

Energy Integrals and Metric Embedding Theory

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For some centrally symmetric convex bodies $K \subset \mathbb{R}^n$, we study the energy integral

$$\sup \int_K \int_K \|x - y\|_r^p d\mu(x) d\mu(y),$$

where the supremum runs over all finite signed Borel measures μ on K of total mass one. In the case where $K = B_q^n$, the unit ball of ℓ_q^n (for $1 < q \leq 2$) or an ellipsoid, we obtain the exact value or the correct asymptotical behavior of the supremum of these integrals. We apply these results to a classical embedding problem in metric geometry. We consider in \mathbb{R}^n the Euclidean distance d_2 . For $0 < \alpha < 1$, we estimate the minimum R for which the snowflaked metric space (K, d_2^α) may be isometrically embedded on the surface of a Hilbert sphere of radius R .

1 Introduction

The study of integrals of the form

$$\int_K \int_K f(x, y) d\mu(x) d\mu(y),$$

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where μ is a Borel measure supported on a compact set $K \subset \mathbb{R}^n$ was mainly motivated by problems in potential theory and geometric measure theory. Usually the function f is given by $f(x, y) = \|x - y\|_2^p$, where p is a real number whose value (or range of values) depends on the problem under study. For example, if $p = 2 - n$ we have the classical Newtonian kernel. Research on this type of integrals for other kernel functions (among them, the case where the exponent p is positive) has its origin in the works of Pólya and Szegő [20] and Schur [24] and have attracted many authors since then (see [2–5, 8, 11–13, 15–18, 21, 25], and the references therein). Besides contributing to the classical potential theory, this study has allowed substantial progress in other areas such as discrepancy theory, metric inequalities and distance sums among others. This work deals with this kind of integrals and its applications to metric geometry and embedding theory.

We denote by $\mathfrak{M}^1(K)$ the set of all finite signed Borel measures on the compact set $K \subset \mathbb{R}^n$ of total mass one. Given a continuous function $f: K \times K \rightarrow \mathbb{R}$ and $\mu \in \mathfrak{M}^1(K)$, the p -energy integral of K given by μ is defined by

$$I_p(\mu, K, f) := \int_K \int_K f(x, y)^p d\mu(x) d\mu(y).$$

The p -maximal energy of K is given by

$$M_p(K, f) = \sup\{I_p(\mu, K, f) : \mu \in \mathfrak{M}^1(K)\}.$$

In this note, we study energy integrals induced by ℓ_r -norms, that is, we consider the distance functions $d_r(x, y) := \|x - y\|_r$. We focus our study on the case where K is a convex and balanced body. Several estimates are obtained for $K = B_q^n$, the unit ball of ℓ_q^n , or in the case that K is an ellipsoid in \mathbb{R}^n .

Alexander and Stolarsky [5], applying geometric arguments, computed the exact value of $M_1(B_2^1, d_2) = M_1([-1, 1], d_2)$. A few years later, Alexander obtained the value of $M_1(B_2^3, d_2)$ in [2] using Archimedes' beautiful theorem on zonal areas (the usually called Hat-Box theorem). But it was Hinrichs et al. who took the big step: they managed to construct a sequence of measures on the ball B_2^n whose marginals w^* -converge to the measure that maximizes the energy integral in the 1-dimensional case. They used this to calculate the precise value of $M_1(B_2^n, d_2)$ for every n . Namely, they showed in [12, Theorem 2.1] that

$$M_1(B_2^n, d_2) = \frac{\pi^{1/2} \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}. \quad (1)$$

We go further in this line, computing or estimating $M_p(K, d_r)$ for different subsets $K \subset \mathbb{R}^n$, different values of p (as treated by Alexander and Stolarsky in [5]), and different

values of r . For our purposes, we use several tools and techniques from functional analysis and Banach space theory, such as the theory of stable measures and p -summing operators.

Let us first introduce the following notation:

$$m_p := M_p(B_2^1, d_2) = M_p([-1, 1], d_2) = \sup\{I_p(\mu, [-1, 1], d_2) : \mu \in \mathfrak{M}^1([-1, 1])\}. \tag{2}$$

For the Euclidean ball, we show the following formula (see Corollary 2.2):

$$M_p(B_2^n, d_2) = m_p \frac{\pi^{1/2} \Gamma(\frac{n+p}{2})}{\Gamma(\frac{p+1}{2})\Gamma(\frac{n}{2})}. \tag{3}$$

In fact, this will be a particular case of a result for ellipsoids in \mathbb{R}^n given in Theorem 2.1. Note that, since $m_1 = 1$ (see [5, Lemma 3.5]), this formula recovers (1). Also, we show in Theorem 2.6 that $M_p(B_2^n, d_r)$ behaves asymptotically as $n^{p/r}$ for $r \in [1, 2)$ and $0 < p < r$.

We consider convex and balanced bodies $K \subset \mathbb{R}^n$ as well. These sets can be seen as the unit ball of \mathbb{R}^n with some norm. In Proposition 2.9, we relate the value of $M_p(K, d_2)$ with a geometric property of K . Loosely speaking, let $W_t(K)$ be the width of K in the direction t (i.e., the distance between the supporting hyperplanes of K orthogonal to t). Then, $M_p(K, d_2)$ can be controlled in terms of the average value of $W_t(K)^p$. If K is the unit ball of an n -dimensional real Banach space $E = (\mathbb{R}^n, \|\cdot\|_E)$, we give in Proposition 2.10 a lower bound for the maximal p -energy $M_p(B_E, d_2)$ which is related with the 2-summing norm of the identity operator from E to ℓ_2^n . This bound is obtained by calculating the supremum of the maximal energies over all the ellipsoids contained in K . For $K = B_q^n$ and $1 < q \leq 2$ we obtain, in Theorem 2.11, that $M_p(B_q^n, d_2)$ behaves asymptotically as $n^{p/q'}$, where $\frac{1}{q} + \frac{1}{q'} = 1$.

Let $K \subset \mathbb{R}^n$ be a compact set. A classical result in metric geometry due to Schoenberg [22] asserts that, for $0 < \alpha < 1$, the snowflaked metric space (K, d_2^α) can be isometrically embedded in the surface of a Hilbert sphere (see Theorem 2.12 for details). Stated in another way, for every compact set $K \subset \mathbb{R}^n$, there exist a number R and a mapping

$$j : (K, d_2^\alpha) \rightarrow (R \cdot S_{\ell_2}, \|\cdot\|_{\ell_2}) \tag{4}$$

that preserves distances. Therefore, it is natural to ask for the least possible R for which there exists a distance preserving mapping j as in (4) (colloquially known as Schoenberg’s radius for the metric space (K, d_2^α)). It was Alexander and Stolarsky [5] who

connected this problem, which is essentially a question in metric geometry, with the classical potential theory. They proved that the least possible radius can be computed from the 2α -maximal energy $M_{2\alpha}(K, \mathcal{d}_2)$ (see also Theorem 2.12).

We use this close connection and our results on energy integrals to compute or estimate these minimum radii for different compact sets K in \mathbb{R}^n . As a consequence, we show in Theorem 2.14 that, for $1 < q \leq 2$ and $0 < \alpha < 1$, the minimum R for which the metric space $(B_q^n, \mathcal{d}_2^\alpha)$ may be isometrically embedded on the surface of a Hilbert sphere of radius R behaves asymptotically as $n^{\alpha/q'}$, where $\frac{1}{q} + \frac{1}{q'} = 1$.

2 Energy Integrals: The Results

2.1 Energy integrals induced by the Euclidean distance

In this section, we prove formula (3). In fact, we will prove a more general result for ellipsoids, for which (3) turns out to be a particular case. An ellipsoid in \mathbb{R}^n is the image of the Euclidean ball by a nondegenerate linear operator. The following theorem gives the value of $M_p(\mathcal{E}, \mathcal{d}_2)$ for \mathcal{E} an ellipsoid. Its proof makes use of some properties of absolutely summing operators. We refer the reader to [9, 10, 19, 27] for definitions and a complete treatment of this subject.

It is important to mention that, for $p \geq 2$, we have $m_p = +\infty$. Indeed, if μ_a is the measure supported on the points $-1, 0$, and 1 with weights $a, 1 - 2a$, and a , respectively, we have $I_p(\mu_a, [-1, 1], \mathcal{d}_2) \rightarrow +\infty$ as $a \rightarrow +\infty$. And, of course, also $M_p(K, \mathcal{d}_2) = +\infty$ for every centrally symmetric convex body. Therefore, we state all our results for the range $0 < p < 2$.

Theorem 2.1. Let $T \in \mathcal{L}(\ell_2^n, \ell_2^n)$ be a bounded linear operator. For $0 < p < 2$, we have

$$M_p(T(B_2^n), \mathcal{d}_2) = m_p \pi_p(T)^p,$$

where $\pi_p(T)$ stands for the absolutely p -summing norm of the operator T . □

For $I \in \mathcal{L}(\ell_2^n, \ell_2^n)$ the identity operator we have, by [9, Theorem 11.10, Exercise 11.24], the equality

$$\pi_p(I)^p = \frac{\pi^{1/2} \Gamma(\frac{n+p}{2})}{\Gamma(\frac{p+1}{2}) \Gamma(\frac{n}{2})}.$$

Using this and Stirling's formula [26] we obtain the following result.

Corollary 2.2. Given $n \in \mathbb{N}$, and $0 < p < 2$, then

$$M_p(B_2^n, d_2) = m_p \frac{\pi^{1/2} \Gamma(\frac{n+p}{2})}{\Gamma(\frac{p+1}{2}) \Gamma(\frac{n}{2})} = n^{p/2} \left(\frac{m_p \pi^{1/2}}{\Gamma(\frac{p+1}{2})} + o(1) \right). \quad \square$$

We remark that we can obtain a direct and rather self-contained proof of Corollary 2.2, which makes no use of absolutely summing operators. Indeed, we can proceed as in the proof of Theorem 2.1 and use (10) instead of Lemma 2.3 and obtain the corollary.

Some comments are in order. The absolutely p -summing norms for operators on Hilbert spaces are all equivalent to the Hilbert–Schmidt norm, with constants depending only on p (as can be deduced from [10, Corollary 3.16]). Therefore, there exist positive constants A_p, B_p such that, for every n and every ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ with orthogonal axes and radii a_1, \dots, a_n , we have

$$A_p(a_1^2 + \dots + a_n^2)^{p/2} \leq M_p(\mathcal{E}, d_2) \leq B_p(a_1^2 + \dots + a_n^2)^{p/2}. \quad (5)$$

We write λ for the normalized surface measure on the sphere S^{n-1} . This is an abuse of notation, since we have a different measure for each n , but there is no risk of confusion. Let $c_p^{(n)}$ be the n -dimensional p th absolute moment defined as follows:

$$c_p^{(n)} = \left(\int_{S^{n-1}} |t_1|^p d\lambda(t) \right)^{1/p}.$$

Although it can easily be obtained by passing to spherical coordinates, we do not need the explicit value of $c_p^{(n)}$ for our purposes. To prove the theorem, we need the following result, which extends (10) and can be found in [27, Lemma 10.5].

Lemma 2.3. Let n be a positive integer, and $T \in \mathcal{L}(\ell_2^n, \ell_2^n)$. For $0 < p < \infty$ we have

$$\|Tx\|_2^p = (c_p^{(n)})^{-p} \int_{S^{n-1}} |\langle x, t \rangle|^p d\nu(t),$$

where ν is the measure given by

$$\int_{S^{n-1}} f(x) d\nu(x) = \int_{S^{n-1}} f\left(\frac{t}{\|Tt\|_2}\right) \|Tt\|_2^p d\lambda(t),$$

for every continuous function on S^{n-1} . □

Note that if ν is the measure defined in the previous lemma, by [27, Proposition 10.4] we have

$$\nu(S^{n-1}) = \int_{S^{n-1}} \|Tt\|_2^p d\lambda(t) = (c_p^{(n)})^p \pi_p(T)^p. \quad (6)$$

Two technical lemmas are also needed. First, let us say that an atomic measure ν on $[-1, 1]$ has *symmetric support* if it is supported in a set of points $-1 \leq p_1 < p_2 < \dots < p_{N-1} < p_N \leq 1$ with $p_j = -p_{N+1-j}$ for every $j = 1, \dots, N$. If moreover $\nu([-1, 1]) = 1$ and $\nu(p_j) = \nu(p_{N+1-j})$ for every $j = 1, \dots, N$ we say that ν is *1-balanced*. Such a measure can be written as $\nu = \sum_{j=1}^N \lambda_j \delta_{p_j}$ on $[-1, 1]$ with $\sum_{j=1}^N \lambda_j = 1$, $-1 \leq p_1 < p_2 < \dots < p_{N-1} < p_N \leq 1$ as above and $\lambda_j = \lambda_{N+1-j}$ for every $j = 1, \dots, N$.

The following lemma shows that 1-balanced measures are enough to compute m_p .

Lemma 2.4. For m_p as in (2), we have

$$m_p = \sup \int_{[-1,1]} \int_{[-1,1]} |u - v|^p d\nu(u) d\nu(v),$$

where the supremum runs over all 1-balanced atomic measures $\nu \in [-1, 1]$. □

Proof. As in [5, Lemma 3.3], it is easy to see that $m_p = \sup I_p(\nu, [-1, 1], \mathcal{d}_2)$, where the supremum runs over all atomic measures $\nu \in [-1, 1]$ of total mass one. Moreover, the supremum can be taken within all atomic measures with symmetric support (adding points with weight zero if necessary).

Thus, to prove the lemma it is enough to show that, among all the measures of total mass one with support in a given symmetric set $\{p_1, \dots, p_N\}$, $I_p(\cdot, [-1, 1], \mathcal{d}_2)$ attains its maximum at a 1-balanced one. Define ϕ the quadratic form given by:

$$\phi(x_1, \dots, x_N) = \sum_{i,j} x_i x_j \mathcal{d}_2(p_i, p_j)^p = \sum_{i,j} x_i x_j \|p_i - p_j\|_2^p,$$

and observe that, if ν is the measure given by $\sum_{j=1}^N \lambda_j \delta_{p_j}$, we have $I_p(\nu, [-1, 1], \mathcal{d}_2) = \phi(\lambda_1, \dots, \lambda_N) = \sum_{i,j} \lambda_i \lambda_j \|p_i - p_j\|_2^p$. By [5, Theorem 3.3], the quadratic form ϕ achieves a unique absolute maximum on the affine hyperplane $\sum_{j=1}^N x_j = 1$. Since the set $\{p_1, \dots, p_N\}$ is symmetric, it is easy to check that $\phi(\lambda_1, \dots, \lambda_N) = \phi(\lambda_N, \dots, \lambda_1)$ for all λ . This, together with the uniqueness of the maximum shows that the measure ν_0 maximizing $I_p(\cdot, [-1, 1], \mathcal{d}_2)$ must be balanced. ■

We do not have a closed formula for m_p . However, it is easy to obtain numerical estimations of this constant. Indeed, by the previous lemma we can compute m_p as the supremum of some atomic measures. Thus, for each finite set of points $\{p_1, \dots, p_N\} \subset [-1, 1]$, the maximum of the quadratic form ϕ in the lemma can be calculated by solving a linear system of equations (see [5, Theorem 3.3]).

We now define, for $t \in S^{n-1}$, the projection $\Pi_t: B_2^n \rightarrow [-1, 1]$ by

$$\Pi_t(x) = \langle x, t \rangle. \quad (7)$$

As in [12], we show how to relate our n -dimensional problem to the 1-dimensional case. We emphasize that the construction of the sequence of measures (see Lemma 2.5) draws heavily on the clever results of Hinrichs et al. [12].

Lemma 2.5. Let e_1 be the canonical unit vector $(1, 0, \dots, 0) \in \mathbb{R}^n$ and ν be a 1-balanced atomic measure on $[-1, 1]$. There exists a sequence of rotation invariant measures $(\eta_k)_{k \geq 1}$ in $\mathfrak{M}_1(B_2^n)$ such that the sequence of projected measures $\bar{\eta}_k := \eta_k \Pi_{e_1}^{-1} \xrightarrow{w^*} \nu$. \square

Proof. Fix any 1-balanced atomic measure $\nu = \sum_{j=1}^N \lambda_j \delta_{p_j}$ on $[-1, 1]$. By the [12, proof of Lemmas 2.8 and 2.9] (identifying the segment $[-1, 1]$ with the diameter D_{e_1} of the ball in the direction e_1) there exists a sequence of rotation invariant measures $(\mu_k)_k$ such that the sequence of projected measures $\bar{\mu}_k \xrightarrow{w^*} \frac{1}{2}(\delta_{-1} + \delta_1)$. For $j = 1, \dots, N$, set $\rho_j := |p_j|$. If $\rho_j > 0$ we define $\mu_k^j \in \mathfrak{M}_1(B_2^n)$ the measure supported in $\rho_j B_2^n$ as:

$$\mu_k^j(A) := \mu_k \left(\frac{1}{\rho_j} A \right),$$

for every Borel set $A \in \rho_j B_2^n$. On the other hand, if $p_j = 0$ we define $\mu_k^j = \delta_{p_j} = \delta_0$. Observe now that, for every index $j = 1, \dots, N$, the measure $\lambda_j \mu_k^j$ is rotation invariant and of total mass λ_j . Moreover, using that $p_j = -p_{N+1-j}$ (the support of ν is symmetric) and the fact that $\lambda_j = \lambda_{N+1-j}$ (ν is 1-balanced) we obtain that the sequence of projected measures

$$\lambda_j \bar{\mu}_k^j \xrightarrow{w^*} \frac{1}{2} (\lambda_j \delta_{p_j} + \lambda_j \delta_{p_{N+1-j}}) = \frac{1}{2} (\lambda_j \delta_{p_j} + \lambda_{N+1-j} \delta_{p_{N+1-j}}).$$

If we set $\eta_k := \sum_{j=1}^N \mu_k^j$ we have that η_k is a rotation invariant measure in $\mathfrak{M}_1(B_2^n)$ and

$$\bar{\eta}_k \xrightarrow{w^*} \sum_{j=1}^N \frac{1}{2} (\lambda_j \delta_{p_j} + \lambda_{N+1-j} \delta_{p_{N+1-j}}) = \nu. \quad \blacksquare$$

Now we are ready to prove our first theorem.

Proof of Theorem 2.1. First observe that

$$M_p(T(B_2^n), d_2) = \sup \int_{B_2^n} \int_{B_2^n} \|Tx - Ty\|_2^p d\mu(x) d\mu(y),$$

where the supremum runs over all finite signed Borel measures μ on B_2^n of total mass one. Fix $\mu \in \mathfrak{M}_1(B_2^n)$ and set

$$I_p(\mu; T) := \int_{B_2^n} \int_{B_2^n} \|Tx - Ty\|_2^p d\mu(x) d\mu(y).$$

By Lemma 2.3, we have

$$\begin{aligned} I_p(\mu; T) &= \int_{B_2^n} \int_{B_2^n} (c_p^{(n)})^{-p} \int_{S^{n-1}} |\langle x - y, t \rangle|^p dv(t) d\mu(x) d\mu(y) \\ &= (c_p^{(n)})^{-p} \int_{S^{n-1}} \left[\int_{B_2^n} \int_{B_2^n} |\langle x - y, t \rangle|^p d\mu(x) d\mu(y) \right] dv(t). \end{aligned}$$

Now we use the notation introduced in (7) to get

$$\begin{aligned} I_p(\mu; T) &= (c_p^{(n)})^{-p} \int_{S^{n-1}} \left[\int_{B_2^n} \int_{B_2^n} |\Pi_t(x) - \Pi_t(y)|^p d\mu(x) d\mu(y) \right] dv(t) \\ &= (c_p^{(n)})^{-p} \int_{S^{n-1}} \left[\int_{-1}^1 \int_{-1}^1 |u - v|^p d\mu \Pi_t^{-1}(u) d\mu \Pi_t^{-1}(v) \right] dv(t) \\ &= (c_p^{(n)})^{-p} \int_{S^{n-1}} I_p(\mu \Pi_t^{-1}, [-1, 1], d_2) dv(t). \end{aligned}$$

Note that $\mu \Pi_t^{-1}$ is also a finite signed Borel measure of total mass one on $[-1, 1]$. Then, we have $I_p(\mu \Pi_t^{-1}, [-1, 1], d_2) \leq m_p$ and

$$\begin{aligned} I_p(\mu; T) &\leq (c_p^{(n)})^{-p} \int_{S^{n-1}} m_p dv(t) = m_p (c_p^{(n)})^{-p} v(S^{n-1}) \\ &= m_p \pi_p(T)^p, \end{aligned}$$

where the last equality follows by (6). This gives $M(T(B_2^n), d_2) \leq m_p \pi_p(T)^p$.

Let us show the reverse inequality. By standard manipulations it is easy to see that if μ is any rotation invariant measure, then, for every $t \in S^{n-1}$, we have $I_p(\mu \Pi_t^{-1}, [-1, 1], d_2) = I_p(\mu \Pi_{e_1}^{-1}, [-1, 1], d_2)$, where $e_1 = (1, 0, \dots, 0)$. Then,

$$\begin{aligned} I_p(\mu; T) &= (c_p^{(n)})^{-p} \int_{S^{n-1}} I_p(\mu \Pi_t^{-1}, d_2) dv(t) = (c_p^{(n)})^{-p} I_p(\mu \Pi_{e_1}^{-1}, d_2) v(S^{n-1}) \\ &= I_p(\mu \Pi_{e_1}^{-1}, [-1, 1], d_2) \pi_p(T)^p, \end{aligned} \tag{8}$$

for any rotation invariant measure μ . Given $\varepsilon > 0$, by Lemma 2.4 there is a 1-balanced measure ζ on $[-1, 1]$ such that $I_p(\zeta, [-1, 1], \mathcal{d}_2) \geq (1 - \varepsilon)m_p$. With the help of Lemma 2.5, we can take a sequence $(\eta_k)_{k \geq 1}$ of rotation invariant measures on B_2^n such that $\eta_k \Pi_{e_1}^{-1} \xrightarrow{w^*} \zeta$. As in [15, Corollary 2.7], it is easy to see that $I_p(\cdot, [-1, 1], \mathcal{d}_2)$ is w^* -sequentially continuous on the set of all Borel measures on D_{e_1} . Hence,

$$M_p(T(B_2^n), \mathcal{d}_2) \geq I_p(\eta_k; T) = I_p(\eta_k \Pi_{e_1}^{-1}, [-1, 1], \mathcal{d}_2) \pi_p(T)^p$$

and the latter tends to $I_p(\zeta, [-1, 1], \mathcal{d}_2) \pi_p(T)^p \geq (1 - \varepsilon)m_p \pi_p(T)^p$. Since ε is arbitrary, we have shown that $M_p(T(B_2^n), \mathcal{d}_2) = m_p \pi_p(T)^p$. ■

2.2 Energy integrals induced by the ℓ_r -norm ($1 \leq r < 2$)

Now, we deal with the estimates of the p -maximal energy of the Euclidean ball induced by the distance functions $d_r(x, y) = \|x - y\|_r$ for $r \in [1, 2)$. We do not analyze the case where $r > 2$ since in this case, (\mathbb{R}^n, d_r) is not a quasihypermetric space (see [15] and the references therein) and then the corresponding energy integrals are not uniformly bounded. Our goal is to prove the following asymptotic behavior of $M_p(B_2^n, d_r)$. Note that for $r = 2$ this is contained in Corollary 2.2.

Theorem 2.6. Let $r \in [1, 2]$ and $p \in (0, r)$, then $M_p(B_2^n, d_r)$ behaves asymptotically as $n^{p/r}$. □

Now we recall the basic property of stable measures (see, e.g., [1, Theorems 6.4.15–6.4.18; 9, Section 24; 19, Lemma 21.1.3]). For any $n \in \mathbb{N}$ and $r \in [1, 2)$, there exists a measure m_r^n (called the r -stable measure) defined on the Borel sets of \mathbb{R}^n such that

$$\|x\|_r^p = c_{r,p}^{-p} \int_{\mathbb{R}^n} |\langle x, w \rangle|^p dm_r^n(w) \tag{9}$$

for all $p \in (0, r)$. Here $c_{r,p}$ is the p th moment of the 1-dimensional stable measure m_r^1 , namely,

$$c_{r,p} = \left(\int_{\mathbb{R}} |w|^p dm_r^1(w) \right)^{1/p}.$$

The exact value of $c_{r,p}$ can be found, for example, in [19, 21.1.2].

For $r = 2$, the measure m_r^n is just the n -dimensional Gaussian measure γ_n . For this measure, we actually have for $x \in \mathbb{R}^n$ and every $0 < p < \infty$,

$$\|x\|_2^p = c_{2,p}^{-p} \int_{\mathbb{R}^n} |\langle x, t \rangle|^p d\gamma_n(t) = b_p^{(n)} \int_{S^{n-1}} |\langle x, t \rangle|^p d\lambda(t) \tag{10}$$

and

$$c_{2,p} = \left(\int_{\mathbb{R}} |w|^p d\gamma_1(w) \right)^{1/p} = 2 \left(\frac{\Gamma(\frac{1+p}{2})}{\Gamma(\frac{1}{2})} \right)^{1/p}; \quad b_p^{(n)} = \frac{\pi^{1/2} \Gamma(\frac{n+p}{2})}{\Gamma(\frac{p+1}{2}) \Gamma(\frac{n}{2})}.$$

The first equality in (10) is just the stability property of Gaussian measures. The second equality follows using spherical coordinates. Note that the equality between the first and the third expressions is just Lemma 2.3 applied to the identity operator in \mathbb{R}^n .

Using r -stable measures it is possible to obtain upper bounds for $M_p(B_E, d_r)$. Note that, in the 1-dimensional case, we have $d_r = d_2$. Therefore, the energies induced on $[-1, 1]$ by all these distance functions obviously agree.

Lemma 2.7. Let E be an n -dimensional Banach space. If $r \in [1, 2]$ and $0 < p < r$, then

$$M_p(B_E, d_r) \leq m_p c_{r,p}^{-p} \int_{\mathbb{R}^n} \|t\|_{E'}^p dm_r^n(t),$$

where m_r^n is the n -dimensional r -stable measure. □

Proof. Let us apply (9) to compute the p -maximal energy of B_E induced by d_r .

$$\begin{aligned} M_p(B_E, d_r) &= \sup_{\mu \in \mathfrak{M}^1(B_E)} \int_{B_E} \int_{B_E} \|x - y\|_r^p d\mu(x) d\mu(y) \\ &= \sup_{\mu \in \mathfrak{M}^1(B_E)} \int_{B_E} \int_{B_E} \left[c_{r,p}^{-p} \int_{\mathbb{R}^n} |(x - y, t)|^p dm_r^n(t) \right] d\mu(x) d\mu(y) \\ &= \sup_{\mu \in \mathfrak{M}^1(B_E)} c_{r,p}^{-p} \int_{\mathbb{R}^n} \left[\int_{B_E} \int_{B_E} |(x - y, t)|^p d\mu(x) d\mu(y) \right] dm_r^n(t) \\ &= \sup_{\mu \in \mathfrak{M}^1(B_E)} c_{r,p}^{-p} \int_{\mathbb{R}^n} \|t\|_{E'}^p \left[\int_{B_E} \int_{B_E} \left\langle x - y, \frac{t}{\|t\|_{E'}} \right\rangle^p d\mu(x) d\mu(y) \right] dm_r^n(t) \\ &= \sup_{\mu \in \mathfrak{M}^1(B_E)} c_{r,p}^{-p} \int_{\mathbb{R}^n} \|t\|_{E'}^p \left[\int_{-1}^1 \int_{-1}^1 |u - v|^p d\mu_{t/\|t\|_{E'}}(u) d\mu_{t/\|t\|_{E'}}(v) \right] dm_r^n(t) \\ &\leq m_p c_{r,p}^{-p} \int_{\mathbb{R}^n} \|t\|_{E'}^p dm_r^n(t). \quad \blacksquare \end{aligned}$$

We can also use (10) to derive an upper bound of $M_p(B_E, d_2)$ using the average over the unit sphere. We will return later to this point because it is possible to obtain some geometric properties of K related with its average width.

We set some useful notation first. Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be sequences of non-negative numbers. If there are positive constants A and B such that $Ab_n \leq a_n \leq Bb_n$

for every n , we write $(a_n)_{n \in \mathbb{N}} \asymp (b_n)_{n \in \mathbb{N}}$. On the other hand, we write $(a_n)_{n \in \mathbb{N}} \preccurlyeq (b_n)_{n \in \mathbb{N}}$ or $(b_n)_{n \in \mathbb{N}} \succcurlyeq (a_n)_{n \in \mathbb{N}}$, in the case there is a positive constant C such that $a_n \leq C b_n$ for every natural number n .

In the following lemma, we present the asymptotic behavior of the average of powers of ℓ_r -norms on the unit sphere.

Lemma 2.8. Given $n \in \mathbb{N}$, $p > 0$, and $1 \leq r < \infty$, we have

$$\int_{S^{n-1}} \|t\|_r^p \, d\lambda(t) \asymp n^{(\frac{1}{r}-\frac{1}{2})p}. \quad \square$$

Proof. We define $\varphi_r^n : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\varphi_r^n(p) = \int_{S^{n-1}} \left[n^{(\frac{1}{2}-\frac{1}{r})} \|t\|_r \right]^p \, d\lambda(t).$$

Then, what we have to prove is that $\varphi_r^n(p) \asymp 1$.

Standard computations show that

$$\int_{\mathbb{R}^n} \|x\|_r^r \, d\gamma_n(x) = \sum_{i=1}^n \int_{\mathbb{R}^n} |x_i|^r \, d\gamma_n(x) = n c_{2,r}^r.$$

Therefore, using spherical coordinates and Stirling's formula we obtain

$$\int_{S^{n-1}} \|t\|_r^r \, d\lambda(t) \asymp n^{1-\frac{r}{2}}$$

or, equivalently,

$$\varphi_r^n(r) = \int_{S^{n-1}} [n^{\frac{1}{2}-\frac{1}{r}} \|t\|_r]^r \, d\lambda(t) \asymp 1.$$

This gives the desired result for the particular case $p=r$.

We now consider $2 \leq r < \infty$. In this case, for every $t \in S^{n-1}$ we have

$$1 = \|t\|_2 \leq n^{(\frac{1}{2}-\frac{1}{r})} \|t\|_r,$$

which gives the lower bound $\varphi_r^n(p) \geq 1$ for every $p > 0$ and shows that φ_r^n is an increasing function of p . As a consequence, for $0 < p \leq r$ we have $1 \leq \varphi(p) \leq \varphi(r) \preccurlyeq 1$.

For $p > r$, note that

$$\|t\|_r \leq n^{(\frac{1}{r}-\frac{1}{p})} \|t\|_p$$

and then

$$[n^{(\frac{1}{2}-\frac{1}{r})} \|t\|_r]^p \leq [n^{(\frac{1}{2}-\frac{1}{r})} n^{(\frac{1}{r}-\frac{1}{p})} \|t\|_p]^p = [n^{(\frac{1}{2}-\frac{1}{p})} \|t\|_p]^p.$$

Therefore,

$$1 \leq \varphi_r^n(p) = \int_{S^{n-1}} [n^{(\frac{1}{2}-\frac{1}{r})} \|t\|_r]^p d\lambda(t) \leq \int_{S^{n-1}} [n^{(\frac{1}{2}-\frac{1}{p})} \|t\|_{\ell_p}]^p d\lambda(t) \asymp 1,$$

which concludes the proof for $2 \leq r < \infty$.

Suppose now that $1 \leq r < 2$. Then, for every $t \in S^{n-1}$ we have

$$n^{1/2-1/r} \|t\|_r \leq 1 \tag{11}$$

and thus $\varphi_r^n(p) \leq 1$ for every $p > 0$. For the reverse inequality, consider first $0 < p < 1$. Since φ_r^n is a decreasing function and $r \geq 1$, we get $1 \asymp \varphi_r^n(r) \leq \varphi_r^n(p)$ as above. For $p \geq 1$, using spherical coordinates and Hölder inequalities we obtain

$$\begin{aligned} \left(\int_{S^{n-1}} \|t\|_r^p d\lambda(t) \right)^{1/p} &\geq \int_{S^{n-1}} \|t\|_r d\lambda(t) \asymp \frac{1}{\sqrt{n}} \int_{\mathbb{R}^n} \|x\|_r d\gamma_n(x) \\ &\geq \frac{1}{\sqrt{n}} \left(\sum_{i=1}^n \left| \int_{\mathbb{R}^n} |x_i| d\gamma_n(x) \right|^r \right)^{1/r} \asymp n^{1/r-1/2}. \end{aligned}$$

This shows that $\varphi_r^n(p) \asymp 1$ and ends the proof. ■

Now we can prove the main result of this section.

Proof of Theorem 2.6. We can apply Lemma 2.7 in the particular case where E is the n -dimensional Euclidean space. Then,

$$M_p(B_2^n, d_r) \leq m_p c_{r,p}^{-p} \int_{\mathbb{R}^n} \|t\|_2^p dm_r^n(t).$$

We can estimate this last integral writing the ℓ_2 -norm as an average on the unit sphere, so

$$\begin{aligned} M_p(B_2^n, d_r) &\leq m_p c_{r,p}^{-p} \int_{\mathbb{R}^n} \|t\|_2^p dm_r^n(t) \\ &= m_p c_{r,p}^{-p} \int_{\mathbb{R}^n} b_p^{(n)} \int_{S^{n-1}} |\langle x, t \rangle|^p d\lambda(t) dm_r^n(t) \\ &= m_p b_p^{(n)} \int_{S^{n-1}} c_{r,p}^{-p} \int_{\mathbb{R}^n} |\langle x, t \rangle|^p dm_r^n(t) d\lambda(t) \\ &= m_p b_p^{(n)} \int_{S^{n-1}} \|w\|_r^p d\lambda(t). \end{aligned}$$

Recall that $b_p^{(n)} = \frac{\pi^{1/2} \Gamma(\frac{n+p}{2})}{\Gamma(\frac{p+1}{2}) \Gamma(\frac{n}{2})} \asymp n^{p/2}$. Thus, by Lemma 2.8, we have

$$M_p(B_2^n, \mathcal{d}_r) \leq n^{p/2} n^{(1/r-1/2)p} = n^{p/r}.$$

To prove the reverse inequality, as in the proof of Theorem 2.2 and using Lemma 2.5, given $\varepsilon > 0$ we can find a sequence of rotation invariant measures $(\eta_k)_k$ such that

$$\begin{aligned} M_p(B_2^n, \mathcal{d}_r) &= \sup_{\mu \in \mathfrak{M}^1(B_2^n)} c_{r,p}^- \int_{\mathbb{R}^n} \|t\|_2^p \left[\int_{-1}^1 \int_{-1}^1 |u-v|^p d\mu_{t/\|t\|_2}(u) d\mu_{t/\|t\|_2}(v) \right] dm_r^n(t) \\ &\geq \limsup_{k \rightarrow \infty} c_{r,p}^- \int_{\mathbb{R}^n} \|t\|_2^p \left[\int_{-1}^1 \int_{-1}^1 |u-v|^p d(\eta_k)_{t/\|t\|_2}(u) d(\eta_k)_{t/\|t\|_2}(v) \right] dm_r^n(t) \\ &\geq m_p(1-\varepsilon) c_{r,p}^- \int_{\mathbb{R}^n} \|t\|_2^p dm_r^n(t) \asymp n^{p/r}. \quad \blacksquare \end{aligned}$$

2.3 Energy integrals on the ball of ℓ_q^n

Given a centrally symmetric convex body K , we have some general upper estimates for the energies induced by the different ℓ_r -norms as in Lemma 2.7. However, it seems to be difficult compute the exact value of $M_p(K, \mathcal{d}_r)$ or its asymptotic behavior. In this section, we deal with ℓ_q -balls for $1 < q < 2$.

A combination of Lemma 2.7 and (10) gives the following result, which has some geometrical interest.

Proposition 2.9. Let $E = (\mathbb{R}^n, \|\cdot\|_E)$ be a real Banach space of dimension n . For $0 < p < 2$, we have

$$M_p(B_E, \mathcal{d}_2) \leq m_p b_p^{(n)} \int_{S^{n-1}} \|t\|_{E'}^p d\lambda(t),$$

where E' denotes the dual of E . □

This proposition has the following geometrical interpretation. Let $W_t = W_t(B_E)$ be the width of B_E in the direction t . This is defined to be the (Euclidean) distance between the supporting hyperplanes of B_E orthogonal to t , and can be computed as

$$W_t = \sup_{x \in B_E} \langle x, t \rangle - \sup_{x \in B_E} \langle x, -t \rangle = 2\|t\|_{E'}.$$

As a consequence, we have established a relationship between $M_p(B_E, \mathcal{d}_2)$ and the expected value of W_t^p , which is a kind of average width of B_E .

Let $E = (\mathbb{R}^n, \|\cdot\|_E)$ be a real Banach space of dimension n , we now give a lower bound for the energy integral $M_p(B_E, \mathcal{d}_2)$ which is related with the 2-summing norm of

the identity operator from E to ℓ_2^n . This bound is obtained by computing the supremum of the maximal energies over all the ellipsoids contained in B_E .

Proposition 2.10. There exist a positive constant C_p such that for every real n -dimensional Banach space $E = (\mathbb{R}^n, \|\cdot\|_E)$ we have

$$C_p \pi_2(i_{E,2} : E \rightarrow \ell_2^n)^p \leq M_p(B_E, \mathbf{d}_2),$$

where $i_{E,2} : E \rightarrow \ell_2^n$ is the formal identity. □

Proof. Observe that, if $S : \ell_2^n \rightarrow E$ is an operator of norm one, then the ellipsoid $S(B_2^n)$ is contained in B_E . Therefore, $M_p(S(B_2^n), \mathbf{d}_2) \leq M_p(B_E, \mathbf{d}_2)$. Using Theorem 2.1, we obtain that $m_p \pi_p(i_{E,2} S : \ell_2^n \rightarrow \ell_2^n)^p = M_p(S(B_2^n), \mathbf{d}_2) \leq M_p(B_E, \mathbf{d}_2)$. Since the absolutely p -summing norms for operators on Hilbert spaces are all equivalent to the 2-summing norm (as can be deduced from [10, Corollary 3.16, Theorem 4.10]) we know there exists a constant A_p which depends only on p such that $A_p \pi_2(i_{E,2} S : \ell_2^n \rightarrow \ell_2^n) \leq \pi_p(i_{E,2} S : \ell_2^n \rightarrow \ell_2^n)$. Thus,

$$\underbrace{A_p^p m_p}_{C_p} \pi_2(i_{E,2} S : \ell_2^n \rightarrow \ell_2^n)^p \leq M_p(B_E, \mathbf{d}_2). \tag{12}$$

Since Equation (12) holds for every norm one operator $S : \ell_2^n \rightarrow E$ we obtain

$$C_p \sup \{ \pi_2(i_{E,2} S : \ell_2^n \rightarrow \ell_2^n)^p : S \in \mathcal{L}(\ell_2^n, E), \|S\| = 1 \} \leq M_p(B_E, \mathbf{d}_2). \tag{13}$$

Now by Kwapien’s test [9, Proposition 11.8] we have that

$$\pi_2(i_{E,2} : E \rightarrow \ell_2^n) = \sup \{ \pi_2(i_{E,2} S : \ell_2^n \rightarrow \ell_2^n) : S \in \mathcal{L}(\ell_2^n, E), \|S\| = 1 \}.$$

Therefore, by (13), we get

$$C_p \pi_2(i_{E,2} : E \rightarrow \ell_2^n)^p \leq M_p(B_E, \mathbf{d}_2).$$

This concludes the proof. ■

We now describe the asymptotical behavior of $M_p(B_q^n, \mathbf{d}_2)$ for $1 < q \leq 2$.

Theorem 2.11. Given $1 < q \leq 2$ and $p > 0$, then $M_p(B_q^n, \mathbf{d}_2)$ behaves asymptotically as $n^{p/q}$, where $\frac{1}{q} + \frac{1}{q'} = 1$. □

Proof. By Proposition 2.9 and Lemma 2.8, we have

$$M_p(B_q^n, \mathbf{d}_2) \leq m_p b_p^{(n)} \int_{S^{n-1}} \|t\|_q^p d\lambda(t) \asymp n^{p/q'}.$$

On the other hand, by Proposition 2.10, we know that $\pi_2(\ell_q^n \rightarrow \ell_2^n)^p \asymp M_p(B_E, \mathbf{d}_2)$. Now, by [19, Lemma 22.4.9] or [7, Theorem 1], we know that $\pi_2(\ell_q^n \rightarrow \ell_2^n)^p \asymp n^{p/q'}$ which gives the lower estimate. ■

Some comments are in order. It should be mentioned that we can avoid the use of Proposition 2.10 for the lower estimate in the previous theorem. Indeed, since $n^{(q-2)/2q} B_2^n \subset B_q^n$, we have

$$n^{p/q'} = n^{p(q-2)/2q} n^{p/2} \asymp n^{p(q-2)/2q} M_p(B_2^n, \mathbf{d}_2) = M_p(n^{(q-2)/2q} B_2^n, \mathbf{d}_2) \leq M_p(B_q^n, \mathbf{d}_2).$$

Since we have obtained the correct asymptotic estimate of $M_p(B_q^n, \mathbf{d}_2)$ by using Propositions 2.9 and 2.10, this says that the bounds of their statement cannot be improved for arbitrary spaces.

Although we have not obtained the asymptotic behavior of $M_p(B_q^n, \mathbf{d}_2)$ for the remaining values of q , we do have certain bounds. Note that, for $q > 2$, we have the inclusions $B_2^n \subset B_q^n \subset n^{1/2-1/q} B_2^n$. Therefore, $n^{p/2} \asymp M_p(B_q^n, \mathbf{d}_2) \asymp n^{p/q'}$. It is interesting to mention that this bounds are the same as the ones that can be obtained using Propositions 2.9 and 2.10.

From Theorems 2.11, 2.1 and Proposition 2.9, we can find a relationship between $M_p(K)$ and the expected value of a random width of K , for K the unit ball of ℓ_q^n or an ellipsoid in \mathbb{R}^n . Namely, suppose $t \in S^{n-1}$ is randomly chosen with uniform distribution in the sphere. Then, for $0 < p < 2$ and $1 < q \leq 2$ we have

$$M_p(B_q^n, \mathbf{d}_2) \asymp n^{p/2} \mathbb{E}(W_t(B_q^n)^p).$$

A similar result holds for ellipsoids. Moreover, in this case, the cited results and (5) give us constants $\tilde{A}_p, \tilde{B}_p > 0$ with the following property. For every $n \in \mathbb{N}$ and every ellipsoid $\mathcal{E} \subset \mathbb{R}^n$ with orthogonal axes and radii a_1, \dots, a_n we have

$$\tilde{A}_p \left(\frac{a_1^2 + \dots + a_n^2}{n} \right)^{1/2} \leq \mathbb{E}(W_t(\mathcal{E})^p)^{1/p} \leq \tilde{B}_p \left(\frac{a_1^2 + \dots + a_n^2}{n} \right)^{1/2}.$$

2.4 Metric embeddings

Uniform, Lipschitz and coarse embeddings of metric spaces into Banach spaces with “good geometrical properties” have found many significant applications, specially in

computer science and topology. The advantages of these embeddings are based on the fact that for spaces with “good properties” one can apply several geometric tools which are generally not available for typical metric spaces. The most significant accomplishments throughout these lines were obtained in the area of approximation algorithms. In this context, the spaces with “good geometrical properties” are mostly a separable Hilbert space (or certain classical Banach spaces, such as ℓ_1).

One of the fundamental problems in metric geometry is the immersion problem, that is, to determine conditions for which a metric space may be isometrically embedded in a Hilbert space. It is well known that not every metric space (X, d) can be isometrically embedded in a Hilbert space. Even if distortion is allowed, there are metric spaces that cannot be embedded in a Hilbert space. The celebrated Assouad’s embedding theorem [6] allows a bi-Lipschitz embedding if we change the metric a little bit. Let (X, d) be a *doubling* metric space and take $\alpha \in (0, 1)$. Assouad’s theorem states that there exists N such that the snowflaked metric space (X, d^α) admits a bi-Lipschitz embedding into \mathbb{R}^N endowed with the Euclidean norm (i.e, in a N -dimensional Hilbert space). Recently, Naor and Neiman [14] proved that the same dimension $N = N(k)$ can be chosen for all $\alpha > \frac{1}{2}$ and all metric spaces with doubling constant at most k . Moreover, the distortion of all the corresponding bi-Lipschitz embeddings is uniformly bounded.

We will concentrate our attention to isometric embeddings for certain snowflaked metric spaces.

In the early 20th century, Wilson [28] investigated those metric spaces which arise from a metric space by taking as its new metric a suitable (one variable) function of the old one. For the metric space (\mathbb{R}, d_2) , he considered the metric transform $f(t) = t^{1/2}$ and showed that the snowflaked metric space $(\mathbb{R}, d_2^{1/2})$ can be isometrically embedded in a separable Hilbert space. In other words, he showed the existence of a distance preserving mapping $j : (\mathbb{R}, d_2^{1/2}) \rightarrow (\ell_2, \|\cdot\|_{\ell_2})$. Some years later, Schoenberg and von-Neumann [23] characterized those functions f for which the metric space $(\mathbb{R}, f(d_2))$ can be isometrically embedded in a Hilbert space. As a particular case they proved that, for $\alpha \in (0, 1)$, the function $f(t) = t^\alpha$ is an appropriate metric transform, generalizing Wilson’s result. Using transcendental means, Schoenberg obtained in [22] the same result for \mathbb{R}^n . He proved that, for $0 < \alpha < 1$, the metric space $(\mathbb{R}^n, d_2^\alpha)$ can also be embedded in ℓ_2 .

In particular, for every compact set $K \subset \mathbb{R}^n$ and every $0 < \alpha < 1$, the snowflaked metric space (K, d_2^α) can be isometrically embedded in the surface of a Hilbert sphere (see Theorem 2.12 for details). In other words, there exist a number R and a distance preserving mapping

$$j : (K, d_2^\alpha) \rightarrow (R \cdot S_{\ell_2}, \|\cdot\|_{\ell_2}). \quad (14)$$

We now focus on finding (or estimating) the Schoenberg radius of $(K, \mathcal{d}_2^\alpha)$, that is, the least possible R for which there exists a distance preserving mapping j as in (14). Alexander and Stolarsky [5] connected the problem of estimating this radius with the calculation of certain energy integrals. We state this relation in the following theorem.

Theorem 2.12. Let $K \subset \mathbb{R}^n$ be a convex body. For $0 < \alpha < 1$, the snowflaked metric space $(K, \mathcal{d}_2^\alpha)$ may be isometrically embedded on the surface of a Hilbert sphere of radius $\sqrt{\frac{M_{2\alpha}(K, \mathcal{d}_2)}{2}}$. Moreover, this is the minimum possible radius. \square

Actually, Alexander and Stolarsky proved this result for $\alpha = \frac{1}{2}$, but their proof works almost line by line for $0 < \alpha < 1$. Similar results can also be found in [17, Theorems 3.1., 3.2., 4.6].

As a consequence of the last theorem and Corollary 2.2, we obtain the minimum R for which the metric space $(B_2^n, \mathcal{d}_2^\alpha)$ may be isometrically embedded on the surface of a Hilbert sphere of radius R .

Theorem 2.13. For $0 < \alpha < 1$, the minimum R for which the metric space $(B_2^n, \mathcal{d}_2^\alpha)$ may be isometrically embedded on the surface of a Hilbert sphere of radius R is

$$\sqrt{\frac{m_p \pi^{1/2} \Gamma(\alpha + \frac{n}{2})}{2\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{n}{2})}} = n^{\alpha/2} \left(\sqrt{\frac{m_p \pi^{1/2}}{2\Gamma(\alpha + \frac{1}{2})}} + o(1) \right). \quad \square$$

On the other hand, combining Theorem 2.11 with Theorem 2.12 we can obtain the asymptotical behavior of the Schoenberg radius for the metric space $(B_q^n, \mathcal{d}_2^\alpha)$.

Theorem 2.14. Let $1 < q \leq 2$ and $0 < \alpha < 1$, the minimum R for which the metric space $(B_q^n, \mathcal{d}_2^\alpha)$ may be isometrically embedded on the surface of a Hilbert sphere of radius R behaves asymptotically as $n^{\alpha/q'}$, where $\frac{1}{q} + \frac{1}{q'} = 1$. \square

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