YOUNG'S (IN)EQUALITY FOR COMPACT OPERATORS

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ABSTRACT. If a, b are $n \times n$ matrices, T. Ando proved that Young's inequality is valid for their singular values: if $p > 1$ and $1/p + 1/q = 1$, then

$$\lambda_k(|ab^*|) \leq \lambda_k \left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q \right) \quad \textit{for all } k.$$

Later, this result was extended for the singular values of a pair of compact operators acting on a Hilbert space by J. Erlijman, D. Farenick and R. Zeng. In this paper we prove that if a, b are compact operators, then equality holds in Young's inequality if and only if $|a|^p = |b|^q$.

1. Introduction

It all boils down to the following elementary inequality named after W. H. Young: if $p > 1$ and $1/p + 1/q = 1$, then for any $\alpha, \beta \in \mathbb{R}^+$,

$$\alpha\beta \leq \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q$$

with equality if and only if $\alpha^p = \beta^q$.

Operator analogues of this elegant fact are considered, following the fundamental paper by T. Ando [1] for $n \times n$ matrices, and an extension for compact operators by J. Erlijman, D. R. Farenick, R. Zeng [5]. If a, b are compact operators on Hilbert space then for all $k \in \mathbb{N}_0$,

$$\lambda_k(|ab^*|) \leq \lambda_k \left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q \right) \tag{1.1}$$

where each eigenvalue is counted with multiplicity. This allows to construct a partial isometry u such that

$$u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q \tag{1.2}$$

for the partial order of operators. Therefore, this raised the natural question of whether

$$\|u|ab^*|u^*\| = \|\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\| \implies |a|^p = |b|^q. \tag{1.3}$$

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It is easy to construct examples where this is false, if $\|\cdot\|$ is the operator norm. But O. Hirzallah and F. Kittaneh showed with an elegant inequality [9] that it is true if a,b are Hilbert-Schmidt operators, and the norm is the Hilbert-Schmidt norm. M. Argerami and D. Farenick [3] obtained the same conclusion for the trace norm $\|\cdot\|_1 = Tr|\cdot|$, assumming $|a|^p, |b|^q$ are nuclear operators.

There has been some attempts to understand the case of equality in (1.2) for compact operators; in 2005, R. Zeng [15] studied the case of operator equality for commuting normal compact injective operators. It was also unclear which class of symmetric norms is adequate to deal with the case of equality in (1.3) -see [3]-. In this paper, we prove that the necessary and sufficient condition to have $|a|^p = |b|^q$ is in fact the equality of all singular numbers in (1.1), which enables us to characterize for exactly which norms the assertion (1.3) above is true, and therefore for which operator ideals.

What follows is Theorem 2.11 below; for the definition of strictly increasing symmetric norm see Definition 2.3 below:

Theorem. Let $a,b \in \mathcal{K}(\mathcal{H})$. If $p>1$ and $1/p+1/q=1$, then the following are equivalent:

(1) $|a|^p = |b|^q$.
(2) $z|ab^*|z^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ for some contraction $z \in \mathcal{B}(\mathcal{H})$
(3) $\|z|ab^*|w\|_\phi = \|\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\|_\phi < \infty$ for a pair of contractions $z,w \in \mathcal{B}(\mathcal{H})$ and $\|\cdot\|_\phi$ a *strictly increasing* symmetric norm.
(4) $\lambda_k(|ab^*|) = \lambda_k\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)$ for all $k \in \mathbb{N}_0$.

Regarding other contexts for the inequality and the case of equality, we now mention briefly some relevant literature. See the paper by R. Zeng [16] for the case of inequality in quaternion matrices, see also the papers by D. Farenick and M. Manjegani regarding the case of equality in finite and semi-finite von Neumann algebras [6, 11]. There is also an extension of the inequality to type III von Neumann algebras given by M. Manjegani in [10]. For reversed Young inequalities see the paper by M. Manjegani and A. Norouzi [12].

2. Young's inequality for compact operators

Let $\mathcal{H}$ be a complex Hilbert space, and let us denote with $\mathcal{B}(\mathcal{H})$ the bounded linear operators acting in $\mathcal{H}$. For $y \in \mathcal{B}(\mathcal{H})$, with $|y| = \sqrt{y^*y}$ we denote the positive square root and then $y = \nu|y|$ is the polar decomposition of y. With $\nu : \overline{\operatorname{Ran}|y|} \to \overline{\operatorname{Ran} y}$ we denote its partial isometry, when necessary, the projection $\nu\nu^*$ onto the closure of the range of y will be denoted by p_y.

In the following lemma we collect some results that will be used throughout this paper (and will help us fix the notation):

Lemma 2.1. *Let $a, b, x \in \mathcal{B}(\mathcal{H})$,*

(1) If $b = \nu|b|$ then $\nu^*\nu$ is the orthogonal projection onto the closure of the range of $|b|$, $|b^*| = \nu|b|\nu^*$ and $\nu\nu^*$ is the orthogonal projection onto the closure of the range of $|b^*|$.
(2) $|ab^*| = \nu||a||b||\nu^*$ and $\nu^*|ab^*|\nu = ||a||b||$.
(3) If p is a projection, then $x = pxp$ implies $x = px$ and in particular $p_x p = p_x$ (equivalently $p_x \leq p$).
(4) If p is a projection, $pxp = p$ and either $x \geq p$ or $0 \leq x \leq p$, then $xp = p$. In particular if $\operatorname{Ran}(p) = span(\xi)$ for some $\xi \in \mathcal{H}$,

$$\langle x\eta, \eta\rangle \geq \langle p\eta, \eta\rangle \text{ for any } \eta \in \mathcal{H}$$

and $\langle x\xi, \xi\rangle = \langle \xi, \xi\rangle$, then $x\xi = \xi$. There is a similar assertion for the other case.

Proof. 1. is trivial, to prove 2. write the polar decompositions $a = u|a|, b = \nu|b|$. Note that $|ab^*|^2 = \nu|b||a|^2|b|\nu^*$; since $\nu^*\nu|b| = |b|$ then $(\nu|b||a|^2|b|\nu^*)^n = \nu(|b||a|^2|b|)^n\nu^*$ for any $n \in \mathbb{N}$, and an elementary functional calculus argument shows that

$$|ab^*| = (\nu|b||a|^2|b|\nu^*)^{1/2} = \nu(|b||a|^2|b|)^{1/2}\nu^* = \nu||a||b||\nu^*.$$

On the other hand, since $\nu\nu^*\nu = \nu$, then $\nu\nu^*|ab^*| = |ab^*| = |ab^*|\nu\nu^*$, therefore from $\nu^*|ab^*|^2\nu = ||a||b||^2$ taking square roots and using a similar argument we derive $\nu^*|ab^*|\nu = ||a||b||$

3. If $pxp = x$ then $\operatorname{Ran} x \subset \operatorname{Ran} p$, therefore $p_x \leq p$ or equivalently $pp_x = p_x$. Multiplying both sides by x gives $x = px$.

4. Assume $x \geq p$ (the case $0 \leq x \leq p$ can be treated in a similar fashion therefore its proof is omitted). Since $x - p \geq 0$, we have, for each $\eta \in \mathcal{H}$,

$$||(x-p)^{1/2}p\eta||^2 = \langle p(x-p)p\eta, \eta\rangle = 0,$$

thus $(x - p)^{1/2}p = 0$ and multipliying by $(x - p)^{1/2}$ on the left we obtain $(x - p)p = 0$ wich shows that $xp = p$. □

2.1. **Singular values.** Denote with $\mathcal{K}(\mathcal{H})$ the compact operators on $\mathcal{H}$. Let $\lambda_k(x)$ ($k \in \mathbb{N}_0$) denote the k-th eigenvalue of the positive compact operator $x \in \mathcal{B}(\mathcal{H})$, arranged in decreasing order,

$$\|x\| = \lambda_0 \geq \lambda_1 \geq \cdots \lambda_k \geq \lambda_{k+1} \geq \cdots$$

where we allow equality because each singular value is counted with multiplicity. Clearly $\lambda_k(f(x)) = |f|(\lambda_k(x))$ for any function defined in $\sigma(x)$.

Remark 2.2. For given $a, b \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{K}(\mathcal{H})$, the min-max characterization of the singular values [14, Theorem 1.5] and Lemma 2.1.2 easily imply that

(1) $\lambda_k(axb) \leq \|a\|\|b\|\lambda_k(x)$,
(2) $\lambda_k(|ab^*|) = \lambda_k(||a||b||)$.

2.2. **Unitarily invariant norms.** For a given vector $a = (a_i)_{i\in\mathbb{N}_0}$ with $a_i \in \mathbb{R}$, we will denote with $a^\downarrow$ the rearrangement of a in decreasing order, that is $a^\downarrow$ is a permutation of a such that

$$a_0^\downarrow \geq a_1^\downarrow \geq \cdots \geq a_k^\downarrow \geq a_{k+1}^\downarrow \geq \cdots .$$

Let $\|\cdot\|_\phi$ stand for a unitarily invariant norm in $\mathcal{B}(\mathcal{H})$, and $\phi : \mathbb{R}_+^{\mathbb{N}_0} \to \mathbb{R}_+$ its associated permutation invariant gauge [14].

Definition 2.3. The norm $\|\cdot\|_\phi$ is *strictly increasing* if, for any given pair of sequences $a = (a_i), b = (b_i)$ such that $0 \leq a_i \leq b_i$ for all $i \in \mathbb{N}_0$, the fact $\phi(a) = \phi(b)$ implies that $a_i = b_i$ for all $i \in \mathbb{N}_0$.

See Hiai's paper [8], it is also property (3) in Simon's paper [13].

Examples of these norms on $\mathcal{K}(\mathcal{H})$ are the Schatten p-norms $1 \leq p < \infty$, and examples of non-strictly increasing norms are the supremum norm and the Ky-Fan norms. Note that we can always define

$$\phi(a) = \sum_{k\geq 0} a_k^\downarrow 2^{-k}, \tag{2.1}$$

which is a strictly increasing norm defined in the whole of $\mathcal{K}(\mathcal{H})$.

Remark 2.4. If $\mathcal{I}_\phi$ is not equivalent to the supremum norm

$$\|x\| = \sup_{\|\xi\|_H=1} \|x\xi\|_H,$$

then $\mathcal{I}_\phi = \{x \in \mathcal{K}(\mathcal{H}) : \|x\|_\phi < \infty\}$ is a proper bilateral ideal in $\mathcal{K}(\mathcal{H})$ according to Calkin's theory. Assume that a symmetric norm has the Radon-Riesz property

$$\|x_n\|_\phi \to \|x\|_\phi \ \ \textit{and} \ \ x_n \to x \ \ \textit{weakly} \ \implies \|x - x_n\|_\phi \to 0$$

(see Arazy's paper [2] on the equivalence for sequences and compact operators). Simon proved in [13] that in that case the norm is strictly increasing according to our definition. *Can this assertion be reversed?*

2.3. **Inequality.**

Remark 2.5. For given $a, b \in \mathcal{K}(\mathcal{H})$ we will always denote

$$\alpha_k = \lambda_k(|a|), \quad \beta_k = \lambda_k(|b|), \quad \gamma_k = \lambda_k(|ab^*|), \quad \delta_k = \lambda_k\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right).$$

Moreover, we will denote

$$|a| = \sum_k \alpha_k a_k,\ |b| = \sum_k \beta_k b_k, \quad |ab^*| = \sum_k \gamma_k p_k, \quad \frac{1}{p}|a|^p + \frac{1}{q}|b|^q = \sum_k \delta_k q_k$$

the spectral decompositions of each operator, with a_k, b_k,etc. one dimensional projections. Note that we allow multiplicity, and if $\gamma_1 = \cdots = \gamma_j$ for some finite j, the election of the first q_j is arbitrary (i.e., it amounts to select an orthonormal base of that span).

Remark 2.6. Concerning $a, b \in \mathcal{K}(\mathcal{H})$, the following was proved in [5] by Erlijman, Farenick and Zeng: for each $k \in \mathbb{N}$,

$$\lambda_k(|ab^*|) \leq \lambda_k\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)$$

hence there exists a partial isometry u such that $q_k = up_ku^*$ and $u^*u = \sum_k p_k$ the projection on the (closure of) the range of $|ab^*|$. Then, for any $a, b \in \mathcal{K}(\mathcal{H})$,

$$u|ab^*|u^* \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

This extended the original result of T. Ando [1] which was stated for positive matrices.

From their result, it can be deduced that the relevant condition to deal with the equality is $\gamma_k = \delta_k$ for all k, to be more precise:

Lemma 2.7. *Let $a, b \in \mathcal{K}(\mathcal{H})$, $p > 1$, $1/p + 1/q = 1$.*

(1) If $|a|^p = |b|^q$, then $\alpha_k^p = \beta_k^q = \gamma_k = \delta_k$ for each $k \in \mathbb{N}_0$.

(2) If either

$$z|ab^*|z^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$$

for some contraction $z \in \mathcal{B}(\mathcal{H})$, or

$$\|z|ab^*|w\|_\phi = \|\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\|_\phi,$$

for a pair of contractions $z, w \in \mathcal{B}(\mathcal{H})$ and a strictly increasing norm, then $\gamma_k = \delta_k$ for each $k \in \mathbb{N}_0$ and

$$u|ab^*|u^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

where u is the partial isometry u of the result in Remark 2.6, i.e $up_k u^* = q_k$ for each k.

Proof. To prove the first assertion, note that clearly $\delta_k = \alpha_k^p = \beta_k^q$. By Lemma 2.1.2, $|ab^*| = \nu|a|^p\nu^*$, and in particular

$$\lambda_k(|ab^*|) \le \lambda_k(|a|^p) = \lambda_k(|b|^q)$$

by Remark (2.2.1). Now since ν is the partial isometry of $|b|$, then also $\nu^*|ab^*|\nu = \nu^*\nu|a|^p\nu^*\nu = |a|^p$ which in turn shows the reversed inequality, and then $\gamma_k = \alpha_k^p = \beta_k^q$ follows.

Regarding 2., note that if equality is attained by a contraction z, then by Remark 2.6 and Remark 2.2.1

$$\gamma_k \le \delta_k = \lambda_k(z|ab^*|z^*) \le \lambda_k(|ab^*|) = \gamma_k.$$

Likewise, if equality is attained for a strictly increasing norm and a pair of contraction z, w, since

$$\lambda_k(z|ab^*|w) \le \lambda_k(|ab^*|) = \gamma_k \le \delta_k = \lambda_k,$$

then $\gamma_k = \delta_k$ for every k. □

2.4. **Equality.** The following result will be crucial to obtain the proof of our main assertion.

Proposition 2.8. *Let $0 \le a, b \in \mathcal{K}(\mathcal{H})$. Let $1 < p < 2$ and $1/p + 1/q = 1$. If*

$$\lambda_k(|ab|) = \lambda_k\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \qquad \textit{for all } k$$

then $\operatorname{Ran}|ba| \subset \overline{\operatorname{Ran} b}$.

Proof. Let $\varepsilon > 0$, let p_b stand for the projection to the closure of the range of b, let $b_\varepsilon = b + \varepsilon(1 - p_b)$, then $b_\varepsilon^q = b^q + \varepsilon^q(1 - p_b) \le b^q + \varepsilon^q$ and $b_\varepsilon^2 = b^2 + \varepsilon^2(1 - p_b)$. Therefore

$$|b_\varepsilon a|^2 = ab_\varepsilon^2 a = ab^2a + \varepsilon^2 a(1 - p_b)a.$$

Let $|ba| = \sum_{k\in\mathbb{N}_0} \gamma_k e_k \otimes e_k$ with $\gamma_k = \lambda_k(|ba|)$ and $\{e_k\}_k$ an orthonormal basis of $\operatorname{Ran}|ba|$. Then since $\gamma_0 = \|ab\| = \|ba\|$, we have $\langle ab^2ae_0, e_0\rangle = \|ba\|^2$ and

$$\begin{aligned}
\varepsilon^2\|(1-p_b)ae_0\|^2 + \|ba\|^2 &\le \|b_\varepsilon a\|^2 = \|ab_\varepsilon\|^2 \le \|\frac{1}{p}a^p + \frac{1}{q}b_\varepsilon^q\|^2 \\
&= \|\frac{1}{p}a^p + \frac{1}{q}b^q + \frac{1}{q}\varepsilon^q(1-p_b)\|^2 \\
&\le \|\frac{1}{p}a^p + \frac{1}{q}b^q + \frac{1}{q}\varepsilon^q\|^2 = (\|\frac{1}{p}a^p + \frac{1}{q}b^q\| + \frac{1}{q}\varepsilon^q)^2 \\
&= (\|ab\| + \frac{1}{q}\varepsilon^q)^2 = \|ab\|^2 + \frac{2}{q}\|ab\|\varepsilon^q + \frac{1}{q^2}\varepsilon^{2q}
\end{aligned}$$

by Remark 2.6 and the hypothesis. Therefore, dividing by ε^2 and letting $\varepsilon \to 0$, since $q > 2$, we conclude that $(1-p_b)ae_0 = 0$ or equivalently, $ae_0 \in \overline{\operatorname{Ran} b}$.

We iterate the argument above for all k such that $\gamma_k = \gamma_0$: let us abuse the notation and assume then that $\gamma_1 < \gamma_0$. Then for all sufficiently small $\varepsilon \le \varepsilon_1$,

$$\gamma_1^2 + \varepsilon^2\|(1-p_b)ae_1\|^2 < \gamma_0^2.$$

Therefore for all such ε, if $Q = e_0 + e_1$ then $\lambda_1\left(Q(\varepsilon^2 a(1-p_b)a + ab^2a)Q\right)$ equals

$$\lambda_1\left(\varepsilon^2\|(a(1-p_b)a)^{1/2}e_1\|^2 e_1 + \gamma_1^2 e_1 + \gamma_0^2 e_0\right) = \gamma_1^2 + \varepsilon^2\|(1-p_b)ae_1\|^2.$$

Now by the same reasons as above (and since $\lambda_k(QAQ) \le \lambda_k(A)$ and $\lambda_k(A + tP) \le \lambda_k(A + t1) = \lambda_k(A) + t$ for $A \ge 0$, $t \in \mathbb{R}_{\ge 0}$ and $P^2 = P = P^*$)

$$\begin{aligned}
\lambda_1\left(Q(\varepsilon^2 a(1-p_b)a + ab^2a)Q\right) &= \lambda_1|Q|ab_\varepsilon|^2Q| \le \lambda_1|ab_\varepsilon|^2 \\
&\le \lambda_1\left(\frac{1}{p}a^p + \frac{1}{q}b_\varepsilon^q\right)^2 \\
&\le \lambda_1\left(\frac{1}{p}a^p + \frac{1}{q}b^q + \frac{1}{q}\varepsilon^q\right)^2 \\
&= \left[\lambda_1\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) + \frac{1}{q}\varepsilon^q\right]^2 \\
&\le \left(\lambda_1|ab| + \frac{1}{q}\varepsilon^q\right)^2 = \gamma_1^2 + \frac{2}{q}\gamma_1\varepsilon^q + \frac{1}{q^2}\varepsilon^{2q}.
\end{aligned}$$

Therefore

$$\gamma_1^2 + \varepsilon^2\|(1-p_b)ae_1\|^2 \le \gamma_1^2 + \frac{2}{q}\gamma_1\varepsilon^q + \frac{1}{q^2}\varepsilon^{2q}$$

and again, dividing by ε and letting $\varepsilon \to 0$, we conclude that $ae_1 \in \overline{\operatorname{Ran} b}$. Proceeding recursively, we conclude that $a(\operatorname{Ran}|ba|) \subset \overline{\operatorname{Ran} b}$. Now if $\xi \in \mathcal{H}$, then $a|ba|\xi \in \overline{\operatorname{Ran}(b)}$, therefore $a^2|ba|\xi = a(a|ba|\xi) \in a\overline{\operatorname{Ran}(b)} \subset$

$\overline{\mathrm{Ran}\,(ab)} = \overline{\mathrm{Ran}\,|ba|}$, and $a^3|ba|\xi = a(a^2|ba|\xi) \in a\overline{\mathrm{Ran}\,|ba|} \subset \overline{\mathrm{Ran}\,(b)}$. Iterating this argument, we arrive to the conclusion that $a^{2n+1}(\mathrm{Ran}\,|ba|) \subset \overline{\mathrm{Ran}\,(b)}$ for all $n \in \mathbb{N}_0$. Using an approximation of $f = \chi_{\sigma(a)}$ by odd functions, we conclude that $p_a(\mathrm{Ran}\,|ba|) = f(a)(\mathrm{Ran}\,|ba|) \subset \overline{\mathrm{Ran}\,(b)}$ where p_a is the projection onto the closure of the range of a. Therefore $|ba|^2\xi = ab^2a\xi = p_a ab^2 a\xi = p_a|ba|^2\xi \subset \overline{\mathrm{Ran}\,(b)}$, which gives $\overline{\mathrm{Ran}\,|ba|} = \overline{\mathrm{Ran}\,(|ba|^2)} \subset \overline{\mathrm{Ran}\,(b)}$. □

The following lemma is well-known, we include a short proof for our particular case (see for instance [4] for the general result):

Lemma 2.9. *Let $0 \le x \in \mathcal{K}(\mathcal{H})$, let $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Then*

$$\langle x^r\xi, \xi\rangle \le \langle x\xi, \xi\rangle^r, \qquad 0 < r < 1$$

with equality iff $x\xi = \langle x\xi, \xi\rangle\xi$. Also

$$\langle x\xi, \xi\rangle^s \le \langle x^s\xi, \xi\rangle, \qquad 1 < s$$

with equality iff $x\xi = \langle x\xi, \xi\rangle\xi$.

Proof. Let $x = \sum x_i p_i$ be a spectral decomposition of x, with $\sum_i p_i = 1$. Let $t_i = \|p_i\xi\|^2$, then $\sum_i t_i = 1$. Using Hölder's inequality for sequences, with $p = 1/r > 1$, we obtain

$$\langle x^r\xi, \xi\rangle = \sum_i x_i^r t_i = \sum_i x_i^r t_i^{1/p}\, t_i^{1/q} \le \left(\sum_i x_i t_i\right)^r \left(\sum_i t_i\right)^{1/q} = \langle x\xi, \xi\rangle^r.$$

Assuming equality, in Hölder's inequality, it must be $x_i t_i = c t_i$ for all i, therefore $x\xi = \langle x\xi, \xi\rangle\xi$. Taking $s = 1/r$ and replacing x with x^s, the proof of the other case ($s > 1$) is straightforward. □

What follows is the statement that tells us that the relevant hypothesis is neither on the operator equality, nor on the norm equality, but the singular numbers equality.

Theorem 2.10. *Assume that $a, b \in \mathcal{K}(\mathcal{H})$, $p > 1$, $1/p + 1/q = 1$. If*

$$\lambda_k(|ab^*|) = \lambda_k\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)$$

for all $k \in \mathbb{N}_0$, then $|a|^p = |b|^q$.

Proof. Since $\lambda_k(|ab^*|) = \lambda_k(|ba^*|)$, exchanging a with b if necessary we can assume that $1 < p \le 2$. It will be easier to deal first with $a, b \ge 0$. We follow the notation of Remark 2.5.

Since $ba^2b = |ab|^2 = \sum_k \gamma_k^2 p_k$, if $p_0 = \xi \otimes \xi$ with $\xi \in \mathcal{H}$ and $\|\xi\| = 1$, then $ba^2b\xi = \gamma_0^2\xi$. Let $\eta = \frac{1}{\gamma_0^2} p_b a^2 b\xi$; then $\eta \in \overline{\operatorname{Ran} b}$ and $b\eta = \xi$. Let p_η be the projection onto $span(\eta)$, then it is easy to check that $\|\eta\|^2 b p_\eta b = p_0$. Observe that

$$ba^2b \geq \gamma_0^2 p_0 = \gamma_0^2 \|\eta\|^2 b p_\eta b. \tag{2.2}$$

Now we have to deal with two cases separately, regarding whether $p = 2$ or $p \neq 2$.

Case $p \neq 2$. By Proposition 2.8, we have $p_b|ba| = |ba|$, but $\operatorname{Ran}|ba| = \operatorname{Ran}(ab)$, hence if we name $\overline{a} = p_b a p_b$, then

$$b\overline{a}^2 b = b p_b a p_b p_b a p_b b = b a p_b a b = b a^2 b \geq \gamma_0^2 \|\eta\|^2 b p_\eta b.$$

Therefore $\overline{a}^2 \geq \gamma_0^2 \|\eta\|^2 p_\eta$ as operators acting on $\mathcal{H}' = \overline{\operatorname{Ran} b}$. Since $1/2 < p/2 < 1$, the operator monotony of $t \mapsto t^{p/2}$ implies that in $\mathcal{H}'$, we have

$$\overline{a}^p \geq \gamma_0^p \|\eta\|^p p_\eta.$$

This also implies

$$\frac{\langle \overline{a}^p \eta, \eta \rangle}{\|\eta\|^2} \geq \gamma_0^p \|\eta\|^p. \tag{2.3}$$

On the other hand, $p_\eta = \|\eta\|^2 p_\eta b^2 p_\eta$, and by Lemma 2.9 with $s = q/2 > 1$,

$$\frac{1}{\|\eta\|^q} p_\eta = \left(\frac{p_\eta}{\|\eta\|^2} \right)^{q/2} = \left(p_\eta b^2 p_\eta \right)^{q/2} \leq p_\eta b^q p_\eta, \tag{2.4}$$

equivalently

$$\frac{1}{\|\eta\|^q} \langle \eta, \eta \rangle \leq \langle b^q \eta, \eta \rangle. \tag{2.5}$$

By Young's numeric inequality

$$\gamma_0 = \gamma_0 \|\eta\| \frac{1}{\|\eta\|} \leq \frac{1}{p} \gamma_0^p \|\eta\|^p + \frac{1}{q} \frac{1}{\|\eta\|^q}. \tag{2.6}$$

Since $1 < p < 2$, the map $t \mapsto t^p$ is operator convex [7, Theorem 2.4], therefore $\overline{a}^p = (p_b a p_b)^p \leq p_b a^p p_b$, hence combining this with (2.3), (2.5) and (2.6) gives

$$\begin{aligned} \gamma_0 &\leq \frac{1}{p} \frac{\langle \overline{a}^p \eta, \eta \rangle}{\|\eta\|^2} + \frac{1}{q} \frac{\langle b^q \eta, \eta \rangle}{\|\eta\|^2} \leq \frac{1}{p} \frac{\langle p_b a^p p_b \eta, \eta \rangle}{\|\eta\|^2} + \frac{1}{q} \frac{\langle b^q \eta, \eta \rangle}{\|\eta\|^2} \\ &= \frac{1}{p} \frac{\langle a^p \eta, \eta \rangle}{\|\eta\|^2} + \frac{1}{q} \frac{\langle b^q \eta, \eta \rangle}{\|\eta\|^2} = \frac{1}{\|\eta\|^2} \left\langle \left(\frac{1}{p} a^p + \frac{1}{q} b^q \right) \eta, \eta \right\rangle \leq \gamma_0 \end{aligned}$$

by the hypothesis on the λ_k.

From here we can derive several conclusions. The first one, since there is equality in Young's numeric inequality (2.6), is that $\gamma_0 = \frac{1}{\|\eta\|^q}$. The second one, since we have equality in (2.4), is that $b^2\eta = \frac{1}{\|\eta\|^2}\eta = \gamma_0^{2/q}\eta$ (Lemma 2.9), therefore $\xi = b\eta = \gamma_0^{1/q}\eta$. The third one, since $0 \leq \frac{1}{p}a^p + \frac{1}{q}b^q \leq \gamma_0 1$ and now

$$\frac{1}{\|\eta\|^2}\left\langle \left(\frac{1}{p}a^p + \frac{1}{q}b^q\right)\eta, \eta\right\rangle = \frac{1}{\|\xi\|^2}\left\langle \left(\frac{1}{p}a^p + \frac{1}{q}b^q\right)\xi, \xi\right\rangle = \gamma_0$$

is that (Lemma 2.1.4)

$$\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right)\xi = \gamma_0\xi$$

and rearranging if necessary the basis of $ker((\frac{1}{p}a^p + \frac{1}{q}b^q) - \gamma_0 1)$, we conclude $p_0 = p_\eta = p_\xi = q_0$. Note that

$$\gamma_0\xi = \frac{1}{p}a^p\xi + \frac{1}{q}b^q\xi = \frac{1}{p}a^p\xi + \frac{1}{q}\gamma_0\xi,$$

which implies that $a^p\xi = \gamma_0\xi$; with a similar argument and since

$$0 \leq \frac{1}{p}\overline{a}^p + \frac{1}{q}b^q \leq \frac{1}{p}p_b a^p p_b + \frac{1}{q}b^q = p_b(\frac{1}{p}p_b a^p p_b + \frac{1}{q}b^q)p_b \leq \gamma_0 p_b \leq \gamma_0 1$$

we deduce that $\overline{a}^p = \gamma_0\xi$ also, therefore $a\xi = \gamma_0^{1/p}\xi$.

We now proceed with an induction argument. Write

$$a = \sum_{\overline{\alpha}_j > \gamma_0^{1/p}} \overline{\alpha}_j \overline{a_j} + \sum_{\alpha_k \leq \gamma_0^{1/p}} \alpha_k a_k,$$

with $a_k, \overline{a}_j$ rank one disjoint projections and $a_k\overline{a}_j = 0$ for all k, j. Then rearranging if necessary $\alpha_0 = \gamma_0^{1/p}$, $a_0 = p_0$. Write similarly

$$b = \sum_{\overline{\beta}_j > \gamma_0^{1/q}} \overline{\beta}_j \overline{b_j} + \sum_{\beta_k \leq \gamma_0^{1/q}} \beta_k b_k, \quad \beta_0 = \gamma_0^{1/q},\ b_0 = p_0.$$

Let $\overline{a} = (1-p_0)a(1-p_0)$ and $\overline{b} = (1-p_0)b(1-p_0)$, then $p_0\overline{a} = p_0\overline{b} = 0$,

$$a = \overline{a} + \gamma_0^{1/p}p_0, \qquad b = \overline{b} + \gamma_0^{1/q}p_0,$$

$$ab = \overline{a}\overline{b} + \gamma_0 p_0, \quad |ab| = |\overline{a}\overline{b}| + \gamma_0 p_0,$$

and

$$\frac{1}{p}\overline{a}^p + \frac{1}{q}\overline{b}^q + \gamma_0 p_0 = \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Therefore

$$\lambda_0\left(\frac{1}{p}\overline{a}^p + \frac{1}{q}\overline{b}^q\right) = \lambda_1\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) = \lambda_1(|ab|) = \lambda_0(|\overline{a}\overline{b}|),$$

and iterating the above construction we arrive to

$$a = \overline{a} + \sum_k \gamma_k^{1/p}p_k, \quad b = \overline{b} + \sum_k \gamma_k^{1/q}p_k$$

with $\overline{a}p_k = \overline{b}p_k = 0$ for each $j, k \in \mathbb{N}_0$. Then

$$\frac{1}{p}a^p + \frac{1}{q}b^q = \sum_k \lambda_k p_k + \frac{1}{p}\overline{a}^p + \frac{1}{q}\overline{b}^q = |ab| + \frac{1}{p}\overline{a}^p + \frac{1}{q}\overline{b}^q = |ab| + T$$

with $T \geq 0$ compact and $T|ab| = 0$. Now $\lambda_k(|ab|) = \lambda_k(\frac{1}{p}a^p + \frac{1}{q}b^q)$ for all k, which means equal eigenvalues with equal (and finite) multiplicities, a fact that forces $T = 0$, therefore $\overline{a} = \overline{b} = 0$, from which the claim $a^p = b^q$ follows for $a, b \geq 0$, assuming $1 < p < 2$.

Case $p = 2$. Let us now return to the case we skipped. From (2.2), we know that $p_b a^2 p_b \geq \gamma_0^2 \|\eta\|^2 p_\eta$ on the whole $\mathcal{H}$, therefore

$$\begin{aligned}\gamma_0 &\leq {}^1\!/_2 \frac{\gamma_0^2\|\eta\|^2}{\|\eta\|^2} + {}^1\!/_2 \frac{1}{\|\eta\|^2} \leq {}^1\!/_2 \frac{\langle p_b a^2 p_b \eta, \eta\rangle}{\|\eta\|^2} + {}^1\!/_2 \frac{\langle b^2\eta, \eta\rangle}{\|\eta\|^2} \\ &= \frac{1}{\|\eta\|^2}\left\langle \left({}^1\!/_2\, p_b a^2 p_b + {}^1\!/_2\, b^2\right)\eta, \eta\right\rangle = \frac{1}{\|\eta\|^2}\left\langle (p_b({}^1\!/_2\, a^2 + {}^1\!/_2\, b^2)p_b)\eta, \eta\right\rangle \\ &= \frac{1}{\|\eta\|^2}\left\langle ({}^1\!/_2\, a^2 + {}^1\!/_2\, b^2)\eta, \eta\right\rangle \leq \gamma_0\end{aligned}$$

since $\eta \in \operatorname{Ran}(b)$. Then from the equality in the numerical inequality (2.6) we derive that $\lambda_0 = \|\eta\|^{-2}$, and $({}^1\!/_2\, a^2 + {}^1\!/_2\, b^2)\eta = \gamma_0\eta$ as before. Since $q = 2$, we have lost the strict inequality in (2.4) regarding b. However, Since now $\langle p_b a^2 p_b \eta, \eta\rangle \geq \gamma_0^2\|\eta\|^2$ must be an equality, from Lemma 2.1.4 we conclude that $p_b a^2 \eta = p_b a^2 p_b \eta = \lambda\eta$ for some positive λ, hence $b^2\eta = (2\gamma_0 - \lambda)\eta$ also. Recalling $1 = \|\xi\|^2 = \|b\eta\|^2 = \langle b^2\eta, \eta\rangle = (2\gamma_0 - \lambda)\|\eta\|^2 = (2\gamma_0 - \lambda)\gamma_0^{-1}$, we obtain $\lambda = \gamma_0$. This tells us that $b\eta = \gamma_0^{1/2}\eta = a\eta$. The rest of the argument follows as in the case of $p < 2$.

Returning to the original statement, if for arbitrary compact a, b, we have equality of singular values, since $\lambda_k(|ab^*|) = \lambda_k(||a||b||)$ (Remark 2.2.2), we obtain $|a|^p = |b|^q$. □

Let us resume all the results in one clear cut statement, the main result of this paper:

Theorem 2.11. *Let $a, b \in \mathcal{K}(\mathcal{H})$. If $p > 1$ and $1/p + 1/q = 1$, then the following are equivalent:*

(1) $|a|^p = |b|^q$.

(2) $z|ab^*|z^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$ *for some contraction* $z \in \mathcal{B}(\mathcal{H})$

(3) $\|z|ab^*|w\|_\phi = \|\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\|_\phi < \infty$ *for a pair of contractions* $z, w \in \mathcal{B}(\mathcal{H})$ *and* $\|\cdot\|_\phi$ *a strictly increasing symmetric norm.*

(4) $\lambda_k(|ab^*|) = \lambda_k\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right)$ *for all* $k \in \mathbb{N}_0$.

Proof. Clearly $1 \Rightarrow 2$ with $z = \nu$ (the partial isometry in the polar decomposition of $b = \nu|b|$). If 2 holds, picking a norm as in equation (2.1), we have $2 \Rightarrow 3$. By Lemma 2.7, we have $3 \Rightarrow 4$ and finally, by Theorem 2.10 it follows that $4 \Rightarrow 1$. □

2.4.1. *Final remarks: equality of operators.* Assume that we have an equality of operators

$$z|ab^*|z^* = \frac{1}{p}|a|^p + \frac{1}{q}|b|^q \tag{2.7}$$

for some contraction $z \in \mathcal{B}(\mathcal{H})$. Then from the previous theorem $|a|^p = |b|^q$ and

$$z|b^*|^q z^* = z\nu|b|^q\nu^* z^* = z|ab^*|^* z^* = |b|^q.$$

Remark 2.12. Let Tr stand for the semi-finite trace of $\mathcal{B}(\mathcal{H})$. Assume for a moment that $Tr|b|^q < \infty$, or equivalently, that $\beta_k = \lambda_k(b) \in \ell_q$. Then

$$Tr(|b^*|^q(1 - z^*z)) = Tr|b^*|^q - Tr(z|b^*|^q) = Tr|b|^q - Tr(z|b^*|^q z^*) = 0,$$

which is only possible if $|b^*|^q = |b^*|^q z^*z$, since z is a contraction and the trace is faithful. Then also

$$zz^*|b|^q = zz^*z|b^*|^q z^* = z|b^*|^q z^* = |b|^q,$$

and

$$|b|^q z = z|b^*|^q z^*z = z|b^*|^q$$

or equivalently $|b|z = z|b^*|$, which can be stated as $bz\nu = \nu zb$. The reader can check that these three conditions

$$1)\ |b^*|z^*z = |b^*|, \qquad 2)\ |b|zz^* = |b|, \qquad 3)\ |b|z = z|b^*|$$

are also sufficient to have equality in (2.7).

This last fact, for z a partial isometry (and with a different proof) was observed [3] by Argerami and Farenick.

We conjecture that these three conditions are also necessary for (2.7) to happen with a contraction z if b is just compact.

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