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Small Furstenberg sets ☆

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ABSTRACT

For α in $(0, 1]$, a subset E of $\mathbb{R}^2$ is called a *Furstenberg set* of type α or F_α-set if for each direction e in the unit circle there is a line segment ℓ_e in the direction of e such that the Hausdorff dimension of the set $E \cap \ell_e$ is greater than or equal to α. In this paper we use generalized Hausdorff measures to give estimates on the size of these sets. Our main result is to obtain a sharp dimension estimate for a whole class of *zero*-dimensional Furstenberg type sets. Namely, for $\mathfrak{h}_\gamma(x) = \log^{-\gamma}(\frac{1}{x})$, $\gamma > 0$, we construct a set $E_\gamma \in F_{\mathfrak{h}_\gamma}$ of Hausdorff dimension not greater than $\frac{1}{2}$. Since in a previous work we showed that $\frac{1}{2}$ is a lower bound for the Hausdorff dimension of any $E \in F_{\mathfrak{h}_\gamma}$, with the present construction, the value $\frac{1}{2}$ is sharp for the whole class of Furstenberg sets associated to the zero dimensional functions $\mathfrak{h}_\gamma$.

1. Introduction

We study dimension properties of sets of Furstenberg type. In particular we are interested in being able to construct very small Furstenberg sets in a given class. We begin with the definition of classical Furstenberg sets.

Definition 1.1. For α in $(0, 1]$, a subset E of $\mathbb{R}^2$ is called *Furstenberg set* of type α or F_α-set if for each direction e in the unit circle there is a line segment ℓ_e in the direction of e such that the Hausdorff dimension ($\dim_H$) of the set $E \cap \ell_e$ is greater than or equal to α. We will also say that such a set E belongs to the class F_α.

It is known [1,2] that $\dim_H(E) \geq \max\{2\alpha, \alpha + \frac{1}{2}\}$ for any F_α-set $E \subseteq \mathbb{R}^2$ and there are examples of F_α-sets E with $\dim_H(E) \leq \frac{1}{2} + \frac{3}{2}\alpha$. Hence, if we denote by

$$\Phi(F_\alpha) = \inf\{\dim_H(E) : E \in F_\alpha\},$$

then

$$\max\left\{\alpha + \frac{1}{2}; 2\alpha\right\} \leq \Phi(F_\alpha) \leq \frac{1}{2} + \frac{3}{2}\alpha, \quad \alpha \in (0, 1]. \tag{1}$$

In [3] the left hand side of this inequality has been extended to the case of more general dimension functions, i.e., functions that are not necessarily power functions [4].

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1.1. Dimension functions and Hausdorff measures

Definition 1.2. A function h will be called *dimension function* if it belongs to the class $\mathbb{H}$ which is defined as follows:

$$\mathbb{H} := \{h : [0, \infty) \to [0 : \infty), \text{non-decreasing, right continuous}, h(0) = 0\}.$$

The important subclass of those $h \in \mathbb{H}$ that satisfy a doubling condition will be denoted by $\mathbb{H}_d$:

$$\mathbb{H}_d := \{h \in \mathbb{H} : h(2x) \le Ch(x) \text{ for some } C > 0\}.$$

Let g, h be two dimension functions. We will say that g is dimensionally smaller than h and write $g \prec h$ if and only if

$$\lim_{x \to 0^+} \frac{h(x)}{g(x)} = 0.$$

A function $h \in \mathbb{H}$ will be called a "zero dimensional dimension function" if $h \prec x^\alpha$ for any $\alpha > 0$. We denote by $\mathbb{H}_0$ the subclass of those functions.

As usual, the h-dimensional (outer) Hausdorff measure $\mathcal{H}^h$ will be defined as follows. For a set $E \subseteq \mathbb{R}^n$ and $\delta > 0$, write

$$\mathcal{H}^h_\delta(E) = \inf\left\{\sum_i h(\text{diam}(E_i)) : E \subset \bigcup_i^\infty E_i, \text{diam}(E_i) < \delta\right\}.$$

The h-dimensional Hausdorff measure $\mathcal{H}^h$ of E is defined by

$$\mathcal{H}^h(E) = \sup_{\delta > 0} \mathcal{H}^h_\delta(E).$$

Recall that the Hausdorff dimension of a set $E \subseteq \mathbb{R}^n$ is the unique real number s characterized by the following properties:

- $\mathcal{H}^r(E) = +\infty$ for all $r < s$.
- $\mathcal{H}^t(E) = 0$ for all $s < t$.

Therefore, to prove that a given set E has dimension s, it is enough to check the preceding two properties, independently if $\mathcal{H}^s(E)$ is zero, finite and positive, or infinite. However, in general it is not true that, given a set E, there is a function $h \in \mathbb{H}$, such that if $g \succ h$ then $\mathcal{H}^g(E) = 0$, and if $g \prec h$, then $\mathcal{H}^g(E) = +\infty$. The difficulties arise from two results due to Besicovitch (see [5] and references therein). The first says that if a set E has null $\mathcal{H}^h$-measure for some $h \in \mathbb{H}$, then there exists a function g which is dimensionally smaller than h and for which still $\mathcal{H}^g(E) = 0$. Symmetrically, the second says that if a *compact* set E has non-σ-finite $\mathcal{H}^h$ measure, then there exists a function $g \succ h$ such that E has also non-σ-finite $\mathcal{H}^g$ measure. These two results imply that if a set E satisfies that there exists a function h such that $\mathcal{H}^g(E) > 0$ for any $g \prec h$ and $\mathcal{H}^g(E) = 0$ for any $g \succ h$, then it must be the case that $0 < \mathcal{H}^h(E)$ and E has σ-finite $\mathcal{H}^h$-measure.

Now consider the set $\mathbb{L}$ of Liouville numbers (see [6]). It is shown in [7] that there are two proper nonempty subsets $\mathbb{L}_0, \mathbb{L}_\infty \subseteq \mathbb{H}$ of dimension functions, such that $\mathcal{H}^h(\mathbb{L}) = 0$ for all $h \in \mathbb{L}_0$ and for all $h \in \mathbb{L}_\infty$, the set $\mathbb{L}$ has non σ-finite $\mathcal{H}^h$-measure. Therefore, this example shows that in the general setting of dimension functions, in order to detect the precise size of a given set, one needs to estimate a dimensional gap starting from a conjectured dimension function, both from above and from below. Precisely, one wants to measure how fast the quotient (or gap) $\frac{h}{g}$ between $g, h \in \mathbb{H}$ grows, whenever $g \prec h$ goes to zero.

1.2. Generalized Furstenberg sets. Statement of the main result

The analogous definition of Furstenberg sets in the setting of dimension functions is the following.

Definition 1.3. Let $\mathfrak{h}$ be a dimension function. A set $E \subseteq \mathbb{R}^2$ is a Furstenberg set of type $\mathfrak{h}$, or an $F_\mathfrak{h}$-set, if for each direction $e \in \mathbb{S}$ there is a line segment ℓ_e in the direction of e such that $\mathcal{H}^\mathfrak{h}(\ell_e \cap E) > 0$.

In [3] we proved that the appropriate dimension function for an $F_\mathfrak{h}$ set E must be dimensionally not much smaller than $\mathfrak{h}^2$ and $\mathfrak{h}\sqrt{\cdot}$ (this is the generalized version of the left hand side of (1)), the latter with some additional conditions on $\mathfrak{h}$. In particular, for the zero-dimensional Furstenberg-type sets E belonging to $F_{\mathfrak{h}_\gamma}$, where $\mathfrak{h}_\gamma \in \mathbb{H}_0$ (see Definition 1.2) is defined by $\mathfrak{h}_\gamma(x) = \frac{1}{\log^\gamma(\frac{1}{x})}$, we showed that $\dim_H(E) \ge \frac{1}{2}$.

In the present work we look at a refinement of the upper bound for the dimension of Furstenberg sets. Since we are looking for upper bounds on a class of Furstenberg sets, the aim will be to explicitly construct a very small set belonging to the given class.

We first consider the classical case of power functions, x^α, for $\alpha > 0$. Recall that for this case, the known upper bound implies that, for any positive α, there is a set $E \in F_\alpha$ such that $\mathcal{H}^{\frac{1+3\alpha}{2}+\varepsilon}(E) = 0$ for any $\varepsilon > 0$. By looking closer at Wolff's arguments, it can be seen that in fact it is true that $\mathcal{H}^g(E) = 0$ for any dimension function g of the form

$$g(x) = x^{\frac{1+3\alpha}{2}} \log^{-\theta}\left(\frac{1}{x}\right), \quad \theta > \frac{3(1+3\alpha)}{2} + 1. \tag{2}$$

Further, that argument can be modified (Theorem 2.4) to sharpen on the logarithmic gap, and therefore improving (2) by proving the same result for any g of the form

$$g(x) = x^{\frac{1+3\alpha}{2}} \log^{-\theta}\left(\frac{1}{x}\right), \quad \theta > \frac{1+3\alpha}{2}. \tag{3}$$

However, this modification will not be sufficient for our main objective, which is to reach the zero dimensional case. More precisely, we will focus at the endpoint $\alpha = 0$, and give a complete answer about the *exact dimension* of a class of Furstenberg sets. We will prove in Theorem 3.3 that, for any given $\gamma > 0$, there exists a set $E_\gamma \subseteq \mathbb{R}^2$ such that

$$E_\gamma \in F_{\mathfrak{h}_\gamma} \quad \text{for } \mathfrak{h}_\gamma(x) = \frac{1}{\log^\gamma\left(\frac{1}{x}\right)} \quad \text{and} \quad \dim_H(E_\gamma) \le \frac{1}{2}. \tag{4}$$

This result, together with the results from [3] mentioned above, shows that $\frac{1}{2}$ is sharp for the class $F_{\mathfrak{h}_\gamma}$. In fact, for this family both inequalities in (1) are in fact the equality $\Phi(F_{\mathfrak{h}_\gamma}) = \frac{1}{2}$.

In order to be able to obtain (4), it is not enough to simply "refine" the construction of Wolff. He achieves the desired set by choosing a specific set as the fiber in each direction. This set is known to have the correct dimension. To be able to reach the zero dimensional case, we need to handle the delicate issue of choosing an analogue zero dimensional set on each fiber. The main difficulty lies in being able to handle *simultaneously* Wolff's construction and the proof of the fact that the fiber satisfies the stronger condition of having positive measure for the correct dimension function.

The paper is organized as follows. In Section 2, for a given $\alpha > 0$, we carefully develop the main construction of small Furstenberg α sets, and obtain dimension estimates for the class F_α, $\alpha > 0$. In Section 3 we show that we can modify the argument of the previous section to include the zero-dimensional functions $\mathfrak{h}_\gamma$ defined in (4). The key ingredient is a lemma on Diophantine approximation regarding the size of zero dimensional fibers proved in Section 4. As usual, we will use the notation $A \lesssim B$ to indicate that there is a constant $C > 0$ such that $A \le CB$, where the constant is independent of A and B. By $A \sim B$ we mean that both $A \lesssim B$ and $B \lesssim A$ hold.

2. Upper bounds for Furstenberg-type sets

In this section we will concentrate on the right hand side of (1). This inequality has been proved by showing that there exists a set E in F_α such that $\mathcal{H}^s(E) = 0$ for any $s > \frac{1+3\alpha}{2}$. However, by the result of Besicovitch cited in the Introduction, this does not necessarily imply that $\mathcal{H}^h(E) = 0$ for any $h \succ x^{\frac{1+3\alpha}{2}}$. Here we refine the arguments of Wolff to show, in Theorem 2.4, that if $\theta > \frac{1+3\alpha}{2}$ there exists a set E in F_α such that for $h_\theta := x^{\frac{1+3\alpha}{2}} \log^{-\theta}\left(\frac{1}{x}\right)$ we have that $\mathcal{H}^{h_\theta}(E) = 0$.

The purpose of this section is twofold. First, we will carefully re-trace Wolff's arguments to show what is the key to be able to sharpen the logarithmic gap to obtain (3) instead of (2). Second, and more importantly, we analyze the proof in detail to understand the nontrivial modification to be performed in Section 3.

We begin with a preliminary lemma about a very well distributed (mod 1) sequence.

Lemma 2.1. *For $n \in \mathbb{N}$ and any real number $x \in [0, 1]$, there is a pair of natural numbers $0 \le j, k \le n-1$, such that*

$$\left| x - \left(\sqrt{2}\frac{k}{n} - \frac{j}{n}\right) \right| \le \frac{\log(n)}{n^2}.$$

This lemma is a consequence of Theorem 3.4 of [8, p. 125], in which an estimate is given about the discrepancy of the fractional part of the sequence $\{n\alpha\}_{n\in\mathbb{N}}$ where α is an irrational of a certain type.

We also need to introduce the notion of G-sets, a common ingredient in the construction of Kakeya and Furstenberg sets.

Definition 2.2. A G-set is a compact set $E \subseteq \mathbb{R}^2$ which is contained in the strip $\{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1\}$ such that for any $m \in [0, 1]$ there is a line segment contained in E connecting $x = 0$ with $x = 1$ of slope m, i.e.

$$\forall m \in [0, 1]\ \exists b \in \mathbb{R} : mx + b \in E, \quad \forall x \in [0, 1].$$

Finally we need some notation for a thickened line.

Definition 2.3. Given a line segment $\ell(x) = mx + b$, we define the δ-tube associated to ℓ as

$$S_\ell^\delta := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1;\ |y - (mx + b)| \le \delta\}.$$

Now we are ready to prove the main result of this section.

Theorem 2.4. *For $\alpha \in (0, 1]$ and $\theta > 0$, define $h_\theta(x) = x^{\frac{1+3\alpha}{2}} \log^{-\theta}(\frac{1}{x})$. Then, if $\theta > \frac{1+3\alpha}{2}$, there exists a set $E \in F_\alpha$ with $\mathcal{H}^{h_\theta}(E) = 0$.*

Proof. Fix $n \in \mathbb{N}$ and let n_j be a sequence such that $n_{j+1} > n_j^j$. We consider T to be the set defined as follows:

$$T = \left\{ x \in \left[\frac{1}{4}, \frac{3}{4}\right] : \forall j \, \exists p, q; \, q \le n_j^\alpha; \left| x - \frac{p}{q} \right| < \frac{1}{n_j^2} \right\}.$$

It can be seen that $\dim_H(T) = \alpha$ (see Section 4, Theorem 4.8).

If $\varphi(t) = \frac{1-t}{t\sqrt{2}}$ and $D = \varphi^{-1}\left([\frac{1}{4}, \frac{3}{4}]\right)$, we have that $\varphi : D \to [\frac{1}{4}, \frac{3}{4}]$ is bi-Lipschitz. Therefore the set

$$T' = \left\{ t \in \mathbb{R} : \frac{1-t}{t\sqrt{2}} \in T \right\} = \varphi^{-1}(T)$$

also has Hausdorff dimension α.

The main idea of our proof is to construct a set for which we have, essentially, a copy of T' in each direction and simultaneously keep some optimal covering property.

Define, for each $n \in \mathbb{N}$,

$$\Gamma_n := \left\{ \frac{p}{q} \in \left[\frac{1}{4}, \frac{3}{4}\right], q \le n^\alpha \right\}$$

and

$$Q_n = \left\{ t : \frac{1-t}{\sqrt{2}t} = \frac{p}{q} \in \Gamma_n \right\} = \varphi^{-1}(\Gamma_n).$$

To count the elements of Γ_n (and Q_n), we take into account that

$$\sum_{j=1}^{\lfloor n^\alpha \rfloor} j \le \frac{1}{2} \lfloor n^\alpha \rfloor (\lfloor n^\alpha \rfloor + 1) \lesssim \lfloor n^\alpha \rfloor^2 \le n^{2\alpha}.$$

Therefore, $\#(Q_n) \lesssim n^{2\alpha}$.

For $0 \le j, k \le n-1$, define the line segments

$$\ell_{jk}(x) := (1-x)\frac{j}{n} + x\sqrt{2}\frac{k}{n} \quad \text{for } x \in [0, 1],$$

and their δ_n-tubes $S_{\ell_{jk}}^{\delta_n}$ with $\delta_n = \frac{\log(n)}{n^2}$. We will use during the proof the notation S_{jk}^n instead of $S_{\ell_{jk}}^{\delta_n}$. Also define

$$G_n := \bigcup_{jk} S_{jk}^n. \tag{5}$$

Note that, by Lemma 2.1, all the G_n are G-sets.

For each $t \in Q_n$, we look at the points $\ell_{jk}(t)$, and define the set $S(t) := \{\ell_{jk}(t)\}_{j,k=1}^n$. Clearly, $\#(S(t)) \le n^2$. But if we note that, if $t \in Q_n$, then

$$0 \le \frac{\ell_{jk}(t)}{t\sqrt{2}} = \frac{1-t}{t\sqrt{2}} \frac{j}{n} + \frac{k}{n} = \frac{p}{q}\frac{j}{n} + \frac{k}{n} = \frac{pj+kq}{nq} < 2,$$

we can bound $\#(S(t))$ by the number of non-negative rationals smaller than 2 of denominator qn. Since $q \le n^\alpha$, we have $\#(S(t)) \le n^{1+\alpha}$. Considering *all* the elements of Q_n, we obtain $\#\left(\bigcup_{t \in Q_n} S(t)\right) \lesssim n^{1+3\alpha}$. Let us define

$$\Lambda_n := \left\{ (x, y) \in G_n : |x - t| \le \frac{\sqrt{2}}{n^2} \text{ for some } t \in Q_n \right\}.$$

Claim 2.5. *For each n, take $\delta_n = \frac{\log(n)}{n^2}$. Then Λ_n can be covered by L_n balls of radius δ_n with $L_n \lesssim n^{1+3\alpha}$.*

To see this, it suffices to set a parallelogram on each point of $S(t)$ for each t in Q_n. The lengths of the sides of the parallelogram are of order n^{-2} and $\frac{\log(n)}{n^2}$, so their diameter is bounded by a constant time $\frac{\log(n)}{n^2}$, which proves the claim.

Back to the proof of the theorem: We can now begin with the recursive construction that leads to the desired set. The starting point F_0 will be essentially any G-set. We only add the assumption that it can be written as

$$F_0 = \bigcup_{i=1}^{M_0} S_{\ell_i^0}^{\delta^0},$$

(the union of M_0 line segments $\ell_i^0 = m_i^0 + b_i^0$ with appropriate orientation and δ^0-thickened). Each F_j to be constructed will be a G-set of the form

$$F_j := \bigcup_{i=1}^{M_j} S_{\ell_i^j}^{\delta^j}, \quad \text{with } \ell_i^j = m_i^j + b_i^j.$$

Having constructed F_j, consider the M_j affine mappings

$$A_i^j : [0, 1] \times [-1, 1] \to S_{\ell_i^j}^{\delta^j} \quad 1 \le i \le M_j,$$

defined by

$$A_i^j \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ m_i^j & \delta^j \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ b_i^j \end{pmatrix}.$$

Here is the key step: by the definition of T, we can choose the sequence n_j to grow as fast as we need (this will not be the case in the next section). For example, we can choose n_{j+1} large enough to satisfy

$$\log\log(n_{j+1}) > M_j \tag{6}$$

and apply A_i^j to the sets $G_{n_{j+1}}$ defined in (5) to obtain

$$F_{j+1} = \bigcup_{i=1}^{M_j} A_i^j(G_{n_{j+1}}).$$

Since $G_{n_{j+1}}$ is a union of thickened line segments, we have that

$$F_{j+1} = \bigcup_{i=1}^{M_{j+1}} S_{\ell_i^{j+1}}^{\delta^{j+1}},$$

for an appropriate choice of M_{j+1}, δ_{j+1} and M_{j+1} line segments ℓ_i^{j+1}. From the definition of the mappings A_i^j and since the set $G_{n_{j+1}}$ is a G-set, we conclude that F_{j+1} is also a G-set. Define

$$E_j := \{(x, y) \in F_j : x \in T'\}.$$

To cover E_j, we note that if $(x, y) \in E_j$, then $x \in T'$, and therefore there exists a rational $\frac{p}{q} \in \Gamma_{n_j}$ with

$$\frac{1}{n_j^2} > \left| \frac{1-x}{x\sqrt{2}} - \frac{p}{q} \right| = |\varphi(x) - \varphi(r)| \ge \frac{|x-r|}{\sqrt{2}}, \quad \text{for some } r \in Q_{n_j}.$$

Therefore $(x, y) \in \bigcup_{i=1}^{M_{j-1}} A_i^{j-1}(\Lambda_{n_j})$, so we conclude that E_j can be covered by $M_{j-1} n_j^{1+3\alpha}$ balls of diameter at most $\frac{\log(n_j)}{n_j^2}$. By our choice of n_j, we obtain that E_j admits a covering by $\log\log(n_j) n_j^{1+3\alpha}$ balls of the same diameter. Therefore, if we set $F = \bigcap_j F_j$ and $E := \{(x, y) \in F : x \in T'\}$ we obtain that

$$\begin{aligned} \mathcal{H}_{\delta_j}^{h_\theta}(E) &\lesssim n_j^{1+3\alpha} \left(\log\log(n_j)\right) h_\theta\left(\frac{\log(n_j)}{n_j^2}\right) \\ &\lesssim n_j^{1+3\alpha} \left(\log\log(n_j)\right) \left(\frac{\log(n_j)}{n_j^2}\right)^{\frac{1+3\alpha}{2}} \log^{-\theta}\left(\frac{n_j^2}{\log(n_j)}\right) \\ &\lesssim \left(\log\log(n_j)\right) \log(n_j)^{\frac{1+3\alpha}{2}-\theta} \lesssim \log^{\frac{1+3\alpha}{2}+\varepsilon-\theta}(n_j) \end{aligned}$$

for large enough j. Therefore, for any $\theta > \frac{1+3\alpha}{2}$, the last expression goes to zero. In addition, F is a G-set, so it must contain a line segment in each direction $m \in [0, 1]$. If ℓ is such a line segment, then

$$\dim_H(\ell \cap E) = \dim_H(T') \ge \alpha.$$

The final set of the proposition is obtained by taking eight copies of E, rotated to achieve *all* the directions in $\mathbb{S}$. $\square$

3. Upper bounds for small Furstenberg-type sets

In this section we will focus on the class F_α at the endpoint $\alpha = 0$. Note that all preceding results involved only the case for which $\alpha > 0$. Introducing the generalized Hausdorff measures, we are able to handle an important class of Furstenberg type sets in F_0.

The idea is to follow the proof of Theorem 2.4. But in order to do that, we need to replace the set T by a generalized version of it. A naïve approach would be to replace the α power in the definition of T by a slower increasing function, like a logarithm. But in this case it is not clear that the set T fulfills the condition of having positive measure for the corresponding dimension function (recall that we want to construct a set in $F_{\mathfrak{h}_\gamma}$). More precisely, we will need the following lemma.

Lemma 3.1. *Let $r > 1$ and consider the sequence $\mathfrak{n} = \{n_j\}$ defined by $n_j = e^{\frac{1}{2}n_{j-1}^{\frac{4}{r}j}}$, the function $\mathfrak{f}(x) = 2\log(x^2)^{\frac{r}{2}}$ and the set*

$$T = \left\{x \in \left[\frac{1}{4}, \frac{3}{4}\right] \setminus \mathbb{Q} : \forall j\, \exists p, q;\, q \leq \mathfrak{f}(n_j);\, \left|x - \frac{p}{q}\right| < \frac{1}{n_j^2}\right\}.$$

Then we have that $\mathcal{H}^{\mathfrak{h}}(T) > 0$ for $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$.

Remark 3.2. We postpone the proof of this lemma to the next section, but we emphasize the following fact: in this case, the construction of this new set T, does not allow us, as in (6), to freely choose the sequence n_j. On one hand we need the sequence to be quickly increasing to prove that the desired set is small enough, but not arbitrarily fast, since on the other hand, we need to impose some control to be able to prove that the fiber has the appropriate *positive* measure.

With this lemma, we are able to prove the main result of this section. We have the next theorem.

Theorem 3.3. *Let $\mathfrak{h} = \frac{1}{\log(\frac{1}{x})}$. There exists a set $E \in F_{\mathfrak{h}}$ such that $\dim_H(E) \leq \frac{1}{2}$.*

Proof. We will use essentially a copy of T in each direction in the construction of the desired set to fulfill the conditions required to be an $F_{\mathfrak{h}}$-set. Let T be the set defined in Lemma 3.1. Define T' as

$$T' = \left\{t \in \mathbb{R} : \frac{1-t}{t\sqrt{2}} \in T\right\} = \varphi^{-1}(T),$$

where φ is the same bi-Lipschitz function from the proof of Theorem 2.4. Then T' has positive $\mathcal{H}^{\mathfrak{h}}$-measure. Let us define the corresponding sets of Theorem 2.4 for this generalized case.

$$\Gamma_n := \left\{\frac{p}{q} \in \left[\frac{1}{4}, \frac{3}{4}\right], q \leq \mathfrak{f}(n)\right\},$$
$$Q_n = \left\{t : \frac{1-t}{\sqrt{2}t} = \frac{p}{q} \in \Gamma_n\right\} = \varphi^{-1}(\Gamma_n).$$

Now the estimate is $\#(Q_n) \lesssim \mathfrak{f}^2(n) = \log^r(n^2) \sim \log^r(n)$, since

$$\sum_{j=1}^{\lfloor \mathfrak{f}(n)\rfloor} j \leq \frac{1}{2}\lfloor \mathfrak{f}(n)\rfloor(\lfloor \mathfrak{f}(n)\rfloor + 1) \lesssim \lfloor \mathfrak{f}(n)\rfloor^2.$$

For each $t \in Q_n$, define $S(t) := \{\ell_{jk}(t)\}_{j,k=1}^n$. If $t \in Q_n$, following the previous ideas, we obtain that

$$\#(S(t)) \lesssim n\log^{\frac{r}{2}}(n),$$

and therefore

$$\#\left(\bigcup_{t\in Q_n} S(t)\right) \lesssim n\log(n)^{\frac{3r}{2}}.$$

Now we estimate the size of a covering of

$$\Lambda_n := \left\{(x, y) \in G_n : |x - t| \leq \frac{\sqrt{2}}{n^2} \text{ for some } t \in Q_n\right\}.$$

For each n, take $\delta_n = \frac{\log(n)}{n^2}$. As before, the set Λ_n can be covered with L_n balls of radius δ_n with $L_n \lesssim n\log(n)^{\frac{3r}{2}}$.

Once again, define F_j, F, E_j and E as before. Now the sets F_j can be covered by fewer than $M_{j-1}n_j\log(n_j)^{\frac{3r}{2}}$ balls of diameter at most $\frac{\log(n_j)}{n_j^2}$. Now we can verify that, since each G_n consist of n^2 tubes, we have that $M_j = M_0 n_1^2 \cdots n_j^2$.

Now we are again at the key point: by the choice of the sequence $\{n_j\}$ in the definition of T, we can also verify that the sequence satisfies the relation $\log n_{j+1} \geq M_j = M_0 n_1^2 \cdots n_j^2$, and therefore we have the bound

$$\dim_H(E) \leq \underline{\dim}_B(E) \leq \underline{\lim}_j \frac{\log\left(\log(n_j)n_j\log(n_j)^{\frac{3r}{2}}\right)}{\log\left(n_j^2\log^{-1}(n_j)\right)} = \frac{1}{2},$$

where $\underline{\dim}_B$ stands for the lower box dimension. Finally, for any $m \in [0, 1]$ we have a line segment ℓ with slope m contained in F. It follows that $\mathcal{H}^{\mathfrak{h}}(\ell \cap E) = \mathcal{H}^{\mathfrak{h}}(T') > 0$. □

We remark that the argument in this particular result is essentially the same needed to obtain the family of Furstenberg sets $E_\gamma \in F_{\mathfrak{h}_\gamma}$ for $\mathfrak{h}_\gamma(x) = \frac{1}{\log^\gamma(\frac{1}{x})}$, $\gamma \in \mathbb{R}_+$, such that $\dim_H(E_\gamma) \leq \frac{1}{2}$ announced in the Introduction.

4. Proof of Lemma 3.1

The purpose of this section is to prove Lemma 3.1. Our proof relies on a variation of a Jarník type theorem on Diophantine approximation. We begin with some preliminary results on Cantor type constructions that will be needed.

4.1. Cantor sets

In this section we introduce the construction of sets of Cantor type in the spirit of [9]. By studying two quantities, the number of children of a typical interval and some separation property, we obtain sufficient conditions on these quantities that imply the positivity of the h-dimensional measure for a test function $h \in \mathbb{H}$.

We will need a preliminary elemental lemma about concave functions. The proof is straightforward.

Lemma 4.1. *Let $h \in \mathbb{H}$ be a concave dimension function. Then*

$$\min\{a, b\} \leq \frac{a}{h(a)}h(b) \quad \text{for any } a, b \in \mathbb{R}_+.$$

Proof. Since h is concave, we have $h\left(\frac{x+y}{2}\right) \leq \frac{h(x)+h(y)}{2}$. We consider two separate cases:

- If $b \geq a$ then $\frac{a}{h(a)}h(b) \geq \frac{a}{h(a)}h(a) = a = \min\{a, b\}$.
- If $a > b$, then $\frac{a}{h(a)}h(b) \geq \frac{b}{h(b)}h(b) = b = \min\{a, b\}$ by concavity of h. □

The following lemma is a natural extension of the "Mass Distribution Principle" to the dimension function setting.

Lemma 4.2 (*h-Dimensional Mass Distribution Principle*)**.** *Let $E \subseteq \mathbb{R}^n$ be a set, $h \in \mathbb{H}$ and μ a probability measure on E. Let $\varepsilon > 0$ and $c > 0$ be positive constants such that for any $U \subseteq \mathbb{R}^n$ with $\operatorname{diam}(U) < \varepsilon$ we have*

$$\mu(U) \leq ch(\operatorname{diam}(U)).$$

Then $\mathcal{H}^h(E) > 0$.

Proof. For any δ-covering we have

$$0 < \mu(E) \leq \sum_i \mu(U) \leq c\sum_i h(\operatorname{diam}(U)).$$

Then $\mathcal{H}^h_\delta > \frac{\mu(E)}{c}$ and therefore $\mathcal{H}^h(E) > 0$. □

Now we present the construction of a Cantor-type set (see Example 4.6 in [9]).

Lemma 4.3. *Let $\{E_k\}$ be a decreasing sequence of closed subsets of the unit interval. Set $E_0 = [0, 1]$ and suppose that the following conditions are satisfied:*

1. *Each E_k is a finite union of closed intervals I_j^k.*
2. *Each level $k - 1$ interval contains at least m_k intervals of level k. We will refer to these as the "children" of an interval.*
3. *The gaps between the intervals of level k are at least of size ε_k, with $0 < \varepsilon_{k+1} < \varepsilon_k$.*

Let $E = \bigcap_k E_k$. Define, for a concave dimension function $h \in \mathbb{H}$, the quantity

$$D_k^h := m_1 \cdot m_2 \cdots m_{k-1} h(\varepsilon_k m_k).$$

If $\underline{\lim}_k D_k^h > 0$, then $\mathcal{H}^h(E) > 0$.

Proof. The idea is to use the version of the mass distribution principle from Lemma 4.2. Clearly we can assume that the property (2) of Lemma 4.3 holds for *exactly* m_k intervals. So we can define a mass distribution on E assigning a mass of $\frac{1}{m_1 \cdots m_k}$ to each of the $m_1 \cdots m_k$ intervals of level k. Now, for any interval U with $0 < |U| < \varepsilon_1$, take k such that $\varepsilon_k < |U| < \varepsilon_{k-1}$. We will estimate the number of intervals of level k that could have non-empty intersection with U. For that, we note the following:

- U intersects at most one I_j^{k-1}, since $|U| < \varepsilon_{k-1}$. Therefore it can intersect at most m_k children of I_j^{k-1}.
- Suppose now that U intersects L intervals of level k. Then it must contain $(L-1)$ gaps of size at least ε_k. Therefore, $L - 1 \le \frac{|U|}{\varepsilon_k}$. Consequently $|U|$ intersects at most $\frac{|U|}{\varepsilon_k} + 1 \le 2\frac{|U|}{\varepsilon_k}$ intervals of level k.

From these two observations, we conclude that

$$\mu(U) \le \frac{1}{m_1 \cdots m_k} \min\left\{m_k, \frac{2|U|}{\varepsilon_k}\right\} = \frac{1}{m_1 \cdots m_k \varepsilon_k} \min\{\varepsilon_k m_k, 2|U|\}.$$

Now, by the concavity of h, we obtain

$$\min\{\varepsilon_k m_k, 2|U|\} \le \frac{\varepsilon_k m_k}{h(\varepsilon_k m_k)} h(2|U|).$$

In addition (also by concavity), h is doubling, so $h(2|U|) \lesssim h(|U|)$ and then

$$\mu(U) \lesssim \frac{\varepsilon_k m_k h(|U|)}{m_1 \cdots m_k \varepsilon_k h(\varepsilon_k m_k)} = \frac{h(|U|)}{m_1 \cdots m_{k-1} h(\varepsilon_k m_k)} = \frac{h(|U|)}{D_k^h}.$$

Finally, if $\underline{\lim}_k D_k^h > 0$, there exists k_0 such $\frac{1}{D_k^h} \le C$ for $k \ge k_0$ and we can use the mass distribution principle with C and $\varepsilon = \varepsilon_{k_0}$. $\square$

Remark 4.4. In the particular case of $h(x) = x^s$, $s \in (0, 1)$ we recover the result of [9], where the parameter s can be expressed in terms of the sequences m_k and ε_k. For the set constructed in Lemma 4.3, we have

$$\dim_H(E) \ge \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \varepsilon_k)}. \tag{7}$$

4.2. Diophantine approximation—Jarník's theorem

The central problem in the theory of Diophantine approximation is, at its simplest level, to approximate irrational numbers by rationals. A classical theorem due to Jarník in this area is the following (see [10]), which provides a result on the size of the set of real numbers that are well approximable. As usual, $d(x, \mathbb{Z})$ denotes the distance from x to the nearest integer.

Theorem 4.5. *For $\beta \ge 2$, define the following set:*

$$B_\beta = \left\{x \in [0, 1] \setminus \mathbb{Q} : d(xq, \mathbb{Z}) < \frac{1}{q^{\beta-1}} \text{ for infinitely many } q \in \mathbb{Z}\right\}.$$

Then $\dim_H(B_\beta) = \frac{2}{\beta}$.

In fact, there are several results (see [11,12]) regarding the more general problem of estimating the size of the set

$$B_\mathfrak{g} := \left\{x \in [0, 1] \setminus \mathbb{Q} : d(xq, \mathbb{Z}) \le \frac{q}{\mathfrak{g}(q)} \text{ for infinitely many } q \in \mathbb{N}\right\}, \tag{8}$$

where $\mathfrak{g}$ is any positive increasing function.

We can therefore see Lemma 3.1 as a result on the size of a set of well approximable numbers. We will derive the proof of Lemma 3.1 from the following proposition, where we prove a lower bound estimate for the set $B_\mathfrak{g}$. The original idea is from [9], Theorem 10.3, where the result for the classical case in the form presented above in Theorem 4.5 is proved. Precisely, for $\mathfrak{h}(x) = \frac{1}{\left(\mathfrak{g}^{-1}\left(\frac{1}{x}\right)\right)^2}$ we will find conditions on $\frac{\mathfrak{h}}{h}$ with $h \prec \mathfrak{h}$ to ensure that $\mathcal{H}^h(B_\mathfrak{g}) > 0$. To simplify the notation of the proof, let $\Delta(h, \mathfrak{h})(x) = \frac{\mathfrak{h}(x)}{h(x)}$ and recall that $\mathbb{H}_d$ is the set of all dimension functions that satisfy a doubling condition.

Proposition 4.6. *Let* $\mathfrak{g}$ *be a positive, increasing function satisfying*

$$\mathfrak{g}(x) \gg x^2 \quad (x \gg 1) \tag{9}$$

and

$$\mathfrak{g}^{-1}(ab) \lesssim \mathfrak{g}^{-1}(a) + \mathfrak{g}^{-1}(b) \quad \text{for all } a, b \geq 1. \tag{10}$$

Define $B_{\mathfrak{g}}$ *as in* (8) *and let* $h \in \mathbb{H}_d$ *be a concave dimension function such that* $h \prec \mathfrak{h}(x) = \frac{1}{\left(\mathfrak{g}^{-1}(\frac{1}{x})\right)^2}$. *Consider a sequence* $\{n_k\}$ *that satisfies:*

(A) $n_k \geq 3\mathfrak{g}(2n_{k-1})$.
(B) $\log(n_k) \leq \mathfrak{g}(n_{k-1})$.

If $\Delta(h, \mathfrak{h})(x) = \frac{\mathfrak{h}(x)}{h(x)} = \frac{1}{h(x)\mathfrak{g}^{-1}(\frac{1}{x})^2}$ *satisfies*

$$\varliminf_k \frac{1}{6^k \mathfrak{g}^2(2n_{k-2})\Delta(h, \mathfrak{h})\left(\frac{1}{\log(n_k)\mathfrak{g}(2n_{k-1})}\right)} > 0, \tag{11}$$

then $\mathcal{H}^h(B_{\mathfrak{g}}) > 0$.

Proof. Define

$$G_q := \left\{x \in [0, 1] \setminus \mathbb{Q} : d(xq, \mathbb{Z}) \leq \frac{q}{\mathfrak{g}(q)}\right\}. \tag{12}$$

For each $q \in \mathbb{N}$, G_q is the union of $q - 1$ intervals (with no rational numbers) of length $2\mathfrak{g}(q)^{-1}$ and two more intervals of length $\mathfrak{g}(q)^{-1}$ at the endpoints of [0, 1]. Now define for each q

$$G'_q := G_q \cap \left(\frac{1}{\mathfrak{g}(q)}, 1 - \frac{1}{\mathfrak{g}(q)}\right).$$

Now, for each $n \in \mathbb{N}$ consider two prime numbers p_1, p_2 such that $n \leq p_1 < p_2 < 2n$ (these can always be chosen for large n, see (13)). We will prove that G'_{p_1} and G'_{p_2} are disjoint and well separated. Note that if $\frac{r_1}{p_1}$ and $\frac{r_2}{p_2}$ are centers of two of the intervals belonging to G'_{p_1} and G'_{p_2}, we have

$$\left|\frac{r_1}{p_1} - \frac{r_2}{p_2}\right| = \frac{1}{p_1 p_2}|r_1 p_2 - r_2 p_1| \geq \frac{1}{4n^2},$$

since $r_1 p_2 - r_2 p_1 \neq 0$. Therefore, taking into account this separation between the centers and the length of the intervals, we conclude that for $x \in G'_{p_1}$ and $y \in G'_{p_2}$,

$$|x - y| \geq \frac{1}{4n^2} - \frac{2}{\mathfrak{g}(n)} \geq \frac{1}{8n^2} \quad (\text{since } \mathfrak{g}(n) \gg n^2).$$

Let $\mathcal{P}_m^n$ be the set of all the prime numbers between m and n and define

$$H_n := \bigcup_{p \in \mathcal{P}_n^{2n}} G'_p.$$

Then H_n is the union of intervals of length at least $\frac{2}{\mathfrak{g}(2n)}$ that are separated by a distance of at least $\frac{1}{8n^2}$.

Now we observe the following: If $n \in \mathbb{N}$, $n \leq p \leq 2n$ and I is an interval with $|I| > \frac{3}{n}$, then at least $\frac{p|I|}{3}$ of the intervals of G'_p are completely contained on I. To verify this last statement, cut I into three consecutive and congruent subintervals. Then, in the middle interval there are at least $\frac{p|I|}{3}$ points of the form $\frac{m}{p}$. All the intervals of G'_p centered at these points are completely contained in I, since the length of each interval of G'_p is $\frac{2}{\mathfrak{g}(p)} < \frac{|I|}{3}$.

In addition, by the Prime Number Theorem, we know that $\#(\mathcal{P}_1^n) \sim \frac{n}{\log(n)}$, so we can find n_0 such that

$$\#(\mathcal{P}_n^{2n}) \geq \frac{n}{2\log(n)} \quad \text{for } n \geq n_0. \tag{13}$$

Hence, if I is an interval with $|I| > \frac{3}{n}$, then there are at least

$$\frac{p|I|}{3}\frac{n}{2\log(n)} > \frac{n^2|I|}{6\log(n)}$$

intervals of H_n contained on I. Now we will construct a Cantor-type subset E of $B_{\mathfrak{g}}$ and apply Lemma 4.3.

Consider the sequence $\{n_k\}$ of the hypothesis of the proposition and let $E_0 = [0, 1]$. Define E_k as the union of all the intervals of H_{n_k} contained in E_{k-1}. Then E_k is built up of intervals of length at least $\frac{1}{\mathfrak{g}(2n_k)}$ and separated by at least $\varepsilon_k = \frac{1}{8n_k^2}$. Moreover, since $\frac{1}{\mathfrak{g}(2n_{k-1})} \geq \frac{3}{n_k}$, each interval of E_{k-1} contains at least

$$m_k := \frac{n_k^2}{6\log(n_k)\mathfrak{g}(2n_{k-1})}$$

intervals of E_k.

Now we can apply Lemma 4.3 to $E = \bigcap E_k$. Recall that ε_k denotes the separation between k-level intervals and m_k denotes the number of children of each of them. Consider $h \prec \mathfrak{h}$, $h \in \mathbb{H}_d$, and recall the notation $\Delta(h, \mathfrak{h})(x) = \frac{\mathfrak{h}(x)}{h(x)}$. Then

$$\begin{aligned} D_k^h &= m_1 \cdot m_2 \cdots m_{k-1} h(\varepsilon_k m_k) \\ &= \frac{6^{-(k-2)} n_2^2 \cdots n_{k-1}^2}{\log(n_2)\cdots\log(n_{k-1})\mathfrak{g}(2n_1)\cdots\mathfrak{g}(2n_{k-2})} h\left(\frac{1}{48\log(n_k)\mathfrak{g}(2n_{k-1})}\right). \end{aligned}$$

Now we note that $n_k \geq \log(n_k)$ and, by hypothesis (A), we also have that $n_k \geq \mathfrak{g}(2n_{k-1})$. In addition, h is doubling, therefore it follows that we can bound the first factor to obtain that

$$D_k^h \gtrsim \frac{6^{-k} n_{k-1}^2}{\mathfrak{g}^2(2n_{k-2})\Delta(h, \mathfrak{h})\left(\frac{1}{\log(n_k)\mathfrak{g}(2n_{k-1})}\right)\left(\mathfrak{g}^{-1}(\log(n_k)) + 2n_{k-1}\right)^2},$$

since, by hypothesis (B), n_k satisfies $\log(n_{k-1}) \leq \mathfrak{g}(n_{k-2})$ and $\mathfrak{g}$ satisfies (10). Now, again by hypothesis (B),

$$D_k^h \geq \frac{1}{6^k \mathfrak{g}^2(2n_{k-2})} \frac{1}{\Delta(h, \mathfrak{h})\left(\frac{1}{\log(n_k)\mathfrak{g}(2n_{k-1})}\right)}.$$

Thus, if

$$\varliminf_k \frac{1}{6^k \mathfrak{g}^2(2n_{k-2})} \frac{1}{\Delta(h, \mathfrak{h})\left(\frac{1}{\log(n_k)\mathfrak{g}(2n_{k-1})}\right)} > 0,$$

then $\mathcal{H}^h(E) > 0$ and therefore $\mathcal{H}^h(B_{\mathfrak{g}}) > 0$. □

The following example not only illustrates this last result, but also will be crucial in the proof of Lemma 3.1.

Example 4.7. Define $\mathfrak{g}_r(x) = e^{x^{\frac{2}{r}}}$ for $r > 0$ and consider the set $B_{\mathfrak{g}_r}$. Then $\mathfrak{h}_r(x) = \frac{1}{\log^r(\frac{1}{x})}$ will be an expected lower bound for the dimension function for the set $B_{\mathfrak{g}_r}$. Consider the family $h_\theta(x) = \frac{1}{\log^\theta(\frac{1}{x})}$ $(0 < \theta < r)$, which satisfy $h_\theta \prec \mathfrak{h}_r$. In this context, $\frac{\mathfrak{h}_r(x)}{h_\theta(x)} = \log^{\theta - r}(\frac{1}{x})$. Define the sequence n_k as follows:

$$n_k = e^{k n_{k-1}^{\frac{2}{r}}}.$$

Clearly the sequence is admissible, since

(A) $n_k \geq 3\mathfrak{g}_r(2n_{k-1})$, and
(B) $\log(n_k) \leq \mathfrak{g}_r(n_{k-1})$.

Then, for the quantity D_k^h defined in Proposition 4.6, we have the following estimate:

$$D_k^h \gtrsim \frac{\left(\log\log(n_k) + n_{k-1}^{\frac{2}{r}}\right)^{r-\theta}}{6^k e^{M n_{k-2}^{\frac{2}{r}}}} \geq \frac{n_{k-1}^{2\frac{r-\theta}{r}}}{6^k e^{M n_{k-2}^{\frac{2}{r}}}}, \quad M = 2^{\frac{2}{r}+1}.$$

Finally, for any $\varepsilon > 0$ and $M > 0$, n_k satisfies, for large k,

$$\frac{n_{k-1}^{\varepsilon}}{6^k e^{M n_{k-2}^{\frac{2}{r}}}} = \frac{e^{\varepsilon k n_{k-2}^{\frac{2}{r}}}}{6^k e^{M n_{k-2}^{\frac{2}{r}}}} = \frac{e^{(\varepsilon k - M) n_{k-2}^{\frac{2}{r}}}}{6^k} \geq 1,$$

so we conclude that $\varliminf_k D_k^h > 0$. Therefore the set $E \subseteq B_{\mathfrak{g}_r}$ constructed in the proof above satisfies $\mathcal{H}^{h_\theta}(E) > 0$ for all $\theta < r$.

4.3. Another Jarník type theorem and proof of Lemma 3.1

For the proof of Lemma 3.1 we will need a different but essentially equivalent formulation of Jarník's theorem. We first recall that in the proof of Theorem 2.4 we used the following theorem.

Theorem 4.8. *Let $\mathfrak{n} = \{n_j\}_j$ be an increasing sequence with $n_{j+1} \geq n_j^j$ for all $j \in \mathbb{N}$. For $0 < \alpha \leq 1$, if $A_\alpha^{\mathfrak{n}}$ is defined as*

$$A_\alpha^{\mathfrak{n}} = \left\{ x \in [0,1] \setminus \mathbb{Q} : \forall j \, \exists \, p, q; \, q \leq n_j^\alpha; \, \left| x - \frac{p}{q} \right| < \frac{1}{n_j^2} \right\},$$

then $\dim_H(A_\alpha^{\mathfrak{n}}) = \alpha$.

For the proofs of Theorems 4.8 and 4.5, we refer the reader to [12–14,10,9].

Now we want to relate the sets $A_\alpha^{\mathfrak{n}}$ and B_β and their generalized versions. It is clear that for any $\alpha \in (0,1]$, we have the inclusion $A_\alpha^{\mathfrak{n}} \subset B_{\frac{2}{\alpha}}$. For $\alpha \in (0,1]$, if $x \in A_\alpha^{\mathfrak{n}}$ then for each $j \in \mathbb{N}$ there exists a rational $\frac{p_j}{q_j}$ with $q_j \leq n_j^\alpha$ such that $|x - \frac{p_j}{q_j}| < n_j^{-2}$, which is equivalent to $|xq_j - p_j| < q_j n_j^{-2}$. Therefore $|xq_j - p_j| \leq q_j^{1-\frac{2}{\alpha}}$. Observe that if there were only finite values of q for a given x, then x has to be rational. For if $q_j = q_{j_0}$ for all $j \geq j_0$, then $|x - \frac{p_j}{q_{j_0}}| \to 0$ and this implies that $x \in \mathbb{Q}$. We conclude then that, for any $x \in A_\alpha^{\mathfrak{n}}$, $d(xq, \mathbb{Z}) < \frac{1}{q^{\frac{2}{\alpha}-1}}$ for infinite many q and therefore $x \in B_{\frac{2}{\alpha}}$. However, since the dimension of $A_\alpha^{\mathfrak{n}}$ coincides with the one of $B_{\frac{2}{\alpha}}$, one can expect that both sets have approximately comparable sizes.

We introduce the following definition, which is the extended version of the definition of the set $A_\alpha^{\mathfrak{n}}$ in Theorem 4.8.

Definition 4.9. Let $\mathfrak{n} = \{n_j\}_j$ be any increasing nonnegative sequence of integers. Let $\mathfrak{f}$ be an increasing function defined on $\mathbb{R}_+$. Define the set

$$A_{\mathfrak{f}}^{\mathfrak{n}} := \left\{ x \in [0,1] \setminus \mathbb{Q} : \forall j \, \exists p, q; \, q \leq \mathfrak{f}(n_j); \, \left| x - \frac{p}{q} \right| < \frac{1}{n_j^2} \right\}.$$

The preceding observation about the inclusion $A_\alpha \subset B_\beta$ can be extended to this general setting. For a given $\mathfrak{g}$ as in the definition of $B_{\mathfrak{g}}$, define $\Gamma_{\mathfrak{g}}(x) = \mathfrak{g}^{-1}(x^2)$. Then the same calculations show that $A_{\Gamma_{\mathfrak{g}}}^{\mathfrak{n}} \subset B_{\mathfrak{g}}$.

We will need a converse relation between those sets, since we want to prove a lower bound for the sets $A_{\mathfrak{f}}^{\mathfrak{n}}$ from the estimates provided in Proposition 4.6.

Lemma 4.10. *Let $\mathfrak{g}$ and $B_{\mathfrak{g}}$ be as in Proposition 4.6. Define $\Gamma_{\mathfrak{g}}(x) = \mathfrak{g}^{-1}(x^2)$. Then, if $\mathfrak{m} = \{m_k\}$ is the sequence defining the set E in the proof of Proposition 4.6, then the set E is contained in $A_{2\Gamma_{\mathfrak{g}}}^{\mathfrak{n}}$, where $\mathfrak{n} = \{n_k\} = \{\mathfrak{g}(m_k)^{\frac{1}{2}}\}$.*

Proof. Recall that in the proof of Proposition 4.6 we define the sets G'_q as a union of intervals of the form $I = \left(\frac{r}{q} - \frac{1}{\mathfrak{g}(q)}; \frac{r}{q} + \frac{1}{\mathfrak{g}(q)}\right) \setminus \mathbb{Q}$. The sets H_n were defined as $H_n := \bigcup_{p \in \mathcal{P}_n^{2n}} G'_p$, where $\mathcal{P}_n^{2n}$ is the set of primes between n and $2n$. We can therefore write

$$H_n := \bigcup I_j^n.$$

Now, given a sequence $\mathfrak{m} = \{m_k\}$, for each k, the set E_k is defined as the union of all the intervals of H_{m_k} that belong to E_{k-1}, where $E_0 = [0,1]$. If $E = \bigcap E_k$, any $x \in E$ is in E_k and therefore in some of the $I_j^{m_k}$. It follows that there exists integers r and q, $q \leq 2m_k$ such that

$$\left| x - \frac{r}{q} \right| < \frac{1}{\mathfrak{g}(q)} < \frac{1}{\mathfrak{g}(m_k)} = \frac{1}{n_k^2}, \quad q \leq 2\mathfrak{g}^{-1}(n_k^2).$$

Therefore $E \subset A_{2\Gamma_{\mathfrak{g}}}^{\mathfrak{n}}$. □

We remark that the above inclusion implies that any lower estimate on the size of E would also be a lower estimate for $A_{2\Gamma_{\mathfrak{g}}}^{\mathfrak{n}}$.

We now conclude the proof of Theorem 3.3 by proving Lemma 3.1:

Proof of Lemma 3.1. Let $\mathfrak{h}(x) = \frac{1}{\log(\frac{1}{x})}$. For $r > 1$, consider the function $\mathfrak{g}_r$, the sequence $\mathfrak{m} = \{m_k\}$ and the set E_r as in Example 4.7. Define $\mathfrak{f} = 2\Gamma_{\mathfrak{g}_r}$, $\mathfrak{n}$ and $A_{\mathfrak{f}}^{\mathfrak{n}}$ as in Lemma 4.10. It follows that $\mathfrak{f}(x) = 2\log(x^2)^{\frac{r}{2}}$, $n_j = e^{\frac{1}{2} n_{j-1}^{\frac{4}{r}j}}$ and

$$A_{\mathfrak{f}}^{\mathfrak{n}} := \left\{ x \in [0,1] \setminus \mathbb{Q} : \forall j \, \exists p, q; \, q \leq \mathfrak{f}(n_j); \, \left| x - \frac{p}{q} \right| < \frac{1}{n_j^2} \right\}.$$

Note that Lemma 4.10 says that the inclusion $E \subset A^{\mathfrak{n}}_{2\Gamma_{\mathfrak{g}}}$ *always* holds, for any defining function $\mathfrak{g}$, where E is the *substantial portion* of the set $B_{\mathfrak{g}}$ (see Proposition 4.6). But we need the positivity of $\mathcal{H}^{\mathfrak{h}}(E)$ to conclude that the set $A^{\mathfrak{n}}_{2\Gamma_{\mathfrak{g}}}$ also has positive $\mathcal{H}^{\mathfrak{h}}$-measure. For the precise choices of $\mathfrak{h}$ and $\mathfrak{g}$, we obtain this last property from Example 4.7. Precisely, $\mathcal{H}^{\mathfrak{h}}(E) > 0$ and therefore the set $A^{\mathfrak{n}}_{\mathfrak{f}}$ has positive $\mathcal{H}^{\mathfrak{h}}$-measure. This concludes the proof of Lemma 3.1 and therefore the set constructed in the proof of Theorem 3.3 fulfills the condition of being an $F_{\mathfrak{h}}$-set. □

As a final remark, we mention that it would be interesting to obtain sharp estimates in terms of generalized Hausdorff measures for any $F_{\mathfrak{h}}$ class, not only for F_α, in the spirit of Theorem 2.4. The intuition here says that the reasonable dimension function for an upper bound for the class $F_{\mathfrak{h}}$ is $\sqrt{\cdot}\,\mathfrak{h}^{\frac{3}{2}}$. Therefore, a nice problem would be to construct an $F_{\mathfrak{h}}$ set of zero $\mathcal{H}^h$-measure for a dimension function h which is "very close" to $\mathfrak{h}$. And, further, to estimate (as in Theorem 2.4) how close to $\mathfrak{h}$ such an h can be in order to still allow us to perform the construction.

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