_2\}$.

Now suppose that $1 \leq i, j \leq 3$ and $i \neq j$. Then, $w_j \notin S_i$ because $v_j \in S_i$ and w_j is not adjacent to it. Since w_j is adjacent to v_i, w_i and w_j are in different connected components of $G - S_i$. Thus, w_i and w_j are not adjacent.

Therefore, $\{v_1, v_2.v_3, w_1, w_2, w_3\}$ induces a 3-sun in G. □

Proposition 5. *Let G be a chordal graph such that $\mathcal{S}(G)$ is Helly. If $L(\mathcal{S}(G))$ is not chordal, then G has an induced k-sun, $k \geq 4$.*

Proof. Let $R : S_1, S_2, \ldots, S_k$, $k \geq 4$, be a chordless cycle of minimum length in $L(\mathcal{S}(G))$. Take $v_i \in S_i \cap S_{i+1}$, $i = 1, \ldots, k-1$, and $v_k \in S_1 \cap S_k$. Since R is chordless, all these vertices are different and form a cycle C in G.

If the vertices of C are pairwise adjacent, let w_1 be a vertex such that $S_1 \subseteq N[w_1]$ and w_1 and the vertices of $\{v_1, v_2, \ldots, v_k\} \setminus \{v_1, v_k\}$ are in different connected components of $G - S_1$. Similarly, for $i = 2, \ldots, k$, let w_i be a vertex such that $S_i \subseteq N[w_i]$ and w_i and the vertices of $\{v_1, v_2, \ldots, v_k\} \setminus \{v_{i-1}, v_i\}$ are in different connected components of $G - S_i$.

Let $1 \leq i, j \leq k$ and $i \neq j$. It is not difficult to see that $w_j \notin S_i$ and that w_i and w_j are in different connected components of $G - S_i$, so w_i is not adjacent to w_j. Therefore, $\{v_1, \ldots, v_k, w_1, \ldots, w_k\}$ induces a k-sun.

If the vertices of C are not pairwise adjacent, take $v_i, v_j \in C$ such that v_i and v_j are not adjacent, and let S be a minimal v_iv_j-separator. S must clearly contain two nonconsecutive vertices of C, let them be v_l and v_m. Without loss of generality, we can assume that $l < m$. Let $A = \{S_n : l + 1 \leq n \leq m\}$. If $|A| \leq 2$, it is clear that the intersection between S and any element of A is not empty. If $|A| \geq 3$, consider the cycle $R' : S_{l+1}, S_{l+2}, \ldots, S_m, S$. Since R' is shorter than R, R' has a chord. As R is chordless, S must be one of the endpoints of the chord. The addition of this chord to R' generates two new cycles. If any of these cycles has length at least four, there is a chord, and again S must be one of the endpoints. We continue this procedure until concluding that the intersection between S and any element of A is not empty. Similarly, let $B = \{S_1, S_2, \ldots, S_k\} \setminus A$ and we can also infer that the intersection between S and any element of B is not empty.

Since $\mathcal{S}(G)$ is Helly, we can take $u_i \in S_i \cap S_{i+1} \cap S$, $i = 1, \ldots, k-1$, and $u_k \in S_1 \cap S_k \cap S$. Then, $\{u_1, u_2, \ldots, u_k\} \subseteq S$ and the vertices in this set are pairwise adjacent. We can find an induced *k-sun* as we did in the case that the vertices of C were pairwise adjacent. □

Theorem 9. *Let G be a graph. Then, G is a hereditary dually chordal graph if and only if G is chordal and without induced suns.*

Proof. Since cycles of length at least four and the suns are not dually chordal, they are not induced subgraphs of a hereditary dually chordal graph.

Now suppose that G is chordal without induced suns. Then, every minimal vertex separator of G is a complete set, clearly inducing a connected subgraph. By Propositions 4 and 5, $\mathcal{S}(G)$ is Helly and $L(\mathcal{S}(G))$ is chordal. Consequently, by Theorem 7, G is dually chordal.

Every induced subgraph of G is also chordal and without induced suns, and hence dually chordal. Therefore, G is a hereditary dually chordal graph. □

5. Concluding remarks

The interpretation of many properties of chordal graphs usually gives properties about dually chordal graphs, as it is the case of the existence of spanning trees such that every clique induces a subtree. The characterizations here exposed show that a higher level of generality, not clearly evidenced by the above mentioned interpretation, can be achieved, thus allowing to introduce new sets such as minimal vertex separators in the theory about dually chordal graphs.

It was particularly curious to us that the characterizations not always require considering all the minimal vertex separators, as we can take a subfamily of minimal vertex separators with $O(|E(G)|)$ cardinality. The condition for every minimal vertex separator of a dually chordal graph to be contained in a neighborhood in terms of chordless cycles, initially unexpected, also attracted our attention, specially when compared with the fact that every minimal vertex separator of a graph being a complete is equivalent to the graph lacking chordless cycles.

It remains to ascertain if the new information can lead to practical applications and if more properties can be extended. For example, it would be interesting to be able to use this to find minimum separators and hence calculate the connectivity of a dually chordal graph in the special case that it has a compatible tree with few leaves.

References

[1] A. Brandstädt, F. Dragan, V. Chepoi, V. Voloshin, Dually chordal graphs, SIAM J. Discrete Math. 11 (1998) 437–455.
[2] G.A. Dirac, On rigid circuit graphs, Abh. Math. Sem. Univ. Hamburg 25 (1961) 71–76.
[3] M. Farber, Characterizations of strongly chordal graphs, Discrete Math. 43 (1983) 173–189.
[4] P. Galinier, M. Habib, C. Paul, Chordal graphs and their clique graphs, Lecture Notes in Comput. Sci. 1017 (1995) 358–371.
[5] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, second ed., in: Annals of Discrete Mathematics, vol. 57, Elsevier, 2004.
[6] M. Gutierrez, Tree-clique graphs, Estudos e comunicações do Instituto de Matemática, Universidade Federal do Rio de Janeiro, vol. 63, 1996, pp. 7–26.
[7] F.R. McMorris, T.J. Warnow, T. Wimer, Triangulating vertex-colored graphs, SIAM J. Discrete Math. 7 (2) (1994) 296–306.
[8] J. Meidanis, J. Setubal, Introduction to Computational Molecular Biology, PWS Publishing, Boston, MA, 1997.
[9] P.J. Slater, A characterization of SOFT hypergraphs, Canad. Math. Bull. 21 (1978) 335–337.
[10] J. Szwarcfiter, C. Bornstein, Clique graphs of chordal graphs and path graphs, SIAM J. Discrete Math. 7 (1994) 331–336.
[11] J.R. Walter, Representations of chordal graphs as subtrees of a tree, J. Graph Theory 2 (1978) 265–267.