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VALUATIONS OF SKEW QUANTUM POLYNOMIALS

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Abstract

In this paper we extend some results obtained by Artamonov and Sabitov for quantum polynomials to skew quantum polynomials and quasi−commutative bijective skew PBW extensions. Moreover, we find a counterexample to the conjecture proposed in [6].

Keywords: Skew PBW extensions, skew quantum polynomials, Ore domains, valuations, completions.

Contents

1 Introduction

This section is divided into three subsections, we recall the definition of $\Gamma-$valuation, valuation and quantum polynomials. We review some fundamental properties of valuations and valuations of quantum polynomials (see [4] and [6]).

1.1 Valuations

Let D be a division ring, D^* the multiplicative group and Γ is a totally ordered group (with additive notation and not necessarily commutative).

Definition 1.1. *A function $\nu : D^* \to \Gamma$ is a $\Gamma-$valuation of D^* if:*

i) ν *is surjective,*

ii) $\nu(ab) = \nu(a) + \nu(b)$,

iii) $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$.

Proposition 1.2. *[14, 9] If ν is a $\Gamma-$valuation of D^*, then:*

1) If $\nu(a) \neq \nu(b)$, then $\nu(a+b) = \min\{\nu(a), \nu(b)\}$.

2) $\Lambda_\nu := \{a \in D; a = 0 \ \text{ or } \ \nu(a) \geq 0\}$ is a subring of D.

3) The group of units $\mathcal{U}_\nu := \{a \in D^*; \nu(a) = 0\}$ is a subgroup of D^*.

4) $\mathcal{W}_\nu := \{a \in D, a = 0 \ \text{ or } \ \nu(a) > 0\}$ is a completely prime ideal of Λ_ν and $\mathcal{W}_\nu = \Lambda_\nu - \mathcal{U}_\nu$.

5) Λ_ν is a local ring with unique maximal ideal $\mathcal{W}_\nu$.

1.2 Valuations with values on $\Gamma \cup \{\infty\}$

Proposition 1.3. *Let Γ be a totally ordered group with additive notation ordere. Then $\Gamma \cup \{\infty\}$ is an ordered additive monoid such that*

$$x + \infty := \infty =: \infty + x, \quad \text{for all } \Gamma \cup \{\infty\},$$

and $\infty > x$ for all $x \in \Gamma$.

Definition 1.4 ([8])**.** *Let R be a ring. By a valuation on R with values in a totally ordered group Γ, the value group, we shall understand a function ν on R with values in $\Gamma \cup \{\infty\}$ subject to the conditions:*

i) $\nu(a) \in \Gamma \cup \{\infty\}$ and ν assumes at least two values,

ii) $\nu(ab) = \nu(a) + \nu(b)$,

iii) $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$.

Proposition 1.5. *[8, 9] If ν is a valuation of R, then:*

1) *$ker\,\nu := \{a \in R; \nu(a) = \infty\}$ is an ideal of R.*

2) *If $\nu(a+b) > \min\{\nu(a), \nu(b)\}$, then $\nu(a) = \nu(b)$.*

3) *$\Lambda_\nu := \{a \in R;\ \nu(a) \geq 0\}$ is a subring of R.*

4) *The group of units $\mathcal{U}_\nu := \{a \in R^*; \nu(a) = 0\}$ is a subgroup of R^*.*

5) *$\mathcal{W}_\nu := \{a \in R, \nu(a) > 0\}$ is an ideal of Λ_ν.*

6) *$ker\,\nu$ is a completely prime ideal of R and $R/ker\,\nu$ is an integral domain.*

Proposition 1.6 ([8])**.** *If ν is a $\Gamma-$valuation of D. Then Γ is abelian, if and only if $\nu(a) = 0$ for all $a \in [D^*, D^*]$.*

1.3 Quantum polynomials

Let D be a division ring with a fixed set α_1, α_2 , ..., α_n, $n \geq 2$, of automorphimsms. Also, we have q_{ij} in D^* for $i, j = 1, 2, \ldots, n$ fix elements, satisfying the relations :

$$q_{ii} = q_{ij}q_{ji} = \boldsymbol{q}_{ijr}\boldsymbol{q}_{jri}\boldsymbol{q}_{rij} = 1$$
$$\alpha_i(\alpha_j(d)) = q_{ij}\alpha_j(\alpha_i(d))q_{ji},$$

where $\boldsymbol{q}_{ijr} = q_{ij}\alpha_j(q_{ir})$ and $d \in D$. We set $\boldsymbol{q} = (q_{ij}) \in \mathscr{M}(n, D)$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Definition 1.7. *The elements q_{ij} of the matrix $\boldsymbol{q}$ are called* ***system of multiparameters****.*

Definition 1.8 (Quantum polynomial ring)**.** *Denote by*

$$\mathcal{O}_{\boldsymbol{q}} := D_{\boldsymbol{q},\alpha}\left[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\right], \tag{1.1}$$

the associative ring generated by D and by elements $x_1^{\pm 1}$, $x_2^{\pm 1}$, ..., $x_r^{\pm 1}$, x_{r+1}, ..., x_n subject to the defining relations

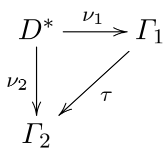
$$x_i x_i^{-1} = x_i^{-1} x_i = 1,\ 1 \leq i \leq r, \tag{1.2}$$
$$x_i d = \alpha_i(d) x_i,\ d \in D,\ i = 1, 2, \ldots, n, \tag{1.3}$$
$$x_i x_j = q_{ij} x_j x_i,\ i, j = 1, 2, \ldots, n. \tag{1.4}$$

Definition 1.9. *Let N be the subgroup in the multiplicative group D^* of the ring D generated by the derived subgroup $[D^*, D^*]$ and by the set of all elements of the form $z^{-1}\sigma_i(z)$ where $z \in R^*$ and $i = 1, \ldots, n$. $\Lambda := D_{\boldsymbol{q},\alpha}[x_1, x_2, \ldots, \ldots, x_n]$ is a general (generic) quantum polynomials ring if the images of all multiparameters q_{ij}, $1 \leq i < j \leq n$, are independent in the multiplicative Abelian group D^*/N.*

The ring $\mathcal{O}_{\boldsymbol{q}}$ is a left and right Noetherian domain, it satisfies Ore Condition and it has a division ring of fractions $F := D_{\boldsymbol{q}}(x_1, \ldots, x_n)$. We consider $\nu : F^* \to \Gamma$ a $\Gamma-$valuation with $\nu(D^*) = 0$.

Theorem 1.10 ([6])**.** *A valuation of a quantum division ring D, is Abelian in the sense that the group Γ is Abelian.*

Definition 1.11 ([4], [6])**.** *Let $\nu_1 : D^* \to \Gamma_1$ and $\nu_2 : D^* \to \Gamma_2$ be two valuations. Set $\nu_1 \geq \nu_2$ if there exists an epimorphism of ordered groups $\tau : \Gamma_1 \to \Gamma_2$ such that $\tau\nu_1 = \nu_2$. It means that the diagram*



is commutative.

Definition 1.12 ([4], [6])**.** *A valuation ν has a maximal rank if τ is an isomorphism in the previous definition.*

Theorem 1.13 ([4])**.** *A valuation $\nu : F^* \to \Gamma$ of a general quantum division ring $\mathcal{O}_q$ is has maximal rank if only if $\Gamma \cong \mathbb{Z}^n$.*

2 Completions of quantum polynomials

In this section $\nu : F^* \to \mathbb{Z}^n$ is a maximal $\mathbb{Z}^n-$valuation.

Definition 2.1 ([6])**.** *Let $\mathscr{F}$ be the set of all maps $f : \mathbb{Z}^n \to k$ and the zero element such that supp $f := \{m \in \mathbb{Z}^n; f(m) \neq 0\}$ is Artinian with respect to the lexicographic order on $\mathbb{Z}^n$.*

Theorem 2.2. *$\mathscr{F}$ is a division ring containing F.*

Proof. See [3] Theorem 3.4 and 3.7. □

Expand the valuation ν to $f \in \mathscr{F}$ in the following way. If $f \in \mathscr{F}$ then $\nu(f)$ the least element from *supp* f.

Definition 2.3 ([6])**.** *The division ring $\mathscr{F}$ is called a completion of F with respect to ν.*

Remark 2.4. If $\mathcal{O} := \{f \in \mathscr{F}; \nu(f) \geq 0\}$ and $\mathbf{m} := \{f \in \mathscr{F}; \nu(f) > 0\}$, then $\mathcal{O}$ is a subring in $\mathscr{F}$ and $\mathbf{m}$ is an ideal in $\mathcal{O}$. Moreover, $\mathcal{O}/\mathbf{m} \cong k$.

Let $\mathbb{R}^n$ be a vector space of all rows $(r_1, \ldots, r_n)$, $r_i \in \mathbb{R}$, of a length n and $\mathbb{R}^n$ is equipped with the lexicographic order.

Theorem 2.5 ([10])**.** *Let $\leq_{\mathbb{Z}^n}$ be a totally order in the additive group $\mathbb{Z}^n$. Then there exists order preserving group embedding $\mathbb{Z}^n \to \mathbb{R}^n$.*

Definition 2.6. *[6] A totally order $\leq_{\mathbb{Z}^n}$ is essentially lexicographic if it belongs to the orbit of the standard embedding of $\mathbb{Z}^n$ in to $\mathbb{R}^n$ under the action of the group $GL(n, \mathbb{Z})$. i.e., if $a, b \in \mathbb{Z}^n$, $a \leq_{\mathbb{Z}^n} b$ if only if $aA \leq bA$ for some fixed A in $GL(n, \mathbb{Z})$ and $\leq$ the lexicographic order.*

Conjecture 2.7 ([6])**.** *A valuation ν is associated to an essentially lexicographic order on $\mathbb{Z}^n$ if and only if $\cap_{i>1} \boldsymbol{m}^i = 0$.*

In the study of this conjecture we obtain the following results partial:

Proposition 2.8. *If $\nu : R \to \Gamma \cup \{\infty\}$ is a valuation of a ring R and Γ is a Archimedean group with $\mathcal{W}_\nu := \{a \in R, \nu(a) > 0\}$, $\inf\{\nu(\mathcal{W}_\nu)\} \neq 0$ and $\bigcap_{i \geq 1} \mathcal{W}_\nu^i := I$, then $\nu(I) = \infty$.*

Proof. Let $A_i := \nu(\mathcal{W}_\nu^i)$ and $\lambda_i := \inf\{A_i\}$ be, then $\lambda_1 < \lambda_2 < \ldots < \lambda_i$ and $i\lambda_1 \leq \lambda_i$, indeed: (by induction over i) as $\inf\{\nu(\mathcal{W}_\nu)\} \neq 0$ then $0 < \lambda_1 \leq \nu(a)$ for all $a \in \mathcal{W}_\nu$, hence $\lambda_1 < 2\lambda_1 \leq \nu(ab)$ for all $a, b \in \mathcal{W}_\nu$, therefore $2\lambda_1 \leq \lambda_2$, suppose that $\lambda_{i-1} < \lambda_i$ and $i\lambda_1 \leq \lambda_i$, then $i\lambda_1 < (i+1)\lambda_1 \leq \lambda_i + \lambda_1 \leq \nu(a) + \nu(b) = \nu(ab)$ for all $a \in \mathcal{W}_\nu^i$ and $b \in \mathcal{W}_\nu$, then, $\lambda_i < \lambda_{i+1}$ and $(i+1)\lambda_1 \leq \lambda_{i+1}$.

Now, suppose there exists $b \in I$ such that $\nu(b) = \lambda < \infty$, so $\lambda_1 < \lambda$ and as Γ is Archimedean there exists an integer m such that $m\lambda_1 > \lambda$, therefore $\lambda \notin A_m$, hence $b \notin \mathcal{W}_\nu^m$, contradicting that $b \in I$. □

Corollary 2.9. *If $\nu : D \to \Gamma \cup \{\infty\}$ is a valuation of a division ring D and Γ is a Archimedean group with $\inf\{\nu(\mathcal{W}_\nu)\} \neq 0$, then $\bigcap_{i \geq 1} \mathcal{W}_\nu^i = 0$.*

Proof. $0 = \nu(1) = \nu(aa^{-1}) = \nu(a) + \nu(a^{-1})$ for all $a \in D^*$, then $\nu(a) < \infty$ for all $a \in D^*$, therefore $\nu(a) = \infty$ if only if $a = 0$. □

Remark 2.10. In the Proposition 2.8 the condition $\inf\{\nu(\mathcal{W}_\nu)\} \neq 0$ can be replaced by $\inf\{\nu(\mathcal{W}_\nu^i)\} \neq 0$ for any $i > 0$ in $\mathbb{N}$.

Example 2.11. If we take lexicographic order on $\mathbb{Z}^2$ the order does not have intersection property: consider $A := \{(x, y) \in \mathbb{Z}^2; (0, 0) < (x, y)\}$ and $nA := \sum_{i=1}^{n} A$ with $n > 0$, then $nA = \{(x, y) \in \mathbb{Z}^2; (0, n) \leq (x, y)\}$. By induction over n: If $n = 2$, then $2A = A \setminus \{(0, 1)\}$, indeed: as $\min\{A\} = (0, 1)$ then $(0, 2) \leq (x, y)$ with $(x, y) \in 2A$, thus $2A \subseteq A \setminus \{(0, 1)\}$. Now, if (x, y)

in $2A$, then $(x, y-1) \in A$, because $x > 0$ or $x = 0$ and $y \geq 2$.

Suppose that $nA = (n-1)A \setminus \{(0, n-1)\}$, as $\min\{nA\} = (0, n)$ then $(0, n+1) \leq (x, y)$ with $(x, y) \in (n+1)A$, thus $(n+1)A \subseteq nA \setminus \{(0, n)\}$. Now, if (x, y) in $(n+1)A$, then $(x, y-1) \in nA$, because $x > 0$ or $x = 0$ and $y \geq n+1$. Consequently $(n+1)A = \{(x, y) \in \mathbb{Z}^2; (0, n+1) \leq (x, y)\}$.

Hence, as $(1, 0) \in nA$ for every $n \geq 1$ since $(0, n) < (1, 0)$, then $(1, 0) \in \cap_{n>0} nA$.

It follows a counterexample to the conjecture, since a lexicographic order is essentially lexicographic.

3 Skew PBW extensions

In this section we recall the definition and some basic properties of skew PBW (Poincaré-Birkhoff-Witt) extensions, introduced in [11]. Some ring-theoretic and homological properties of these class of noncommutative rings have been studied in [12].

Definition 3.1. *Let R and A be rings. We say that A is a skew PBW extension of R (also called a $\sigma - PBW$ extension of R) if the following conditions hold:*

(i) $R \subseteq A$.

(ii) *There exists finitely many elements $x_1, \ldots, x_n \in A$ such A is a left R-free module with basis*

$$\mathrm{Mon}(A) := \{x^u = x_1^{u_1} \cdots x_n^{u_n} \mid u = (u_1, \ldots, u_n) \in \mathbb{N}^n\}.$$

In this case it also says that A is a left polynomial ring over R with respect to $\{x_1, \ldots, x_n\}$ and $Mon(A)$ is the set of standard monomials of A. Moreover, $x_1^0 \cdots x_n^0 := 1 \in Mon(A)$.

(iii) *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that*

$$x_i r - c_{i,r} x_i \in R. \tag{3.1}$$

(iv) *For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that*

$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n. \tag{3.2}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n \rangle$.

Proposition 3.2. *Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exists an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r),$$

for each $r \in R$.

Proof. See [11], Proposition 3. □

The previous proposition gives the notation and the alternative name given for the skew PBW extensions.

Definition 3.3. *Let A be a skew PBW extension.*

(a) *A is quasi-commutative if the conditions (iii) and (iv) in Definition 3.1 are replaced by*

  (iii') *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that*
  $$x_i r = c_{i,r} x_i. \tag{3.3}$$

  (iv') *For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that*
  $$x_j x_i = c_{i,j} x_i x_j. \tag{3.4}$$

(b) *A is bijective if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.*

Definition 3.4. *Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 3.2.*

(i) *For $u = (u_1, \dots, u_n) \in \mathbb{N}^n$, $\sigma^u := \sigma_1^{u_1} \cdots \sigma_n^{u_n}$, $|u| := u_1 + \cdots + u_n$. If $v = (v_1, \dots, v_n) \in \mathbb{N}^n$, then $u + v := (u_1 + v_1, \dots, u_n + v_n)$.*

(ii) *For $X = x^u \in \mathrm{Mon}(A)$, $\exp(X) := u$ and $\deg(X) := |u|$.*

(iii) *If $f = c_1 X_1 + \cdots + c_t X_t$, with $X_i \in Mon(A)$ and $c_i \in R - \{0\}$, then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.*

Theorem 3.5. *Let A be a left polynomial ring over R w.r.t. $\{x_1, \dots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions hold:*

(a) *For every $x^u \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$ there exist unique elements $r_u := \sigma^u(r) \in R - \{0\}$ and $p_{u,r} \in A$ such that*
  $$x^u r = r_u x^u + p_{u,r}, \tag{3.5}$$
  where $p_{u,r} = 0$ or $\deg(p_{u,r}) < |u|$ if $p_{u,r} \neq 0$. Moreover, if r is left invertible, then r_u is left invertible.

(b) *For every* $x^u, x^v \in \mathrm{Mon}(A)$ *there exist unique elements* $c_{u,v} \in R$ *and* $p_{u,v} \in A$ *such that*

$$x^u x^v = c_{u,v} x^{u+v} + p_{u,v}, \tag{3.6}$$

where $c_{u,v}$ *is left invertible,* $p_{u,v} = 0$ *or* $\deg(p_{u,v}) < |u + v|$ *if* $p_{u,v} \neq 0$.

Proof. See [11], Theorem 7. □

Proposition 3.6. *Let* A *be a skew* PBW *extension of a ring* R. *If* R *is a domain, then* A *is a domain.*

Proof. See [12]. □

The next theorem characterizes the quasi-commutative skew PBW extensions.

Theorem 3.7. *Let* A *be a quasi-commutative skew* PBW *extension of a ring* R. *Then,*

(i) *A is isomorphic to an iterated skew polynomial ring of endomorphism type, i.e.,*

$$A \cong R[z_1; \theta_1] \cdots [z_n; \theta_n].$$

(ii) *If A is bijective, then each endomorphism θ_i is bijective,* $1 \leq i \leq n$.

Proof. See [12]. □

Corollary 3.8. *Let* A *be a bijective and quasi-commutative skew* PBW *extension of a ring* R. *If* R *is a left Ore domain, then* A *is a left Ore domain.*

Proof. By Theorem 3.7, A is isomorphic to an iterated skew polynomial ring of automorphism type over a left Ore domain R. □

Theorem 3.9. *Let* A *be an arbitrary skew* PBW *extension of* R. *Then,* A *is a filtered ring with filtration given by*

$$F_m := \begin{cases} R & \text{if} \quad m = 0 \\ \{f \in A \mid \deg(f) \leq m\} & \text{if} \quad m \geq 1 \end{cases} \tag{3.7}$$

and the corresponding graded ring $Gr(A)$ *is a quasi-commutative skew* PBW *extension of* R. *Moreover, if* A *is bijective, then* $Gr(A)$ *is a quasi-commutative bijective skew* PBW *extension of* R.

Proof. See [12]. □

Theorem 3.10 (Hilbert Basis Theorem)**.** *Let* A *be a bijective skew* PBW *extension of* R. *If* R *is a left (right) Noetherian ring then* A *is also a left (right) Noetherian ring.*

Proof. See [12]. □

3.1 Skew quantum polynomials

In this subsection we recall the definition and some basic properties of skew quantum polynomials ring over R, introduced in [12].We mention some results generalized for valuations of skew quantum polynomials and bijective and quasi-commutative skew PBW extension.

Definition 3.11. *Let R be a ring with matrix of parameters $q := [q_{ij}] \in M_n(R)$, $n \geq 2$, such that $q_{ii} = 1 = q_{ij}q_{ji} = q_{ji}q_{ij}$ for each $1 \leq i, j \leq n$ and suppose also that is given a system $\sigma_1, \dots, \sigma_n$ of automorphisms of R. The skew quantum polynomials ring over R, denoted by*

$$R_{\boldsymbol{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n], \tag{3.8}$$

is defined whit the following conditions:

i) $R \subseteq R_{\boldsymbol{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$,

ii) $R_{\boldsymbol{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$ *is a free left $R-$module with basis* $\{x^u; x^u = x_1^{u_1} \cdots x_n^{u_n}, u_i \in \mathbb{Z}, 1 \leq i \leq r$ *and* $u_i \in \mathbb{N}$ *for* $r+1 \leq i \leq n\}$,

iii) The $x_1, \dots, x_n$ elements satisfy the defining relations

$$x_i x_i^{-1} = 1 = x_i^{-1} x_i,\ 1 \leq i \leq r, \tag{3.9}$$

$$x_i x_j = q_{ji} x_j x_i\ 1 \leq i, j \leq n, \tag{3.10}$$

$$x_i r = \sigma_i(r) x_i,\ r \in R\ y\ 1 \leq i \leq n. \tag{3.11}$$

When all automorphisms are trivial, we write $R_{\boldsymbol{q}}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$ and this ring is called *the ring of quantum polynomials over R*. If $R = K$ is a field, then $K_{\boldsymbol{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$ is the *algebra of skew quantum polynomials*. For trivial automorphisms we get the *algebra of quantum polynomials simply.*

If $r = n$, $R_{\boldsymbol{q},\sigma}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is called the *n-multiparametric skew quantum torus over R*, when all automorphisms are trivial, is called the *$n-$multiparametric quantum torus over R*. If $r = 0$, $R_{\boldsymbol{q},\sigma}[x_1, \dots, x_n]$ is called the *n-multiparametric skew quantum space over R*, when all automorphisms are trivial is called *$n-$multiparametric quantum space over R.*

The algebra of quantum polynomials can be defined as a quasi-commutative bijective skew PBW extension of the r-multiparameter quantum torus, or also, as a localization of a quasi-commutative bijective skew PBW extension.

Theorem 3.12. *$R_{\boldsymbol{q},\sigma}[x_1,\dots,x_n] \cong R[z_1;\theta_1]\cdots[z_n;\theta_n]$, where*

i) $\theta_1 = \sigma_1$,
ii) $\theta_i : R[z_1;\theta_1]\cdots[z_{i-1};\theta_{i-1}] \to R[z_1;\theta_1]\cdots[z_{i-1};\theta_{i-1}]$,
iii) $\theta_i(z_i) = q_{ij}z_i, 1 \leq i < j \leq n, \theta_i(r) = \sigma_i(r)$ for $r \in R$.

In particular, $R_{\boldsymbol{q}}[x_1,\dots,x_n] \cong R[z_1]\cdots[z_n;\theta_n]$.

Proof. See [12]. □

Theorem 3.13. *$R_{\boldsymbol{q},\sigma}[x_1^{\pm 1},\dots,x_r^{\pm 1},x_{r+1}\dots,x_n]$ is a ring of fractions of $B := R_{\boldsymbol{q},\sigma}[x_1,\dots,x_n]$ with respect to the multiplicative subset*

$$S = \{rx^u; r \in R^*, x^u \in Mon\{x_1,\dots,x_r\}\},$$

i.e,

$$R_{\boldsymbol{q},\sigma}[x_1^{\pm 1},\dots,x_r^{\pm 1},x_{r+1}\dots,x_n] \cong S^{-1}B.$$

Proof. See [12]. □

Remark 3.14. Let $Q^{r,n}_{\boldsymbol{q},\sigma}(R) := R_{\boldsymbol{q},\sigma}[x_1^{\pm 1},\dots,x_r^{\pm 1},x_{r+1},\dots,x_n]$ and R be a left (right) Noetherian ring, then $Q^{r,n}_{\boldsymbol{q},\sigma}(R)$ is left (right) Noetherian by Theorem 3.10. Moreover, if R is a domain, then $Q^{r,n}_{\boldsymbol{q},\sigma}(R)$ is also a domain by Theorem 3.6. Thus, if R is a left (right) Noetherian domain, then $Q^{r,n}_{\boldsymbol{q},\sigma}(R)$ is a left (right) Ore domain.

Thus, $Q^{r,n}_{\boldsymbol{q},\sigma}(R)$ has a total division ring of fractions

$$Q(Q^{r,n}_{\boldsymbol{q},\sigma}(R)) \cong Q(A) := \sigma(R)(x_1,\dots,x_n),$$

where $\sigma(R)(x_1,\dots,x_n)$ denotes the rational fractions of $A := \sigma(R)\langle x_1,\dots,x_n\rangle$.

3.2 Some properties

Definition 3.15. *Let N be the subgroup in the multiplicative group R^* of the ring R generated by the derived subgroup $[R^*,R^*]$ and by the set of all elements of the form $z^{-1}\sigma_i(z)$ where $z \in R^*$ and $i = 1,\dots,n$.*

Remark 3.16. N is a normal subgroup in R^*.

Definition 3.17. *If the images of q_{ij} with $1 \leq i < j \leq n$ are independent in the multiplicative Abelian group $\bar{R} = R^*/N$ then, $R_{\boldsymbol{q},\sigma}[x_1^{\pm 1},\dots,x_r^{\pm 1},x_{r+1},\dots,x_n]$ is a generic skew quantum polynomials ring.*

Remark 3.18. If n=2 in $R_{\boldsymbol{q},\sigma}[x_1^{\pm 1},\dots,x_r^{\pm 1},x_{r+1},\dots,x_n]$, of the previous definition $q = q_{12}$ is not a root of unity.

Proposition 3.19. *For each $a \in R^*$ and σ endomorphism over R, $\sigma^k(a) = an$ with $k \in \mathbb{N}$ and $n \in N$.*

Proof.

$$\begin{aligned}\sigma^k(a) &= a\left(a^{-1}\sigma(a)\right)\left((\sigma(a))^{-1}\sigma^2(a)\right)\dots\left((\sigma^{k-1}(a))^{-1}\sigma^k(a)\right)\\ &= an,\ \text{with } n\in N. \end{aligned} \tag{3.12}$$

□

Proposition 3.20. *If $u, v \in \mathbb{Z}^r \times \mathbb{N}^{n-r}$ and $\lambda, \mu \in R^*$, then*

(1) $x_i x^u = \left(\prod_{j=1}^n q_{ji}^{u_j}\right) n_u \cdot x^u x_i$, *for some $n_u \in N$ and for any $1 \le i \le n$.*

(2) $(x^u)(x^v) = \left(\prod_{i<j} q_{ji}^{u_j v_i}\right) n_{u+v} \cdot x^{u+v}$, *with $n_{u+v} \in N$.*

(3) $(\lambda x^u)(\mu x^v) = \lambda\mu \left(\prod_{i<j} q_{ji}^{u_j v_i}\right) n' \cdot x^{u+v}$, *with $n' \in N$.*

Proof. Applying the Proposition 3.19 and note that $x_i x_j^{-1} = q_{ji}^{-1} x_j^{-1} x_i$ with $1 \le j \le r$. □

Proposition 3.21. *Let $f := \sum_{u\in\mathbb{Z}} \lambda_u x^u$ be in $R_{\boldsymbol{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$ and x_i with $1 \le i \le r$.*

(1) *If $\lambda_u \in R$, then*

$$x_i f x_i^{-1} = \sum_{u\in\mathbb{Z}^n} \sigma_i(\lambda_u)\lambda'_u x^u,$$

where $\lambda'_u := \left(\prod_{j=1}^n q_{ji}^{u_j}\right) n_u \in R^*$.

(2) *If $\lambda_u \in R^*$, then*

$$x_i f x_i^{-1} = \sum_{u\in\mathbb{Z}^n} \lambda'_u x^u,$$

where $\lambda'_u \in R^*$.

Proof. (1) Note that $N \subseteq R^*$ and

$$\begin{aligned} x_i f x_i^{-1} &= \sum \sigma_i(\lambda_u) x_i x^u x^{-i}\\ &= \sum_{u\in\mathbb{Z}^n} \sigma_i(\lambda_u)\left(\prod_{j=1}^n q_{ji}^{u_j}\right) n_u x^u, \end{aligned}$$

where $n_u \in N$.

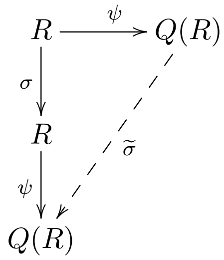
(2) By item (1), $\sigma_i(\lambda_u)\lambda'_u \in R^*$.

□

Remark 3.22. If $Q(Q^{r,n}_{\boldsymbol{q},\sigma}(R))$ exists and G denotes the multiplicative subgroup in $Q(Q^{r,n}_{\boldsymbol{q},\sigma}(R))^*$ generated by R^* and $x_1, ..., x_n$. Then $R^* \lhd G$ and G/R^* is a free abelian group with the base $x_1R^*, \dots, x_nR^*$.

Proposition 3.23. *Let R be a left Ore domain and σ automorphisms over R, then σ can be extended to $Q(R)$ and is also an automorphism.*

Proof. By universal property we have the following commutative diagram:



where ψ, σ are injective and $\widetilde{\sigma}\left(\frac{a}{b}\right) = \frac{\sigma(a)}{\sigma(b)}$ for $a, b \neq 0 \in R$. Therefore, $\psi \circ \sigma$ is injective and so is $\widetilde{\sigma}$.

If $\frac{a}{b} \in Q(R)$, then $\frac{a}{b} = \psi(b)^{-1}\psi(a) = \psi(\sigma(b_0))^{-1}\psi(\sigma(a_0))$ for $a_0, b_0 \neq 0 \in R$, consequently,

$$\begin{aligned} \frac{a}{b} &= \psi(\sigma(b_0))^{-1}\psi(\sigma(a_0)) \\ &= \widetilde{\sigma}(\psi(b_0))^{-1}\widetilde{\sigma}(\psi(a_0)) \\ &= \widetilde{\sigma}(\psi(b_0)^{-1}\psi(a_0)) \\ &= \widetilde{\sigma}\left(\frac{a_0}{b_0}\right). \end{aligned}$$

□

Theorem 3.24. *Let R be a left Ore domain and $S = R - \{0\}$, then*

$$S^{-1}(R_{\boldsymbol{q},\sigma}[x_1, \dots, x_n]) \cong Q(R)_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}[x_1, \dots, x_n],$$

where $\widetilde{\boldsymbol{q}} = \left(\frac{q_{ij}}{1}\right) \in \mathscr{M}(n, Q(R))$.

Proof. By Theorem 3.12 $R_{\boldsymbol{q},\sigma}[x_1, \dots, x_n] \cong R[z_1; \theta_1] \cdots [z_n; \theta_n]$, with each θ_i bijective. Thus, if $S = R - \{0\}$ then

$$\begin{aligned} S^{-1}\left(R_{\boldsymbol{q},\sigma}[x_1, \dots, x_n]\right) &\cong S^{-1}\left(R[z_1; \theta_1] \cdots [z_n; \theta_n]\right) \\ &\cong S^{-1}(R)\,[z_1; \widetilde{\theta_1}] \cdots [z_n; \widetilde{\theta_n}] \\ &= Q(R)\,[z_1; \widetilde{\theta_1}] \cdots [z_n; \widetilde{\theta_n}] \end{aligned}$$

where

$$\begin{aligned} \widetilde{\theta_1} : Q(R) &\to Q(R) \\ \frac{a}{b} &\mapsto \widetilde{\theta_1}\left(\frac{a}{b}\right) = \frac{\theta_1(a)}{\theta_1(b)} = \frac{\sigma_1(a)}{\sigma_1(b)} = \widetilde{\sigma_1}\left(\frac{a}{b}\right), \end{aligned}$$

and

$$\widetilde{\theta_i} : Q\left(R\right)[z_1;\widetilde{\theta_1}]\cdots[z_{i-1};\widetilde{\theta_{i-1}}] \to Q\left(R\right)[z_1;\widetilde{\theta_1}]\cdots[z_{i-1};\widetilde{\theta_{i-1}}]$$

with

$$\widetilde{\theta_i}\left(\frac{a}{b}\right) = \widetilde{\sigma_i}\left(\frac{a}{b}\right) \text{ y } \widetilde{\theta_j}\left(z_i\right) = \frac{q_{ij}}{1} z_i.$$

Therefore,

$$S^{-1}(R_{\boldsymbol{q},\sigma}[x_1,\ldots,x_n]) \cong Q(R)_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}[x_1,\ldots,x_n],$$

where $\widetilde{\boldsymbol{q}} = \left(\frac{q_{ij}}{1}\right) \in \mathscr{M}(n, Q(R))$. $\square$

Proposition 3.25. *Let R be a left Ore domain , there exists*

$$\phi : R_{\boldsymbol{q},\sigma}[x_1^{\pm 1},\ldots,x_n^{\pm 1}] \to Q(R)_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$$

an injective ring homomorphism.

Proof. Let $B_R := R_{\boldsymbol{q},\sigma}[x_1,\ldots,x_n]$ and $B_{Q(R)} := Q(R)_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}[x_1,\ldots,x_n]$ be, by Theorem 3.13 $R_{\boldsymbol{q},\sigma}[x_1^{\pm 1},\ldots,x_n^{\pm 1}] \cong S_1^{-1}B_R$ with $S_1 = \{rx^u; r \in R^*, x^u \in Mon\{x_1,\ldots,x_n\}\}$, and $Q(R)_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}[x_1^{\pm 1},\ldots,x_n^{\pm 1}] \cong S_{1'}^{-1}B_{Q(R)}$ with $S_{1'} = \{rx^u; r \in Q(R)^*, x^u \in Mon\{x_1,\ldots,x_n\}\}$.

Now, consider the following diagram of ring homomorphisms:

$$\begin{array}{ccccc} R & \hookrightarrow & R_{\boldsymbol{q},\sigma}[x_1,\ldots,x_n] & \xrightarrow{\psi_1} & S_1^{-1}B_R \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \varphi} \\ Q(R) & \hookrightarrow & Q(R)_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}[x_1,\ldots,x_n] & \xrightarrow{\psi_{1'}} & S_{1'}^{-1}B_{Q(R)} \end{array}$$

where ψ is the injection for the localization of R to the total ring fractions $Q(R)$, ψ' the injection determined by the isomorphism of Theorem 3.24 where $\psi'(ax^u) = \frac{a}{1}x^u$, and ψ_1, $\psi_{1'}$ injections determined by the localizations for B_R and $B_{Q(R)}$ respectively.

As $\psi'(S_1) \subseteq S_{1'}$, then $\psi_{1'}(\psi'(S_1)) \subseteq \psi_{1'}(S_{1'}) \subseteq \left(S_{1'}^{-1}B_{Q(R)}\right)^*$, therefore, by universal property there exists φ. If $f = \sum a_u x^u \in R_{\boldsymbol{q},\sigma}[x_1,\ldots,x_n]$ and $rx^v \in S_1$ then,

$$
\begin{aligned}
\varphi\left(\frac{f}{rx^v}\right) &= \varphi\left(\frac{\sum a_u x^u}{rx^v}\right)\\
&= \psi_{1'}(\psi'(rx^v))^{-1}\psi_{1'}\left(\psi'\left(\sum a_u x^u\right)\right)\\
&= \psi_{1'}\left(\frac{r}{1}x^v\right)^{-1}\psi_{1'}\left(\sum \frac{a_u}{1}x^u\right)\\
&= \frac{\frac{1}{1}}{\frac{r}{1}x^v}\frac{\sum \frac{a_u}{1}x^u}{\frac{1}{1}}\\
&= \frac{\sum \frac{a_u}{1}x^u}{\frac{r}{1}x^v}\\
&= \frac{\psi'(f)}{\psi'(rx^v)}.
\end{aligned}
$$

Also, φ is injective by ψ' and $\psi_{1'}$ are injective.

□

Need the following result for the subsequent theorem:

Proposition 3.26. *Let R be a ring and $S \subset R$ a multiplicative subset. If $Q := S^{-1}R$ exists, then any finite set $\{q_1, \ldots, q_n\}$ of elements of Q posses a common denominator, i.e., there exists $r_1, \ldots, r_n \in R$ and $s \in S$ such that $q_i = \frac{r_i}{s}, 1 \leq i \leq n$.*

Proof. See [13], Lemma 2.1.8. □

Theorem 3.27. *Let R be a left Ore domain, then $Q(Q^{n,n}_{\boldsymbol{q},\sigma}(R)) \cong Q(Q^{n,n}_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}(Q(R))$.*

Proof. With the notation of the proof in the Proposition 3.25 consider the following diagram of ring homomorphisms

$$
\begin{array}{ccc}
S_1^{-1}B_R & \xrightarrow{\psi_2} & Q(S_1^{-1}B_R)\\
\varphi\downarrow & & \downarrow\varphi'\\
S_{1'}^{-1}B_{Q(R)} & \xrightarrow{\psi_{2'}} & Q(S_{1'}^{-1}B_{Q(R)})
\end{array}
$$

where ψ_2, $\psi_{2'}$ are injections determined by the localizations of $S_1^{-1}B_R$ and $S_{1'}^{-1}B_{Q(R)}$ respectively and φ the injection of the Proposition 3.25.

By Remark 3.14, $S_1^{-1}B_R$ and $S_{1'}^{-1}B_{Q(R)}$ are domain, now, if $\frac{p_1}{q_1}, \frac{p_2}{q_2} \in S_1^{-1}B_R$ with $\frac{p_1}{q_1} \neq 0$, then $p_1 \neq 0$ and there exist $f_1 \neq 0$ and $f_2 \in B_R$ such that $f_1p_1 = f_2p_2$. Then, $\frac{f_1q_1}{1}\frac{p_1}{q_1} = \frac{f_1p_1}{1} = \frac{f_2q_2}{1} = \frac{f_2q_2}{1}\frac{p_2}{q_2} \neq 0$, therefore $S_1^{-1}B_R$ is a Ore domain, similarly it has to $S_{1'}^{-1}B_{Q(R)}$. Thus, if $S_2 = S_1^{-1}B_R - \{0\}$ and $S_{2'} = S_{1'}^{-1}B_{Q(R)} - \{0\}$ as $\varphi(S_2) \subseteq S_{2'}$, then

$\psi_{2'}(\varphi(S_2)) \subseteq \psi_{2'}(S_{2'}) \subseteq \left(Q(S_{1'}^{-1}B_{Q(R)})\right)^*$, hence, by universal property there exists φ' injective ring homomorphism.

Note that if $f, g \in B_R$ and $ax^u, bx^b \in S_1$, then

$$\frac{\frac{f}{ax^u}}{\frac{g}{bx^v}} = \left(\frac{g}{bx^v}\right)^{-1}\frac{f}{ax^u} = \frac{bx^v}{g}\frac{f}{ax^u} = \frac{f'}{g'}$$

and

$$\frac{f'}{g'} = \frac{1}{g'}\frac{f'}{1} = \left(\frac{g'}{1}\right)^{-1}\frac{f'}{1} = \frac{\frac{f'}{1}}{\frac{g'}{1}},$$

where $f', g' \in B_R$ by Remark 3.14 with $r = 0$. Similarly is obtained for $Q(S_{1'}^{-1}B_Q(R))$.

Therefore,

$$\begin{aligned}
\varphi'\left(\frac{f}{g}\right) &= \psi_{2'}\left(\varphi\left(\frac{g}{1}\right)\right)^{-1}\psi_{2'}\left(\varphi\left(\frac{f}{1}\right)\right)\\
&= \psi_{2'}\left(\frac{\psi'(g)}{\frac{1}{1}}\right)^{-1}\psi_{2'}\left(\frac{\psi'(f)}{\frac{1}{1}}\right)\\
&= \frac{\frac{1}{1}}{\psi'(g)}\frac{\psi'(f)}{\frac{1}{1}}\\
&= \frac{\psi'(f)}{\psi'(g)}.
\end{aligned}$$

Now, if $f, 0 \neq g \in S'_{1'}B_{Q(R)}$, applying Theorem 3.26 must be

$$\begin{aligned}
\frac{f}{g} &= \frac{\sum \frac{a_u}{b_u}x^u}{\sum \frac{c_v}{d_v}x^v} = \frac{\frac{1}{s}\sum \frac{a'_u}{1}x^u}{\frac{1}{s'}\sum \frac{c'_v}{1}x^v} = \left(\sum \frac{c'_v}{1}x^v\right)^{-1}\left(\frac{1}{s'}\right)^{-1}\frac{1}{s}\sum \frac{a'_u}{1}x^u\\
&= \left(\sum \frac{c'_v}{1}x^v\right)^{-1}\left(\frac{s'}{1}\frac{1}{s}\right)\sum \frac{a'_u}{1}x^u = \left(\sum \frac{c'_v}{1}x^v\right)^{-1}\left(\frac{r'}{r}\right)\sum \frac{a'_u}{1}x^u\\
&= \left(\sum \frac{c'_v}{1}x^v\right)^{-1}\left(\frac{1}{r}\frac{r'}{1}\right)\sum \frac{a'_u}{1}x^u = \left(\frac{r}{1}\sum \frac{c'_v}{1}x^v\right)^{-1}\left(\frac{r'}{1}\sum \frac{a'_u}{1}x^u\right)\\
&= \left(\sum \frac{rc'_v}{1}x^v\right)^{-1}\left(\sum \frac{r'a'_u}{1}x^u\right)\\
&= \frac{\sum \frac{r'a'_u}{1}x^u}{\sum \frac{rc'_v}{1}x^v} = \frac{\psi'(f')}{\psi'(g')}\\
&= \varphi\left(\frac{f'}{g'}\right).
\end{aligned}$$

where $f' = \sum(r'a'_u)x^u$ y $g' = \sum(rc'_v)x^v$, then φ is surjective. Hence $Q(Q^{n,n}_{\boldsymbol{q},\sigma}(R)) \cong Q(Q^{n,n}_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}(Q(R)))$.

□

3.3 Valuations of skew quantum polynomials.

Theorem 3.28. *Let R be a left Ore domain and $\nu : Q(Q^{n,n}_{\boldsymbol{q},\sigma}(R))^* \to \Gamma$ is a valuation with $\nu(Q(R)^*) = 0$, then Γ is Abelian.*

Proof. $Q(R)$ is a division ring and $Q(Q^{n,n}_{\boldsymbol{q},\sigma}(R)) \cong Q(Q^{n,n}_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}(Q(R)))$, by Theorem 1.10. Γ is Abelian. □

Corollary 3.29. *Let R be a left Ore domain, $\nu : Q(Q^{n,n}_{\boldsymbol{q},\sigma}(R))^* \to \Gamma$ a valuation with $\nu(Q(R)^*) = 0$ and $Q^{n,n}_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}(Q(R))$ generic, then Γ is Abelian.*

Theorem 3.30. *Let R be a left Ore domain, a valuation $\nu : Q(Q^{n,n}_{\boldsymbol{q},\sigma}(R))^* \to \Gamma$ with $\nu(Q(R)^*) = 0$ and $Q^{n,n}_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}(Q(R))$ generic. The valuation ν has maximal rank if only if $\Gamma \cong \mathbb{Z}^n$.*

Proof. By Theorem 3.27. $Q(Q^{n,n}_{\boldsymbol{q},\sigma}(R)) \cong Q(Q^{n,n}_{\widetilde{\boldsymbol{q}},\widetilde{\sigma}}(Q(R)))$ with $Q(R)$ a division ring, by Theorem 1.13 the valuation ν has maximal rank if only if $\Gamma \cong \mathbb{Z}^n$. □

3.4 Valuations of skew PBW extension.

Theorem 3.31. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a bijective and quasi-commutative skew PBW extension of a ring R. If R is a left Ore domain and $\nu : Q(A)^* \to \Gamma$ a valuation with $\nu(Q(R)^*) = 0$, then Γ is Abelian*

Proof. By Theorem 3.8 A is an Ore domain then, $Q(A)$ exists and is a division ring, by Remark 3.14. $Q(A) \cong Q(Q^{r,n}_{\boldsymbol{q},\sigma}(R))$ (in particular $r = 0$) and by Theorem 3.28 Γ is abelian. □

Corollary 3.32. *Let A be a bijective skew PBW extension of a ring R. If R is a left Ore domain and $\nu : Q(Gr(A))^* \to \Gamma$ a valuation with $\nu(Q(R)^*) = 0$, then Γ is Abelian.*

Proof. By Theorem 3.9 $Gr(A)$ is bijective and quasi-commutative. □

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