Journal of Algebra 470 (2017) 91–121

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On syzygies over 2-Calabi–Yau tilted algebras ☆

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A R T I C L E   I N F O

Article history:
Received 26 January 2016
Available online 8 September 2016
Communicated by Luchezar L. Avramov

Keywords:
Syzygy
Cohen–Macaulay module
2-Calabi–Yau tilted algebra
Cluster-tilted algebra
Igusa–Todorov function
Triangulated surface
Punctured polygon

A B S T R A C T

We characterize the syzygies and co-syzygies over 2-Calabi–Yau tilted algebras in terms of the Auslander–Reiten translation and the syzygy functor. We explore connections between the category of syzygies, the category of Cohen–Macaulay modules, the representation dimension of algebras and the Igusa–Todorov functions. In particular, we prove that the Igusa–Todorov dimensions of d-Gorenstein algebras are equal to d.

For cluster-tilted algebras of Dynkin type $\mathbb{D}$, we give a geometric description of the stable Cohen–Macaulay category in terms of tagged arcs in the punctured disc. We also describe the action of the syzygy functor in a geometric way. This description allows us to compute the Auslander–Reiten quiver of the stable Cohen–Macaulay category using tagged arcs and geometric moves.

☆ The first author was supported by CONICET, PICT 2013-0799 ANPCyT. Both authors were supported by the National Science Foundation grant DMS-1254567, by the Universidad Nacional de Mar del Plata, and by the University of Connecticut.

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http://dx.doi.org/10.1016/j.jalgebra.2016.08.035
0021-8693/

1. Introduction

The concept of a 2-Calabi–Yau tilted algebra is a natural generalization of the concept of a cluster-tilted algebra. A cluster-tilted algebra is the endomorphism algebra of a cluster-tilting object in the cluster category of a hereditary algebra. The 2-Calabi–Yau tilted algebras are obtained by replacing the cluster category by a 2-Calabi–Yau triangulated category. Examples of 2-Calabi–Yau tilted algebras that are not cluster-tilted are the Jacobian algebras of the quivers with potential arising from a cluster algebra of non-acyclic type. In particular, this includes the Jacobian algebras from triangulated surfaces other than the disc with 0, 1, or 2 punctures and the annulus without punctures. Cluster categories and cluster-tilted algebras were introduced in [11,14,12]. The generalization to 2-Calabi–Yau categories was given in [24]. Cluster algebras were defined in [19] and their quivers with potentials in [17]. For triangulated surfaces, the Jacobian algebras were introduced in [3,25].

From now on, let B be a 2-Calabi–Yau tilted algebra. We study the stable category $\underline{\mathrm{CM}}(\mathrm{B})$ of Cohen–Macaulay modules over B. A B-module M is (maximal) Cohen–Macaulay if $\mathrm{Ext}^i_{\mathrm{B}}(M, \mathrm{B}) = 0$, for all $i > 0$. It was proved in [24] that B is d-Gorenstein, with $d = 0$ or 1, and that the stable category of Cohen–Macaulay modules is 3-Calabi–Yau. We denote by $\mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$ the set of indecomposable objects in $\underline{\mathrm{CM}}(\mathrm{B})$. Let Ω be the syzygy functor and τ the Auslander–Reiten translation. Our first main result gives the following characterization.

Theorem 1. *Let M be an indecomposable* B*-module. Then the following statements are equivalent.*

(1) *M is a non-projective syzygy,*
(2) $M \in \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$,
(3) $\Omega^2 \tau M \simeq M$,
(4) *M is non-projective and $\Omega^{-2} M \simeq \tau M$.*

We obtain the following corollary on selfinjective 2-Calabi–Yau tilted algebras.

Corollary 1. *Suppose that* B *is selfinjective and such that the Nakayama functor has finite order m. Let M be an indecomposable, non-projective* B*-module. Then $\tau^{2m} M \cong M$. In particular, if* B *is symmetric then $\tau^2 M \cong M$.*

The representation dimension of cluster-tilted algebras has been studied in [20] for the special case of cluster concealed algebra, thus cluster-tilted algebras whose corresponding cluster-tilting object is preprojective. The authors of [20] have shown that in this case the representation dimension is at most 3. Cluster concealed algebras may or may not be tame and, on the other hand, tame cluster-tilted algebras need not be cluster concealed. Using results of [9] and [29], we extend the result to tame cluster-tilted algebras.

Corollary 2. *The representation dimension of a tame cluster-tilted algebra is at most 3.*

We also study the two Igusa–Todorov functions ϕ and ψ. These functions were introduced in [22] to study the relation between the representation dimension and the finitistic dimension of an algebra. The supremum of ϕ, or ψ respectively, is called the ϕ-dimension, respectively ψ-dimension, of the algebra. We obtain the following result in a more general setting, namely for d-Gorenstein algebras. This result has been proved independently in [27].

Theorem 2. *For any d-Gorenstein algebra Λ, ϕ-dim $\Lambda = \psi$-dim $\Lambda = d$.*

Thus for 2-Calabi–Yau tilted algebras, both ϕ-dim and ψ-dim are at most 1.

In the second part of the paper, we give a geometric realization of the stable Cohen–Macaulay category $\underline{\mathrm{CM}}(\mathrm{B})$ for all 2-Calabi–Yau tilted algebras from surfaces without punctures as well as for those from the disc with one puncture. The unpunctured case corresponds to the gentle algebras introduced in [3] and the case of the punctured disc to the cluster-tilted algebras of type $\mathbb{D}$. While the unpunctured case is very simple, the case of the punctured disc is less so and requires a refinement of the combinatorial model of the cluster category of [31]. In [16], it was proved that in this case, $\underline{\mathrm{CM}}(\mathrm{B})$ is equivalent to a union of stable module categories of selfinjective cluster-tilted algebras. We describe the indecomposable objects in $\underline{\mathrm{CM}}(\mathrm{B})$ in Theorem 5.10 as arcs in the punctured disc and we also describe the action of the syzygy functor Ω on these arcs in Theorem 5.7. Furthermore, we construct the Auslander–Reiten quiver of $\underline{\mathrm{CM}}(\mathrm{B})$ in terms of arcs and elementary moves.

The paper is organized as follows. In section 2, we recall basic facts about Gorenstein algebras and prove our result on the Igusa–Todorov dimensions. Section 3 is devoted to the study of 2-Calabi–Yau tilted algebras and contains Theorem 1 and its corollaries. The geometric realizations of the stable Cohen–Macaulay categories are given in section 4 for the unpunctured case and in section 5 for the punctured disc.

2. Gorenstein Artin algebras

Let R be a commutative artinian ring. In this section we consider Λ an Artin R-algebra, and $\mathrm{mod}\,\Lambda$ the category of finitely generated right Λ-modules. Given a Λ-module M, denote the kernel of a projective cover by ΩM. ΩM is called the syzygy of M. We let $\Omega^0 M = M$. The projective dimension of M, proj.dim M, is the minimal length among all finite projective resolutions of M, and M has infinite projective dimension if there is no finite projective resolution. Similarly, the injective dimension inj.dim M, is the minimal length among all finite injective resolutions of M, and M has infinite injective dimension if there is no finite injective resolution. We denote by D the duality functor $\mathrm{Hom}_R(-, J)$ where J is the injective envelope of the direct sum of the simple modules, [6, II.3]. When Λ is a finite dimensional k-algebra over an algebraically closed field, there exists a quiver Q such that Λ is the quotient of the path algebra kQ by an admissible ideal I. In this case, we denote by e_i the idempotent associated to the lazy path at the vertex i, and by $P(i) = e_i\Lambda$ the corresponding indecomposable projective module, by $I(i) = D(\Lambda e_i)$ the indecomposable injective module and by $S(i)$ the simple top of $P(i)$, see for example [4, Chapter III] or [32, Chapter 2].

Definition 2.1. An Artin algebra Λ is said to be *Gorenstein of dimension* d (d-Gorenstein) if $\operatorname{proj.dim} D(\Lambda) = \operatorname{inj.dim} \Lambda = d < \infty$.

Definition 2.2. A Λ-module M is said to be *Cohen–Macaulay* if $\operatorname{Ext}^i_\Lambda(M, \Lambda) = 0$ for all $i > 0$. A Λ-module N is *Co-Cohen–Macaulay* if $\operatorname{Ext}^i_\Lambda(D\Lambda, N) = 0$ for all $i > 0$.

Denote by $\mathrm{CM}(\Lambda)$ the full subcategory of $\operatorname{mod}\Lambda$ of Cohen–Macaulay modules. The category $\mathrm{CM}(\Lambda)$ is an exact subcategory, it is Frobenius and its projective–injective objects are precisely the projective modules in $\operatorname{mod}\Lambda$. Correspondingly, the category of Co-Cohen–Macaulay modules $\mathrm{CoCM}(\Lambda)$ is Frobenius and its projective–injective objects are the injective modules in $\operatorname{mod}\Lambda$. The stable category $\underline{\mathrm{CM}}(\Lambda)$ is a triangulated category whose inverse shift is given by the usual syzygy operator in $\operatorname{mod}\Lambda$, Ω_Λ. Dually, the category $\underline{\mathrm{CoCM}}(\Lambda)$ is a triangulated category whose shift is given by the usual co-syzygy operator in $\operatorname{mod}\Lambda$, Ω^{-1}_Λ. There are triangle equivalences between $\underline{\mathrm{CM}}(\Lambda)$, $\underline{\mathrm{CoCM}}(\Lambda)$ and the singularity category given by the localization $\mathcal{D}^b(\Lambda)\mathcal{S}^{-1}$, where $\mathcal{S}$ is the set of morphisms whose cone lies in $\mathcal{K}^b(proj\ \Lambda)$. See [13, Section 4.8].

Remark 2.3. The Auslander–Reiten (AR) translations τ, τ^{-1} in $\operatorname{mod}\Lambda$ induce quasi-inverse triangle equivalences $\tau : \underline{\mathrm{CM}}(\Lambda) \to \underline{\mathrm{CoCM}}(\Lambda)$ and $\tau^{-1} : \underline{\mathrm{CoCM}}(\Lambda) \to \underline{\mathrm{CM}}(\Lambda)$, see [8, Chapter X].

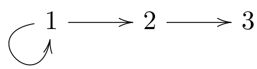
2.1. Igusa–Todorov functions

The Igusa–Todorov functions, ϕ and ψ, were introduced in order to study the relation between the representation dimension and the finitistic dimension in the context of Artin algebras. In a sense, they generalize the concept of projective dimension, meaning that for a module M of finite projective dimension we have $\operatorname{proj.dim} M = \phi(M) = \psi(M)$. The important change occurs when the projective dimension of a module is not finite. We only give the definitions of Igusa–Todorov functions. For further details, see [22]. Let K_0 be the abelian group generated by the symbols $[M]$ for each isomorphism class M in $\operatorname{mod}\Lambda$, modulo the relations $[M \oplus N] = [M] + [N]$ and $[P] = 0$ for P projective. Define the linear morphism $L : K_0 \to K_0$ via $L[M] = [\Omega M]$, where Ω is the syzygy operator. For M in $\operatorname{mod}\Lambda$, consider the direct sum $M = \bigoplus_{i=1}^{k} M_i$, where all M_i are indecomposable. Denote by $L^0(M) = \langle [M_1], \dots, [M_k] \rangle$ the subgroup of K_0 generated by the symbols $[M_1], \dots, [M_k]$, then denote by $L^1(M) = L(L^0[M])$, and so on.

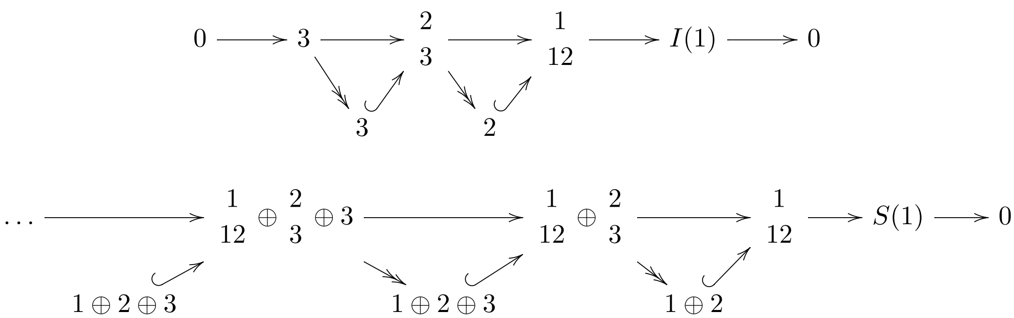
Definition 2.4. The *first Igusa–Todorov function* is defined by $\phi(M) = t$, where t is the smallest integer such that $L : L^{t+s}(M) \to L^{t+s+1}(M)$ is an isomorphism for all $s \geqslant 0$.

Definition 2.5. The *second Igusa–Todorov function* is given by $\psi(M) = \phi(M) + k$, where k is the largest finite projective dimension in a summand of the module $\Omega^{\phi(M)} M$.

Example 2.6. Let Λ be the algebra given by kQ/rad^2, where Q is the quiver



We compute $\phi(M)$ and $\psi(M)$ for the module $M = I(1) \oplus S(1)$. We have the following projective resolutions:



Note that $[S(3)] = 0$, since $S(3)$ is projective. The sequence of abelian groups is given by:

$$\begin{aligned} L^0(M) &= \langle [I(1)], [S(1)] \rangle \xrightarrow{L} \langle [S(2)], [S(1)] + [S(2)] \rangle \xrightarrow{L} \langle [S(1)] + [S(2)] \rangle \\ &\xrightarrow{L} \langle [S(1)] + [S(2)] \rangle \to \cdots \end{aligned}$$

Starting from the second syzygy, the rank of the abelian groups is one. So, $\phi(M) = 2$. And $\psi(M) = 3$, because proj.dim $S(2) = 1$.

The ϕ-dimension and ψ-dimension of an algebra Λ are given by $\phi\text{-dim}\,\Lambda = \sup\{\phi(M) : M \in \text{mod}\,\Lambda\}$ and $\psi\text{-dim}\,\Lambda = \sup\{\psi(\Lambda) : M \in \text{mod}\,\Lambda\}$.

Now consider Λ a d-Gorenstein Artin algebra. As stated before, Ω_Λ is the inverse shift in the stable category $\underline{\text{CM}}(\Lambda)$, so it is a bijective map on the non-projective d-th syzygies in mod Λ. We deduce the following.

Theorem 2.7. *Let Λ be a d-Gorenstein Artin algebra, then* $\phi\text{-dim}\,\Lambda = \psi\text{-dim}\,\Lambda = d$.

Remark 2.8. This result has been obtained independently in [27].

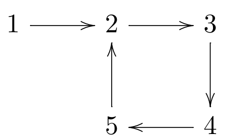
Proof. Let Λ be as in the hypothesis and let M be a Λ-module written as a direct sum of indecomposables $M = \oplus_{i=1}^{r} M_i$. Consider a projective resolution

$$0 \to \Omega^d M \to P_{d-1} \to \cdots \to P_0 \to M \to 0.$$

It is known from homological algebra that $\mathrm{Ext}^{d+j}(M, \Lambda) \simeq \mathrm{Ext}^{j}(\Omega^d M, \Lambda)$ for $j > 0$. By hypothesis, $\mathrm{inj.dim}\, \Lambda = d$, then for all $j > 0$ we have $\mathrm{Ext}^{d+j}(-, \Lambda) = 0$. Therefore, $\Omega^d M$, and all its summands, belong to $\mathrm{CM}(\Lambda)$. Since Ω acts as the inverse shift on the triangulated category $\underline{\mathrm{CM}}(\Lambda)$, the morphism L is injective over the subgroup $\mathcal{S} = \langle \mathrm{ind}\, \underline{\mathrm{CM}}(\Lambda) \rangle$ of K_0. For $j \geqslant d$, $S_j = \langle [\Omega^j M_1], \ldots, [\Omega^j M_r] \rangle$ is a subgroup of $\mathcal{S}$ of finite rank. The morphism L is such that $L(S_j) \cong S_{j+1}$, and, since it is injective over S_j for all $j \geqslant d$, it preserves the rank from d on. According to the definition, $\phi(M) \leqslant d$. Also, $\phi(D\Lambda) = \mathrm{proj.dim}\, D\Lambda = d$, then ϕ-dim $\Lambda = d$. Now, we analyze the ψ-dim. Let M be a Λ-module and let $\phi(M) = t \leqslant d$. By definition $\psi(M) = \phi(M) + \mathrm{proj.dim}\, Z$, where Z is a summand of $\Omega^t M$ with finite projective dimension. Then $N = \Omega^{d-t} Z$ is a summand of $\Omega^d M$. We have that N belongs to $\mathrm{CM}(\Lambda)$. If N is projective, then $\mathrm{proj.dim}\, Z \leqslant d - t$ and this implies $\psi(M) \leqslant d$. If N is not projective, thus N is nonzero in $\underline{\mathrm{CM}}$, then N has infinite projective dimension, which is impossible because we are assuming that $\mathrm{proj.dim}\, Z$ is finite. Therefore, $\psi(M) \leqslant d$ and, since $\psi(D\Lambda) = d$, this yields ψ-dim $\Lambda = d$. □

Remark 2.9. In [21] it is proved that an Artin algebra is selfinjective if and only if the ϕ and ψ dimensions are zero. For $d > 0$, the converse of Theorem 2.7 is not true, as we see in the following example due to Mata [27].

Example 2.10. Let Q be the following quiver, and $A = kQ/rad^2$.



This algebra is not 1-Gorenstein because $\mathrm{proj.dim}\, I(2) = \infty$. The only indecomposable non-projective modules are $I(2)$ and the simple modules $S(j)$ for $j \neq 1$. Among these modules, the indecomposable first syzygies are the simple modules and they have infinite projective dimension. The morphism L permutes the modules $\{S(j) : j \neq 1\}$. Then, ϕ-dim $A = \psi$-dim $A = 1$.

3. 2-Calabi–Yau tilted algebras

The main result of this section is the characterization of the modules in the categories $\underline{\mathrm{CM}}(\mathrm{B})$ and $\underline{\mathrm{CoCM}}(\mathrm{B})$ for a 2-Calabi–Yau tilted algebra B in terms of the functors τ, τ^{-1}, Ω and Ω^{-1}. As an immediate corollary, we obtain that $\Omega(\mathrm{mod}\, \mathrm{B}) = \mathrm{CM}(\mathrm{B})$. First, let us recall definitions and results about 2-Calabi–Yau categories and 2-Calabi–Yau tilted algebras. Let $\mathcal{C}$ be a k-linear triangulated category. For X, Y in $\mathcal{C}$ we have $\mathrm{Ext}^n(X, Y) = \mathrm{Hom}(X, Y[n])$.

Definition 3.1. Let k be an algebraically closed field. A k-linear triangulated category $\mathcal{C}$ with split idempotents, suspension functor [1] and finite dimensional Hom-spaces is called *2-Calabi–Yau* if

$$D\,\mathrm{Ext}_{\mathcal{C}}(X, Y) \simeq \mathrm{Ext}_{\mathcal{C}}(Y, X).$$

We will often write 2-CY instead of 2-Calabi–Yau. Examples of 2-CY categories are the cluster categories defined in [11,14,1]. For more information about 2-CY categories we refer to [2].

Definition 3.2. Let $\mathcal{C}$ be a 2-CY category. An object $T \in \mathcal{C}$ is called *cluster-tilting* if T is basic and $\mathrm{add}\, T = \{X \in \mathcal{C} : \mathrm{Ext}_{\mathcal{C}}(X, T) = 0\}$.

Definition 3.3. The endomorphism algebra of a cluster-tilting object $\mathrm{End}_{\mathcal{C}}(T)$ is called a *2-Calabi–Yau tilted algebra.*

It was shown in [12] that there is an equivalence of categories

$$F : \mathcal{C}/\,\mathrm{add}\, T[1] \to \mathrm{mod}\,\mathrm{End}_{\mathcal{C}}(T) \tag{1}$$

given by $X \mapsto \mathrm{Hom}_{\mathcal{C}}(T, X)$. This equivalence maps $\mathrm{add}\, T$ to the projective $\mathrm{End}_{\mathcal{C}}(T)$-modules and $\mathrm{add}\, T[2]$ to the injective $\mathrm{End}_{\mathcal{C}}(T)$-modules.

It has been proved in [24] that every 2-CY tilted algebra is Gorenstein of dimension at most one. Consequently, the projective dimension of a module can only be 0, 1 or infinity. And the same holds for the injective dimension.

Lemma 3.4. *Let* B *be a 2-CY tilted algebra and let* $M \in \mathrm{mod}\,\mathrm{B}$. *Let* $P_1 \to P_0 \xrightarrow{h} M \to 0$ *be a minimal projective presentation. Then, there is an exact sequence*

$$\tau^{-1}P_0 \xrightarrow{f} \tau^{-1}M \xrightarrow{g} P_1 \to P_0 \xrightarrow{h} M \to 0.$$

The induced epimorphism $\overline{g} : \tau^{-1}M \to \Omega^2 M$ *is an isomorphism if and only if* $f = 0$.

Proof. Consider an object X, without summands in $\mathrm{add}\, T[1]$, in $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(T, X) = M$. It has been shown in [24] that there is a triangle in $\mathcal{C}$

$$T_1 \to T_0 \xrightarrow{\tilde{h}} X \to T_1[1]$$

where $T_0, T_1 \in \mathrm{add}\, T$, and $\tilde{h}$ is a minimal $\mathrm{add}\, T$-approximation. Applying $\mathrm{Hom}_{\mathcal{C}}(T, -)$, we obtain

$$\mathrm{Hom}(T,T_0[-1]) \to \mathrm{Hom}(T,X[-1]) \to \mathrm{Hom}(T,T_1) \to \mathrm{Hom}(T,T_0) \xrightarrow{h} \mathrm{Hom}(T,X) \to 0$$

which gives the following exact sequence in $\operatorname{mod} \mathrm{B}$

$$\tau^{-1}P_0 \xrightarrow{f} \tau^{-1}M \xrightarrow{g} P_1 \to P_0 \to M \to 0,$$

where $P_1 \to P_0 \to M \to 0$ is a minimal projective presentation and every minimal projective presentation arises this way. By the exactness of the sequence, the induced epimorphism $\overline{g} : \tau^{-1}M \to \Omega^2 M$ is an isomorphism if and only if $f = 0$. □

We are now ready to state our first main result. We denote by $\operatorname{ind} \underline{\mathrm{CM}}(\mathrm{B})$ the set of indecomposable objects in $\underline{\mathrm{CM}}(\mathrm{B})$. Abusing notation, we will denote the class of a B-module M in the category $\underline{\mathrm{CM}}(\mathrm{B})$ also by M.

Theorem 3.5. *Let* B *be 2-CY tilted algebra. Let M and N be indecomposable modules in* $\operatorname{mod} \mathrm{B}$*. Then the statements* (a1)–(a4) *(respectively* (b1)–(b4)*) are equivalent*

(a1) *M is a non-projective syzygy,*
(a2) $M \in \operatorname{ind} \underline{\mathrm{CM}}(\mathrm{B})$,
(a3) $\Omega^2 \tau M \simeq M$,
(a4) *M is non-projective and $\Omega^{-2}M$ is isomorphic to τM.*

(b1) *N is a non-injective co-syzygy,*
(b2) $N \in \operatorname{ind} \underline{\mathrm{CoCM}}(\mathrm{B})$,
(b3) $\Omega^{-2}\tau^{-1}N \simeq N$,
(b4) *N is non-injective and $\Omega^2 N$ is isomorphic to $\tau^{-1}N$.*

Proof. (a1)⇒(a2).

Let M be an indecomposable non-projective syzygy. Then there is a short exact sequence $0 \to M \to P_0 \to N \to 0$ in $\operatorname{mod} \mathrm{B}$. Applying $\mathrm{Hom}_{\mathrm{B}}(-, \mathrm{B})$ yields an exact sequence

$$\cdots \to \mathrm{Ext}^1_{\mathrm{B}}(P_0, \mathrm{B}) \to \mathrm{Ext}^1_{\mathrm{B}}(M, \mathrm{B}) \to \mathrm{Ext}^2_{\mathrm{B}}(N, \mathrm{B}) \to \cdots .$$

The first term is zero because P_0 is projective and the last term is zero because $\operatorname{inj.dim} \mathrm{B} \leqslant 1$, by the Gorenstein condition. Then $\mathrm{Ext}^1_{\mathrm{B}}(M, \mathrm{B}) = 0$, so M is in $\mathrm{CM}(\mathrm{B})$, and, since M is not projective, then $M \in \operatorname{ind} \underline{\mathrm{CM}}(\mathrm{B})$.

(a2)⇒(a3). Let $M \in \operatorname{ind} \underline{\mathrm{CM}}(\mathrm{B})$. Since M is not projective, $\tau M \neq 0$. Consider the exact sequence given in Lemma 3.4, but for the module τM

$$\tau^{-1}P_0 \xrightarrow{f} M \xrightarrow{g} P_1 \to P_0 \to \tau M \to 0.$$

So the epimorphism $\overline{g} : M \to \Omega^2 \tau M$ is an isomorphism if and only if $f = 0$. The morphism f belongs to $\mathrm{Hom}_{\mathrm{B}}(\tau^{-1}P_0, M)$. From the Auslander–Reiten formulas, and the fact that $\operatorname{inj.dim} P_0 \leqslant 1$, we have

$$\mathsf{D}\,\mathrm{Ext}^1_{\mathrm{B}}(M, P_0) \simeq \underline{\mathrm{Hom}}_{\mathrm{B}}(\tau^{-1}P_0, M) \simeq \mathrm{Hom}_{\mathrm{B}}(\tau^{-1}P_0, M). \tag{2}$$

But since $M \in \underline{\mathrm{CM}}(\mathrm{B})$, we have $\mathrm{Ext}^1_{\mathrm{B}}(M, P_0) = 0$, and thus f must be zero and $M \simeq \Omega^2\tau M$.

(a3)⇒(a1). If $M \simeq \Omega^2\tau M$, then M cannot be projective because $\tau M \neq 0$, and M is a syzygy.

(a2)⇒(a4). First notice that the proof of the equivalence (b1)⇔(b2)⇔(b3) is similar to (a1)⇔(a2)⇔(a3), using the dual of Lemma 3.4. If $M \in \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$, then by Remark 2.3, $\tau M \in \mathrm{ind}\,\underline{\mathrm{CoCM}}(\mathrm{B})$. Using (b2)⇔(b3), we have $\Omega^{-2}M \simeq (\Omega^{-2}\tau^{-1})\tau M \simeq \tau M$.

(a4)⇒(a2). If $\tau M \simeq \Omega^{-2}M$ and M is not projective, then τM is a co-syzygy and is not injective. Hence, $\tau M \in \mathrm{ind}\,\underline{\mathrm{CoCM}}(\mathrm{B})$ and, by Remark 2.3, $M \in \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$. The proof of equivalence (b4) and the other statements of part (b) is similar. □

The equivalence of (a1) and (a2) shows that $\mathrm{CM}(\mathrm{B}) = \Omega(\mathrm{mod}\,\mathrm{B})$, and the set $\mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$ of indecomposable objects in the stable category can be identified with the set of non-projective indecomposable syzygies in $\mathrm{mod}\,\mathrm{B}$. This holds for all 1-Gorenstein Artin algebras. It is known that, for a d-Gorenstein algebra, M is Cohen–Macaulay if and only if M is a d-th syzygy, see [7, Proposition 6.20]. Each module over a d-Gorenstein algebra either has infinite projective dimension or has projective dimension at most d. In this context, as explained in [7], the modules called Cohen–Macaulay in these notes are the modules of Gorenstein dimension zero in [5, Proposition 3.8], and Gorenstein-projective modules in [18, Chapter 10]. We use the terminology Cohen–Macaulay and Co-Cohen–Macaulay, following [8].

Denote by $\mathcal{N} = \mathsf{D}\mathrm{Hom}_\Lambda(-,\Lambda)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ the Nakayama functor. If Λ is a selfinjective k-algebra then the functors $\Omega^{-2}\tau$ and $\mathcal{N}$ induce naturally isomorphic functors $\underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda$ on the stable category. We say that the Nakayama functor has *finite order* if there exists $m > 0$ such that the functor $\mathcal{N}^m$ is naturally isomorphic to the identity.

We get the following:

Corollary 3.6. *If a 2-CY tilted algebra* B *is selfinjective and the Nakayama functor* $\mathcal{N}_{\mathrm{B}}$ *has finite order* m*, then* $\tau^{2m}M \simeq M$ *for all* M *in* $\underline{\mathrm{mod}}\,\mathrm{B}$*. In particular, if* B *is symmetric then* $\tau^2 M \cong M$*.*

Proof. Since B is selfinjective, we have $\underline{\mathrm{CM}}(\mathrm{B}) = \underline{\mathrm{mod}}\,\mathrm{B}$. Therefore Theorem 3.5 (a4) implies that $\Omega_{\mathrm{B}}^{-2}M = \tau_{\mathrm{B}}M$ for all $M \in \underline{\mathrm{mod}}\,\mathrm{B}$. Now if $\mathcal{N}$ has order m, we get $M = \mathcal{N}^m M = (\Omega^{-2}\tau)^m M = \tau^{2m}M$. The last statement of the corollary follows because for symmetric algebras the Nakayama functor is the identity. □

Remark 3.7. Examples of symmetric 2-CY tilted algebras are the Jacobian algebras arising from triangulations of closed surfaces. In this case Corollary 3.6 was already obtained in [33] and [26].

We have seen that for 2-CY tilted algebras, the indecomposable syzygies are the same as the indecomposable CM-modules. In [29], Artin algebras with a finite number of syzygies are called torsionless-finite. In this work, the author proves that for torsionless-finite Artin algebras the representation dimension is at most three. On the other hand, in [9] the authors prove that when B is a tame cluster-tilted algebra, CM(B) has finitely many indecomposable modules. We thus obtain the following result.

Corollary 3.8. *If* B *is a* CM *finite 1-Gorenstein algebra, then its representation dimension is at most three. In particular, if* B *is a tame cluster-tilted algebra, then its representation dimension is at most three.*

Remark 3.9. The representation dimension of cluster-concealed algebras, a class of cluster-tilted algebra which overlaps with the tame ones, was studied in [20]. The authors compute the representation dimension by constructing an explicit Auslander generator.

4. Geometric description of CM modules: the unpunctured case

An important class of 2-CY tilted algebras are those arising from surface triangulations. In this case, the 2-CY category is the generalized cluster category introduced in [1]. It is described in terms of arcs and elementary moves in [14] for Dynkin type $\mathbb{A}$, in [31] for Dynkin type $\mathbb{D}$, in [10] for surfaces without punctures and in [28] for surfaces with punctures. Starting with a marked surface and a triangulation, one gets the bound quiver of the 2-CY tilted algebra as adjacency quiver of the triangulation. The dimension vector of representations are given by the intersection number of curves with the arcs that form the triangulation, and the AR translation is given by elementary moves and change of tags in the surface. Motivated by these ideas, our goal is to find a geometric description of $\underline{\mathrm{CM}}(\mathrm{B})$ and of the Ω-action in terms of curves.

In [23], the author computes $\mathrm{CM}(A)$ where $A = kQ/I$ is a gentle algebra. The indecomposable modules in $\underline{\mathrm{CM}}(A)$ are given by the non-projective indecomposable summands of $\mathrm{rad}\, P(a)$, where a is a vertex in a cycle $\alpha_1 \dots \alpha_n$ such that $\alpha_t\alpha_{t+1} \in I$ for all t. In the following example, we refer to the 2-CY tilted algebras from unpunctured surfaces introduced in [3]. These algebras are gentle, so the CM and CoCM modules can be described easily.

Example 4.1. Geometric description of $\underline{\mathrm{CM}}(\mathrm{B})$ for a 2-CY tilted algebra arising from a triangulated surface without punctures.

We have $M \in \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$ if and only if M is a summand of $\mathrm{rad}\, P(j)$, where j is a vertex lying in a 3-cycle $i \xrightarrow{\alpha_i} j \xrightarrow{\alpha_j} k \xrightarrow{\alpha_k} i$. Such 3-cycles are in bijection with internal triangles in the surface triangulation. We write the projective modules as string modules

$$P(i) : \beta_i^{-1} \leftarrow i \;\rightarrow\; j \;\rightarrow\; \beta_j$$

$$P(j) : \beta_j^{-1} \leftarrow j \;\rightarrow\; k \;\rightarrow\; \beta_k$$

$$P(k) : \beta_k^{-1} \leftarrow k \;\rightarrow\; i \;\rightarrow\; \beta_i$$

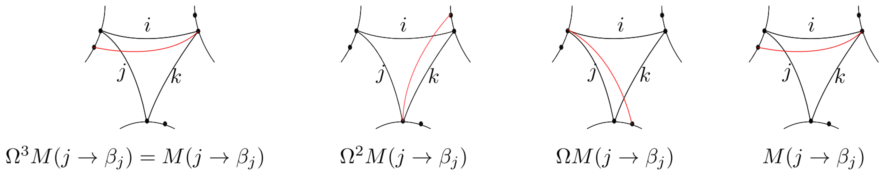


Fig. 1. Arcs that correspond to indecomposable $\underline{\text{CM}}$ modules in a gentle algebra arising from a surface. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

where β_i, β_j and β_k are paths in the quiver. We also use the notation $M(i \to \beta_i)$ for the module associated to the string $i \to \beta_i$. An internal triangle formed by arcs i, j, k in the surface induces a periodic projective resolution:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & P(i) & \longrightarrow & P(k) & \longrightarrow & P(j) \to M(j \to \beta_j) \to 0 \\ & \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\ M(j \to \beta_j) & & & M(i \to \beta_i) & & M(k \to \beta_k) & \end{array}$$

All possible indecomposable $\underline{\text{CM}}$ modules appear in that way. For a triangulated unpunctured surface, each internal triangle gives a periodic projective resolution with period 3 and uniserial modules as syzygies. The dimension vector for these modules is given by the crossing number in the red arcs in Fig. 1. The module $M(j \to \beta_j)$ is associated to the arc γ obtained from the arc i by performing an elementary move counter-clockwise on one of the endpoints of i, in such way that the arc γ crosses the arc j. By Remark 2.3, the modules in $\underline{\text{CoCM}}$ are associated to the AR translated of the arcs in Fig. 1, and can be obtained performing an elementary move in τ (clockwise) direction at the endpoints.

Remark 4.2. An alternative way to prove this description of $\underline{\text{CM}}$(B) would be to use the formula of [15] for Ext^1 over Jacobian algebras of unpunctured surfaces in terms of different types of crossings in the surface.

5. Geometric description of CM modules: the punctured disc

In this section we describe $\underline{\text{CM}}$(B) for the disk with one puncture, or equivalently, for cluster-tilted algebras of Dynkin type $\mathbb{D}$. This gives an answer to [23, Remark 3.7] in this case. This situation is much more complicated than the unpunctured case, and we will need to distinguish between 3 types of triangulations.

The geometric definition of the cluster category of type $\mathbb{D}$ was given in [31]. The reader can also see [32, Section 3.3]. Since our surface now has a puncture, we have to work with tagged arcs. If α is an arc that is incident to the puncture, then it gives rise to two tagged arcs, one is tagged *plain* and will still be denoted by α and the other is tagged *notched* and will be denoted by $\alpha^{\bowtie}$. The underlying curve for both α and $\alpha^{\bowtie}$ is the same. All possible quivers for a cluster-tilted algebra type $\mathbb{D}_n$ arise from

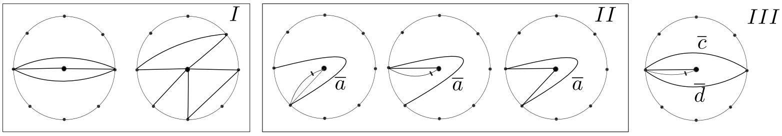


Fig. 2. Partial triangulations of type I (framed left), of type II (framed center) and of type III (right).

triangulations of a punctured disc with n marked points on the boundary. There is a correspondence between tagged arcs and indecomposable objects in the cluster category, and this leads to a correspondence between tagged arcs that are not in the triangulation and indecomposable B-modules. If γ is an arc, we denote by X_γ the corresponding object in the cluster category. Let M_γ be the indecomposable module corresponding to γ if γ is not in the triangulation, and let $M_\gamma = 0$ otherwise. The AR translation in the cluster category is realized as follows, $\tau X_\gamma = X_\delta$ where δ is obtained from γ by moving the endpoints to their clockwise neighbors along the boundary, and changing tag if γ is incident to the puncture. The AR translation in $\operatorname{mod} \mathrm{B}$ is inherited from the AR translation in the cluster category. Let a be an arc in the triangulation, the module associated to the arc $\tau^{-1}(a)$ is the indecomposable projective $P(a)$, and the module associated to the arc $\tau(a)$ is the indecomposable injective $I(a)$.

We say that two arcs incident to the puncture are *consecutive* if one of them is incident to a marked point q and the other is incident to $\tau(q)$. In the following definition, we introduce 3 types of triangulations which are illustrated in Fig. 2.

Definition 5.1. We say that a triangulation on the punctured disc is of

(1) *type I* if there are three or more arcs, or two non-consecutive plain arcs, incident to the puncture,
(2) *type II* if there are precisely two consecutive arcs incident to the puncture, or if there are two isotopic arcs incident to the puncture, one plain and one notched, and a third arc $\overline{a}$ as in Fig. 2,
(3) *type III* if there are two arcs incident to the puncture and the same marked point, and two arcs $\overline{c}$ and $\overline{d}$ as in Fig. 2.

All possible quivers for a cluster tilted algebra of type $\mathbb{D}$ arise from a triangulation of type I, II or III. Note however, that not all triangulations are of one of these types. In the following lemmas, we give a necessary condition for an object X_γ to be associated to a module $M_\gamma \in \mathrm{CM(B)}$. Our principal tool is the following consequence of the AR formulas.

Remark 5.2. Let M be a B-module.

(a) $M \notin \mathrm{CM(B)}$ if and only if $\mathrm{Hom_B}(\tau^{-1}P(t), M) \neq 0$ for some $t \in Q_0$. In particular, $\tau^{-1}P(t) \notin \mathrm{CM(B)}$.
(b) If $\operatorname{proj.dim} M = 1$ then $M \notin \mathrm{CM(B)}$.

Proof. Part (a) follows from the definition of CM(B) and equation (2) in section 3. Part (b) holds because if proj.dim $M = 1$, then $\mathrm{Ext}^1_{\mathrm{B}}(M, \mathrm{B}) \cong D\mathrm{Hom}_{\mathrm{B}}(\mathrm{B}, \tau M) \cong \tau M \neq 0$ since M is not projective. □

Because of the equivalence (1) of section 3, to prove that M_γ is not in CM(B) is enough to show that $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T_t, X_\gamma) \neq 0$ and there is at least one non-zero morphism in this Hom-space that does not factor through add τT. In the figures, we will often symbolize the object X_a, associated to an arc a in the triangulation, by a circled a. For an arc a in the triangulation, the associated object X_a in the cluster category is equal to $\tau T_a = T_a[1]$, where T_a is an indecomposable summand of the cluster tilting object T. Also, to say that the arc δ crosses the arc $a \in \mathcal{T}$ is equivalent to saying that $\mathrm{Ext}_{\mathcal{C}}(X_a, X_\delta) \neq 0$. Then, $\mathrm{Hom}_{\mathcal{C}}(T_a, X_\delta) \simeq \mathrm{Hom}_{\mathcal{C}}(X_a[-1], X_\delta) \simeq \mathrm{Hom}_{\mathcal{C}}(X_a, X_\delta[1]) \neq 0$. Thus, the objects X_δ such that the arc δ crosses the arc a are precisely those in the support of $\mathrm{Hom}_{\mathcal{C}}(T_a, -)$.

5.1. Triangulations of type II and III and the set $\underline{\mathrm{CM}}_\clubsuit$

We first study these two types because they are easier. Type I is studied in the next subsection.

Lemma 5.3.

(a) *Let $\mathcal{T}$ be a triangulation of type II and let $\overline{a} \in \mathcal{T}$ be as in Fig. 2. Let γ be an arc that crosses $\overline{a}$. If $M_\gamma \in \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$ then, in the AR quiver, M_γ lies at one of the positions marked with a $\diamond$ in Fig. 3.*

(b) *Let $\mathcal{T}$ be a triangulation of type III and let $\overline{c}, \overline{d} \in \mathcal{T}$ be as in Fig. 2. Let γ be an arc that crosses $\overline{c}$ or $\overline{d}$. If $M_\gamma \in \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$ then, in the AR quiver, M_γ lies at one of the positions marked with a $\clubsuit$ or $*$ in Fig. 4.*

Proof. (a) We prove the lemma for the right case in Fig. 2. A similar analysis can be done for the other cases, resulting in the same relative positions for M_γ. The objects X_δ such that δ is in the support of $\mathrm{Hom}_{\mathcal{C}}(T_{\overline{a}}, -)$ are: $\tau^{-1}T_b$, $T_{\overline{a}}$, $T_{\overline{c}}$, those in $\diamond$ position and those in the area A in Fig. 3. The objects in A are in the support of $\mathrm{Hom}_{\mathcal{C}}(T_{\overline{a}}, -)$ and of $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T_{\overline{a}}, -)$. Then, for X_δ in A there exists a non-zero morphism $f \in \mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T_{\overline{a}}, X_\delta)$ and, since A does not contain summands of τT, f does not factor through τT. This implies that M_δ is not in CM(B). The objects $T_{\overline{a}}$ and $T_{\overline{c}}$ are associated to the projectives $P(\overline{a})$ and $P(c)$, so they are not in ind $\underline{\mathrm{CM}}$(B). The object $\tau^{-1}T_b$ is associated to $\tau^{-1}P(b)$, and by Remark 5.2, this module is not in CM(B). The only remaining positions are those labeled by a $\diamond$.

(b) The objects X_δ in A_1 are in the support of $\mathrm{Hom}_{\mathcal{C}}(T_{\overline{d}}, -)$ and of $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T_{\overline{d}}, -)$, and by the same argument as in type II, we see that $M_\delta \notin \mathrm{ind}\ \underline{\mathrm{CM}}(\mathrm{B})$, for all X_δ in A_1.

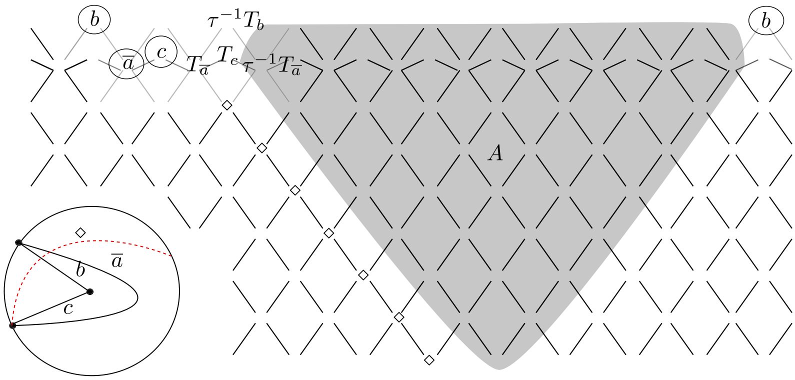


Fig. 3. Lemma 5.3 for a triangulation of type II. The position of $T_{\overline{a}}$, T_b and T_c make $\diamond$ the only admissible positions for arcs such that γ crosses $\overline{a}$ and M_γ can be in ind $\underline{\mathrm{CM}}$.

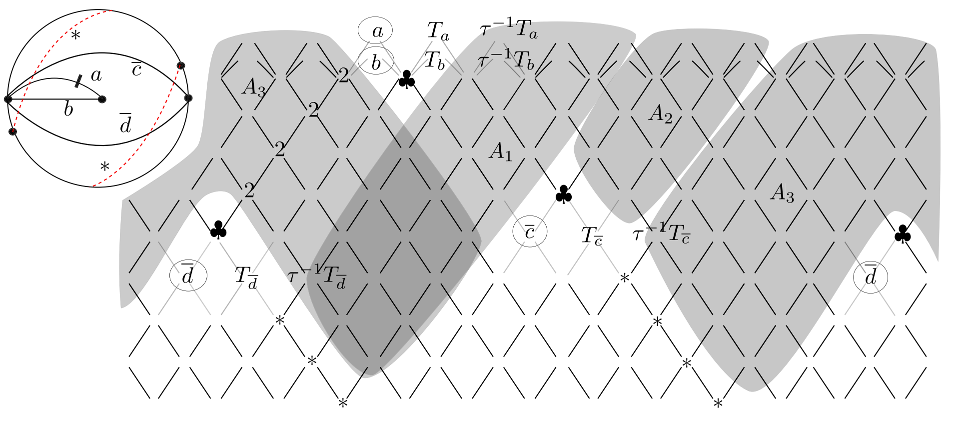


Fig. 4. Lemma 5.3 for a triangulation of type III. The position of T_a, T_b, $T_{\overline{c}}$ and $T_{\overline{d}}$ make $*$ and ♣ the only admissible positions for arcs such that γ crosses $\overline{c}$ or $\overline{d}$ and $M_\gamma \in$ ind $\underline{\mathrm{CM}}$.

The objects X_δ in A_2 are in the support of $\mathrm{Hom}_{\mathcal{C}}(T_a \oplus T_b, -)$ and of $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T_a \oplus \tau^{-1}T_b, -)$, and by the same argument as in type II, we see that $M_\delta \notin \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$ for all X_δ in A_2.

The area A_3 does not contain summands of τT. All X_δ in A_3 satisfy $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T_{\overline{c}}, X_\delta) \neq 0$. For the objects X_δ not labeled 2 in Fig. 4, we can apply the same argument as before to prove that the associated M_δ are not in ind $\underline{\mathrm{CM}}(\mathrm{B})$. Let X be an indecomposable object whose position in the AR quiver is labeled 2 in Fig. 4. This X is such that $\dim\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T_{\overline{c}}, X) = 2$ and, in this two-dimensional space, the subspace of morphisms that factor through τT has dimension at most one. Then, there is a non-zero morphism in $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T_{\overline{c}}, X)$ that does not factor through τT, so X is not associated to a module in CM(B). Therefore, $M_\delta \notin \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$ for all X_δ in A_3. The only remaining positions are those labeled by ♣ and $*$. □

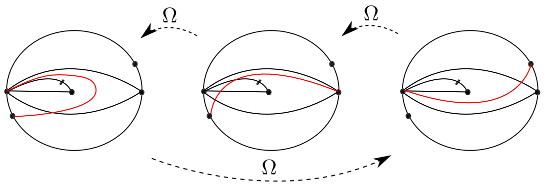


Fig. 5. Arcs associated to the modules in $\underline{\mathrm{CM}}_{\clubsuit} \subset$ ind $\underline{\mathrm{CM}}(\mathrm{B})$ and the action of Ω.

The objects labeled by $\clubsuit$ in Fig. 4 are associated to non-projective indecomposable syzygies, since they are summands of the radical of $P(a)$, $P(\overline{c})$ and $P(\overline{d})$. We will denote the set of these modules by $\underline{\mathrm{CM}}_{\clubsuit}$. It is easy to check that they are the syzygies in a periodic projective resolution. The corresponding arcs are shown in Fig. 5.

5.2. Triangulations of type I and the set $\underline{\mathrm{CM}}_{\odot}$

In this subsection, we determine all the non-projective syzygies arising from the area that contains the puncture in a triangulation of type I. First, we need to introduce some notation. Recall that the AR translation τ corresponds to moving endpoints clockwise along the boundary and changing the tag at the puncture. If q is a marked point on the boundary, we denote by $\tau(q)$ the next marked point in the clockwise direction.

Definition 5.4. In a triangulation of type I, a marked point q is called *colored* if it is on the boundary and q or $\tau(q)$ is adjacent to the puncture.

A triangle is *internal* if none of its sides is a boundary segment. Given a triangulation of type I, label from 1 to m, advancing clockwise, the arcs incident to the puncture. We consider indices modulo m. If the arcs $k, k+1$, for $k \in \{1, \dots, m\}$, are sides of an internal triangle Δ, we label the third side of Δ by $\overline{k}$.

Let $\mathcal{E}$ be the set of all internal triangles which have the puncture as a vertex. Two triangles in $\mathcal{E}$ are called *consecutive* if they share an edge. If all triangles at the puncture are internal, the set $\mathcal{E}$ has exactly m triangles. In this case all colored points are labeled red and blue, and we set $d = m$. If not all triangles at the puncture are internal, then $\mathcal{E}$ can be decomposed as $\mathcal{E} = \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_l$ and each $\mathcal{E}_i = \{\Delta^i_1, \dots, \Delta^i_{t_i}\}$ is such that Δ^i_{j+1} and Δ^i_j are consecutive, Δ^i_{j+1} follows Δ^i_j in clockwise order and $\mathcal{E}_i$ is maximal in the sense that if there is a triangle Δ in $\mathcal{E}$ consecutive to a triangle in $\mathcal{E}_i$, then Δ is in $\mathcal{E}_i$. For this case set $d_i = |\mathcal{E}_i| - 1$ and $d = \sum d_i$. For example, in the left picture in Fig. 6, we have $m = 6$ because there are 6 arcs incident to the puncture, and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1 = \{[1, \overline{6}, 6]\}$ and $\mathcal{E}_2 = \{[3, \overline{2}, 2], [4, \overline{3}, 3], [5, \overline{4}, 4]\}$, thus $d = 2$.

Definition 5.5. A colored point q is *red* if $\tau(q)$ is not the last vertex in the last triangle in a set $\mathcal{E}_i$. A colored point q is *blue* if q is not the first vertex of a first internal triangle in a set $\mathcal{E}_i$.

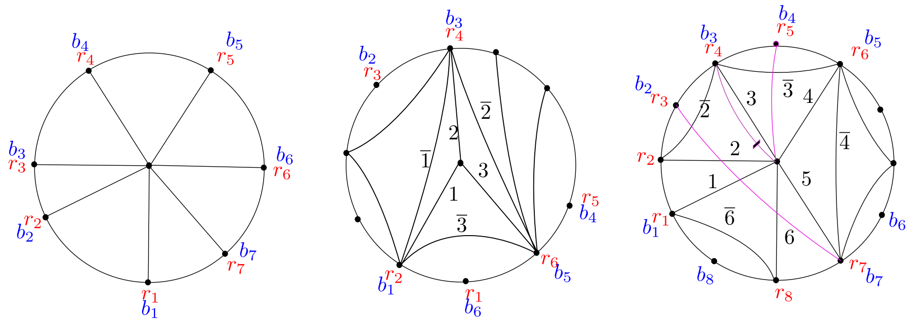


Fig. 6. Triangulations of type I with colored points, from right to left cases (1), (2), (3). On the left example, arcs associated to the modules $M(r_7, b_2)$, $M(r_5, b_4)$ and $M(r_4, b_3)$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

In other words, there are the following three possibilities for a colored point q.

(rb) q is *red and blue* if q is adjacent to the puncture but q is not the first vertex of the first triangle in a set $\mathcal{E}_i$, or $\tau(q)$ is adjacent to the puncture but $\tau(q)$ is not the last vertex of the last triangle in a set $\mathcal{E}_i$;

(r) q is *red but not blue* if q is the first vertex of the first triangle in a set $\mathcal{E}_i$; in this case q and $\tau^{-1}(q)$ are incident to the puncture;

(b) q is *blue but not red* if $\tau(q)$ is the last vertex of the last triangle in a set $\mathcal{E}_i$; in this case $\tau(q)$ and $\tau^2(q)$ are incident to the puncture.

There are $m + d$ red points, and $m + d$ blue points. In the following, we consider indices modulo $m + d$. We shall define a labeling of the colored points, assigning the labels $r_1, r_2, \ldots, r_{m+d}$ to the red points and the labels $b_1, b_2, \ldots, b_{m+d}$ to the blue points. These labels will be determined by the choice of the labels r_1 and b_1; the other labels will follow the order of the red or blue points along the boundary in clockwise direction. The labels r_1 and b_1 are assigned as follows.

(1) If there is no internal triangles (thus $\mathcal{E} = \emptyset$), take any red–blue point and label it r_1 and b_1;
(2) If all triangles at the puncture are internal (thus $\mathcal{E} = \mathcal{E}_1$), take any red–blue point and label it r_1 and b_{m+d};
(3) If there are internal and non-internal triangles incident to the puncture (thus $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_l$ with $l \geq 1$), let q be the first vertex in the first triangle of $\mathcal{E}_1$, and label the point $\tau^{-1}(q)$ by $r_1 b_1$, and the point q by r_2. Note that q is red but not blue and $\tau^{-1}(q)$ is red and blue.

See the examples in Fig. 6. Note that in the case (1) every marked point on the boundary is red and blue and each is labeled r_i, b_i, for some i. In case (2), again all colored points are red and blue, but then each one is labeled r_i, b_{i-1}, for some i.

In case (3), the sequence of monochromatic points in clockwise order along the boundary is an alternating sequence of red points and blue points. Moreover the labels of the points that are both red and blue are of the form r_ib_i, for some i, if the previous monochromatic point is blue, and it is of the form r_ib_{i-1}, for some i, if the previous monochromatic point is red. Moreover, in each of the sets $\mathcal{E}_j$, the first vertex is red but not blue, all other vertices except for the last are red and blue and have labels r_ib_{i-1}, and the last vertex is red and blue and has label r_ib_i.

Let q and s be two marked points on the boundary such that $s \neq q$ and $s \neq \tau(q)$, denote by $\gamma(q,s)$ the arc starting at q, going around the puncture in τ (= clockwise) direction and ending at s. If $s = \tau(q)$ then $\gamma(q,s)$ is a boundary segment. Denote by $\gamma^{\bowtie}(q,q)$ the notched arc and by $\gamma(q,q)$ the plain arc incident to q and the puncture.

Now consider the special case where the marked points are colored points. Suppose first that r_i and b_j are labels in two different marked points. Denote by $M(r_i,b_j)$ the module associated to the arc $\gamma(r_i,b_j)$, if the arc $\gamma(r_i,b_j) \notin \mathcal{T}$, and let $M(r_i,b_j) = 0$ otherwise. Next suppose that r_i and b_j are labels in the same marked point q. If $\gamma(q,q) \in \mathcal{T}$, then let $M(r_i,b_j)$ be the module associated to $\gamma^{\bowtie}(r_i,b_j)$. In the other case, let $M(r_i,b_j)$ be the module associated to $\gamma(r_i,b_j)$. For example, see in Fig. 6, $M(r_4,b_3) = M_{\gamma^{\bowtie}(r_4,b_3)}$ and $M(r_5,b_4) = M_{\gamma(r_5,b_4)}$.

The modules denoted by $M(r_i,b_j)$ will be important throughout this section. First let us study the modules $M(r_i,b_i)$ and $M(r_i,b_{i+1})$, $1 \le i \le m+d$.

Lemma 5.6. *Let $i \in \{1,2,\dots,m+d\}$. Then*

(a) *$M(r_i,b_{i+1})$ is projective.*
(b) *$M(r_i,b_i)$ is projective or there exists an arc $\overline{k} = \gamma(q,s)$ such that $M(r_i,b_i) = M_{\gamma(q,\tau^{-1}(s))}$.*

Proof. (a) Let q be the marked point labeled r_i. Suppose first that q is the first vertex in an internal triangle $\Delta \in \mathcal{E}_t$. Let k be the arc in Δ from q to the puncture and let $\overline{k} = \gamma(q,s)$ be the arc in Δ that is not incident to the puncture. Then $\tau^{-1}(s)$ and s are the next colored points, their blue labels are b_i and b_{i+1} respectively, and so $M(r_i,b_{i+1}) = M_{\overline{k}}$ is zero, hence projective.

Now suppose that $\tau(q)$ is the first vertex in an internal triangle $\Delta \in \mathcal{E}_t$. Let $\overline{k} = \gamma(\tau(q),s)$ be the arc in Δ that is not incident to the puncture. The colored points after q are $\tau(q)$ and $\tau^{-1}(s)$. If q is not adjacent to the puncture, then the blue labels of $\tau(q)$ and $\tau^{-1}(s)$ are b_i and b_{i+1}, respectively. If q is adjacent to the puncture then q is labeled r_ib_i and $\tau^{-1}(s)$ is labeled b_{i+1}. In both cases, $\gamma(r_i,b_{i+1}) = \tau^{-1}(\overline{k})$ and $M(r_i,b_{i+1}) = P(\overline{k})$ is projective.

It remains the case where there both q and $\tau(q)$ are incident to the puncture and $\tau(q)$ is not a vertex of an internal triangle. Then q is labeled r_ib_i and the blue label of $\tau(q)$ is b_{i+1}. Hence the curve $\gamma(r_i,b_{i+1})$ is a boundary segment and thus $M(r_i,b_{i+1})$ is zero, hence projective.

(b) Suppose $M(r_i, b_i)$ is not projective and let q be the marked point labeled r_i. Suppose first that $\tau(q)$ is the last vertex in an internal triangle Δ in $\mathcal{E}_t$. Then q is labeled r_i, b_{i-1} and the blue label of $\tau(q)$ is b_i. Then, the curve $\gamma(r_i, b_i)$ is a boundary segment and $M(r_i, b_i)$ is zero, hence projective.

Now suppose that both q and $\tau(q)$ are incident to the puncture. It follows from the construction of the labels that q is labeled $r_i b_i$. Thus $M(r_i, b_i) = M_{\gamma^{\bowtie}(q,q)} = P(k)$, where k is the arc from $\tau(q)$ to the puncture. This is a contradiction to our assumption that $M(r_i, b_i)$ is not projective.

It remains the case where q is the first vertex of an internal triangle Δ. Let $\overline{k} = \gamma(q, s)$ be the side of Δ that is not incident to the puncture. Then the point $\tau^{-1}(s)$ is labeled b_i and $M(r_i, b_i) = M_{\gamma(q,\tau^{-1}(s))}$. $\square$

We are now ready for the first main result of this section. The following theorem determines most of the modules in $\underline{\mathrm{CM}}(\mathrm{B})$ over configurations of type I and describes the action of Ω in geometric terms. We define the set $\underline{\mathrm{CM}}_{\odot}$ by

$$\underline{\mathrm{CM}}_{\odot} = \{M(r_i, b_j) \mid i = 1, 2, \ldots, m + d, j = i + 2, \ldots, i - 1\}.$$

Theorem 5.7. *With the above notation, if $j \in \{i + 2, ..., i - 1\}$, then*

$$\Omega M(r_i, b_j) \simeq M(r_{j-1}, b_i). \tag{3}$$

In particular, the map $\Omega\colon \underline{\mathrm{CM}}_{\odot} \to \underline{\mathrm{CM}}_{\odot}$ is a bijection and $\underline{\mathrm{CM}}_{\odot} \subset \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$.

Proof. We prove the statement by analyzing the possible relative positions of the points r_i, b_i and r_{j-1}, b_j where i is arbitrary and $j = i+1, \ldots, i-1$. The possibilities are listed in Fig. 7. The first row of that figure shows possibilities for r_i and b_i and the second row shows possibilities for r_{j-1} and b_j. The last row illustrates three special cases.

Choosing a module $M(r_i, b_j)$ corresponds to either choosing one of the three cases 1, 2, 3 of the first row and one of the three cases a, b, c of the second row, or choosing one of the three cases d, e, f of the third row. In each case the syzygy $\Omega M(r_i, b_j)$ is easily computed and the results are shown in Figs. 8–11. We present the computation in detail for the cases (1c), (2b) and (d), and leave the other to the reader. Notice that the formula also works for the configuration of type I with only two arcs incident to the puncture; that case is also left to the reader.

Throughout the proof we use the notation β_k for the maximal non-zero path starting at the vertex $k' \neq k$ such that there is an arrow $\overline{k} \to k'$. Note that k' is unique if it exists.

Case (1c): the module $M(r_i, b_j)$ is given by the string $k \to \cdots \to k+t \leftarrow \overline{k+t} \to \beta_{k+t}$ and top $M(r_i, b_j) = S(k) \oplus S(\overline{k+t})$, where k is the first arc crossed by $\gamma(r_i, b_j)$ and $k + t$ is the last arc incident to the puncture that is crossed by $\gamma(r_i, b_j)$. The projective cover $P(k) \oplus P(\overline{k+t})$ of $M(r_i, b_j)$ consist of a uniserial projective $P(k)$ given by the string $k \to k+1 \to \cdots \to k-2$ and a projective $P(\overline{k+t})$ given by the string $\beta^{-1}_{k+t-1} \leftarrow$

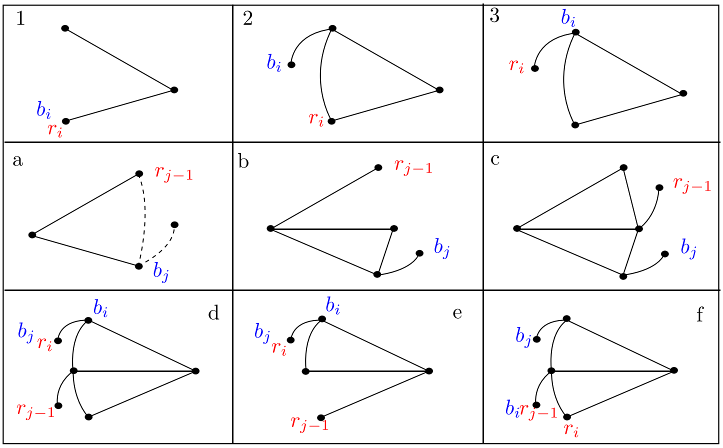


Fig. 7. Possible relative positions of r_i, r_{j-1}, b_i and b_j.

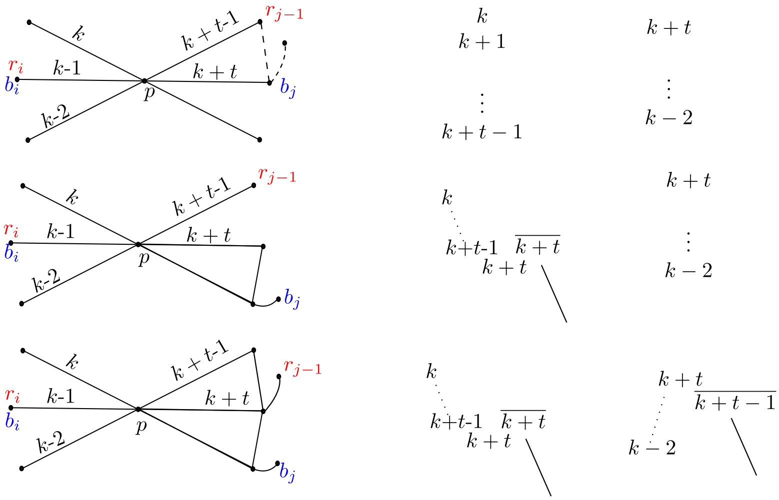


Fig. 8. Cases (1a), (1b) and (1c). The left column shows part of the triangulation, the middle column, the module $M(r_i, b_j)$ and the right column shows $\Omega M(r_i, b_j) = M(r_{j-1}, b_i)$.

$\overline{k+t-1} \leftarrow k+t \leftarrow \overline{k+t} \rightarrow \beta_{k+t}$. Therefore, the syzygy $\Omega M(r_i, b_j)$ is given by the string $\beta^{-1}_{k+t-1} \leftarrow k+t \rightarrow k+t+1 \rightarrow \cdots \rightarrow k-2$, this module is denoted by $M(r_{j-1}, b_i)$, see Fig. 8.

Case (2b): the module $M(r_i, b_j)$ is given by the string $k \rightarrow \cdots \rightarrow k+t \leftarrow \overline{k+t} \rightarrow \beta_{k+t}$ and top $M(r_i, b_j) = S(k) \oplus S(\overline{k+t})$, where k is the first arc crossed by $\gamma(r_i, b_j)$ and $k+t$ is the last arc incident to the puncture that is crossed by $\gamma(r_i, b_j)$. The projective cover $P(k) \oplus P(\overline{k+t})$ of $M(r_i, b_j)$ consists of a projective $P(\overline{k+t})$ given by the string $\beta^{-1}_{k+t} \leftarrow \overline{k+t} \rightarrow k+t$ and a projective $P(k)$ supported on the paths $k \rightarrow \overline{k-1} \rightarrow \beta_{k-1}$ and $k \rightarrow k+1 \rightarrow \cdots \rightarrow k-1 = k \rightarrow \overline{k-1} \rightarrow k-1$ so the dimension vector of $P(k)$ is $(d_i)_{i \in Q_0}$ with $d_i = 1$ for $i = k, k+1, \ldots, k-1, \overline{k-1}$ and for all i in the

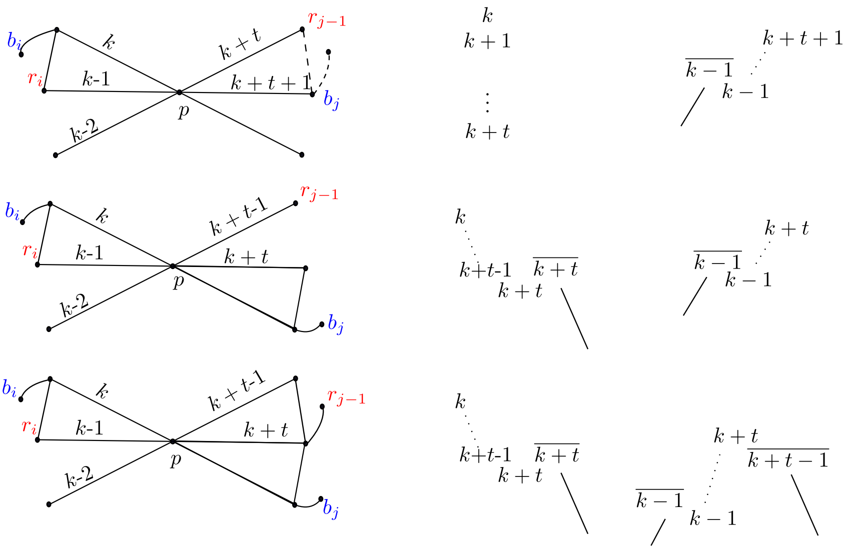


Fig. 9. Cases (2a), (2b) and (2c). The left column shows part of the triangulation, the middle column, the module $M(r_i, b_j)$ and the right column shows $\Omega M(r_i, b_j) = M(r_{j-1}, b_i)$.

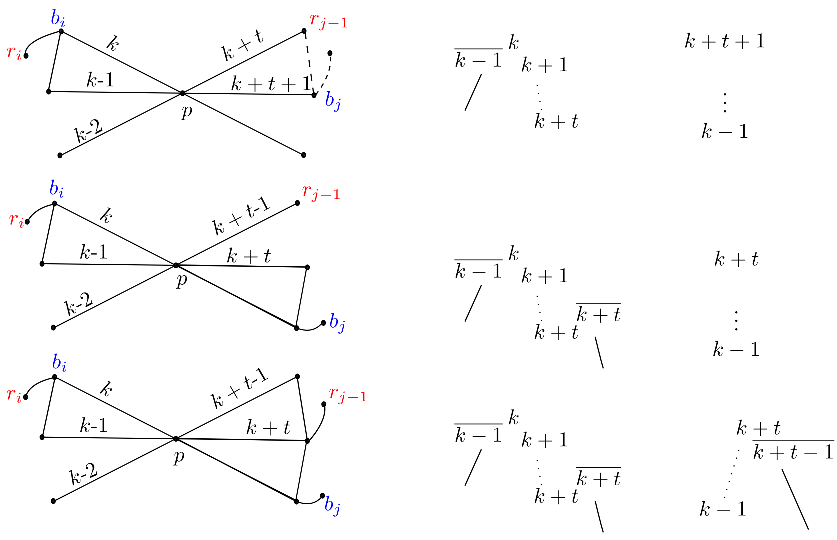


Fig. 10. Cases (3a), (3b) and (3c). The left column shows part of the triangulation, the middle column, the module $M(r_i, b_j)$ and the right column shows $\Omega M(r_i, b_j) = M(r_{j-1}, b_i)$.

support of β_{k-1}, and $d_i = 0$ for all other i. Then $\Omega M(r_i, b_j)$ is given by the string $\beta_{k-1}^{-1} \leftarrow \overline{k-1} \rightarrow k-1 \leftarrow k-2 \leftarrow \cdots \leftarrow k+t$. Thus, $\Omega M(r_i, b_j) = M(r_{j-1}, b_i)$. See Fig. 9.

The last row in Fig. 7 contains three special cases, where the projective is not like in the other cases.

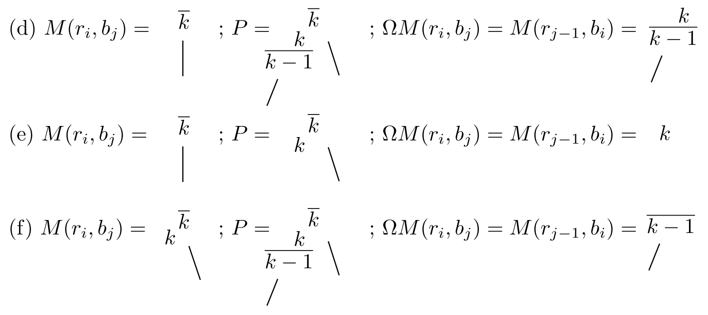


Fig. 11. Cases (d), (e) and (f).

Case (d): the module $M(r_i, b_j)$ is given by the string $\overline{k} \to \beta_k$ and top $M(r_i, b_j) = S(\overline{k})$. The projective cover $P(\overline{k})$ is a biserial projective given by the string $\beta_{k-1}^{-1} \leftarrow \overline{k-1} \leftarrow k \leftarrow \overline{k} \to \beta_k$. Then, $\Omega M(r_i, b_j)$ is given by the string $k \to \overline{k-1} \to \beta_{k-1}$, so $\Omega M(r_i, b_j) = M(r_{j-1}, b_i)$. This shows equation (3).

In order to show the last statement of the theorem, note that Ω is bijective in the set $\underline{\mathrm{CM}}_\odot$ because, for the indices under consideration, the map $(r_i, b_j) \to (r_{j-1}, b_i)$ is bijective. Each pair (r_i, b_j) has a unique predecessor (r_j, b_{i+1}) and the indices satisfy the condition $i + 1 \in \{j + 2, \dots j - 1\}$ since $j \neq i + 1$ and $j \neq i$. Moreover, we see that for each pair (r_i, b_j) the module $M(r_i, b_j)$ is not zero and the projective cover $P(M(r_i, b_j))$ is different from $M(r_i, b_j)$. Thus $M(r_i, b_j)$ is not projective and its syzygy is not trivial. This shows that $\underline{\mathrm{CM}}_\odot \subset \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$. □

5.3. Full description of the category $\underline{\mathrm{CM}}(\mathrm{B})$

In Theorem 5.7, we gave an explicit description of the subset $\underline{\mathrm{CM}}_\odot$ of $\mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$. Although this is the most interesting part of $\underline{\mathrm{CM}}(\mathrm{B})$, it is in general not all of it. In this subsection, we describe $\underline{\mathrm{CM}}(\mathrm{B})$ completely. Before formulating our main result, we need two preparatory lemmas.

Let $\mathcal{T}$ be a triangulation of type I and let Δ be an internal triangle in $\mathcal{T}$ incident to the puncture. As usual we denote the sides of Δ by $k, k+1$ and $\overline{k}$, where $\overline{k}$ is the side opposite to the puncture.

Lemma 5.8. *Let $\overline{k} = \gamma(q, s)$ be an arc as above and let γ be an arc that crosses the arcs $\overline{k} = \gamma(r, s)$ and $\tau^{-1}(\overline{k})$. If $M_\gamma \in \mathrm{CM}(\mathrm{B})$ then $\gamma = \gamma(\tau^{-1}(s), q)$.*

Proof. We work in the AR quiver of the cluster category $\mathcal{C}$, see Fig. 12. The relative positions of $T_{\overline{k}}, T_k, T_{k+1}$ are indicated in the figure. The condition that the arc γ crosses $\overline{k}$ and $\tau^{-1}(\overline{k})$ translates to the condition that $\mathrm{Hom}_{\mathcal{C}}(T_{\overline{k}}, X_\gamma) \neq 0$ and $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1} T_{\overline{k}}, X_\gamma) \neq 0$. It then follows from the structure of the AR quiver that X_γ must lie in the shaded region in Fig. 12. Suppose first that $X_\gamma \in A_1$ and let f be a non-zero morphism in $\mathrm{Hom}_{\mathcal{C}}(\tau^{-1} T_{\overline{k}}, X_\gamma)$, then it cannot factor through τT because, by the position of

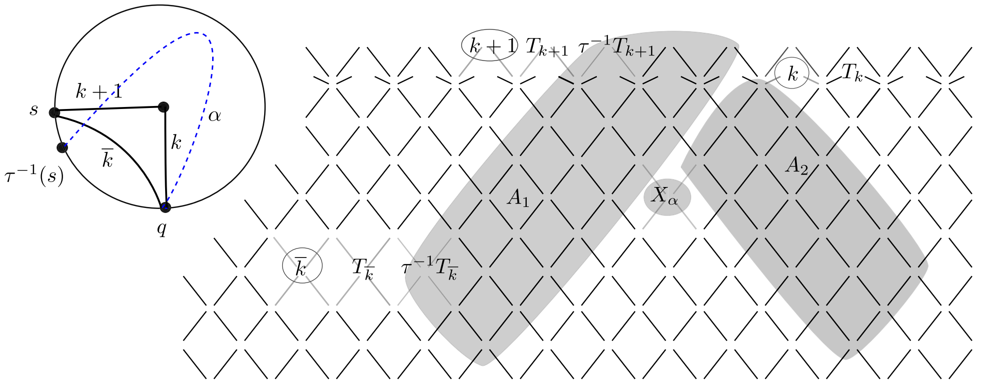


Fig. 12. Proof of Lemma 5.8. The arc $\gamma(\tau^{-1}(s), q)$ is denoted by α in the figure.

$T_{\overline{k}}$, A_1 does not contain summands of τT. Thus, there is a non-zero morphism in $\text{Hom}_{\text{B}}(\tau^{-1}P(\overline{k}), M_\gamma)$ and $M_\gamma \notin \text{CM(B)}$ by Remark 5.2 (a). Now suppose that $X_\gamma \in A_2$ and let f be a non-zero morphism in $\text{Hom}_{\mathcal{C}}(\tau^{-1}T_{k+1}, X_\gamma)$, then it cannot factor through τT because, by the position of $T_{\overline{k}}$ and T_{k+1}, A_1 and A_2 do not contain summands of τT. Thus, there is a non-zero morphism in $\text{Hom}_{\text{B}}(\tau^{-1}P(k+1), M_\gamma)$ and $M_\gamma \notin \text{CM(B)}$ by Remark 5.2 (a). The only other object associated to an arc crossing the arcs $\overline{k}$ and $\tau^{-1}(\overline{k})$ is $X_{\gamma(\tau^{-1}(s),q)}$. The lemma follows. □

5.3.1. The sets $\mathfrak{X}_k$ and $\mathfrak{M}_k$

Now, we set the last results needed to give a complete geometric description of $\underline{\text{CM}}$(B). Recall the Definition 5.1. Let $\mathcal{T}$ be a triangulation and let $\overline{k}$ be an arc such that

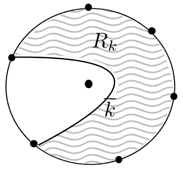
(a) $\overline{k}$ is a side of an internal triangle such that the other two sides are incident to the puncture, if $\mathcal{T}$ is of type I,
(b) $\overline{k}$ is the arc $\overline{a}$ in Fig. 2 when $\mathcal{T}$ is of type II,
(c) $\overline{k}$ is one of the arcs $\overline{c}$ or $\overline{d}$ in Fig. 2 when $\mathcal{T}$ is of type III.

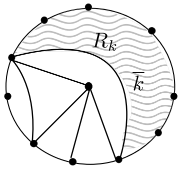
By cutting along $\overline{k} = \gamma(q, s)$, we obtain two pieces of the disc. Denote by R_k the piece that does not contain the puncture. See the Fig. 13.

Let $\mathfrak{X}_k$ be the set of objects X_γ in the cluster category such that γ lies in R_k, or $\tau(\gamma)$ lies in R_k. The objects in $\mathfrak{X}_k$ lie in the shaded region in Fig. 13. The objects $X_\gamma \in \mathfrak{X}_k$ such that $\gamma \notin R_k$ and $\tau(\gamma) \in R_k$ are the objects marked with a bullet together with $T_{\overline{k}}$. Let $\mathfrak{M}_k$ be the set of modules M_γ such that $X_\gamma \in \mathfrak{X}_k$.

5.3.2. The set $\underline{\text{CM}}_\Delta$

Recall that we denote by $\mathcal{E}$ the set of all internal triangles that are incident to the puncture. We now consider the other internal triangles. For every internal triangle $\delta \notin \mathcal{E}$, we construct 3 arcs by performing an elementary move in counterclockwise direction to each of the sides of δ such that the resulting arc runs through the interior of δ. We





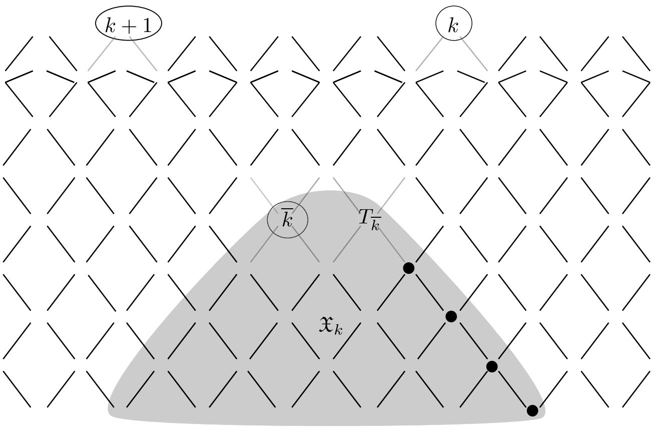


Fig. 13. On the right Region R_k shaded. On the left, the region shaded in the AR quiver shows the objects in $\mathfrak{X}_k$.

denote by $\underline{\mathrm{CM}}_\Delta$ the set of all modules M_γ for which there exists an internal triangle $\delta \notin \mathcal{E}$ such that the arc γ is one of the three arcs given by the above construction with respect to δ. As we have seen in section 4, we have $\underline{\mathrm{CM}}_\Delta \subset \operatorname{ind} \underline{\mathrm{CM}}(\mathrm{B})$. In fact M_γ is a radical summand of the projective $P(a)$ where a is the side of δ that is crossed by γ.

Lemma 5.9. *If a module $M_\gamma \in \mathfrak{M}_k$ is a non-projective syzygy then $M_\gamma \in \underline{\mathrm{CM}}_\Delta$.*

Proof. Case (i): Let M_γ be an indecomposable non-projective syzygy such that γ lies entirely in R_k for some $\overline{k} = \gamma(q,s)$. Since $\overline{k} \in \mathcal{T}$, the only arcs $x \in \mathcal{T}$ such that the projective module $P(x) = M_{\tau^{-1}(x)}$ is supported in the vertices corresponding to arcs in R_k are those that lie entirely in R_k and those starting at s. Since M_γ is a syzygy, there is a monomorphism $u\colon M_\gamma \to P = \oplus_{i=1}^n P(x_i)$ such that each arc x_i lies on R_k or has s as an endpoint. Suppose that an indecomposable summand $P(x)$ of P is associated to an arc that does not lie in R_k. Let N be the maximal submodule of $P(x)$ such that its corresponding arc δ_N lies in R_k, then N is a summand of $\mathrm{rad}P(\overline{k})$. The morphism u factors through $(P/P(x)) \oplus N$ and M_γ can be considered a submodule of $(P/P(x)) \oplus P(\overline{k})$ via $M_\gamma \hookrightarrow (P/P(x)) \oplus N \hookrightarrow (P/P(x)) \oplus P(\overline{k})$. So, without loss of generality, we can assume that all the arcs x_i lie in R_k. Now, define the triangulated polygon Π_k by gluing to R_k, along $\overline{k}$, a triangle with two boundary segments as the other sides. Thus, Π_k has a triangulation $\widetilde{\mathcal{T}}$ that is formed by a triangulated region $\widetilde{R_k}$, analogous to R_k, and an extra triangle, see Fig. 14. For each arc j in R_k, denote by $\tilde{j}$ the analogous arc in $\widetilde{\mathcal{T}}$. Denote by $\widetilde{\mathrm{B}}$ the cluster-tilted algebra of type $\mathbb{A}$ arising form the triangulation $\widetilde{\mathcal{T}}$, [3,14]. Since M_γ is not projective over B, γ is not of the form $\tau^{-1}(a)$ for an arc $a \in \mathcal{T}$. This implies that $\tilde{\gamma}$ is not $\tau^{-1}(\tilde{a})$ for an arc in $\widetilde{\mathcal{T}}$. Then, $M_{\tilde{\gamma}}$ is not projective over $\widetilde{\mathrm{B}}$. Define the induced morphism of representations $\widetilde{u} = (u_{\tilde{j}}) : M_{\tilde{\gamma}} \hookrightarrow \oplus_{i=1}^n P(\widetilde{x}_i)$ by $u_{\tilde{j}} = u_j$, for all $\tilde{j} \in \widetilde{\mathcal{T}}$. By construction, $\widetilde{u}$ is a monomorphism in $\operatorname{mod} \widetilde{\mathrm{B}}$. Then, $M_{\tilde{\gamma}}$ is an

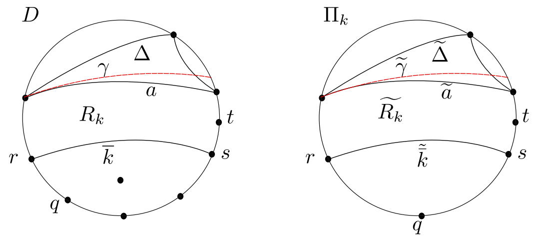


Fig. 14. Original punctured disc D and polygon Π_k.

indecomposable non-projective syzygy and, by Section 4, it has to be a summand of the radical of an indecomposable projective module $P(\widetilde{a})$, where $\widetilde{a}$ is a side of an internal triangle $\widetilde{\Delta}$. Lifting the arc $\widetilde{\gamma}$ to $\mathcal{T}$, we obtain that M_γ is a summand of the radical of a projective module $P(a)$, where a is a side of the lifted internal triangle Δ in R_k. Thus $M_\gamma \in \underline{\mathrm{CM}}_\Delta$.

Case (ii): Now, consider an arc γ that starts at q, crosses $\overline{k}$, continues in τ direction and ends at a vertex in R_k, and let M_γ be the associated module. By Remark 2.3, since $M_\gamma \in \operatorname{ind} \underline{\mathrm{CM}}(\mathrm{B})$ we have $\tau M_\gamma \in \operatorname{ind} \underline{\mathrm{CoCM}}(\mathrm{B})$ and $\tau M_\gamma = M_{\tau(\gamma)}$ is a non-injective co-syzygy in mod B. Observe that $\tau(\gamma)$ lies in R_k. In particular $\tau(\gamma) \neq \overline{k}$ because M_γ is not projective. Applying the same ideas as in case (i), using injective modules and taking $M_{\tau(\gamma)}$ as a co-syzygy, we find an epimorphism $h = (h_j)_{j\in Q} : \oplus_{i=1}^{l} I(x_i) \to M_{\tau(\gamma)}$ such that each arc x_i lies in R_k. Again, $h_j \neq 0$ only for j in R_k, and now h_j is a surjective linear map. Considering the induced morphism $\widetilde{h}$ in mod $\widetilde{\mathrm{B}}$, we have that $M_{\widetilde{\tau(\gamma)}}$ is an indecomposable co-syzygy over mod $\widetilde{\mathrm{B}}$. Then, applying Remark 2.3 to the geometric interpretation in Section 4, we conclude that $M_{\widetilde{\tau(\gamma)}}$ is a summand of $I(\widetilde{a})/S(\widetilde{a})$, where $\widetilde{a}$ is a side of an internal triangle $\widetilde{\Delta}$ in Π_k. Lifting the arc $\widetilde{\tau(\gamma)}$ to D, we obtain that $M_{\tau(\gamma)}$ is a summand of $I(a)/S(a)$, where a, the lift of $\widetilde{a}$, is a side of the internal triangle Δ. Then $\tau^{-1}M_{\tau(\gamma)} = M_\gamma$ is a summand of the radical of $P(a)$. □

5.3.3. The main result

We are now ready for the full description of $\underline{\mathrm{CM}}(\mathrm{B})$. The following result, together with the morphisms description in the next subsection, gives a complete description of the category $\underline{\mathrm{CM}}(\mathrm{B})$ in geometric terms.

Theorem 5.10. *Let* B *be a cluster-tilted algebra of type* $\mathbb{D}$ *and let* $\mathcal{T}$ *be the corresponding triangulation of the punctured disk. Let t be the number of internal triangles which do not have the puncture as a vertex. Then, the indecomposable modules in* $\underline{\mathrm{CM}}(\mathrm{B})$ *are precisely*

(1) $\underline{\mathrm{CM}}_\odot \cup \underline{\mathrm{CM}}_\Delta$ *if* $\mathcal{T}$ *is of type I,*
(2) $\underline{\mathrm{CM}}_\Delta$ *if* $\mathcal{T}$ *is of type II,*
(3) $\underline{\mathrm{CM}}_\clubsuit \cup \underline{\mathrm{CM}}_\Delta$ *if* $\mathcal{T}$ *is of type III.*

In particular, the number of indecomposable modules in $\underline{\mathrm{CM}}(\mathrm{B})$ *is*

$$\begin{array}{ll} (m+d)(m+d-2)+3t & \text{if } \mathcal{T} \text{ is of type I,} \\ 3t & \text{if } \mathcal{T} \text{ is of type II,} \\ 3(t+1) & \text{if } \mathcal{T} \text{ is of type III.} \end{array}$$

Proof. (1) Let $\mathcal{T}$ be of type I, and let $\gamma = \gamma(q,s) \notin \mathcal{T}$ be an arc such that $M_\gamma \in \underline{\mathrm{CM}}(\mathrm{B})$ is not zero. Suppose that $M_\gamma \notin \underline{\mathrm{CM}}_\odot$. Then, according to the definition of $\underline{\mathrm{CM}}_\odot$, there are 4 possible cases.

(i) The point q is uncolored or q is blue but not red.
(ii) The point s is uncolored or s is red but not blue.
(iii) The point q is labeled r_i and the point s is labeled b_{i+1}.
(iv) The point q is labeled r_i and the point s is labeled b_i.

(i) Suppose first that q is blue but not red. By condition (r) of Definition 5.5, the triangulation $\mathcal{T}$ contains an arc a from $\tau(q)$ to the puncture and an arc b from $\tau^2(q)$ to the puncture. If q is equal to s, then either $M_\gamma = P(a)$ if γ is notched, or $M_\gamma = \tau^{-1}P(b)$ if γ is plain. In the first case, M_γ is zero in $\underline{\mathrm{CM}}(\mathrm{B})$, because it is projective, and in the second case, $M_\gamma \notin \mathrm{CM}(\mathrm{B})$, by Remark 5.2(a). This is a contradiction, thus q must be different from s. Again by condition (r) of Definition 5.5, the point $\tau(q)$ is a last vertex in a last triangle $[k, \overline{k}, k+1]$ in a set $\mathcal{E}_i$ and the arc $\gamma(\tau^2(q), \tau^2(q)) = k+2$ belongs to $\mathcal{T}$. The module M_γ is defined by a string $\beta_k \leftarrow \overline{k} \leftarrow k+1 \rightarrow w$, where w is a walk. The string module $\tau^{-1}P(k+2) = M(\beta_k \leftarrow \overline{k})$ is a submodule of M_γ. Therefore there is a non-zero morphism in $\mathrm{Hom}_{\mathrm{B}}(\tau^{-1}P(k+2), M_\gamma)$, and by Remark 5.2(a), M_γ is not in $\mathrm{CM}(\mathrm{B})$, a contradiction.

Now suppose that q is uncolored. By Definition 5.4, this means that neither q nor $\tau(q)$ is incident to the puncture in $\mathcal{T}$. Since $\mathcal{T}$ is a triangulation, this implies that the arc from q to the puncture and the arc from $\tau(q)$ to the puncture cross an arc in $\mathcal{T}$. This implies that there exists an arc $\overline{k} = \gamma(x,y) \in \mathcal{T}$ that is not incident to the puncture such that $q, \tau(q) \in R_k \setminus \{x,y\}$. If γ or $\tau(\gamma)$ lies entirely inside R_k then $M_\gamma \in \mathfrak{M}_k$ and Lemma 5.9 implies that $M_\gamma \in \underline{\mathrm{CM}}_\Delta$. On the other hand, if both γ and $\tau(\gamma)$ do not lie in R_k then both γ and $\tau(\gamma)$ cross $\overline{k} = \gamma(x,y)$ and $q, \tau(q) \notin \{x,y\}$. In other words, γ crosses both $\overline{k}$ and $\tau^{-1}(\overline{k})$ and $q \notin \{x, y, \tau^{-1}(x), \tau^{-1}(y)\}$. Then Lemma 5.8 implies that M_γ is not in $\mathrm{CM}(\mathrm{B})$, a contradiction.

(ii) Suppose first that s is red but not blue. By condition (b) of Definition 5.5, the triangulation $\mathcal{T}$ contains an arc a from s to the puncture and an arc b from $\tau^{-1}(s)$ to the puncture. If q is equal to s, then γ is an arc from q to the puncture, and since γ is not in the triangulation, we see that γ must be notched. Then $M_\gamma = I(b)$. But since the projective dimension of $I(b)$ is 0 or 1, we see that M_γ is projective or, by Remark 5.2(b), $M_\gamma \notin \mathrm{CM}$. In both cases $M_\gamma \notin \mathrm{ind}\,\underline{\mathrm{CM}}(\mathrm{B})$. This is a contradiction, thus q must be different from s. Then, in the AR-quiver of the cluster category, the object X_γ lies in

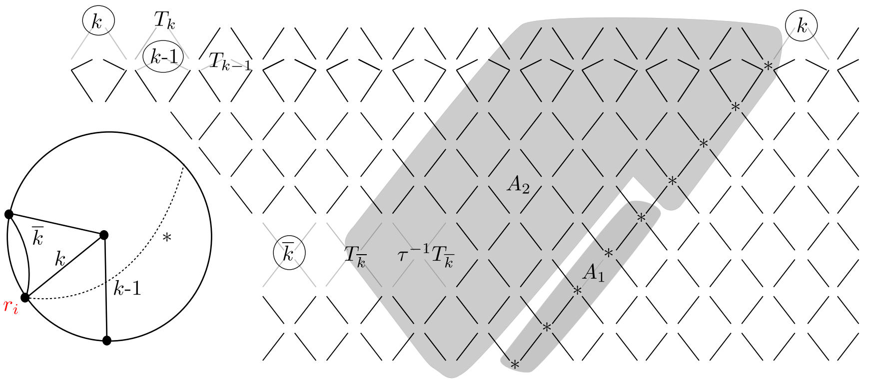


Fig. 15. Proof of Theorem 5.10(1).

one of the positions labeled by a $*$ in Fig. 15. From the relative position of $T_{\overline{k}}$ and the $*$, it follows that there is a nonzero morphism $f \in \mathrm{Hom}_{\mathcal{C}}(\tau^{-1}T_{\overline{k}}, X_\gamma)$. Moreover, because of the relative position of $T_{\overline{k}}$, the shaded region A_2 do not contain any summands of τT, and because of the relative position of T_k, the shaded region A_1 do not contain any summands of τT. This implies that the morphism f does not factor through τT, and therefore f induces a nonzero morphism in $\mathrm{Hom}_{\mathrm{B}}(\tau^{-1}P(\overline{k}), M_\gamma)$. Now Remark 5.2(a) implies that M_γ is not in CM(B), a contradiction.

Now suppose that s is uncolored. Replacing q by s in the argument in the case (i) for q uncolored, we see that if $M_\gamma \notin \underline{\mathrm{CM}}_\Delta$ then there exists an arc $\overline{k} = \gamma(x, y) \in \mathcal{T}$ such that γ crosses both $\overline{k}$ and $\tau^{-1}(\overline{k})$ and $s \notin \{x, y, \tau^{-1}(x), \tau^{-1}(y)\}$. Then Lemma 5.8 implies that $\gamma = \gamma(q, s) = \gamma(\tau^{-1}(y), x)$ and thus $s = x$, a contradiction.

(iii) Let $\gamma = \gamma(q, s)$ and suppose q is labeled r_i and s is labeled b_{i+1}. Whenever $M_\gamma = M(r_i, b_{i+1})$, we conclude from Lemma 5.6(a) that M_γ is projective and thus M_γ is zero in $\underline{\mathrm{CM}}$(B), a contradiction.

If γ is not incident to the puncture then $M_\gamma = M(r_i, b_{i+1})$, a contradiction.

Suppose that γ is incident to the puncture. If γ is plain then $M_\gamma = 0$ if $\gamma \in \mathcal{T}$ and $M_\gamma = M(r_i, b_{i+1})$ otherwise. Both cases yield a contradiction. On the other hand, if γ is notched, let γ° denote the plain arc homotopic to γ. Then if $\gamma^\circ \in \mathcal{T}$, we have $M_\gamma = M(r_i, b_{i+1})$, a contradiction. And if $\gamma^\circ \notin \mathcal{T}$, then $\tau(\gamma)$ is the plain arc from $\tau(q)$ to the puncture, and since q is a colored point, we get $\tau(\gamma) \in \mathcal{T}$. This implies that M_γ is projective and thus M_γ is zero in $\underline{\mathrm{CM}}$(B), a contradiction.

(iv) Now let $\gamma = \gamma(q, s)$ and suppose q is labeled r_i and s is labeled b_i. Whenever $M_\gamma = M(r_i, b_i)$, we conclude from Lemma 5.6(a) that $M_\gamma \in \underline{\mathrm{CM}}_\Delta$ and we are done.

If γ is not incident to the puncture then $M_\gamma = M(r_i, b_i)$.

Suppose that γ is incident to the puncture. Since $q = s$ is a red and blue point with label $r_i b_i$, remark (3) following Definition 5.5 implies that q and $\tau(q)$ are incident to the puncture in $\mathcal{T}$. Thus if γ is plain then $\gamma \in \mathcal{T}$ and if γ is notched then $\tau(\gamma) \in \mathcal{T}$. In the first case $M_\gamma = 0$ and in the second case M_γ is projective, so in both cases we get a contradiction. This completes the proof of statement (1).

(2) Let $\mathcal{T}$ be a triangulation of type II and assume that we are in the case studied in Fig. 3. Let M_γ be a non-projective syzygy. If $X_\gamma \in \mathfrak{X}_a$ then, by Lemma 5.9, $M_\gamma \in \underline{\mathrm{CM}}_\Delta$. If $X_\gamma \notin \mathfrak{X}_a$ then, by Lemma 5.3, X_γ does not lie in the shaded area A in Fig. 3. Thus, $X_\gamma \in \{\tau^{-1}T_b, T_b, \tau T_b, \tau^2 T_c, \tau T_c, T_c\}$ so the module M_γ is injective or projective, this is impossible by Remark 5.2(b). The remaining two triangulations of type II can be analyzed in the same manner.

(3) Let $\mathcal{T}$ be a triangulation of type III. Let M_γ be a non-projective syzygy. If $X_\gamma \in \mathfrak{X}_c \cup \mathfrak{X}_d$ then, by Lemma 5.9, $M_\gamma \in \underline{\mathrm{CM}}_\Delta$. Suppose that $X_\gamma \notin \mathfrak{X}_c \cup \mathfrak{X}_d$ then, by Lemma 5.3, $X_\gamma \notin A_1 \cup A_2 \cup A_3$, see Fig. 4. The only positions for X_γ under the assumption are those labeled $\clubsuit$ in said figure. The remaining objects correspond to arcs a and b in $\mathcal{T}$ or summands T_a and T_b of the cluster tilting object T, and the corresponding modules are zero in $\underline{\mathrm{CM}}(\mathrm{B})$. Therefore $M_\gamma \in \underline{\mathrm{CM}}_\clubsuit \cup \underline{\mathrm{CM}}_\Delta$. □

5.3.4. Auslander–Reiten quiver

The $\underline{\mathrm{CM}}$ category of cluster tilted-algebras of type $\mathbb{D}$ has been studied in an abstract way in [16]. Our approach is different, we find the modules and their projective resolutions explicitly. Moreover, we can compute the AR quiver of the triangulated category $\underline{\mathrm{CM}}$ in terms of (relative) elementary moves, analogous to the elementary moves defined in [31].

According to [16, Theorem 4.9], the stable category $\underline{\mathrm{CM}}(\mathrm{B})$ is equivalent to a union of stable categories $\underline{\mathrm{mod}}S$ where S is a selfinjective cluster-tilted algebra of type $\mathbb{D}$ or $\mathbb{A}_3$. In these cases, the only stable categories having non-trivial morphisms are those arising from cluster-tilted algebras of type $\mathbb{D}_n$, where $n \geqslant 4$. In our setting, the objects $M \in \mathrm{ind}\,\mathrm{CM}(\mathrm{B})$ such that $\mathrm{Hom}_{\underline{\mathrm{CM}}}(M, -)$ is not trivial are the modules in $\underline{\mathrm{CM}}_\odot$. Recall that $\underline{\mathrm{CM}}_\odot = \{M(r_i, b_j) \colon j \in \{i+2, \ldots, i-1\}\}$ and the notation refers to color-labeled endpoints, red r_i and blue b_j.

Definition 5.11. A *relative elementary move* is a composition of elementary moves $\gamma_0 \to \gamma_1 \to \ldots \to \gamma_{l-1} \to \gamma_l$ such that

(1) the arcs γ_0 and γ_l are associated to modules in $\underline{\mathrm{CM}}_\odot$,
(2) the arcs γ_i, $i \in 1, \ldots, l-1$ are not associated to modules in $\underline{\mathrm{CM}}_\odot$ and they are not arcs at the puncture,
(3) one of the endpoints of γ_0 is fixed under all the elementary moves.

In the language of cluster-tilting categories, a relative elementary move is a composition of irreducible morphisms in a sectional path of the AR quiver, such that only the first and the last object are associated to modules in $\underline{\mathrm{CM}}_\odot$. There are two types of relative elementary moves, defined as follows and described in Fig. 16.

- *Red elementary move:* Start with an arc $\gamma(r_i, b_j)$ associated to a module $M(r_i, b_j) \in \underline{\mathrm{CM}}_\odot$ and make elementary moves in τ^{-1} direction, keeping the vertex labeled by

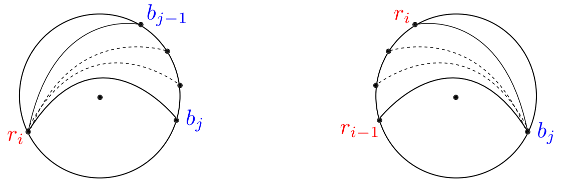


Fig. 16. Red and blue elementary moves in $\underline{\mathrm{CM}}_{\odot}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

b_j fixed, until the arc $\gamma^*(r_{i-1}, b_j)$ associated to the module $M(r_{i-1}, b_j) \in \underline{\mathrm{CM}}_{\odot}$ is reached.

- *Blue elementary move:* Start with an arc $\gamma^*(r_i, b_j)$ associated to a module $M(r_i, b_j) \in \underline{\mathrm{CM}}_{\odot}$ and make elementary moves in τ^{-1} direction, keeping the vertex labeled by r_i fixed, until the arc $\gamma(r_i, b_{j-1})$ associated to the module $M(r_i, b_{j-1}) \in \underline{\mathrm{CM}}_{\odot}$ is reached.

The relative elementary moves represent non-zero morphisms in mod B. Moreover, such morphisms are irreducible in the category $\underline{\mathrm{CM}}_{\odot}$ and this fact leads to a nice description of the category in geometric terms.

Proposition 5.12. *Let $\mathcal{T}$ be a triangulation of type I. The AR quiver of the category $\underline{\mathrm{CM}}_{\odot}$ can be described as follows:*

(1) *There is a bijection between relative elementary moves and irreducible morphisms.*
(2) *The AR translation is given by $\widetilde{\tau}(\gamma(r_i, b_j)) = \gamma(r_{i+1}, b_{j+1})$.*

Proof. The correspondence between arcs and indecomposable modules in the set $\underline{\mathrm{CM}}_{\odot}$ was studied in Section 5.2. Let us prove that the relative elementary moves are associated to irreducible morphisms. We will study the case of red elementary moves, the proof for blue elementary moves is similar. Let T be the cluster tilting object associated to the triangulation $\mathcal{T}$. Assume that the red elementary move is a composition of (at least) two elementary moves $\gamma(r_i, b_j) \to \gamma_1 \to \ldots \to \gamma_{l-1} \to \gamma(r_{i-1}, b_j)$. According to Definition 5.5 this occurs in two cases:

(1) The vertices labeled r_{i-1} and r_i lie in a last triangle $\Delta = [k, \overline{k}, k+1]$ of a set $\mathcal{E}_t$. The situation is shown in the left picture in Fig. 17, where the blue dotted line shows the allowed positions for the label b_j with $j \in \{i+2, \ldots, i-1\}$. The objects $X_1, \ldots, X_{l-1}$ are not summands of $\tau T \oplus T$ since the arcs $\tau(\gamma_1), \ldots, \tau(\gamma_{l-1})$ cross the arc $k+2$, and the arcs $\gamma_1, \ldots, \gamma_{l-1}$ cross the arc $\overline{k}$.

(2) The vertices labeled r_{i-1} and r_i lie in an internal triangle $\Delta = [k, \overline{k}, k+1]$ of a set $\mathcal{E}_t$ and such triangle is not the last in τ order. The case is shown in the right picture in Fig. 17, where the blue dotted line shows the allowed positions for the label b_j with $j \in \{i+2, \ldots, i-1\}$. The objects $X_1, \ldots, X_{l-1}$ are not summands of $\tau T \oplus T$ since the arcs $\tau(\gamma_1), \ldots, \tau(\gamma_{l-1})$ cross the arc $k+1$, and the arcs $\gamma_1, \ldots, \gamma_{l-1}$ cross the arc $\overline{k}$.

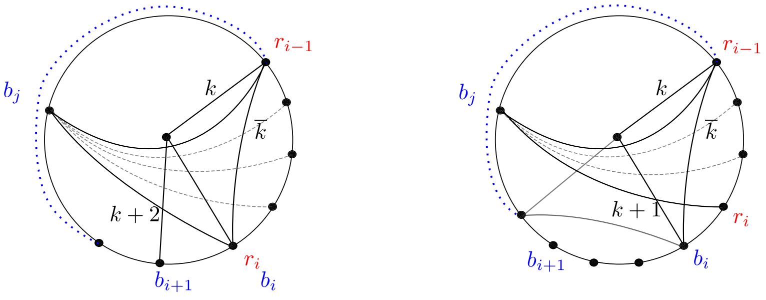


Fig. 17. Possible red elementary moves. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

This implies that the irreducible modules appearing in the composition defined by the red elementary move are non-zero and non-projective. Thus, the composition is a non-zero morphism that is part of a sectional path in the AR quiver of $\underline{\text{mod}}\,\text{B}$. Also, only the first modules and the last module of the sectional path belong to $\underline{\text{CM}}_{\odot}$. It follows that the composition is an irreducible morphism in $\underline{\text{CM}}_{\odot}$. Finally, if the red elementary move is already a single elementary move, the corresponding morphism is irreducible in $\underline{\text{CM}}_{\odot}$ trivially.

The equivalence of categories $\underline{\text{mod}}\, S_{(m+d)(m+d-2)} \simeq \underline{\text{CM}}_{\odot}$ establishes the structure of the AR quiver. An arc $\gamma(r_i, b_j)$ is a source of one or two relative elementary moves. When $\gamma(r_i, b_j)$ is a source of two moves, that is when $j \notin \{i+1, i+2\}$ and $i \notin \{j+1, j+2\}$, there is a red elementary move $\gamma(r_i, b_j) \to \gamma(r_{i-1}, b_j)$ and a blue elementary move $\gamma(r_i, b_j) \to \gamma(r_i, b_{j-1})$. Analogously, the arc $\gamma(r_{i-1}, b_{j-1})$ is a sink of two elementary moves $\gamma(r_{i-1}, b_j) \to \gamma(r_{i-1}, b_{j-1})$ and $\gamma(r_i, b_{j-1}) \to \gamma(r_{i-1}, b_{j-1})$. The knitting algorithm produced by the geometric moves generates the AR quiver and the translation is given by $\widetilde{\tau}(\gamma(r_i, b_j)) = \gamma(r_{i+1}, b_{j+1})$. □

The result is consistent with the following homological argument. Recall that the category $\underline{\text{CM}}(\text{B})$ is triangulated and the shift [1] is given by the formal inverse of the syzygy operator $(\Omega)^{-1}$. Also, from [30, Proposition I.2.3], is known that the category has Serre functor $\mathbb{S}$ which is related with the AR translation $\widetilde{\tau}$ via $\mathbb{S} = \widetilde{\tau}[1]$. According to [24, Section 3.3] the category $\underline{\text{CM}}(\text{B})$ is 3-CY, this means that $\mathbb{S} = (\Omega)^{-3}$. Combining both results, we have $\mathbb{S} = (\Omega)^{-3} = \widetilde{\tau}(\Omega)^{-1}$, so $\widetilde{\tau} = (\Omega)^{-2}$. In particular, for each $M(r_i, b_j) \in \underline{\text{CM}}_{\odot}$, we can use Theorem 5.7 to obtain $\widetilde{\tau} M(r_i, b_j) = M(r_{i+1}, b_{j+1})$.

Example 5.13. Consider the algebra defined by the triangulation $\mathcal{T}$ of type I in the upper part of Fig. 18. The AR quiver of $\underline{\text{CM}}(\text{B}) = \underline{\text{CM}}_{\odot}$ is given by the red and blue elementary moves shown in the figure. Labeling the arcs in $\mathcal{T}$ and computing the intersection number is easy to reconstruct the associated module representations.

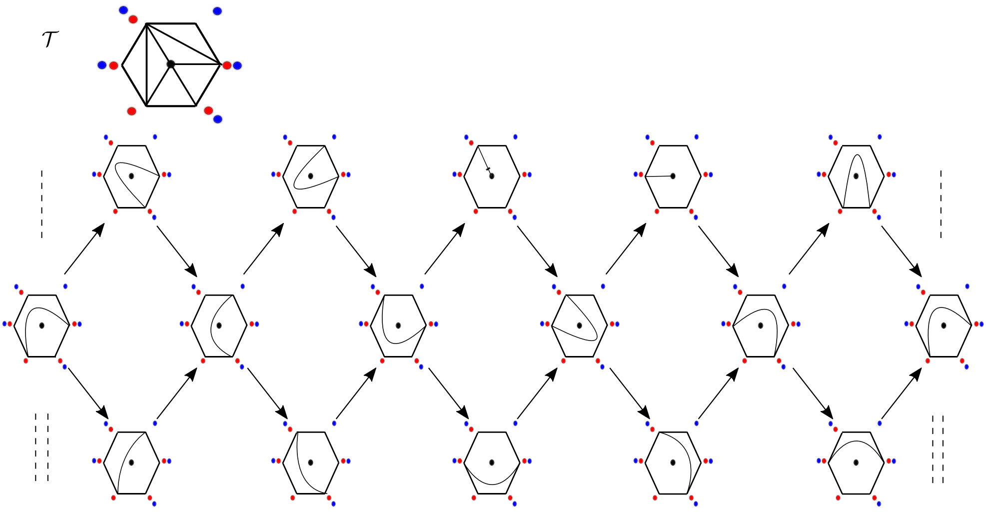


Fig. 18. Geometric realization of the AR quiver of $\underline{\mathrm{CM}}_{\odot}$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

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