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> the danger lies in this, that when we have succeeded in thoroughly mastering a technique, we are very liable to be mastered *by* her
>
> Piero Sraffa (D3/12/4:15).

1. Introduction

The concept and role of capital are unquestionably among the principal sources of controversy in economic theory and the long list of authoritative economists involved includes scholars like Böhm-Bawerk, J.B. Clark, Hayek, Knight, Hicks, Samuelson, Solow, Pasinetti, and Garegnani in different periods.

While the specific issues differ in the course of the various disputes, their common root can be found in the *ambiguity* that surrounds the conception of capital and its role in the neoclassical-marginalist theory of value and distribution. As this ambiguity still remains unresolved, new disputes periodically arise when attempts are made to define a "new" concept of marginal product of capital and prove its equality, in equilibrium, with the rate of interest.

The analysis begins in Section 2 with a discussion of the ambiguity in question. It is argued that a major source of misunderstanding and confusion is the fact that capital, which actually means the *amount of purchasing power* making it possible to finance the costs of production, has often been understood as a synthesis of the *capital goods* employed as *inputs* in the production process. This gave rise to the false impression – and hope – that these goods could be aggregated into a single input called "capital".

In a nutshell, the inability to keep detached two different objects, namely capital and capital goods, has given rise to a purely ideal

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conception, a sort of Holy Grail of economic theory, which can be called "aggregate capital", a factor of production to be employed together with and with the same role as labour and land. If such an aggregate capital existed, then its marginal product, in equilibrium, would be equal to the rate of interest, just as the marginal products of labour and land would be, respectively, equal to the rates of wage and rent.

In fact, however, unlike labour and land, capital is not an input but simply an amount of value that allows firms to finance their costs of production. While capital goods *are* instead inputs, their employment cannot be aggregated into a homogenous mass without illegitimate "hyper-simplification".

In listing the inputs, there is thus no possibility of considering anything other than the vector of capital goods alongside labour services and the use of natural resources. As argued in Section 3, however, the fact that capital goods are both complementary to other inputs and highly specialised poses serious obstacles to the construction of a production function that has capital goods as independent variables and whose partial derivatives could be used to determine income distribution.

Sections 4 and 5 examine a recent attempt made by Paul Samuelson, also jointly with his pupil Erko Etula (while reference is made here in particular to Samuelson 2007, see also Samuelson and Etula 2006 and Etula 2008), to use the "Master Function" (MF) – a production function that includes the vector of capital goods among its arguments – in order to determine income distribution by means of what the author calls the function's "non-neoclassical" marginal products. The conclusion drawn here is that the attempt is unsuccessful, at least in the general case including the employment of physically heterogeneous capital goods. As a result, contrary to the authors' claims, the explanation of income distribution by means of the MF's partial derivatives cannot be accepted.

The demonstration of this begins by considering the stationary framework in which Samuelson embeds the analysis and showing that the conditions allowing the differentiability of the MF are generally incompatible with those ensuring the uniformity of returns on the supply prices of the capital goods that are reproduced in stationary equilibrium. A non-stationary Arrow–Debreu framework is then examined and it is shown that in this case, the number of methods whose coexistence allows the full employment of the given initial endowments is not enough to obtain a differentiable MF.

2. On the notion of capital once again

According to one widespread view, capital seems to be two things at the same time, both an amount of purchasing power and a vector of commodities, i.e. capital goods. In other words, capital seems conceivable either as value or in physical terms, more or less in the same way as GDP, for example, can be expressed in either nominal or real terms.

The primary aim of this section is to disprove this view by showing that value capital, on the one hand, and capital goods, on the other hand, are not in general two sides of the same coin but two different things. As we shall see, the distinction between value capital and capital goods is not merely a matter of the way in which the theorist chooses to represent capital but rather reflects a difference in nature and role.

Let us start from the beginning. Production takes time, namely inputs are employed before outputs are obtained. In accordance with a standard representation of production, the case can therefore be imagined in which a set of inputs – i.e. various commodities (capital goods), different kinds of labour services and the use of different sorts of natural resources – are employed in a certain process in period t and a set of outputs – commodities – are obtained as a result in period $t + 1$.

It can be stated in terms of a standard notation[1] that a vector of quantities of inputs a_t is employed in t and a vector of quantities of outputs b_{t+1} is obtained in $t + 1$. If p_a is the (row) price vector of inputs, then $p_a \ldots a_t$ represents the total production cost. Similarly, if p_b is the (row) price vector of outputs, then $p_b \ldots b_{t+1}$ is the amount of revenue (and the difference $\pi = p_b \ldots b_{t+1} - p_a \ldots a_t$ stands for profit).

If it is assumed that inputs are bought onto the market in the period in which they are employed and that outputs are sold in the period in which they are obtained, the costs and revenues of the same process do not manifest themselves simultaneously. Entrepreneurs, therefore, cannot use revenues to finance costs because costs and revenues are related to different market days. Capital is what allows entrepreneurs to *buy* inputs on the market in period t and it is, therefore, an amount of purchasing power. Subsequently, in period $t + 1$, when the outputs are sold, revenues reimburse the capital with a profit.[2]

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1 In particular, we refer to the notation introduced by Malinvaud (1953).

2 As the reader will have noticed, this is the notion of capital found, among others, in Marx with the money–commodities–money triad. A sum of money M, i.e. purchasing power, is initially turned into an amount (or vector) of commodities, C. This is done directly, in the case of merchants' capital, or indirectly, by buying the inputs that produce the commodities, in the case of industrial capital. The commodities are then turned back into a sum of money M'. This is because capital "is not spent, is merely advanced" (Marx 1909, Vol. 1, p. 166) and therefore returns to the capitalist augmented by the profit or "surplus-value". See Marx (1909, Vol. 1., pp. 163–73).

If the costs can instead be paid – totally or partially – on the same market day as the outputs are sold, then no capital is needed in that payment. This is what happens with the cost of labour, for example, if it is assumed that wages are paid *post factum*, i.e. in the moment in which the outputs are obtained and sold. Another example is provided by the models in which all markets, for both current and future delivery, are assumed to open for a single instant,[3] so that the inputs and outputs of the same process are traded simultaneously. In this case too, costs can be met directly out of revenues and no advance financing by capital is needed.

Two observations follow from the above. First, capital is an object of the same kind as costs and revenues. Capital is an amount of purchasing power, i.e. an amount of exchange value, and as such *it is not an input*. Unlike labour and land, it is not in a technical relationship with outputs, as clearly shown by the fact that while it is always possible to express the employment of labour and land in "technical units" – i.e. in such a way as to have a non-ambiguous relationship between each of them and the amount of output[4] – this possibility does not exist for the employment of capital. The problem is not simply that value is not a technical unit of measure but rather that capital is not an input. The lack of a technical unit of measure for it is simply a consequence of this fact.[5]

Second, capital goods, which are better referred to as means of production in order to avoid any ambiguity, *are inputs*. In the absence of specific assumptions, however, they cannot be regarded as the physical counterpart of capital or as what capital is spent on or invested in. Capital is spent to purchase all the inputs in vector a_t, which includes means of production but also the production services of various sorts of labour and natural resources. Capital is invested in financing the costs of production of certain outputs, totally or partially.

Means of production can be regarded as the real assets into which capital is converted[6] only on some ad hoc assumptions. In particular, it can be

3 In particular, such an assumption characterises Arrow–Debreu equilibrium models.

4 To give just one example, if the employment of labour is expressed – as it should be – in terms of labour hours, then an increase in the employment of labour brings about an increase in the amount of output, *ceteris paribus*. If it is instead expressed as the sum of the heights of all workers, then the relationship between labour employment and output is ambiguous, as no general conclusion can be drawn about the effect of an increase in workers' total height on output. Then, height is not a technical unit of measure for labour.

5 It is clear that if capital were an input, its technical unit of measure could be deduced directly from the observation of reality, as is the case for all true inputs.

6 This approach is adopted, for example, in Fratini (2013a) to study the effects of a change in the rate of interest on the supply of savings.

assumed that (i) wages and rents are paid at the end of the production process or that (ii) wages are regarded in physical terms "as the fuel for the engines or the feed for the cattle" (Sraffa 1960, p. 9) and rents do not enter into the costs of production. From a history-of-economic-thought point of view, we may say that assumption (ii) has a classical flavour, while assumption (i) is typically neoclassical,[7] although in both cases, capital is used to buy a set of commodities.[8]

These assumptions have certainly helped to generate the ambiguity mentioned at the beginning of this section and in particular to spread the erroneous idea that capital is an input and can be conceived in both "aggregate" and "disaggregate" terms. It should now be clear, however, that capital is the amount of purchasing power that makes it possible to finance production costs (totally or partially) and must not be confused with the means of production or capital goods, which play a different role. This distinction should be kept in mind even – or especially – when the value of the means of production is the only part of the costs financed by capital.

There is no shortage of claims in the twentieth-century literature on capital theory that the problem is one of expressing capital as a single magnitude. On the one hand, this is somewhat surprising because capital *is* a single magnitude, namely an amount of purchasing power. On the other, if the real problem is – as the present authors believe – one of expressing a vector of capital goods as an amount of "aggregate capital" in order to regard it as an input on the same footing as labour and land, then it is not simply a "problem" but an impossible task, as a vector cannot be expressed by a scalar. Various attempts in this direction can be mentioned, from the average period of production of Jevons and Böhm-Bawerk to the "Meccano sets" of Swan (1956) and the "jelly" of Samuelson (1962), that will be discussed in the next section. As is known, none of these attempts has worked. The possibility of "synthesising" or "aggregating" capital goods, in

7 The neoclassical theory of distribution tends – at least in its initial formulations – to see wages and rents in the same terms as profits (interests). As a result, since profits appear in the same moment as outputs are sold, it is also assumed that wages and rents are paid in that moment.

8 It is worth noting that in these two cases, the transformation of capital into commodities does not have the same meaning as the Marxian M–C–M'. The C in Marx's expression is not, in fact, a vector of inputs but rather a vector of outputs that is sold for the amount of money M'. The Marxian transformation of M into C – and then of C into M' – therefore requires no ad hoc assumptions and is decidedly general. On the contrary, the conversion of capital into a vector of means of production necessitates either assumption (i) or assumption (ii).

general, into a single factor of production is nothing more than an illusion.[9]

3. Marginal equalities and capital goods

There is no need here to enter into an analysis of the meaning and role in neoclassical theory of marginal productivity and its equality with the price of inputs. Those interested are referred to the extensive literature already existing on these matters.[10] For our present purposes, it will suffice to recall very briefly just a few points.

If, given the technical conditions of production, the quantity of a certain output can be expressed as a differentiable function of the quantities of inputs employed in its production, then the equalities between the marginal products of the latter and their relative prices in terms of output (marginal equalities) are the first-order conditions of a standard profit-maximisation problem. Marginal equalities have, thus, been used by neoclassical theory as a possible basis for the claim that the demand for inputs depends on their relative prices and a supply-and-demand equilibrium can, therefore, be attained through their adjustment. This is, indeed, the way in which distributive variables – interpreted as factor

9 Capital goods can clearly be aggregated in various ways, including their weight and the quantities of labour they embody. The point is that the result of the aggregation cannot be regarded as an input or a factor of production. To be more precise, let us assume that there are many techniques, labelled $\theta = \alpha, \beta, \gamma, \ldots$, each producing the same final output (consumption good) and denoted as $y\theta$ and $k\theta \in \Re_{+}n$, respectively, the net product and the vector of capital goods, both understood per unit of labour. The aggregation of capital goods consists in turning the vector $k\theta$ into a scalar $s\theta$. In other words, it consists in finding a vector $v \in \Re n$ such that $v{\cdot}k\theta = s\theta$. This aggregation is, however, problematic in many respects.

First of all, given two techniques α and β, one of the following cases may very well happen: (i) $s\alpha = s\beta$ but $y\alpha \neq y\beta$, (ii) $s\alpha \neq s\beta$, but $y\alpha = y\beta$ or (iii) $s\alpha > s\beta$, but $y\alpha < y\beta$. It is clear in these cases that aggregation brings about a loss of relevant information about the relationship between inputs and output: $s\theta$ does not provide enough information to explain $y\theta$. Second, if the price vector is used as vector v so that $s\theta = p{\cdot}k\theta$, new problems arise. With r as the rate of interest, it is possible to have (iv) $ds\theta/dr > 0$ and (v) $s\alpha > s\beta$ if $r = r'$ and $s\alpha < s\beta$, if $r = r''$, with $r' \neq r''$. (It should be noted that (iv) is called "reverse capital deepening", while (v) has no name.) In conclusion, it is thus impossible, in general, to say that one technique is more capital-intensive than another in anything other than tautological terms.For the problems arising from the aggregation of capital goods, see also the analysis presented in Zambelli (2004).

10 Just to give a reference, we can mention Kurz and Salvadori (1995, pp. 428–32).

prices – are determined according the neoclassical-marginalist theory.[11]

As has been known since the publication of Wicksell's *Lectures* ([1901] 1934), capital is not an input and therefore cannot appear among the independent variables of a production function, or at least not if this function is viewed exclusively as the expression of the technical conditions of production. Various attempts have been made, however, to obtain an indirect or "surrogate" marginal product of capital. In these cases, a variation in the rate of interest is usually assumed with changes in the methods of production in use and in the price system arising as a result. There are, thus, variations both in the quantity of output and in the investment of capital – with a given employment of labour – and the ratio between them has been interpreted as a "marginal product of capital". Moreover, if an equilibrium position is taken as the starting point and changes in the price system due to the variation of distribution are overlooked, this particular marginal product of capital proves to be equal to the rate of interest, thus giving the false impression of a marginal equality (see, for example, Malinvaud 1953, pp. 260–1). There is again no need to discuss this point here.[12] Suffice it to recall that if price changes are admitted, this ratio may very well be negative,[13] thus frustrating any attempt to interpret it as a "marginal product".

Unlike capital, capital goods are inputs and their quantities can, therefore, appear among the independent variables of a production function. The problems in this case, however, concern the partial derivatives of the function.

The first arises due to the complementarity of capital goods with one another and/or with other inputs, especially labour. A well-known example used by many economists in the past is that of the shepherd and his crook. A shepherd is not a shepherd without a crook and a crook is useful only in the hands of a shepherd. In this case, increasing the number of shepherds employed each day while the number of crooks remains

11 Needless to say, an important role is played in this theory by the principle of decreasing marginal productivity. This implies, first, that the profit function is concave, so that the first-order condition, i.e. marginal equality, is necessary and sufficient for the maximisation of profits, and, second, that there is an inverse relationship between the employment of an input and its price, the quantities employed and the prices of the other inputs being constant.

12 On the weakness of this position, see in particular Pasinetti (1969), Garegnani (1984) and Fratini (2013b).

13 In Fratini (2010), for instance, the possibility of a monotonically decreasing schedule of the investment of capital associated with a non-monotonic behaviour of the curve of the (physical) net product per worker is shown.

unchanged brings about no rise in output (lambs) because the additional workers cannot control the flock without crooks. As a result, the marginal product of labour, with a given set of capital goods, would be zero.

The way to circumvent this problem devised in various debates on capital theory[14] is to assume (i) the possibility of using different kinds of crook (longer or shorter) and (ii) the existence of an "aggregate capital" capable of remaining constant while the crooks vary in number and kind. This "aggregate capital" would thus appear in the production function instead of the crooks. As stated at the end of the previous section, however, no such synthetic expression of capital goods can exist.

The longer or shorter crooks assumed in the above argument lead us to the second problem. Many capital goods are specialised inputs and, as a result, different methods of producing the same commodity usually employ different kinds of capital goods. The best-known theoretical representation of this is unquestionably the model put forward by Samuelson (1962) with a final output (consumption goods) and as many heterogeneous capital goods, α, β, γ,…, as available techniques. Given a technique, there is just one kind of capital good which, together with labour, permits the production of the final output and its own replacement, whereas every change in the technique adopted involves a change of the quality of the capital goods employed.[15] Since different techniques imply different amounts of net product per unit of labour, an increase in this quantity cannot take place without a change in the kind of capital goods employed, while the marginal product of a specific capital good is still zero.[16]

We are, therefore, back at the above case of shepherds and crooks, the difference being that the focus is now on a specific kind of crook rather than on labour. Unsurprisingly, Samuelson tried to solve the problem in the way already outlined, i.e. by means of a "surrogate homogenous capital" – described as a sort of "jelly" – capable of standing as an argument in a "surrogate production function" together with labour. As is known, however, this did not work.

14 See, for example, Hicks (1932) and Robertson (1931). For a reconstruction of this debate, see Trabucchi (2011).

15 As Samuelson himself states (1962, p. 196): "No alchemist can turn one capital good into another. [Capital good] [a]lpha needs labour to work with in a fixed proportion: more than its critical proportion of labour will yield nothing extra; take away either input, while holding the other input at the previously proper proportion, and you lose all the product that has resulted from the combined does of the two inputs".

16 As has been insistently stressed by Garegnani (e.g. in 2007, p. 581–2) and Petri (2011, p. 381), the fact that marginal products are generally zero, hence capital goods' rental prices and possibly the real wage rate are zero, questions the plausibilty that income distribution is determined by factors' marginal products.

Since no real "aggregate" capital exists, Samuelson's jelly was nothing other than the value of the capital goods employed and, therefore, a magnitude dependent on prices and distribution. Pasinetti, Garegnani and other scholars were then able to prove the possibility for this model of results such as "reverse capital deepening" and "reswitching", which not only prevent the construction of the surrogate production function but also contradict the standard neoclassical "tale".[17]

This brings us up to the late 1960s. The following points should now be clear: (i) there can be no marginal product of value capital because it is not an input; (ii) there can be no marginal product of "aggregate" capital because a scalar cannot properly represent a vector and this magnitude therefore does not exist; (iii) serious difficulties arise in defining meaningful (strictly positive) marginal products when the quantities of the various capital goods are included among the independent variables of a production function.[18] As a result, differentiable production functions and marginal equalities disappeared from the neo-Walrasian general equilibrium theory, even though they did go on to play an important part in macroeconomic theories[19] and in "the vulgar theories of textbooks" (Hahn 1975, p. 363).

Despite all these difficulties, Samuelson and Etula have recently made a further attempt to express output as a differentiable function of the quantities of the different inputs employed in production so as to obtain something that may appear similar to marginal equalities at first sight. As we shall see in the next section, the innovation of this approach with respect to the foregoing lies in the fact that their conception of the marginal product of inputs is based not on the *substitution of the methods in use* but rather on the change in the proportions in which the methods *already in use* are employed. As will be shown, however, in cases where capital goods are employed in production, this necessitates the simultaneous use of so many

17 In particular, as Samuelson wrote (1966, p. 568), according to the "tale" told by Jevons and Böhm-Bawerk, an increase in the rate of interest should bring about the use of less "roundabout" or "mechanised" techniques, i.e. techniques that involve a smaller net product per unit of labour. Thanks to that debate, it is known that the very opposite may well occur.

18 There is, of course, no mathematical difficulty in doing this. It is possible to write a Cobb–Douglas or a CES production function $y_{t+1} = f(a_t)$ whose domain is the set of non-negative vectors of inputs a_t or a differentiable transformation function $\varphi(a_t, b_{t+1})$. These functions, however, overlook important aspects connected with the employment of capital goods in production, namely their complementarity and specialisation.

19 It is, in fact, known that the current mainstream macroeconomic theory is actually general equilibrium theory with some very restrictive assumptions imposed (e.g. just one agent, just one commodity,…).

methods for the same output that it is something very hard to justify with both stationary and non-stationary relative prices.

4. The Samuelson–Etula Master Function and its marginal products

The MF was recently developed by Samuelson and Etula (2006) as an attempt to define the marginal products of inputs when there is a discrete number of alternative methods for the production of commodities (both of consumption and of capital goods) and to use these marginal products in order to determine income distribution. In their words:

> We define … a novel cornered Master Function whose Newtonian derivatives do determine … the competitive supply demand market-clearing equilibrium distribution pricing (Samuelson and Etula 2006, p. 333; see also Samuelson 2007, pp. 245–6).[20]

The logic behind the MF can be briefly summarised as follows (see Garegnani 2007, pp. 579–85). Given consumers' demands and the supply of endowments, both of primary factors and of capital goods, it is assumed that there is a sufficiently large number of methods whose simultaneous application allows the full employment equilibrium by the adjustment of prices and distributive variables.[21] Even when the production of capital goods is considered, the construction assumes that the different methods employ the *same kind* of inputs in *different proportions*. It is thus possible to change the proportions in which the different methods are used so as to

20 Actually, Samuelson's and Etula's apparently "novel" attempt is not so new. The idea on which the MF is based can already be found in Samuelson (1959) and, particularly, in Kurz and Salvadori (1992, pp. 232–5). In this last contribution, Kurz and Salvadori have built a function that very much resembles the MF, whose marginal product of labour can be determined by means of the same mathematical tools used by Samuelson and Etula (i.e. the theory of linear programming). Kurz and Salvodori do not use their construction to determine income distribution by marginal-productivy theory; however, they use it to provide a formal explanantion of Ricardo's view of the capital-accumulation process in an economy that produces corn by means of labour and land only, along the lines of Passinetti (1960).

However, both in Samuelson (1959) and in Kurz and Salvadory (1992), the analysis is restricted to economies that do not employ capital goods in production. Kurz and Salvadori are very clear "that with heterogeneous capital goods no production function can be constructed" (p. 232). We shall see in the follwing section the reasons for this.

21 In fact, the authors offer no description of the working of the market for inputs and the equality of supply and demand appears to be an assumption rather than the result of the market mechanism.

Table 1. Production methods without capital goods

Method	Labour	Land	Corn
a	$L^{(a)}$	$T^{(a)}$	$C^{(a)}$
b	$L^{(b)}$	$T^{(b)}$	$C^{(b)}$
c	$L^{(c)}$	$T^{(c)}$	$C^{(c)}$
d	$L^{(d)}$	$T^{(d)}$	$C^{(d)}$

keep the quantity of all the inputs employed constant but one, and hence to calculate the marginal product of the input in question.

Let us now proceed by steps to give a clearer idea of the aims – and limitations – of the MF. We shall first consider the production of a consumption good, corn (C), by means of labour (L) and land (T) of uniform quality. The methods available are summarised in Table 1.

The procedure to derive the MF follows Samuelson (and Etula). The problem is to choose the levels of activity y_i, $i = a, b, c, d$ that maximise the production of C subject to the full employment of labour and land[22]. We thus have

$$\mathbf{P1}: \quad \text{Max}_{y_i}: \; C = \sum_{i=a}^{d} C^{(i)} y_i$$

subject to:

$$\sum_{i=a}^{d} L^{(i)} y_i = L \tag{P1.1}$$

$$\sum_{i=a}^{d} T^{(i)} y_i = T \tag{P1.2}$$

$$y_i \geq 0 \; \forall \; i = a, b, c, d \tag{P1.3}$$

where (L) and (T) stand, respectively, for the supplied endowments of labour and land (regarded here as exogenous variables).

22 Of course, in a capitalist economy, nobody consciously maximises total output (there is no "Central Planner"). However, as it should be clearer when we examine P1's dual problem below, D1, maximisation of corn production is the counterpart of the employment of cost-minimising methods of production, which is in turn the outcome of profit maximisation.

The set Ω is now defined as the set of all endowment vectors: $\omega = [L, T]$, such that P1 has a solution. Then, for each ω, there is a vector of activity levels $y(\omega)$ that is a solution of P1. The MF is defined as follows:

$$C = MF\ (L, T) = \sum_{i=a}^{d} C^{(i)} y(\omega) \tag{1}$$

Given that P1 has two constraints (P1.1) and (P1.2),[23] it is known from the theory of linear programming (see Dantzig 1951, p. 341) that the vector of activity levels has at most two positive components.[24] The vector $y(\omega)$ with exactly two positive components is called a non-degenerate vector (as is known, the vectors in Ω will be generally non-degenerate). The subset $\Omega^{a,b}$ is now defined such that, for all $\omega \in \Omega^{a,b}$, the vector $y(\omega)$ is non-degenerate and $y_i(\omega) > 0,\ \forall\ i = a, b$ and 0 otherwise. In this case, the MF is

$$C = MF\ (L, T) = \sum_{i=a}^{b} C^{(i)} y_i(\omega) \tag{2}$$

And the components $y_a(\omega)$ and $y_b(\omega)$ of $y(\omega)$ must satisfy the following conditions:

$$A \times y\ (\omega) = \begin{bmatrix} L^{(a)} & L^{(b)} \\ T^{(a)} & T^{(b)} \end{bmatrix} \begin{bmatrix} y_a(\omega) \\ y_b(\omega) \end{bmatrix} = \begin{bmatrix} L \\ T \end{bmatrix} \tag{3}$$

If the inverse of matrix A, $A^{-1} = \begin{bmatrix} L^{(a)} & T^{(a)} \\ L^{(b)} & T^{(b)} \end{bmatrix}^{-1}$ is defined as $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, then the solution to (3) can be expressed as

$$\begin{bmatrix} y_a(\omega) \\ y_b(\omega) \end{bmatrix} = \begin{bmatrix} A_{11} L + A_{12} T \\ A_{21} L + A_{22} T \end{bmatrix} \tag{4}$$

23 Apart from the non-negative constraints (P1.3).

24 In the general case, if n is the number of methods and m the number of constraints, with $n > m$, the vector of activity levels will have at most m positive components.

And the substitution of (4) into (2) makes it possible to arrive explicitly at an expression of the MF in terms of the endowments L and T:

$$MF\ (L, T) = C^a\ [A_{11}L + A_{12}T] + C^b[A_{21}L + A_{22}T] \tag{5}$$

Finally, it is possible to obtain the marginal products of labour ($\frac{\partial C}{\partial L}$) and land ($\frac{\partial C}{\partial T}$) from condition (5):

$$\begin{aligned} \frac{\partial C}{\partial L} &= A_{11}\ C^a + A_{21}C^b \\ \frac{\partial C}{\partial T} &= A_{21}\ C^a + A_{22}C^b \end{aligned} \tag{6}$$

It should be noted that, while the MF generally exists, the MF is a differentiable function and the marginal products can be calculated only when the vector of activity levels $y(\omega)$ is non-degenerate, as the matrix A^{-1} will not exist if $y(\omega)$ is a degenerate vector.

Now, there is a dual-minimising problem (D1) associated with P1 in terms of price variables, whose number of variables will be equal to P1's number of constraints (2) while its number of constraints will coincide with P1's number of variables (4). We thus have

$$\mathbf{D1}: \qquad \text{Min}_{w,r}\ : wL + rT$$

subject to:

$$wL^{(a)} + rT^{(a)} \geq C^{(a)} \tag{D1.1}$$

$$wL^{(b)} + rT^{(b)} \geq C^{(b)} \tag{D1.2}$$

$$wL^{(c)} + rT^{(c)} \geq C^{(c)} \tag{D1.3}$$

$$wL^{(d)} + rT^{(d)} \geq C^{(d)} \tag{D1.4}$$

$$w, r \geq 0 \tag{D1.5}$$

where w and r stand, respectively, for the rate of real wages and the rate of land rent in terms of corn, which is taken as the numéraire. It is known from duality that the employment of methods (i) and (ii) at positive levels means that only (D1.1) and (D1.2) will be satisfied with equality signs as "break-even conditions" (Samuelson 2007, p. 253). Equations (D1.3)–(D1.4) will instead be satisfied as strict inequalities, indicating that the employment of those methods will entail entrepreneurial loses. It is, therefore, possible to use the subset of break-even conditions (D1.1)–(D1.2) directly to determine the distributive variables w and r. If A^t denotes the transposed of matrix A, then,

$$A^t \times p = \begin{bmatrix} L^{(a)} & T^{(a)} \\ L^{(b)} & T^{(b)} \end{bmatrix} \begin{bmatrix} w \\ r \end{bmatrix} = \begin{bmatrix} C^{(a)} \\ C^{(b)} \end{bmatrix} \tag{7}$$

Given that $(A^t)^{-1} = \left(A^{-1}\right)^t$, the solution to (6) is

$$\begin{bmatrix} w \\ r \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} C^{(a)} \\ C^{(b)} \end{bmatrix} = \begin{bmatrix} A_{11}C^a + A_{21}C^b \\ A_{21}C^a + A_{22}C^b \end{bmatrix} \tag{8}$$

Comparison of the solutions to (6) and (8), thus, leads to the condition that the marginal product of each factor is equal to its rate of remuneration.

$$\begin{aligned} \frac{\partial C}{\partial L} &= w \\ \frac{\partial C}{\partial T} &= r \end{aligned} \tag{9}$$

The conclusion is as follows. When the consumption good is produced by means of primary factors *alone*, the purpose of the MF appears to be attained: in this very particular case, the differentiability of the function seems to allow us to conclude that income distribution is determined by the principle of marginal productivity.

5. The inclusion of capital goods in the MF

As seen in Section 3, consideration of heterogenous capital goods entails particular difficulties for marginal productivity theory. As will now be shown, problems also arise with the MF and its marginal products. We shall start by considering the case, examined in Samuelson (2007, pp. 255–62), of a stationary economy where two capital goods are produced and employed,[25] and then go on to examine the issue under non-stationary prices so as to confirm that difficulties also emerge under this framework.

25 In their first development of the MF, Samuelson and Etula (2006) referred exclusively to the case with just one kind of capital good. In that framework, as will become clear later, the problems we intend to show in the present paper cannot arise. For this reason, we consider directly the other case, with two kinds of capital goods, addressed by Samuelson in his (2007) article.

5.1 Stationary conditions

Let us consider an economy where two commodities, corn and iron, are produced by means of labour and two circulating capital goods, seed corn and iron. Corn is, thus, both a capital good and the only consumption good. As in the primary-factors-only case, the quantities of corn (and iron) produced are considered in gross terms. Table 2 summarises the alternative methods available.

The new linear programming problem, P2, is as follows:

$$\mathbf{P2}: \qquad \text{Max}_{y_i}: \; Q = \sum_{i=a}^{d} Q^{(i)} y_i$$

subject to:

$$\sum_{i=a}^{g} L^{(i)} \; y_i = L \tag{P2.1}$$

$$\sum_{i=a}^{g} K_1{}^{(i)} \; y_i = K_1 \tag{P2.2}$$

$$\sum_{i=a}^{g} K_2{}^{(i)} \; y_i = K_2 \tag{P2.3}$$

$$\sum_{i=e}^{g} F^{(i)} \; y_i = K_2 \tag{P2.4}$$

$$y_i \geq 0, \;\; i = a, b, c, d, e, f, g \tag{P2.5}$$

(P2.1) – (P2.3) entail the full employment of labour and of the given endowments of seed corn and iron, and are hence analogous to conditions

Table 2. Production methods with heterogeneous capital goods

Method	Labour	Seed corn	Iron capital	Corn produced	Iron produced
a	$L^{(a)}$	$K_1{}^{(a)}$	$K_2{}^{(a)}$	$Q^{(a)}$	0
b	$L^{(b)}$	$K_1{}^{(b)}$	$K_2{}^{(b)}$	$Q^{(b)}$	0
c	$L^{(c)}$	$K_1{}^{(c)}$	$K_2{}^{(c)}$	$Q^{(c)}$	0
d	$L^{(d)}$	$K_1{}^{(d)}$	$K_2{}^{(d)}$	$Q^{(d)}$	0
e	$L^{(e)}$	$K_1{}^{(e)}$	$K_2{}^{(e)}$	0	$F^{(e)}$
f	$L^{(f)}$	$K_1{}^{(f)}$	$K_2{}^{(f)}$	0	$F^{(f)}$
g	$L^{(g)}$	$K_1{}^{(g)}$	$K_2{}^{(g)}$	0	$F^{(g)}$

(P1.1) – (P1.2) of P1. Condition (P2.4) is instead specific to the stationary context now examined. It establishes that the gross production of iron must be equal to the initial endowment of iron and thus entails the stationary nature of the economic system.[26]

As in P1, the set Ω is defined as the set of all vectors $\omega = [L, K_1, K_2]$ such that P2 has a solution. For each ω, there is a vector of activity levels $y(\omega)$ that is a solution of P2. The corresponding MF in this case is

$$Q = MF\ (L, K_1, K_2) = \sum_{i=a}^{d} Q^{(i)} y_i(\omega) \tag{10}$$

It is known that in this case, the vector $y(\omega)$ has at most four positive components and will be non-degenerate if it has exactly four positive components. It should be recalled that the MF will be differentiable in ω if $y(\omega)$ is a non-degenerate vector. Let us now define the set $\Omega^{c,d,e,f}$ as the subset of Ω such that $y_i(\omega) > 0,\ \ i = c, d, e, f$ and 0 otherwise to obtain the following MF:

$$Q = MF\ (L, K_1, K_2) = \sum_{i=c}^{f} Q^{(i)} y_i(\omega) \tag{11}$$

with:

$$B \times y(\omega) \equiv \begin{bmatrix} L^{(c)} & L^{(d)} & L^{(e)} & L^{(f)} \\ K_1^{(c)} & K_1^{(d)} & K_1^{(e)} & K_1^{(f)} \\ K_2^{(c)} & K_2^{(d)} & K_2^{(e)} & K_2^{(f)} \\ 0 & 0 & F^{(e)} & F^{(f)} \end{bmatrix} \times \begin{bmatrix} y_c(\omega) \\ y_d(\omega) \\ y_e(\omega) \\ y_f(\omega) \end{bmatrix} = \begin{bmatrix} L \\ K_1 \\ K_2 \\ K_2 \end{bmatrix} \tag{12}$$

By solving system (12) and substituting the $y_i(\omega),\ \ i = c, d, f, g$ in (11), it is possible to obtain the MF whose partial derivatives should determine income distribution in the stationary economy on the same footing as in the primary-factors-only case.

26 Once again, we follow Samuelson, who derives the MF by maximising corn gross production rather than corn net production. Under their hypotheses, however, both procedures are equivalent. The reason is that if the employment of corn capital is exogenously given as is the case in Samuelson's argument (because the endowment of corn is given and must be fully employed), and the endowment of corn must be reproduced by the stationary assumption, then gross and net outputs differ by a constant and the maximisation of gross corn production is equivalent to the maximisation of net corn.

As we shall see, however, this solution cannot be accepted for at least two different reasons.

First, the determination of the marginal products by means of the MF *necessitates* the assumption that the different methods employ the same capital goods in different proportions. As argued here in Section 3, however, the employment of different methods of production will generally require the employment of capital goods of a different kind. In this case, the marginal product of a single capital good will generally be zero, as will the marginal contribution of an additional worker not given the necessary equipment to work with.[27] It must in any case be acknowledged that Samuelson's assumption that only a *discrete* number of methods (seven in our example) employ the same capital goods appears to be weaker than the assumption that there is *continuum* of methods employing the same capital goods in all possible proportions, as is the case with a (traditional) differentiable production function of the form $y = f(k_1, k_2,\ldots,k_n)$.

The second reason can be seen in relation to the dual problem of P2, namely D2, the minimisation of gross production costs for corn.

$$\mathbf{D2}: \qquad \mathrm{Min}_{w,\sigma_1,\sigma_2,\,\pi}: \; wL + \sigma_1 K_1 + (\sigma_2 - \pi)K_2$$

subject to:

$$wL^{(a)} + \sigma_1 K_1{}^{(a)} + \sigma_2 K_2{}^{(a)} \geq Q^{(a)} \tag{D2.1}$$
$$wL^{(b)} + \sigma_1 K_1{}^{(b)} + \sigma_2 K_2{}^{(b)} \geq Q^{(b)} \tag{D2.2}$$
$$wL^{(c)} + \sigma_1 K_1{}^{(c)} + \sigma_2 K_2{}^{(c)} \geq Q^{(c)} \tag{D2.3}$$
$$wL^{(d)} + \sigma_1 K_1{}^{(d)} + \sigma_2 K_2{}^{(d)} \geq Q^{(d)} \tag{D2.4}$$
$$wL^{(e)} + \sigma_1 K_1{}^{(e)} + \sigma_2 K_2{}^{(e)} \geq \pi F^{(e)} \tag{D2.5}$$
$$wL^{(f)} + \sigma_1 K_1{}^{(f)} + \sigma_2 K_2{}^{(f)} \geq \pi F^{(f)} \tag{D2.6}$$
$$wL^{(g)} + \sigma_1 K_1{}^{(g)} + \sigma_2 K_2{}^{(g)} \geq \pi F^{(g)} \tag{D2.7}$$

where w is the real wage and π the price of iron in terms of corn, while σ_1 and σ_2 are the gross rental prices of seed corn and iron, respectively. It is known from duality that – for $[L, K_1, K_2] \in \Omega^{\mathrm{c,d,e,f}}$ – the break-even conditions consist of the set of equations (D2.3)–(D2.6), namely those that allow the employment of methods (c)–(f), while the remaining constraints will be satisfied as strict inequalities since, by construction, methods (a), (b), and (g) are not employed at positive levels.

27 In Section 3, fn. 14, we have seen that Samuelson (1962) himself was very aware of this fact when he built his Surrogate Production Function.

These four conditions are sufficient at first sight to determine the four prices w, π, σ_1 , and σ_2. If B^t is the transpose of matrix B, the four prices are determined by the following conditions:

$$B^t \times p \equiv \begin{bmatrix} L^{(c)} & K_1^{(c)} & K_2^{(c)} & 0 \\ L^{(d)} & K_1^{(d)} & K_2^{(d)} & 0 \\ L^{(e)} & K_1^{(e)} & K_2^{(e)} & F^{(e)} \\ L^{(f)} & K_1^{(f)} & K_2^{(f)} & F^{(f)} \end{bmatrix} \times \begin{bmatrix} w \\ \sigma_1 \\ \sigma_2 \\ -\pi \end{bmatrix} = \begin{bmatrix} Q^{(c)} \\ Q^{(d)} \\ 0 \\ 0 \end{bmatrix} \qquad (13)$$

This is not the case, however. To see this, it should be first recalled that, due to the stationary assumption, both seed corn and iron *must* necessarily be reproduced in equilibrium, with the implication that both capital goods should yield the same return on their supply prices (cost of production). Otherwise, only the capital good that yields the highest return will be reproduced in the following period, contrary to what is implied by the condition of stationariness.

This consideration implies the *need* to add an additional equation imposing the required uniformity of returns on supply prices, namely:

$$1 + i^* \equiv \sigma_1 = \frac{\sigma_2}{\pi} \qquad (14)$$

where i^* is the effective return on the supply price of both capital goods.[28]

28 To see more clearly the problems raised by the non-fulfilment of condition (14), consider the following example: let i_1 and i_2 be the (net) rates of return on the supply prices of K_1 and K_2, respectively. Since corn is the numéraire, condition $\sigma_1 = 1 + i_1$ holds for the case of corn and condition $\frac{\sigma_2}{\pi} = 1 + i_2$ for the case of iron. Let us further define p_1^D and p_2^D as the *demand*, or *selling*, prices of capital goods 1 and 2, respectively. These are the *maximum* prices investors are willing to pay to buy K_1 and K_2 and must be equal, in equilibrium, to the present value of the sum of the future yields of each capital good. Investors will be indifferent between buying K_1 or K_2 as long as the return on demand prices is the same. Now, consider the case where the solution to system (13) implies $i_1 > i_2$. In this situation, for K_1, demand and supply prices coincide, namely $1 = p_1^D = \frac{\sigma_1}{1+i_1}$. For capital good 2, on the other hand, its demand price must satisfy the following condition: $p_2^D \leq \frac{\sigma_2}{1+i_1}$. Otherwise, nobody will be willing to invest in this capital good. But given that $i_1 > i_2$, this means that $p_2^D < \pi$. In other words, the fact that $i_1 > i_2$ implies that the price at which investors are willing to buy K_2 is not sufficient to cover production costs, and hence the capital good will not be reproduced in the following period, violating the assumption of stationariness.The implication is that condition (14) should be added to allow the reproduction of both capital goods.

It should then be noted, however, that the system (13)–(14) is a system of five equations in four unknowns, and hence *does* not *generally admit a solution*. System (13) is, in fact, a linear system that will generally admit one and only one solution, and there is no reason to hope that this solution will also satisfy condition (14).

The implication is that there is, in general, no system of stationary prices that is compatible with the simultaneous use of the four methods *(c)–(d)–(e)–(f)* at positive levels, and therefore *a differentiable Master Function will not generally exist.*

The following two points deserve to be stressed. The first is that when faced with this issue, Samuelson adopts what seems to be a contradictory attitude. On the one hand, he dismisses the problem as irrelevant: he does not see the need to add the additional condition (14) to his system of "break-even" equations, but this is simply because he wrongly identifies the lack of uniformity of returns on the supply prices of the capital goods with the lack of uniformity of their own rates of interest (2007, pp. 260–1), a divergence that could only emerge, however, as a result of including price changes in the definition of the equilibrium, but which are instead *ruled out* here by the assumption of stationary prices.[29] On the other hand, in a footnote, he comes very close to admitting the problem, since he accepts that, due to the arbirtiraly given endowments of capital goods, there will be a divergency in the returns on their costs of produciton, divergence that will in its turn cause the compostion of capital to change in the follwing periods. "Generically", he writes:

> for most exogenous $(K_1/L,\ K_2/L)$ endowments, $r_1^* \neq r_2^*$! [i.e. $i_1^* \neq i_2^*$, in our notation]. So to speak, this serves the economy *to leave the stationary state* and proceed with generalized Ramsey (1928) dynamics (Samuelson 2007, p. 258, fn. 5, emphasis added)

It remains largely implicit in this passage that relative prices and income distribution will generally not remain stationary under these circumstances. But if the non-uniformity of returns implies, on the one hand, that the economy will be forced to *leave* stationariness and supposedly follow an intertemporal equilibrium path, it is hard, on the other, to see on what grounds Samuelson can claim that a system like

29 For a detailed discussion about the difference between commodities' own rates of interest and the returns on the supply prices of capital goods, see Garegnani (2003, Appendix II [A]).

(13) *does* determine "a stationary maintained" supply-and-demand equilibrium (Samuelson 2007, p. 260).[30]

The second observation is the following: in a comment on the MF, Garegnani (2007, pp. 584–5) has already noticed that Samuelson's orginal system of "break-even conditions" presents the problem of not satisfying the required uniformity of returns. However, our point here is considerably stronger, since what we have shown is that *the problem cannot be solved*: the missing Equation (14) cannot be added to the system of equations without causing the whole construction to collapse. In fact, we have just seen that, when the attempt is made to include Equation (14) among the equilibrium conditions, the MF *will not be a differentiable function,* hence the "non-neoclassical" marginal products will not exist. It seems thus hard to escape the conclusion that the MF is unable to explain income distribution in this more general case.

5.2 Non-stationary prices

It might appear that the problem addressed in the previous section is due to the specific stationary character of the economy considered there. As we shall now see, however, difficulties arise in a *non-stationary-intertemporal* framework too.

Let us consider an intertemporal equilibrium over three periods $t = 0, 1, 2$. Production takes place during periods $t = 0$ and $t = 1$, and consumption in periods $t = 1$ and $t = 2$. The possibilities of production are the same as those considered in the stationary framework of the previous section: there are two capital goods, seed corn and iron, produced by means of themselves and labour, and corn is the only consumption good. The methods available for production are those already described in Table 2. Production is considered in gross terms.

For each of the periods in which production takes place, there is a MF that emerges as the result of a maximisation problem of gross corn production. In particular, we have the following problem for the period $t = 1$:

$$\mathbf{P3}: \qquad \text{Max } Q_1 = \sum_{i=a}^{d} Q_1^{(i)} y_{i,0}$$

30 See also Samuelson (2007, p. 259), where the author claims that a system of equations like (13) is both "necessary and sufficient for characterising competitive distribution equilibrium".

subject to:

$$\sum_{i=a}^{g} L^{(i)} y_{i,0} = L_0 \tag{P3.1}$$

$$\sum_{i=a}^{g} K_1^{(i)} y_{i,0} = K_{1,0} \tag{P3.2}$$

$$\sum_{i=a}^{g} K_2^{(i)} y_{i,0} = K_{2,0} \tag{P3.3}$$

$$\sum_{i=e}^{g} F_1^{(i)} y_{i,0} = F_1 \tag{P3.4}$$

$$y_{i,0} \geq 0, \quad i = a, b, c, d, e, f, g \tag{P3.5}$$

where the variable Q_t is the amount of corn produced in period $t-1$ and consumed in t (e.g. Q_1 is consumed in $t=1$); $y_{i,t}$ the activity level of method i employed in period t; L_t the amount of labour available in period t; K_{ht} the endowment of the capital good of kind h (i.e. $h=1$ for seed corn, $h=2$ for iron) available for employment in period t; and F_t the quantity of iron produced during period $t-1$ and delivered one period later (in t). Conditions (P3.1)–(P3.3) thus stand for the full employment of labour and of the endowments of seed corn and iron, respectively, in period $t=1$. Condition (P3.4) is instead the market-clearing condition for the iron produced during the first period. Given the non-stationary framework now considered, it may well be the case that the quantity of corn produced during the period is different from the initial endowment of seed corn, i.e. $F_1 \neq K_{2,0}$. Conditions (P3.5) are the usual non-negative constraints imposed on the activity levels y_i in the first period.

From P3, it is possible to derive the Master Function MF_1 that corresponds to the first period:

$$Q_1 = MF_1\left(L_0,\ K_{1,0},\ K_{2,0}, F_1\right) = \sum_{i=a}^{d} Q_1^i y_{i,0} \tag{15}$$

Let us now turn to P4, the problem faced in the second period, when there is no production of iron:

$$P\mathbf{4}: \qquad \text{Max } Q_2 = \sum_{i=a}^{d} Q_1^i y_{i,1}$$

subject to:

$$\sum_{i=a}^{d} L_1^{(i)} y_{i,1} = L_1 \tag{P4.1}$$

$$\sum_{i=a}^{d} K_1^{(i)} y_{i,1} = K_{1,1} \tag{P4.2}$$

$$\sum_{i=a}^{d} K_2^{(i)} y_{i,1} = K_{2,1} \tag{P4.3}$$

$$y_{i,1} \geq 0,\ i = a,\ b,\ c,\ d \tag{P4.4}$$

Conditions (P4.1)−(P4.3) represent the full employment of labour and of the initial endowments of seed corn and iron in the second period, while (P4.4) represents the non-negative constraints on levels of activity. It should be noted that in problem P4, there is no condition analogous to (P3.4). The reason for this should be clear. There is no production in the last period ($t = 2$) and therefore no reason to undertake the production of capital goods in the previous one ($t = 1$).

The MF corresponding to the second period, namely MF_2, can be derived from P4:

$$Q_2 = MF_2\left(L_0,\ K_{1,1},\ K_{2,1}\right) = \sum_{i=a}^{d} Q_1^i y_{i,1} \tag{16}$$

Now, for the same reasons addressed in the previous sections, it is known that MF_1 will be differentiable if *exactly* four methods at positive levels are employed in the first period, whereas MF_2 will be differentiable if there are exactly three methods employed in the second period. As is known, however, this means that exactly four conditions must hold as break-even conditions while three break-even conditions must hold for P4. The remaining conditions will hold as strict inequalities, indicating that their use will not be profitable.

Let us then assume that the solution to P3 is such that methods *(c)–(d)–(e)–(f)* are employed at positive levels, i.e. $y_{i,0} > 0,\ i = c,\ d,\ e, f$, and 0 otherwise, while the solution to P4 entails the employment of methods *(a)–(b)–(c)*, i.e. $y_{i,1} > 0\ i = a,\ b,\ c$, and 0 otherwise. The differentiability of both MF_1 and MF_2 requires the following break-even conditions to hold simultaneously:

$$
\begin{aligned}
L_0^{(c)} w_0 + K_1^{(c)} p_{1,0} + K_2^{(c)} p_{2,0} &= Q^{(c)} p_{1,1} \\
L_0^{(d)} w_0 + K_1^{(d)} p_{1,0} + K_2^{(d)} p_{2,0} &= Q^{(d)} p_{1,1} \\
L_0^{(e)} w_0 + K_1^{(e)} p_{1,0} + K_2^{(e)} p_{2,0} &= F^{(e)} p_{2,1} \\
L_0^{(f)} w_0 + K_1^{(f)} p_{1,0} + K_2^{(f)} p_{2,0} &= F^{(f)} p_{2,1} \\
L_1^{(a)} w_1 + K_1^{(a)} p_{1,1} + K_2^{(a)} p_{2,1} &= Q^{(c)} p_{1,2} \\
L_1^{(b)} w_1 + K_1^{(b)} p_{1,1} + K_2^{(b)} p_{2,1} &= Q^{(c)} p_{1,2} \\
L_1^{(c)} w_1 + K_1^{(c)} p_{1,1} + K_2^{(c)} p_{2,1} &= Q^{(c)} p_{1,2}
\end{aligned}
\tag{17}
$$

where w_t is for the present value of the wage rate in period t and $p_{h,t}$ the present value of commodity h in period t.

Now, if the value of seed corn is taken as the numéraire, i.e. $p_{1,0} = 1$, system (17) is a linear system of seven equations in six unknowns: $w_0, p_{2,0}, w_1, p_{1,1}, p_{1,2}, p_{2,1}$. In other words, the system will again be generally overdetermined, with the implication that the marginal products of MF_1 and MF_2 will again generally not exist. It should be noted, however, that while in the case of the framework examined in the previous section, the over-determinacy is due to the fact that the arbitrarily given endowments of capital goods inputs are *incompatible* with the stationary conditions there assumed, the intertemporal setting with non-stationary prices *is* consistent with the arbitrarily given initial endowment. System (17) is overdetermined, however, because the number of methods that must be in use in order for the *MF* to be differentiable is larger than the number of methods that generally allows the full employment of the initial endowments.[31]

6. Conclusions

Can neoclassical theory dispense with marginal productivity? There is no doubt that Arrow and Debreu's proof of equilibrium existence is completely independent of it. As for multiplicity and stability, they are issues of such complexity that it is not clear whether the presence of

31 Our analysis, thus, seems to confirm a conjecture made by Opocher (2008), who "doubted" (p. 109) that the MF could be actually extended to cover the case of heterogeneous capital.

differentiable production functions can be of any help.[32] Differentiability is required, however, for local comparative statics, the only kind that can be applied if multiple equilibria cannot be ruled out. Moreover, all the neoclassical theories of growth (both endogenous and exogenous) and most of mainstream macroeconomics derive their results (and policy prescriptions) on the basis of production functions and marginal equalities.

The possibility of using these tools is unquestionably important enough to have attracted the attention of one of the most important neoclassical authors, namely Paul Samuelson, the founder of the modern neo-Walrasian approach together with Hicks. Furthermore, he made not just one but two different attempts – following opposite approaches – to justify marginal equalities analytically.

The first was based on the surrogate production function over 50 years ago, when Samuelson tried to obtain a differentiable "surrogate" production function by the aggregation of heterogeneous capital goods into a single input: an amount of "jelly". However, as pointed out in the reconstruction of the debate provided in Sections 2 and 3, capital goods cannot generally be aggregated and treated as a single production factor without the serious possibility of paradoxical results arising. In particular, further problems arise if the aggregation is performed by using prices, as in Samuelson's case.

The second is far more recent and instead regards the possibility of having marginal products for the individual capital goods as well as the original inputs. As argued in Section 3, if marginal productivity is associated, as is usually the case, with a change in the methods of production in use, then the high degree of specialisation of capital goods makes it impossible to have an economically meaningful marginal product for each of them, because a change in the methods in use entails a change in the kinds of capital goods employed. The attempt made by Samuelson (together with Erkko Etula) to circumvent this problem, therefore, consists of basing marginal productivity, as shown in Sections 4 and 5, on the coexistence – rather than change – of different methods of production. While it does not eliminate the need to assume that the different methods employ the same capital goods, the assumption that there is a *continuum* of methods that employ the same capital goods in different proportions is considerably weakened, since the MF only assumes the existence of *discrete* – and

32 Fratini (2013a) provides a discussion of the set of hypotheses required in order to claim that reswitching is a possible source of equilibrium instability. Moreover, it is shown in Fratini (2007) for an overlapping generation model that some multiple equilibria are due to reswitching. Reswitching appears to be impossible, however, in the case of differentiable production functions, although the point is not crystal clear (see also Hatta, 1976).

comparatively few – methods of production with capital goods in common.

As often happens in the history of the marginalist theory, while this approach appears to work quite well in the primary-inputs-only case (Section 4), difficulties arise when capital goods are taken into consideration. In particular, as shown in Section 5, the coexistence of so many methods for the same commodity as to have a differentiable Samuelson–Etula MF is impossible – in the sense that it is non-optimal – in the case both of stationary relative prices and of Arrow–Debreu intertemporal prices. Therefore, once again, the marginal productivity theory does not work.

In conclusion, it should be pointed out that above and beyond the problems of the MF, this attempt as well as all the other contemporary neoclassical efforts that still rely on marginal productivity theory to explain value and distribution are in any case useful because they offer an opportunity to reopen the debate on capital and thus to examine and clarify points that may have been overlooked or inadequately addressed in the existing literature on capital.

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Abstract

The paper addresses the ambiguity that surrounds the conception of capital and its role in neoclassical price-and-distribution theory. The difficulties encountered in the various attempts to define the marginal product either of capital or of a capital good are recalled and the conclusion is drawn that neither concept appears theoretically sound. This historical reconstruction is combined with critical discussion of the recent attempt by Paul Samuelson to determine income distribution by means of the "Master Function", a device previously developed and presented by Samuelson himself with Erkko Etula, and its "non-neoclassical" marginal products. Rather than the existence of a continuum of alternative technical possibilities, this construction assumes the simultaneous use of a discrete number of methods of production for the same commodity. Even though each technique employs the inputs in fixed proportions, the coexistence of various techniques permits the full employment of an arbitrarily given vector of input endowments. As is shown here, however, the coexistence of methods required for the differentiability of the Master Function can take place, if heterogenous capital goods are used in production, neither in the case with stationary relative prices nor in the non-stationary Arrow–Debreu framework.

Keywords

Capital, capital goods, marginal productivity, Master Function, Samuelson