GAUSSIAN BEHAVIOR OF QUADRATIC IRRATIONALS

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Abstract. We study the probabilistic behavior of the continued fraction expansion of a quadratic irrational number, when weighted by some "additive" cost. We prove asymptotic Gaussian limit laws, with an optimal speed of convergence. We deal with the underlying dynamical system associated with the Gauss map, and its weighted periodic trajectories. We work with analytic combinatorics methods, and mainly bivariate Dirichlet generating functions; we use various tools, from number theory (the Landau Theorem), probability (the Quasi-Powers Theorem), or dynamical systems: our main object of study is the (weighted) transfer operator, that we relate with the generating functions of interest. The present paper exhibits a strong parallelism between periodic trajectories and rational trajectories. We indeed extend the general framework which has been previously described by Baladi and Vallée for rational trajectories. However, our extension to quadratic irrationals needs deeper functional analysis properties.

1. Introduction

1.1. **General framework.** This paper studies particular quadratic irrationals (rqi in shorthand) defined by purely periodic continued fraction expansions. When a cost is defined on continued fraction expansions, notably in an "additive" way, each rqi number inherits the cost defined on its period, and the paper focus on such costs, with a "moderate growth". Besides, we consider the usual notion of size that is defined in Number Theory contexts for a rqi number, and closely related to the fundamental

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unit of the associated quadratic field, and to the length of the corresponding closed geodesic in the modular surface.

Our general framework is then described as follows: we consider the set $\mathcal{P}$ of rqi numbers, the subset $\mathcal{P}_N$ of rqi numbers with a size at most N, and the restriction C_N to $\mathcal{P}_N$ of an additive cost C defined on $\mathcal{P}$. We perform a probabilistic analysis of the cost C_N when the set $\mathcal{P}_N$ is endowed with the uniform probability. We prove in Theorem 1.1 that the cost C_N asymptotically follows a Gaussian law (when $N \to \infty$). Moreover, Theorem 1.1 exhibits precise asymptotic expansions for its expectation and its variance. Both are of order $\Theta(\log N)$. Theorem 1.2 provides a computable description of the constants that appear in the expansions. The main term of the expectation is already well studied, but, to the best of our knowledge, the estimates on the variance are new, and this is the first study which exhibits a limit Gaussian law in such a context.

All these results are first proven within the unrestricted framework, when there do not exist "constraints" on the continued fraction expansion of the rqi numbers. But we also present in Section 5 similar results that hold for rqi numbers with bounded digits. These results combine the general methods of this paper with previous results that are specific to the constrained case, described in [11, 27, 4]. In the same section, we also study an important (non additive) cost, namely the Lévy constant, and obtain asymptotic expansions for its expectation and its variance. Finally, we discuss the occurrence of local limit laws, according to the type of costs (lattice or non-lattice).

The central object in our analysis is a bivariate Dirichlet generating function $P(s, w)$ where the complex parameter s "marks" the size and the complex parameter w "marks" the cost. Our general methodology is described along three main steps:

Step 1. Relate the series $P(s, w)$ to the weighted transfer operator $\mathbf{H}_{t,w}$, associated with the underlying dynamical system defined by the Gauss map (with $t = s/2$).

Step 2. Transfer the analytic properties of the operator $\mathbf{H}_{t,w}$ (acting on convenient functional spaces) to obtain analytic properties of $P(s, w)$ about:

– its dominant singularity (when s is close to the real axis),

– and its polynomial growth (when s is far from the real axis).

Step 3. Extract asymptotics of coefficients (with the Landau Theorem) and exhibit a quasi-powers behavior for the coefficients. With the Quasi-Powers Theorem, this entails the Gaussian law.

There are strong similarities with the approach introduced by Baladi and Vallée in [1], who perform a similar probabilistic analysis on rationals, with an analogous series $Q(s, w)$. This explains the strong similarity between the two results: the Gaussian law obtained in the present paper, together with the precise asymptotic expansion of the expectation and the variance, is of the same type as in the rational case.

Even though Step 3 is the same in the two analyses, the present rqi framework introduces new important issues in the first two steps. In Step 1, the periodicity phenomenon leads to several Dirichlet series – not only the initial series $P(s, w)$– which take into account the "primitivity" vs the "non-primitivity" of the period (as it is described in Section 2.5). In Step 2, we deal with two different functional spaces:

– when the parameter s is close to the real axis, we relate the bivariate series $P(s, w)$ to *traces* of operators. We then need spaces of analytic functions, where such traces are well-defined, as it is shown in the works of Mayer [18].

– when the parameter s is far from the real axis, the useful space is the space $\mathcal{C}^1(\mathcal{I})$ of functions on the unit interval, as in [1]. Here, the rqi framework leads us to adapt results due to Pollicott and Sharp to obtain polynomial growth (see Theorem 3.5).

Finally, whereas the Baladi-Vallée analysis in [1] uses a unique Dirichlet series $Q(s, w)$ and "stays" in the functional space $\mathcal{C}^1(\mathcal{I})$, the more involved present analysis deals with various bivariate generating functions and two different functional spaces. Each generating function, and each functional space plays a specific role. The occurrence of the trace in the present study explains the resemblances and the differences between constants which appear in the asymptotic estimates of the expectation and the variance, in the rqi case and in the rational case. This is made precise in Section 4.4.

1.2. **Continued fractions and quadratic irrational numbers.** Every real number $x \in]0, 1[$ admits a continued fraction expansion of the form

$$x = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cdots}}}$$

denoted as $x = [m_1, m_2, \ldots]$. Here, the coefficients m_k are positive integers known as partial quotients and also called digits. Rational numbers have a finite continued fraction expansion. When the number x is irrational, the continued fraction expansion is infinite and x is completely determined by the whole sequence (m_k) of its digits.

In 1770, Lagrange proved that a number is quadratic irrational if and only if its continued fraction expansion is eventually periodic. We are mainly interested here in particular quadratic irrational numbers, whose continued fraction expansion is purely periodic. Such numbers are called *reduced* (rqi in shorthand). This paper is devoted to their study: in particular, we describe the probabilistic behavior of such numbers via "costs" that are defined via their continued fraction expansion.

Size. If $p(x)$ is the length of the smallest period of a rqi number x, then the number x is defined by the relation $[m_1, m_2, \ldots, m_k + x] = x$, with $k = p(x)$ and is denoted by $\langle m_1, m_2, \ldots, m_p \rangle$. This relation rewrites as a quadratic polynomial equation of the form $Ax^2 + Bx + C = 0$ with a triple (A, B, C) of relatively prime integers. Then x belongs to the quadratic field $\mathbb{Q}(\sqrt{\Delta})$ where $\Delta = B^2 - 4AC$ is the discriminant of the polynomial, and one associates with x the fundamental unit $\epsilon(x) > 1$ of this quadratic field. This fundamental unit $\epsilon(x)$ plays, for rqi numbers, the same role as the denominator for rationals, and it defines a natural notion of size.

In this paper, we consider the set $\mathcal{P}$ of rqi numbers x, together with its finite[1] subsets $\mathcal{P}_N$ formed with rqi numbers of size $\epsilon(x)$ at most N, defined as

(1.1)

$$\mathcal{P} := \{x \in \mathcal{I} \mid x \text{ is a rqi number } \}, \qquad \mathcal{P}_N := \{x \in \mathcal{P} \mid \ \epsilon(x) \leq N\}\,.$$

Cost on digits. Any non zero map $c : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is called a *digit-cost*. With a given digit-cost c, we associate the *cost* $C(x)$ of a rqi number x defined as the "total cost" on the (smallest) period of x, namely

$$C(x) := c(m_1) + \cdots + c(m_p), \quad \text{ if } x = \langle m_1, m_2, \ldots m_p \rangle \tag{1.2}$$

[1]The finiteness of $\mathcal{P}_N$ is proven at the end of Section 2.3.

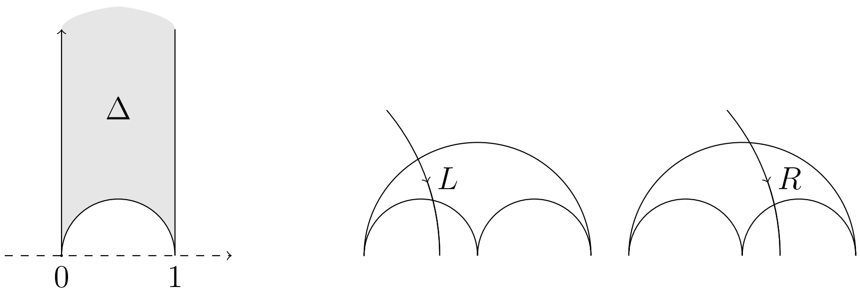

FIGURE 1. On the left, the fundamental tile Δ of the Farey tessellation. On the right, the labeling of a portion of a curve that intersects a triangle.

This defines a non zero cost $C : \mathcal{P} \to \mathbb{R}_{\geq 0}$. Moreover, we restrict ourselves to digit-costs c that satisfy $c(m) = O(\log m)$ and are said to be of *moderate growth*[2].

There are three main examples of digit-costs c that can be treated by our methods, namely: the unit cost $c = 1$; the characteristic function χ_n of a given digit n; the length ℓ of the binary expansion of integers, defined as $\ell(m) := \lfloor \log_2 m \rfloor + 1$. The associated costs C are natural and interesting. For $c = 1$, $C(x)$ coincides with the period-length $p(x)$. For $c = \chi_n$, $C(x)$ equals the number of digits equal to n in the smallest period of x. Finally, for $c = \ell$, $C(x)$ is the number of binary digits needed to store the rqi x.

Interpretation in the hyperbolic plane. We now describe the classical geometric interpretation of the size $\epsilon(x)$ and the digits $m_j(x)$ of a rqi number x in the hyperbolic plane.

We first recall the coding of a curve in the hyperbolic plane $\mathbb{H} := \{z = x + iy \in \mathbb{C} \mid y > 0\}$. When $\mathbb{H}$ is endowed with the usual metric $ds = |dz|/y$, the geodesics are the semi-circles centered on the real axis together with the vertical lines. Consider the hyperbolic triangle Δ with the three cusps $i\infty$, 0 and 1, represented in Fig. 1. Together with all the triangles $h(\Delta)$ which are the transforms of Δ with $h \in SL_2(\mathbb{Z})$, it defines the Farey tessellation of $\mathbb{H}$ represented in Fig. 2. When a "good" oriented curve γ of $\mathbb{H}$ goes through such a triangle $h(\Delta)$ in a general position, there are two possibilities for the position of the curve $h(\Delta) \cap \gamma$ with respect to the three cusps $h(i\infty)$, $h(0)$ and $h(1)$: if there is only one cusp on the right of the curve (and then two cusps on the left), we code the portion of the curve by R; otherwise, we code it by L (see Fig. 1).

[2]The reasons for such a restriction are explained at the end of Section 3.2.

We now return to a rqi x, and consider its minimal even period, of length $e(x)$, with $e(x) := p(x)$ if $p(x)$ is even, and length $e(x) := 2p(x)$ if $p(x)$ is odd. First, if $x \in]0,1[$ is a rqi number, its conjugate $\bar{x}$ satisfies $\bar{x} < -1$ and

$$x = \langle m_1, m_2, \ldots, m_e \rangle \quad \Longrightarrow \quad -1/\bar{x} = \langle m_e, m_{e-1}, \ldots, m_2, m_1 \rangle \,.$$

The geodesic $\gamma(x)$ which links $\bar{x}$ to x (with this orientation) intersects the imaginary axis at $it(x)$. This defines two oriented curves, the curve $\gamma_+(x)$ that links $it(x)$ to x, and the curve $\gamma_-(x)$ that links $it(x)$ to $\bar{x}$. As Series [25] and Pollicott [21] show, the coding of the geodesics "copies" the continued expansions of the associated rqi numbers, and the codings of the two curves $\gamma_+(x)$ and $\gamma_+(-1/\bar{x})$ are the respective periodic infinite words

$$\left(L^{m_1} R^{m_2} L^{m_3} \ldots R^{m_e}\right)^{\mathbb{N}}, \qquad \left(L^{m_e} R^{m_{e-1}} L^{m_{e-2}} \ldots R^{m_1}\right)^{\mathbb{N}} \,.$$

The close connection that relates the codings of the two curves $\gamma_+(-1/\bar{x})$ and $-\gamma_-(x)$ finally entails a coding for the concatenation $\gamma(x)$ of the two curves $(-\gamma_-(x))$ and $\gamma_+(x)$ as the bi-infinite periodic word of period $R^{m_1} L^{m_2} R^{m_3} \ldots L^{m_e}$ (see Fig. 2).
Moreover, the length of the "primitive" part of the geodesic $\gamma(x)$ – associated with the coding $L^{m_1} R^{m_2} L^{m_3} \ldots R^{m_e}$ of the minimal even period – equals $2 \log \epsilon(x)$, where $\epsilon(x)$ is the fundamental unit $\epsilon(x)$ associated with the rqi x.

These interpretations of the size $\epsilon(x)$ and the digits $m_j(x)$ of a rqi x on its geodesic $\gamma(x)$ provide a geometric framework for the whole present study.

1.3. **Statement of the main results.** We study the probabilistic behavior of the continued fraction expansion of a rqi number, with respect to some cost C. When the set $\mathcal{P}_N$ is endowed with the uniform probability $\mathbb{P}_N$, the restriction C_N of the cost C to $\mathcal{P}_N$ becomes a random variable. Our first main result exhibits the asymptotic Gaussian behavior for the sequence (C_N).

Theorem 1.1. *Consider the set $\mathcal{P}$ of the reduced quadratic irrational numbers x, endowed with the size ϵ. With a non zero cost of moderate growth, associate the additive cost C on the set $\mathcal{P}$. Then, the following holds on the set $\mathcal{P}_N$ of reduced quadratic irrational numbers x with $\epsilon(x) \leq N$, for $N \to \infty$:*

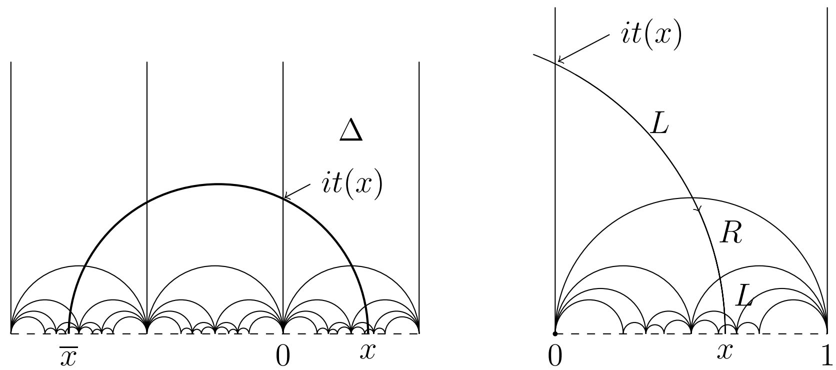

FIGURE 2. Farey tessellation and continued fractions. Here, we consider $x := \phi^{-1}$ (here ϕ is the golden ratio) and the geodesic which links x to its conjugate $-\phi$ as it goes through the Farey tessellation. On the right, the coding of x in terms of R and L.

(i) *There are four constants, two dominant constants $\mu(c)$ and $\nu(c)$ that are strictly positive, and two subdominant constants, $\mu_1(c)$ and $\nu_1(c)$, for which the expectation $\mathbb{E}_N[C]$ and the variance $\mathbb{V}_N[C]$ satisfy the following asymptotic estimates, for some $\beta > 0$,*

$$\mathbb{E}_N[C] = \mu(c) \log N + \mu_1(c) + O(N^{-\beta}),$$

$$\mathbb{V}_N[C] = \nu(c) \log N + \nu_1(c) + O(N^{-\beta}).$$

(ii) *Moreover, the distribution of C is asymptotically Gaussian,*

$$\mathbb{P}_N \left[x \mid \frac{C(x) - \mu(c) \log N}{\sqrt{\nu(c) \log N}} \leq t \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du + O\left(\frac{1}{\sqrt{\log N}} \right).$$

Remark that the speed of convergence towards the Gaussian law is optimal, of the same order $(\log N)^{-1/2}$ as in the usual Central Limit Theorem.

1.4. **Main constants of the analysis.** The next theorem gives a precise description of the four constants that appear in Theorem 1.1. We gather here, various results of different types. The first two items (i) and (ii) are classical; items (iii) and (iv) show a strong analogy with results in [1].

Theorem 1.2. *Consider the digit-cost c of moderate growth associated with the additive cost C. The following holds:*

(i) *Consider the dynamical system related to the Gauss map that underlies the continued fraction expansion, and the (weighted) transfer operator $\mathbf{H}_{t,w}$ which associates with a function f the function*

$\mathbf{H}_{t,w}[f]$, *defined as*

$$\mathbf{H}_{t,w}[f](x) := \sum_{m\geq 1} \frac{e^{w\,c(m)}}{(m+x)^{2t}} f\left(\frac{1}{m+x}\right). \tag{1.3}$$

When it acts on the functional space $\mathcal{A}_\infty(\mathcal{V})$ *defined in* (3.4), *the operator* $\mathbf{H}_{t,w}$ *possesses, for* (t,w) *close to* $(1,0)$, *a dominant eigenvalue denoted as* $\lambda(t,w)$.

(*ii*) *The two partial derivatives of the map* $(t,w) \mapsto \lambda(t,w)$ *at* $(t,w) = (1,0)$ *are related to the entropy* $\mathcal{E}$ *of the system together with the mean value* $\mathbb{E}[c]$ *of the cost with respect to the Gauss density* ψ : $x \mapsto (1/\log 2)(1/(1+x))$,

$$-\lambda'_t(1,0) = \mathcal{E} = \frac{\pi^2}{6\log 2}, \quad \lambda'_w(1,0) = \mathbb{E}[c] = \sum_{m\geq 1} c(m) \int_{1/m}^{1/(m+1)} \psi(x)dx\,.$$

(*iii*) *Consider the map* $w \mapsto \sigma(w)$ *that is defined from the equation* $\lambda(\sigma(w),w) = 1$ *and satisfies* $\sigma(0) = 1$, *together with the two mappings*

$$U(w) := 2(\sigma(w)-1), \qquad V(w) := \log\left(\frac{\lambda'_t(1,0)}{\lambda'_t(\sigma(w),w)}\right) - \log\sigma(w)\,.$$

The constants of Theorem 1.1 are expressed with the first two derivatives of U *and* V, *namely,*

$$\mu(c) = U'(0) = \frac{2}{\mathcal{E}}\,\mathbb{E}[c], \quad \nu(c) = U''(0), \quad \mu_1(c) = V'(0), \quad \nu_1(c) = V''(0).$$

(*iv*) *The dominant constants* $\mu(c), \nu(c)$ *coincide with their analogs that occur in the analysis of rational trajectories described in* [1].

We first observe the equality $\mu(c) = (2/\mathcal{E})\,\mathbb{E}[c]$ that entails the estimate $\mathbb{E}_N[C] \sim \mathbb{E}_N[p]\cdot\mathbb{E}[c]$. This estimate may be compared to the ergodic relation that holds on almost any generic trajectories. This leads to the following (informal) statement "The periodic trajectories behave on average as the generic trajectories behave almost everywhere", which holds in a very general context of "dynamical analysis".

All the constants that appear in Theorem 1.1 are computable. In particular, the constants $\mathbb{E}[c]$ related to the three costs of interest are

$$\mathbb{E}[c] = 1, \quad \mathbb{E}[\chi_n] = \frac{1}{\log 2}\log\left[\frac{(n+1)^2}{n(n+2)}\right], \quad \mathbb{E}[\ell] = \frac{1}{\log 2}\prod_{i=1}^{\infty}\log\left(1+\frac{1}{2^k}\right).$$

In many natural situations, the mean value $\mathbb{E}[c]$ admits an explicit expression, and this is thus the same for the constant $\mu(c)$ itself.

The situation is completely different for the other three constants, as there are no longer close expressions for the second (and the third) derivatives of the mapping $(t, w) \mapsto \lambda(t, w)$ at $(1, 0)$. However, these derivatives are computable in polynomial time with methods that were first described in [11] and [5], then further developed in [16]. All these methods are based on a very natural idea: use the truncated Taylor expansion of analytic functions to approximate the operator by matrices (that now act on polynomials); compute the spectrum of the matrices with classical tools of linear algebra (in finite dimensions). This should provide approximates of the (upper part of) the spectrum of the operator (and this is proven to be true).

1.5. **Description of the methods.** The present work deals with methods that come from analytic combinatorics, in the same lines as in the book of Flajolet and Sedgewick [8]. We associate with the set $\mathcal{P}$, endowed with the size ϵ and the cost C, a bivariate generating function; in the present number theory framework, it is of Dirichlet type,

$$P(s, w) := \sum_{x \in \mathcal{P}} \exp(wC(x))\, \epsilon(x)^{-s} \ .$$

Cumulative costs over the set $\mathcal{P}_N$ are written as sums of coefficients, and we notably define

$$S_w^{[C]}(N) := \sum_{\substack{x \in \mathcal{P} \\ \epsilon(x) \leq N}} \exp(wC(x))\, ;$$

the moment generating of the cost C on $\mathcal{P}_N$ is then written as the quotient of such sums,

$$\mathbb{E}_N[\exp(wC)] = \frac{S_w^{[C]}(N)}{S_0^{[C]}(N)}\, .$$

The distribution of the cost C on $\mathcal{P}_N$ is indeed studied here with this moment generating function: we use the Quasi-Powers Theorem [12] that states that an asymptotic Gaussian law for C on $\mathcal{P}_N$ holds as soon as the moment generating function has a "uniform quasi-powers" form (when w is a complex number near 0).

We thus need precise asymptotic estimates on the sums $S_w^{[C]}(N)$ of the series $P(s, w)$; analytic combinatorics principles relate them to the analytic properties of the series $P(s, w)$. The strong tool which operates this transfer here, between analytic properties of $P(s, w)$ and asymptotic properties of its coefficients, is the Landau Theorem proven by Landau in [15]. The analytical properties that are needed for $s \mapsto P(s, w)$ are of two types: a precise knowledge of its singularities (here, its poles, located near the real axis) together with a good knowledge of its behavior for $|\Im s| \to \infty$ (here, polynomial growth for $|\Im s| \to \infty$).

We have then to study the series $P(s,w)$. This will be done with "dynamical analysis", which is already used in many works, in particular in the analog rational study (see for instance in [1, 29, 28]). The idea is to use the underlying dynamical system, associated with the continued fraction transformation (the Gauss map), and relate the series $P(s,w)$ to the (weighted) transfer operator of the dynamical system described in (1.3). It is then needed to study the transfer operator itself, for w close to 0, and various values of s.

Here, as already mentioned in Section 1.1, we need two different functional spaces: the first one, useful when s is close to the real axis, is the set $\mathcal{A}_\infty(\mathcal{V})$ of analytic functions (defined in (3.4)) where the transfer operator admits dominant spectral properties and a well-defined trace (in the sense of Grothendieck); this study provides a precise knowledge of the singularities of $s \mapsto P(s,w)$ located near the real axis; the second functional space, useful for $|\Im s| \to \infty$, is the set of $\mathcal{C}^1$ functions, where the transfer operator admits bounds à la Dolgopyat [6], which entail the polynomial growth of $P(s,w)$ for $|\Im s| \to \infty$.

1.6. **Comparison with the Baladi-Vallée approach.** As already mentioned in Section 1.1, our methods are strongly similar to those that are used for rational trajectories in [1]. The paper [1] deals with the generating function of the rational set $\mathcal{Q} := \mathbb{Q} \cap [0,1[$, endowed with the denominator size $q(x)$, namely

$$Q(s,w) := \sum_{x \in \mathcal{Q}} \exp(wC(x))\, q(x)^{-s}\,. \tag{1.4}$$

The series is directly expressed with the weighted transfer operator $\mathbf{H}_{t,w}$ introduced in (1.3), as the equality $Q(s,w) = (I - \mathbf{H}_{t,w})^{-1}[1](0)$ holds[3], with $t = s/2$. This explains why the first two steps of the rational study are easier than in the present one, that is more involved due to the three issues described in Section 1.1.

[3]This is not completely exact, as the final operator $\mathbf{F}_{t,w}$ also intervenes (see Section 4.4).

Along Step 3, the main tools are essentially the same, namely, the Landau Theorem[4], and the Quasi-Powers Theorem.

1.7. **Comparison with already known results.**

Plain periodic trajectories. The periodic trajectories of dynamical systems are very well studied, by means of various zeta series, as Ruelle introduced it in his pioneering work [24]. In the case of the Gauss map, the Selberg zeta series is a powerful tool that is well-adapted to plain periodic trajectories (without a cost C), and there is a close connection exhibited by Mayer in [17] between the plain Dirichlet series $P(s) := P(s, 0)$, the Selberg zeta series and the zeta Riemann function. This leads to the asymptotic estimate of the cardinality $|\mathcal{P}_N| \sim [(3 \log 2)/\pi^2] \cdot N^2$. The study of the remainder term is more involved and was performed by Boca in [3] who obtains an error term of order $O_\varepsilon(N^{7/4+\varepsilon})$ for any $\varepsilon > 0$. His proof relies on precise estimates on the distribution of pairs (x, y) of coprime integers that satisfy $xy = 1 \mod q$ (for an integer q), together with classical results on the Riemann zeta function. Following this approach, Ustinov [26] improves the estimate of the error term and obtains an order $O(N^{3/2} \log^4 N)$.

Our series $P(s, w)$ can be viewed as a "twisted zeta Selberg series", where the twist is brought by the cost C; many geometric properties of the Selberg zeta series "disappear" with this twist and it is not clear how to extend Boca's and Ustinov's methods to weighted periodic trajectories.

Weighted periodic trajectories. To the best of our knowledge, Parry and Pollicott in [20], then Pollicott in [21] were the first to study weighted periodic trajectories, with analytic combinatorics methods. Pollicott indeed introduces a weighted generating function, and, as he only performs average-case analysis, he may use Tauberian Theorems that provide estimates of the mean values, as in Theorem 1.1(i), but with only the first (dominant) term, without remainder terms. Later on, Faivre in [7] used slightly different methods, and obtained the same results.

With similar methods, Kelmer [14] studies the set $\mathcal{P}_N^{(n)}$ of rqi numbers whose alternate sum of partial quotients is fixed and equal to n. He considers the restriction of a function $f \in \mathcal{C}([0, 1])$ to the set $\mathcal{P}_N^{(n)}$ and proves that, for n fixed and $N \to \infty$, its mean value tends to the mean value

[4]The paper [1] does not use the "ready for use" version of the Landau Theorem described in Section 4.2, and, as in many other works (see for instance [22]), it proves the analog result "by hands".

$\mathbb{E}[f]$ of f with respect to the Gauss density ψ. He does not make precise the speed of convergence.

Remainder terms? One may expect more precise asymptotic estimates, for the mean value (estimates with remainder terms), or for the variance, (with even a rough estimate); such estimates are also needed for a distributional study. They appear to be based on a more precise knowledge of the weighted transfer operator $\mathbf{H}_{t,w}$ for $|\Im t| \to \infty$. This better knowledge was indeed brought by works of Dolgopyat, notably in [6], first for *unweighted* transfer operators associated with Markovian dynamical systems with a *finite* number of branches. Then, Pollicott and Sharp have used Dolgopyat's result (see for instance [22]), and performed various average-case analyses, where they obtain estimates for various mean values (related to restricted periodic trajectories) with remainder terms.

Dolgopyat's results were further adapted in [1] to the present case of interest – a weighted transfer operator associated with a Markovian dynamical system with a *infinite denumerable* number of branches–. Then, in [1], Baladi and Vallée have used this extension of Dolgopyat's result for distributional studies of weighted rational trajectories and derived asymptotic Gaussian laws. To the best of our knowledge, the present work is the first which is devoted to distributional studies of weighted periodic trajectories.

1.8. **Plan of the paper.** Section 2 presents the main objects, and the general methodology; it introduces the various generating functions of interest, describes their relations and their first properties. Section 3 introduces the weighted transfer operator, the two useful functional spaces $\mathcal{A}_\infty(\mathcal{V})$ and $\mathcal{C}^1(\mathcal{I})$, and describes their role in the analysis. It relates analytic properties of the generating functions to fine properties of the weighted transfer operator: bounds à la Dolgopyat in the space $\mathcal{C}^1(\mathcal{I})$, and properties of the traces in the space $\mathcal{A}_\infty(\mathcal{V})$. As our work needs a precise (and long) study of traces, we perform it in the Annex and just summarize the main results in Section 3. Section 4 transfers analytic properties of the bivariate generating functions into probabilistic properties of the cost: it uses two main tools (the Landau Theorem, then the Quasi-Powers Theorem) to transfer analytic properties of the bivariate generating functions into probabilistic properties of the cost, notably the asymptotic Gaussian law (Theorem 1.1). It also performs a precise study of the main constants of interest (Theorem 1.2). In Section 5, we explain how to "transfer" to periodic trajectories some results that are already proven on rational trajectories. We study in particular the case of "constrained" periodic trajectories. The paper ends with an Annex, which is devoted to a self-contained study of traces of operators. It is needed in Section 3 and may be of independent interest for people that are not specialists of this theory.

The results that are already known and directly used in the paper are labeled with letters, whereas the new results that are proven in the paper are labeled with numbers.

2. Presentation of the main objects

The present section introduces the main actors: first, in Section 2.1, the underlying dynamical system associated with the Gauss map, together with the main properties of its branches (Section 2.2). We then recall well-known facts about irrational quadratic numbers and fundamental units in Section 2.3 and describe the costs of interest (of moderate growth) (Section 2.4). Then, Section 2.5 introduces the various (bivariate) generating functions of interest. The initial generating function $P(s,w)$ involves quadratic numbers, and thus deals with a primitivity condition that is difficult to manage. We first "suppress" this primitivity condition, then we change of size: we replace the initial size ϵ by another (closely related) size α, which satisfies multiplicative properties. We finally obtain a series $E(s,w)$. The end of the Section describes the first easy properties of these generating functions and explains why it suffices to deal with $E(s,w)$ throughout the rest.

2.1. The Euclidean dynamical system.

The continued fraction expansion encodes the trajectories of the dynamical system associated with the Gauss map $T : \mathcal{I} \to \mathcal{I}$,

$$\mathcal{I} := [0,1], \qquad T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \qquad \text{for } x \neq 0, \;\; T(0) = 0\,. \tag{2.1}$$

Here, $\lfloor x \rfloor$ is the integer part of x. The trajectory

$$\mathcal{T}(x) = (T(x), T^2(x), \ldots, T^k(x), \ldots)$$

of an irrational x never meets 0 and is encoded by the infinite sequence

$$(m_1(x), m_2(x), \ldots, m_k(x), \ldots) \text{ with } m_k(x) := m(T^{k-1}(x)), \;\; m(x) := \left\lfloor \frac{1}{x} \right\rfloor.$$

The map T is a piecewise complete interval map, with its set $\mathcal{H}$ of inverse branches,

$$\mathcal{H} = \left\{ h_m \mid x \mapsto \frac{1}{m+x}; \quad m \geq 1 \right\} .$$

For any $k \geq 1$, the set $\mathcal{H}^k$ of inverse branches of T^k gathers LFTs (linear fractional transformations) of the form

$$h = h_{m_1} \circ h_{m_2} \circ \cdots \circ h_{m_k} . \tag{2.2}$$

By definition, the depth of a LFT $h \in \mathcal{H}^k$ (denoted by $|h|$) is equal to k. The set

$$\mathcal{H}^+ := \bigcup_{k \geq 1} \mathcal{H}^k$$

gathers the inverse branches of any strictly positive depth. For $h \in \mathcal{H}^+$, the interval $h(\mathcal{I})$ is called the fundamental interval relative to h. For a LFT h described in (2.2), the interval $h(\mathcal{I})$ gathers reals x that have their i-th partial quotients equal to m_i for any $i \in [1..k]$. For any $k \geq 1$, the fundamental intervals $h(\mathcal{I})$ for $h \in \mathcal{H}^k$ form a topological partition of the unit interval $\mathcal{I}$.

2.2. **First properties of the dynamical system.** Here, we describe the main properties (of geometric flavor) satisfied by the branches of the set $\mathcal{H}^+$ defined in Section 2.1 and their derivatives. The intervals of interest are $\mathcal{I} := [0, 1]$ and $\mathcal{J} := [-1/4, 9/4]$, the last one intervenes because of Property $(P3)$ associated with $r = 5/4$.

$(P1)$ There is a constant L (usually called the distortion constant – here equal to $\sup(8/3, \exp(20/3))$– for which the two inequalities hold for any inverse branch $h \in \mathcal{H}^+$,

$$|h''(x)| \leq L\,|h'(x)|, \quad \forall x \in \mathcal{J}, \qquad \frac{1}{L} \leq \frac{|h'(x)|}{|h'(y)|} \leq L, \quad \forall (x, y) \in \mathcal{J}^2 .$$

$(P2)$ The map T is piecewise strongly expanding. There is a contraction ratio ρ equal to ϕ^{-2} for which the following inequalities hold and involve the fixed point x_h of the inverse branch h and the distortion constant L, namely

$$|h'(x_h)| \leq \rho^{|h|}, \qquad |h'(x)| \leq L\rho^{|h|} \quad \forall x \in \mathcal{J}, \qquad \forall h \in \mathcal{H}^+ .$$

$(P3)$ Consider the disk $\mathcal{V}_r$ of center 1 and radius r. For $r \in]1, (1+\sqrt{5})/2[$, there exists $\tilde{r} < r$ for which the following holds, for any branch $h \in \mathcal{H}^+$:

(i) It is analytic on $\mathcal{V}_r$ and the inclusion $h(\mathcal{V}_r) \subset \mathcal{V}_{\tilde{r}}$ holds.

(*ii*) The analytic extension $\underline{h}$ of the map $x \mapsto |h'(x)|$ is non zero on $\mathcal{V}_r$ and the function $z \mapsto \log \underline{h}(z)$ is analytic on $\mathcal{V}_r$.

We consider in the following the case $r = 5/4$ and denote $\mathcal{V} := \mathcal{V}_{5/4}$.

($P4$) The two series

$$\sum_{h\in\mathcal{H}} \underline{h}^s(z), \qquad \sum_{h\in\mathcal{H}} \log \underline{h}(z)\, \underline{h}^s(z),$$

are absolutely convergent for any z in the closure of the disc $\overline{\mathcal{V}}$ and on the compact subsets of the half-plane $\Re s > 1/2$.

($P5$) The UNI (Uniform Non Integrability) condition holds. It is described with the "distance" $\Delta(h,k)$ between two inverse branches h and k of same depth, (introduced in [1]),

$$\Delta(h,k) := \inf_{x\in\mathcal{I}} |\Psi'_{h,k}(x)|, \quad \text{with} \quad \Psi_{h,k}(x) := \log \frac{|h'(x)|}{|k'(x)|},$$

and the measure $J(h,\eta)$ of the "ball" of center h and radius $\eta > 0$,

$$J(h,\eta) := \sum_{\substack{k\in\mathcal{H}^n,\\ \Delta(h,k)\le\eta}} |k(\mathcal{I})|, \qquad (\text{for } h \in \mathcal{H}^n).$$

The condition UNI states the existence of a constant K for which

$$J(h,\rho^{an}) \le K\rho^{an}, \quad \forall a\,(0<a<1), \quad \forall n, \quad \forall h \in \mathcal{H}^n.$$

There is also a technical statement about the second derivatives $\Psi''_{h,k}(x)$

$$Q := \sup\{|\Psi''_{h,k}(x)| \mid n \ge 1\,, h,k \in \mathcal{H}^n, x \in \mathcal{I}\} < \infty.$$

2.3. **Quadratic irrational numbers.** Quadratic irrational numbers are the only numbers x for which the trajectory $\mathcal{T}(x)$ is eventually periodic, and a *reduced* quadratic irrational number (rqi in shorthand) is associated with a purely periodic trajectory. A rqi x is then completely defined by its (smallest) period $p(x)$ which defines a *primitive* cycle (of the same length).

The set of rqi numbers coincides with the set of fixed points x_h of LFTs $h \in \mathcal{H}^+$. All the LFTs which determine a given rqi number x are the powers of a given LFT which is primitive. This defines a bijection between the set $\mathcal{P}$ of rqi numbers and the set of primitive LFTs, together with a decomposition of $\mathcal{H}^+$ as a disjoint sum,

$$\mathcal{H}^+ = \bigcup_{k\ge1} \mathcal{P}^{(k)}, \quad \mathcal{P}^{(k)} := \{h^k,\, h \in \mathcal{P}\}. \tag{2.3}$$

A fundamental quantity associated with a rqi number x is the product of all the shifted $T^i(x)$ along its period. If h is the primitive LFT which defines $x = x_h$, one considers

$$\alpha(x) := \prod_{i=0}^{p(x)-1} T^i(x) = |h'(x_h)|^{1/2} .$$

Here, the fundamental unit $\epsilon(x)$ is chosen as the size for the rqi x (see Section 1.2). It is related to the product $\tilde{\alpha}(x)$ of all the shifted $T^i(x)$ along the *minimal even* period of length $e(x)$ (already mentioned in Section 1.2), and the following equality holds

$$\epsilon(x) = \tilde{\alpha}(x)^{-1}, \qquad \tilde{\alpha}(x) := \prod_{i=0}^{e(x)-1} T^i(x) ,$$

that also translates as

$\epsilon(x) = \alpha(x)^{-r(x)}$ with $r(x) := 1$ for even $p(x)$, and $r(x) := 2$ for odd $p(x)$.

It is then natural (and useful) to extend the definition of α, ϵ, r from the set $\mathcal{P}$ of rqi numbers to the set $\mathcal{H}^+$. For any $h \in \mathcal{H}^+$ (primitive or not), we define $\alpha(h)$ as

(2.4)
$$\alpha(h) := \prod_{i=0}^{|h|-1} T^i(x_h) = |h'(x_h)|^{1/2} ,$$

(2.5)
$$\epsilon(h) := \alpha(h)^{-r(h)}, \text{ with } r(h) := 1 \text{ for even } |h|, \text{ and } r(h) := 2 \text{ for odd } |h| .$$

With this extension, α, ϵ are now defined on the set $\mathcal{H}^+$ and α satisfies the multiplicative property[5] $\alpha(h^k) = (\alpha(h))^k$. Now, the set $\mathcal{H}^+$ is endowed with the size ϵ (closely related to α) and a cost c.

The rational trajectories are classically studied with the denominator size: one associates with $h \in \mathcal{H}^+$ the denominator $q(h) = |h'(0)|^{-1/2}$. The distortion constant L introduced in Property $(P1)$ of Section 3.1 now relates the two sizes ϵ and q, defined on $\mathcal{H}^+$, via the inequality

$$L^{-1/2}\,\epsilon(h)^{1/2} \le L^{-1/2}\,\alpha(h)^{-1} \le q(h) \le L^{1/2}\,\alpha(h)^{-1} \le L^{1/2}\,\epsilon(h) ,$$

that also proves that the set $\mathcal{P}_N$ of rqi numbers with size $\epsilon(x) \le N$ is finite, and satisfies $|\mathcal{P}_N| = O(N^2)$.

[5]Remark that this is not the case for ϵ.

2.4. **Cost of moderate growth.** A digit-cost $c : \mathbb{N}^* \to \mathbb{R}_{\geq 0}$ is also a mapping $c : \mathcal{H} \to \mathbb{R}_{\geq 0}$, defined by the equality $c(h_m) := c(m)$. We next extend c to $\mathcal{H}^+$ in an additive way and define

$$c : \mathcal{H}^+ \to \mathbb{R}_{\geq 0} \quad \text{with} \quad c(h) := \sum_{i=1}^{k} c(h_i) \qquad \text{if } h = h_1 \circ h_2 \circ \ldots \circ h_k \, .$$

We now consider costs of moderate growth[6].

Definition 2.1. A digit-cost c is of moderate growth if there exists (A, B) with $A \geq 0, B \geq 0$ and $A + B > 0$ for which

$$c(m) \leq A \log m + B \qquad \text{for all } m \geq 1 \, . \tag{2.6}$$

In the present rqi framework, a precise comparison between the various generating functions needs a precise study of these costs (which was not needed in previous studies, for instance in [1]). In particular, Relation (2.7) between costs and derivatives of inverse branches will be central in the proof of Proposition 2.5.

Lemma 2.2. *For a digit-cost c of moderate growth, there exists a pair (K, d) of strictly positive numbers for which*

$$e^{c(h)} \leq |h'(x_h)|^{-d}, \quad e^{c(h)} \leq K\, |h'(0)|^{-d} \qquad \textit{for any } h \in \mathcal{H}^+ \, . \tag{2.7}$$

The cost is said to be of exponent d.

Proof. For any h of depth 1, of the form $h = h_m$, the equality $2 \log m = |\log |h'(0)||$ holds and Relation (2.6) writes

$$c(h) \leq (A/2) |\log |h'(0)|| + B \qquad \text{for all } h \in \mathcal{H} \, .$$

We first prove that the previous relation extends to any $h \in \mathcal{H}^+$ as

$$c(h) \leq (A/2) |\log |h'(0)|| + B|h| \qquad \text{for all } h \in \mathcal{H}^+ \, , \tag{2.8}$$

where $|h|$ is the depth of the LFT h. One indeed performs an easy induction; when $c = h_1 \circ h_2$, one has

$$c(h_1 \circ h_2) = c(h_1) + c(h_2) \leq \frac{A}{2} |\log |h_1'(0)|| + \frac{A}{2} |\log |h_2'(0)|| + B\,|h_1| + B\,|h_2| \, .$$

Now, for any $x \in [0, 1]$, the inequality $|h'(0)| \geq |h'(x)|$ holds, and

$$|\log |h_1'(0)|| + |\log |h_2'(0)|| \leq |\log |h_1'(h_2(0))|| + |\log |h_2'(0)|| = |\log |h'(0)|| \, .$$

Relation (2.8) is now proven, and implies

$$c(h) \leq (A/2) |\log |h'(x_h)|| + B|h|.$$

[6] We will explain in Section 3.2 why we restrict ourselves to this class of costs.

Using Property $(P2)$, with $A := 2a, B := -b\log\rho$, now entails the bound

$$e^{c(h)} \le \exp[-a\log|h'(x_h)|]\exp[B|h|] = |h'(x_h)|^{-a}\cdot\rho^{-b|h|} \le |h'(x_h)|^{-(a+b)}\,.$$

With the distortion constant L, this entails the second inequality with $K = L^{a+b}$.

□

2.5. **Generating functions of interest.** When the set $\mathcal{P}$ of rqi numbers is endowed with the size ϵ and an additive cost C that comes from a digit-cost c, its basic generating function is a Dirichlet weighted generating function of the form

$$P(s,w) := \sum_{x\in\mathcal{P}} \exp(wC(x))\,\epsilon(x)^{-s}\,, \tag{2.9}$$

where the variable w "marks" the cost C and the variable s "marks" the size ϵ.

We now define two variants of series P, namely series Z and series E, that are closely related to the initial series P and easier to deal with. The series $Z(s,w)$ "avoids the primitivity condition", whereas the series $E(s,w)$ deals with size α instead of size ϵ.

Series Z**.** To avoid the primitivity condition, we "replace" the series $P(s,w)$ by the "total" series

$$Z(s,w) := \sum_{h\in\mathcal{H}^+} \exp(wc(h))\,\epsilon(h)^{-s}\,, \tag{2.10}$$

and the decomposition (2.3) leads to the decomposition

$$Z(s,w) = P(s,w) + A(s,w)\,, \tag{2.11}$$

with $A(s,w) := \sum_{k\ge2} P_k(s,w)$ and $P_k(s,w) := \sum_{h\in\mathcal{P}^{(k)}} \exp(wc(h))\cdot\epsilon(h)^{-s}$.

Proposition 2.5 will prove that the "main" term in (2.11) is brought by $P(s,w)$.

Series E**.** We wish to deal with size α (instead of size ϵ). Size α indeed fulfills multiplicative properties described in Section 2.3. This is why we introduce the series

$$Y(s,w) := \sum_{k\ge1} Y_k(s,w)\,,$$

(2.12)

with $$Y_k(s,w) := \sum_{h\in\mathcal{H}^k} \exp(wc(h))\,\alpha(h)^{2s} = \sum_{h\in\mathcal{H}^k} \exp(wc(h))\,|h'(x_h)|^{s}\,.$$

We also write $Z(s,w)$ as

$$Z(s,w) := \sum_{k\geq 1} Z_k(s,w), \qquad Z_k(s,w) := \sum_{h\in\mathcal{H}^k} \exp(wc(h))\,\epsilon(h)^{-s}\,.$$

The notion of minimal even period leads to Relation (2.5) between ϵ and α. This entails the following relations between the series $Z_k(s,w)$ and $Y_k(s,w)$,

$$Z_k(s,w) = Y_k(s/2,w) \ \ (\text{for } k \text{ even}), \quad Z_k(s,w) = Y_k(s,w) \ \ (\text{for } k \text{ odd})\,,$$

and leads to the decomposition

$$Z(s,w) = E(s,w) + O(s,w)\ , \tag{2.13}$$

where the even and the odd part of $Z(s,w)$ are related to components $Y_k(s,w)$ as

$$E(s,w) = \sum_{\substack{k\,\text{even}\\ k\geq 2}} Y_k(s/2,w)\,, \quad O(s,w) = \sum_{\substack{k\,\text{odd}\\ k\geq 1}} Y_k(s,w)\,. \tag{2.14}$$

Proposition 2.5 will prove that the “main” term in (2.13) is brought by $E(s,w)$.

Sets $\Gamma(a)$**.** Each term of the generating functions involves a product $|h'(x)|^s\exp(wc(h))$. With a cost c (of moderate growth) of exponent d and a real $a>0$, we associate the sets $\mathcal{C}_a$ and $\Gamma(a)$,

$$\mathcal{C}_a := \{(\sigma_0,\nu_0) \mid \sigma_0 > 1/2,\, \nu_0>0,\, \sigma_0 - d\nu_0 = a\}\,,$$

$$\Gamma(a) := \{(t,w) \mid \Re t > \sigma_0,\, |\Re w| < \nu_0, \quad (\sigma_0,\nu_0)\in\mathcal{C}_a\}\,, \tag{2.15}$$

that are well-adapted to the study of such products.

In the sequel, we will deal with $\Gamma(a)$ or some of its subsets of the form $\mathcal{S}\times\mathcal{W}$ where $\mathcal{W}$ is a complex neighborhood of 0 of the form $\{|w| \ \leq \nu_0\}$ and $\mathcal{S}$ is a (part of a) vertical strip of the form $\{s \mid \Re t > 1-\delta_0\}$. Such a domain is contained in $\Gamma(a)$ as soon as the inequality $1-\delta_0-d\nu_0 > a$ holds.

Series Y**.** We have introduced the three series P, Z, E, together with series Y. The series Y will be a useful reference in this context, as we now explain:

Lemma 2.3. *Consider the series $X(s,w)$ with $X\in\{P,Z,E\}$ and a pair (s,w) with $\nu := \Re w$ and $\sigma := \Re s$.*

(*a*) *The following relations hold between X and Y,*

$$|X(s,w)| \leq Y(\sigma/2,|\nu|), \quad |P_k(s,w)| \leq Y((k/2)\sigma, k|\nu|)\,.$$

(*b*) *The following bound holds for each component* $Y_k(s,w)$ *on each* $\Gamma(a)$, *and involves the distortion constant* L,

$$|Y_k(t,w)| \le L\,\rho^{(a-1)k} \qquad \text{for any } k \ge 1\,. \tag{2.16}$$

Proof. (*a*) The first bound is due to the inequality $\epsilon(h)^{-\sigma} \le \alpha(h)^\sigma = |h'(x_h)|^{\sigma/2}$ that holds for $\sigma > 0$. The second bound uses the multiplicative property of size α, namely $\alpha(h^k) = \alpha(h)^k$.

(*b*) For a cost c of exponent d, and a pair (t,w) with $\sigma := \Re t$ and $\nu := \Re w$, Lemma 2.2 yields

$$|Y_k(s,w)| \le Y_k(\sigma, |\nu|) = \sum_{h\in\mathcal{H}^k} e^{|\nu| c(h)} |h'(x_h)|^\sigma \le \sum_{h\in\mathcal{H}^k} |h'(x_h)|^{\sigma - d|\nu|}\,. \tag{2.17}$$

Moreover, with Properties ($P1$) and ($P2$), one has

$$\sum_{h\in\mathcal{H}^k} |h(0)-h(1)| = 1, \quad \sum_{h\in\mathcal{H}^k} |h'(0)| \le L, \quad |h'(x_h)| \le \rho^k \qquad \text{for } h \in \mathcal{H}^k\,;$$

This entails the bound

$$\sum_{h\in\mathcal{H}^k} |h'(x_h)|^a \le \rho^{(a-1)k} L \sum_{h\in\mathcal{H}^k} |h(0)-h(1)| \le L\,\rho^{(a-1)k}\,,$$

and finally the bound (2.16) when $(t,w) \in \Gamma(a)$. □

2.6. **Analytical equivalence between generating functions.** The inequality

$$Y_k(1,0) \ge \frac{1}{L} \sum_{h\in\mathcal{H}^k} |h(1)-h(0)| = \frac{1}{L} \qquad \text{for any } k \ge 1 \tag{2.18}$$

entails that the series that defines $E(s,w)$ is divergent at $(2,0)$. The point $(2,0)$ plays an important role, and we now wish to compare more precisely the three series of interest P, Z, E near the vertical line $\Re s = 2$. We first introduce a definition:

Definition 2.4. Two Dirichlet series $A(s,w)$ and $B(s,w)$ are analytically equivalent if, for any neighborhood $\mathcal{W}_0$ of $w=0$ small enough, there are two reals δ_0 and δ_1 with $\delta_1 > \delta_0 > 0$ for which the following holds,

(*a*) For any $w \in \mathcal{W}$, each function $s \mapsto A(s,w)$, $s \mapsto B(s,w)$ is analytic on the half plane $\mathcal{P}_0 := \{s \mid \Re s > 2 + \delta_0\}$ and bounded on the set $\overline{\mathcal{P}}_0 \times \overline{\mathcal{W}}_0$;

(*b*) For any $w \in \mathcal{W}$, the difference $D(s,w) := A(s,w) - B(s,w)$ is analytic on the vertical strip $\mathcal{S}_1 := \{s \mid |\Re s - 2| < \delta_1\}$ and bounded on $\overline{\mathcal{S}}_1 \times \overline{\mathcal{W}}_0$.

Next Proposition 2.5 shows that the three series of interest are all analytically equivalent.

Proposition 2.5. *Consider a cost of moderate growth. Then the three generating functions $P(s,w), Z(s,w)$ and $E(s,w)$ are analytically equivalent.*

Proof. We use Lemma 2.3 and the series Y plays an important role in the proof.

Condition (a). Consider the neighborhood $\mathcal{W}_0 := \{w \mid |w| < \nu_0\}$. The domain $\mathcal{P}_0 \times \mathcal{W}_0$ is a subset of $\Gamma(a)$ with $a > 1$ as soon as $\delta_0 > 2d\nu_0$. Moreover, as soon as $(s/2, w)$ belongs to $\Gamma(a)$ with $a > 1$, Lemma 2.3 entails that $E(s,w)$, $P(s,w)$ and $Z(s,w)$ all satisfy Condition (a).

Condition (b) *for the difference* $Z(s,w) - P(s,w)$. This difference is described in (2.11). For $k \geq 2$, we use Lemma 2.3 together with bound (2.17) that entail the inequality

$$|P_k(s,w)| \leq P_k(\sigma,\nu) \leq Y((k/2)\sigma, k|\nu|) \leq \sum_{h\in\mathcal{H}^+} |h'(x_h)|^{(k/2)(\sigma-2d|\nu|)} .$$

Assume now that $(s/2,w) \in \Gamma(a)$ with $a > 1/2$. This means that $\sigma - 2d|\nu|$ is at least $\gamma > 1$ and let $\gamma = 1 + 3\beta$ with $\beta > 0$. Then, for any $k \geq 2$, the inequality $(k/2)\gamma \geq (k-1)\beta + 1$ holds and entails the bound

$$|h'(x_h)|^{(k/2)(\sigma-2d|\nu|)} \leq |h'(x_h)|^{(k-1)\beta} \cdot |h'(x_h)| ,$$

and thus, in the same vein as in the proof of Lemma 2.3, the bound,

$$Y((k/2)\sigma, k|\nu|) \leq L \sum_{n\geq 1} \rho^{n(k-1)\beta} , \qquad \text{for any } k \geq 2 .$$

Finally, as soon as $(s/2,w) \in \Gamma(a)$ with $a > 1/2$, one obtains

$$\begin{aligned} |Z(s,w) - P(s,w)| &\leq \sum_{k\geq 2} Y((k/2)\sigma, k|\nu|) \leq L \sum_{n\geq 1}\sum_{k\geq 1} \rho^{nk\beta} \\ &= L \sum_{n\geq 1} \frac{\rho^{n\beta}}{1-\rho^{n\beta}} \leq L \frac{\rho^{\beta}}{(1-\rho^{\beta})^2} . \end{aligned}$$

Now, consider $\mathcal{W}_0 := \{w \mid |w| < \nu_0\}$. The domain $\mathcal{S}_1 \times \mathcal{W}_0$ is a subset of $\Gamma(a)$ for $a > 1/2$ as soon as $\delta_1 < 1 - 2d\nu_0$. If furthermore $\nu_0 < 1/(4d)$ then we can choose $\delta_1 > \delta_0 > 2d\nu_0$, and Condition (b) holds for $Z(s,w) - P(s,w)$ for $\mathcal{W}_0$.

Condition (b) *for the difference* $Z(s,w) - E(s,w)$. This difference $Z(s,w) - E(s,w)$ is equal to the odd series $O(s,w)$. The bound (2.16) implies that $O(s,w)$ is absolutely convergent and uniformly bounded when

$(s/2, w) \in \Gamma(a)$ with $a > 1/2$. Then, for any neighborhood $\mathcal{W}_0$ with $\nu_0 < 1/(4d)$, Condition (b) holds for the same choice of δ_1, δ_0 as before. □

Then, with the previous proposition, it will be sufficient to study $E(s, w)$ around $\Re s = 2$. Next Section 3 will be then devoted to this task.

3. Generating functions and weighted transfer operators

Beginning with the initial generating function $P(s, w)$, we have related $P(s, w)$ to $E(s, w)$. In order to study more deeply the series $E(s, w)$, we introduce our main tool, the weighted transfer operator $\mathbf{H}_{t,w}$, and further relate the series $E(s, w)$ to the transfer weighted operator $\mathbf{H}_{t,w}$ and its iterates $\mathbf{H}_{t,w}^k$ with $t = s/2$.

In Section 3.1, we present the transfer operators and make precise the two functional spaces on which they act. We describe the first properties of these operators in Section 3.2. Then Section 3.3 explains the role of each functional space. The space $\mathcal{A}_\infty(\mathcal{V})$ is a space of analytic functions where the traces of operators is well defined; we obtain a *precise* alternative expression of the series $E(s, w)$ in terms of traces in Section 3.4; we then deduce precise information on $E(s, w)$ when $|\Im s|$ is bounded (see Sections 3.5 and 3.6) and, in particular, a precise study of its poles in Section 3.5. However, the norms of the operators –when they act on $\mathcal{A}_\infty(\mathcal{V})$– exponentially grow with respect to $|\Im s|$; this is why the space $\mathcal{A}_\infty(\mathcal{V})$ is not well adapted to the study of $E(s, w)$ for large $|\Im s|$. In Section 3.7, we then deal with the space $\mathcal{C}^1(\mathcal{I})$, where we use only *approximate* relations between the series $E(s, w)$ and the norms of transfer operators. Estimates à la Dolgopyat – of polynomial growth with respect to $\Im s$– are valid on the norms of the operators on the functional space $\mathcal{C}^1(\mathcal{I})$ and they entail that $E(s, w)$ is of polynomial growth for $|\Im s| \to \infty$. Finally, with Proposition 3.3, we will return from $E(s, w)$ to $P(s, w)$, and obtain a precise knowlege of the initial series $P(s, w)$ for any complex s (and w near 0) in Section 3.8.

3.1. **Weighted transfer operator and functional spaces.** We now introduce our main object, the weighted transfer operators (via the component transfer operators).

The *component weighted transfer operator* $\mathbf{H}_{[h],t,w}$ relative to an inverse branch $h \in \mathcal{H}^+$ involves the map $\underline{h}$ defined in $(P3)$,

$$\mathbf{H}_{[h],t,w}[f](x) := \exp(wc(h))\, \underline{h}^t(z)\, f \circ h(z)\,. \tag{3.1}$$

The sum of the weighted component operators $\mathbf{H}_{[h],t,w}$ taken over the set $\mathcal{H}$ defines the weighted transfer operator

$$\mathbf{H}_{t,w} := \sum_{h \in \mathcal{H}} \mathbf{H}_{[h],t,w}\,. \tag{3.2}$$

This is the main dynamical object of the study, as an extension of the density transformer of the system, obtained when $(t, w) = (1, 0)$.
Due to the additivity of the cost, and the multiplicativity of the derivative (or its associated mapping $\underline{h}$), the k-th iterate of the operator $\mathbf{H}_{t,w}$ has exactly the same expression as the operator itself, with a sum that is now taken over $\mathcal{H}^k$,

$$\mathbf{H}^k_{t,w} = \sum_{h \in \mathcal{H}^k} \mathbf{H}_{[h],t,w}\,.$$

As in [1], the quasi-inverse plays a central role. Here, we will deal with the even quasi-inverse

$$\mathbf{E}_{t,w} := \sum_{\substack{k \text{ even} \\ k \geq 2}} \mathbf{H}^k_{t,w} = \mathbf{H}^2_{t,w}(I - \mathbf{H}^2_{t,w})^{-1}\,. \tag{3.3}$$

In the present study, there are two useful functional spaces on which these operators act, that we now describe.

***The space* $\mathcal{A}_\infty(\mathcal{V})$.** Generating functions deal with fixed point x_h of each branch h. As we will see, traces of operators, *as soon as they are well-defined*, provide a good tool to deal with fixed points. This is why we are led to study the operators when they act on spaces of analytic functions.

We consider the disk $\mathcal{V} = \{z \in \mathbb{C} : |z - 1| < 5/4\}$ described in Property $(P3)$ of Section 2.2 and we deal with the Banach space

$$\mathcal{A}_\infty(\mathcal{V}) := \{f : \overline{\mathcal{V}} \to \mathbb{C} \mid f \text{ continuous},\ f|_{\mathcal{V}} \text{ analytic}\} \tag{3.4}$$

endowed with the sup-norm $||f||_{\mathcal{V}} := \sup\{|f(z)| \mid z \in \overline{\mathcal{V}}\}$. This is a Banach space. With Property $(P3)$, we associate with any $h \in \mathcal{H}^+$ the function $\underline{h}$ which is the analytic extension of the function $|h'(x)|$ to the disk $\mathcal{V}$. Then, with Property $(P3)$ of Section 2.2, the component weighted transfer operator $\mathbf{H}_{[h],t,w}$ acts on the Banach space $\mathcal{A}_\infty(\mathcal{V})$. Moreover, the equality that holds for (t, w), with $t = \sigma + i\tau$ and $\nu = \Re w$,

$$|\mathbf{H}_{[h],t,w}[f](z)| = \exp(\nu c(h))\, |\underline{h}(z)|^\sigma\, \exp(\tau \arg \underline{h}(z))\, f \circ h(z) \tag{3.5}$$

shows that the norm $||\mathbf{H}_{[h],t,w}||_{\mathcal{V}}$ is of exponential growth with respect to $\tau := \Im t$.

The space $\mathcal{C}^1(\mathcal{I})$**.** This is why $\mathcal{A}_\infty(\mathcal{V})$ is not adapted to our study when $\tau := \Im t$ is large, where we expect a polynomial growth with respect to τ. However, the exponential term $\exp(\tau \arg \underline{h}(z))$ disappears when z is real. This is why we also deal with the Banach space $\mathcal{C}^1(\mathcal{I})$ of continuously differentiable functions on the unit interval $\mathcal{I}$ with the usual norm $\|f\|_{(1,1)} := \|f\|_0 + \|f'\|_0$ where $\|f\|_0 := \sup\{|f(x)| \mid x \in \mathcal{I}\}$. In this case, the transfer operator $\mathbf{H}_{[h],t,w}$ involves the absolute value of the derivative, satisfies

$$\mathbf{H}_{[h],t,w}[f](x) = e^{wc(h)}\,|h'(x)|^t\, f \circ h(x)\,, \tag{3.6}$$

and is proven to act on this space. We will adapt Dolgopyat-Baladi-Vallée results which entail polynomial growth of the norm of the operator $\mathbf{H}_{t,w}$ when $|\Im t|$ becomes large.

3.2. **First properties of the operator $\mathbf{H}_{t,w}$.** We now study the main properties of the operator $\mathbf{H}_{t,w}$, when the pair (t,w) belongs to the sets defined in (2.15), and when it acts either on $\mathcal{A}_\infty(\mathcal{V})$ or $\mathcal{C}^1(\mathcal{I})$. We also study the analyticity of the mapping $(t,w) \mapsto \mathbf{H}_{t,w}$ on each space.

Proposition 3.1. *Consider a digit-cost c of moderate growth and the associated sets $\Gamma(a)$ (defined in (2.15)). The following holds:*

- *(a)* [Case $a > 1/2$]. *For any $(t,w) \in \Gamma(a)$, the operator $\mathbf{H}_{t,w}$ acts on $\mathcal{A}_\infty(\mathcal{V})$ and on $\mathcal{C}^1(\mathcal{I})$. The map $(t,w) \mapsto \mathbf{H}_{t,w}$ is analytic on $\Gamma(a)$.*
- *(b)* [Case $a > 1$]. *For any $(t,w) \in \Gamma(a)$, the quasi-inverse $(I - \mathbf{H}_{t,w})^{-1}$ acts on $\mathcal{A}_\infty(\mathcal{V})$ and on $\mathcal{C}^1(\mathcal{I})$. The map $(t,w) \mapsto (I - \mathbf{H}_{t,w})^{-1}$ is analytic on $\Gamma(a)$.*
- *(c)* [Useful bounds]. *The bounds (3.7) and (3.8) hold on $\Gamma(a)$ for any $a > 1/2$.*

Proof. With Lemma 2, the following inequality holds

$$\mathbf{H}^n_{\sigma,\nu}[\mathbf{1}](-1/4) = \sum_{h \in \mathcal{H}^n} e^{|\nu| c(h)} |h'(-1/4)|^\sigma \le K^{|\nu|} L^\sigma \sum_{h\in\mathcal{H}^n} |h'(0)|^{\sigma - d|\nu|}\,. \tag{3.7}$$

With $z \in \mathcal{V}$, and $f \in \mathcal{A}_\infty(\mathcal{V})$, the bound

$$\left|\mathbf{H}^n_{t,w}[f](z)\right| \le e^{\pi|\tau|}\, \mathbf{H}^n_{\sigma,\nu}[\mathbf{1}](-1/4)\, ||f||_{\mathcal{V}}$$

holds, and, with (3.7), this ensures that, for any $(t, w) \in \Gamma(a)$ with $a > 1/2$, the operator $\mathbf{H}_{t,w}$ acts on $\mathcal{A}_\infty(\mathcal{V})$; similar arguments, together with Lasota-Yorke bounds (see [1]), show that $\mathbf{H}_{t,w}$ acts on $\mathcal{C}^1(\mathcal{I})$.

Moreover, the bound obtained in (3.7) entails the inequality

(3.8)

$$\mathbf{H}^n_{\sigma,\nu}[\mathbf{1}](-1/4) \leq K^{|\nu|}\, L^{a+\sigma}\, \rho^{(a-1)n} \quad \text{for any } (t,w) \in \Gamma(a), \ \text{ for any } n \geq 1\,.$$

Then,together with Property $(P1)$, and for $(t, w) \in \Gamma(a)$ with $a > 1$, the quasi-inverse $(I - \mathbf{H}_{t,w})^{-1}$, and its variants, acts on $\mathcal{A}_\infty(\mathcal{V})$ and on $\mathcal{C}^1(\mathcal{I})$. □

We now explain why we restrict ourselves to costs of moderate growth. Consider a cost c that is *not* of moderate growth. In this case, the ratio $c(m)/\log m$ is not bounded, and, for any $\nu > 0$, and any $\epsilon > 0$, the inequality $\nu c(m) > (1 + \epsilon) \log m$ holds for a subsequence of integers m; the series $\mathbf{H}_{1,\nu}[1](0)$ is thus divergent for any $\nu > 0$. Then, there does not exist any neighborhood $\mathcal{S} \times \mathcal{W}$ of $(1, 0)$ on which the operator $\mathbf{H}_{t,w}$ is well-defined. However, the existence of such a neighborhood is crucial in all our study, notably for the Implicit Function Theorem used in Theorem 1.2 (iii), Dolgopyat's bounds (Theorem A) and Quasi-Powers Theorem (Theorem C).

3.3. **Role of each functional space.** The expression of $\alpha(h)$ given in (2.4) in terms of derivatives entails an alternative expression for $Y_k(t, w)$ (introduced in Eq. (2.12)), that involves the component operators $\mathbf{H}_{[h],t,w}$ defined in (3.6) as

$$Y_k(t, w) = \sum_{h \in \mathcal{H}^k} \mathbf{H}_{[h],t,w}[1](x_h)\,.$$

Here, each component operator is evaluated at a point that depends on h, namely the fixed point x_h of h. This is the main difference with previous studies (rational trajectories for instance) where the same "x" is used for each component operator: there, the analog of $Y_k(t, w)$ is expressed with the k-th iterate of $\mathbf{H}_{t,w}$.

Far from the real axis. In Section 3.7, we deal with the space $\mathcal{C}^1(\mathcal{I})$, where there are interesting Dolgopyat-Baladi-Vallée bounds on the k-iterates of the operator for large $|\Im s|$ (established by Dolgopyat in [6], and extended by Baladi and Vallée in [1]). We thus need to relate (but not exactly)

$$Y_k(s, w) \quad \text{and various} \quad \mathbf{H}^\ell_{s,w}[f](x) \quad \text{for } \ell \leq k, \text{ and some pairs } (f, x)\,,$$

when the operator acts on the functional space $\mathcal{C}^1(\mathcal{I})$.

Near the real axis, or in the intermediate region. Here, we need exact alternative expressions, and we deal with the space $\mathcal{A}_\infty(\mathcal{V})$ defined in (3.4). In this space, as we will show in the next Section, the trace of each weighted component transfer operator is well-defined and admits expression in terms of $\alpha(h)$. Due to Proposition 2.5 that relates $P(s,w)$ and $E(s,w)$, we are then led to introduce the even quasi-inverse $\mathbf{E}_{t,w}$ (with $t = s/2$), already described in (3.3), together with its trace. We then obtain an alternative expression for the series $E(s,w)$, given in Proposition 3.2. This will be useful to find the main singularities of $E(s,w)$ near the real axis, described in Proposition 3.3 of Section 3.5 and study $E(s,w)$ in the intermediary region (see Proposition 3.4 in Section 3.6).

3.4. **Trace of the even quasi-inverse.** In [9], Grothendieck proved that the usual formulae that hold for the trace of a matrix are also valid in a precise framework that is described in the Annex. As it is proven there, the space $(\mathcal{A}_\infty(\mathcal{V}), \| \|_\mathcal{V})$, defined in (3.4) provides an instance of such a framework.

First, as it is proven in Proposition 6.4(a), the trace of each component operator $\mathbf{H}_{[h],t,w}$ is well defined there, and is expressed in terms of $\alpha(h) = |h'(x_h)|^{1/2}$ and depth $|h|$ of the branch h,

$$\operatorname{Tr} \mathbf{H}_{[h],t,w} = \alpha(h)^{2t} \frac{\exp(wc(h))}{1 - (-1)^{|h|}\alpha(h)^2} .$$

Then, each component of the series $E(s,w)$, relative to a branch h of even depth, can be written as

$$\alpha(h)^{2t} \exp(wc(h)) = \operatorname{Tr} \mathbf{H}_{[h],t,w} - \operatorname{Tr} \mathbf{H}_{[h],t+1,w} .$$

Second, for $(t,w) \in \Gamma(a)$ with $a > 1$, and as it is proven in Proposition 6.4(c), the trace of the even quasi-inverse $\mathbf{E}_{t,w}$ defined in (3.3) is also well-defined and equals the sum of the traces $\operatorname{Tr} \mathbf{H}_{[h],t,w}$ taken over all the inverse branches $h \in \mathcal{H}$ with even depth. Then, introducing the Dirichlet series $\underline{E}(s,w)$,

$$\underline{E}(s,w) := \operatorname{Tr} \mathbf{E}_{(s/2),w} , \tag{3.9}$$

the following relation holds,

$$E(s,w) = \underline{E}(s,w) - \underline{E}(s+2,w) . \tag{3.10}$$

The next proposition relates the two series $E(s,w)$ and $\underline{E}(s,w)$.

Proposition 3.2. *The Dirichlet series $E(s,w)$ and $\underline{E}(s,w)$ are analytically equivalent.*

Proof. Proposition 6.4 provides in (6.6) the explicit expression of $\underline{E}(s,w)$. The maximal value for $h \mapsto \alpha(h)$ over $\mathcal{H}^+$ is obtained for $h = h_1$ and equals ρ^2 with ρ defined in Property $(P2)$. Letting $K := 1-\rho^2$, this entails the bound,

$$|\underline{E}(s,w)| \leq (1/K) Y(\sigma/2, |\nu|), \qquad \text{for } \sigma = \Re s, \nu = \Re w\,.$$

Condition (a). With bound (2.16), the series $\underline{E}(s,w)$ satisfies Condition (a) when $(s/2, w)$ belongs to $\Gamma(a)$ with $a > 1$.

Condition (b). The difference $E(s,w) - \underline{E}(s,w)$ is equal to $\underline{E}(s+2,w)$, is a shift to the left of $\underline{E}(s,w)$. This proves with bound (2.16), that $\underline{E}(s+2,w)$ is analytic and uniformly bounded as soon as the pair $((s+2)/2, w)$ belongs to $\Gamma(a)$ with $a > 1$. This means that the pair $(s/2, w)$ belongs to $\Gamma(a)$ with $a > 1/2$. Then, Condition (b) holds under the same conditions as in Proposition 2.5.

□

3.5. **Study of $E(s,w)$ near the real axis.** With Proposition 3.2, we are then led to study the trace $\underline{E}(s,w)$ of the even quasi-inverse. Proposition 6.4 proves that the map $(t,w) \mapsto \operatorname{Tr} \mathbf{E}_{t,w}$ is analytic on $\Gamma(a)$ with $a > 1$. We now study an analytic continuation of this map on a neighborhood of $(1,0)$. This will be done in the Annex, notably in Sections 6.4, 6.5, 6.6. Section 6.4 describes the spectral decomposition of the operators $\mathbf{H}_{t,w}$ near the real axis, and relates it to the spectral decomposition of the even quasi-inverse $\mathbf{E}_{t,w}$. We recall here the main steps:

For real pairs $(t,w) \in \Gamma(a)$ (with $a > 1/2$), the operator $\mathbf{H}_{t,w}$ is compact and satisfies strong positive properties that entail the existence of dominant spectral objects, in the same vein as the Perron-Frobenius properties. For $(t,w) \in \Gamma(a)$, with $a > 1/2$, the dominant spectral objects defined for real pairs may be extended with analytic perturbation of the dominant part of the spectrum when the pair (t,w) is close to a real pair. More precisely, the Annex introduces in (6.8) the spectral decomposition of the operator $\mathbf{H}_{t,w}$ that holds when (t,w) is in a neighborhood of $(1,0)$,

$$\mathbf{H}_{t,w} = \lambda(t,w)\mathbf{P}_{t,w} + \mathbf{N}_{t,w}\,,$$

where $\lambda(t,w)$ is the dominant eigenvalue, and the spectral radius of $\mathbf{N}_{t,w}$ is less than 1. This decomposition gives rise to the spectral decomposition of the even quasi-inverse given in (6.12)

$$\mathbf{E}_{t,w} = \mathbf{H}^2_{t,w}(I - \mathbf{H}^2_{t,w})^{-1} = \frac{\lambda^2(t,w)}{1-\lambda^2(t,w)}\mathbf{P}_{t,w} + \mathbf{N}^2_{t,w}(I - \mathbf{N}^2_{t,w})^{-1}.$$

Now, Section 6.5 of the Annex studies the trace of this quasi-inverse. It explains how the previous decomposition gives rise to the spectral decomposition of $\operatorname{Tr}\mathbf{E}_{t,w}$ given in (6.13),

$$\operatorname{Tr}\mathbf{E}_{t,w} = \frac{\lambda^2(t,w)}{1-\lambda^2(t,w)} + \operatorname{Tr}\mathbf{N}^2_{t,w}(I - \mathbf{N}^2_{t,w})^{-1},$$

and leads to the analytic continuation of $(t,w) \mapsto \mathbf{E}_{t,w}$. Moreover, Section 6.6 of the Annex describes the constants that are involved in this decomposition.

Using Proposition 3.2, we now transfer these results to the generating function $E(s,w)$ when (s,w) is close to the point $(2,0)$.

Proposition 3.3. *There is a domain $\mathcal{T}_2 := \mathcal{S}_2 \times \mathcal{W}_2$, formed with a neighborhood $\mathcal{W}_2$ of 0, and a rectangle $\mathcal{S}_2 := \{s = \sigma + i\tau \mid |\Re s - 2| < \delta_2,\ |\tau| < \tau_2\}$ where the following holds:*

- (a) *For each $w \in \mathcal{W}_2$, the mapping $s \mapsto E(s,w)$ is meromorphic on $\mathcal{S}_2$.*
- (b) *It has a unique pole simple on $\mathcal{S}_2$, located at $s_w := 2\sigma(w)$ where $\sigma(w)$ is defined by the equation $\lambda(\sigma(w),w) = 1$, $\sigma(0) = 1$.*
- (c) *The residue $v(w)$ of $s \mapsto E(s,w)$ at $s = s_w$ equals $-1/\lambda'_t(\sigma(w),w)$. The product $(s - s_w)E(s,w)$ is bounded on $\overline{\mathcal{T}}_2$.*

Proof. Sections 6.5 and 6.6 describe the analog of Properties (a), (b) and (c) for $\operatorname{Tr}\mathbf{E}_{t,w}$ and introduce the domains $\mathcal{S}_1$ and $\mathcal{W}_1$ and their product $\mathcal{T}_1 := \mathcal{S}_1 \times \mathcal{W}_1$. We now return from $t = s/2$ to s and the change $(s/2) \mapsto s$ brings the factor 2 in the residue. It transforms the domain $\mathcal{S}_1$ into its homothetic $\mathcal{S}_2 := 2\mathcal{S}_1$ and we let $\mathcal{T}_2 := \mathcal{S}_2 \times \mathcal{W}_1$. We now use the property of Section 6.6 which entails that the product $(s - s_w)\operatorname{Tr}\mathbf{E}_{s/2,w} = (s - s_w)\underline{E}(s,w)$ is bounded on $\overline{\mathcal{T}}_2$. Then, Properties (a), (b) and (c) are proven for $\underline{E}(s,w)$. Finally, with Proposition 3.2, we return to $E(s,w)$ and conclude the proof. □

3.6. **In the middle.** This section studies the behavior of $E(s,w)$ when s is in the intermediary region, and belongs to the union of two rectangles $\{s \mid |\Re s - 2| < \delta, \tau_2 \le |\tau| \le \tau_4\}$. The bound τ_2 comes from the domain $\mathcal{T}_2$ of Proposition 3.3, and the bound τ_4 will be provided later on by the domain $\mathcal{T}_4$ of Theorem 3.5 described in the next section. The behavior of the mapping $(t,w) \mapsto \operatorname{Tr} \mathbf{E}_{t,w}$ in the intermediary region is related to the behavior of the even quasi-inverse $\mathbf{E}_{t,w}$ there, which is itself based on the following spectral property of the plain operator $\mathbf{H}_t := \mathbf{H}_{t,0}$ when t belongs to the vertical line $\Re s = 1$ (see, for instance, the papers by Faivre [7] or Vallée [27]).

Proposition A. *On the punctured vertical line $\{\Re t = 1, t \neq 0\}$, the spectral radius $R(t)$ of the operator $\mathbf{H}_t$ (when acting on $\mathcal{A}_\infty(\mathcal{V})$) is strictly less than 1.*

Using analytic perturbation of parameters (t,w), the following result proves that $\underline{E}(s,w)$, and thus $E(s,w)$, remains bounded in the intermediary region.

Proposition 3.4. *For any pair (τ_2, τ_4) with $0 < \tau_2 < \tau_4$, there exists a domain $\mathcal{T}_3 := \mathcal{S}_3 \times \mathcal{W}_3$ formed with a neighborhood $\mathcal{W}_3$ of $w = 0$, and a union $\mathcal{S}_3$ of two rectangles, $\mathcal{S}_3 := \{s \mid |\Re s - 2| \leq \delta_3, \tau_2 \leq |\Im s| \leq \tau_4\}$ together with a bound $M_3 > 0$, for which the series $E(s,w)$ satisfies $|E(s,w)| \leq M_3$ on the domain $\overline{\mathcal{T}}_3$.*

Proof. We first study the function $(t,w) \mapsto \operatorname{Tr} \mathbf{E}_{t,w}$. Proposition A together perturbation theory of finite parts of the spectrum imply that, for any two fixed positive numbers, say $0 < \tau_2 < \tau_4$, there is a domain $\mathcal{T}_3 := \mathcal{R}_3 \times \mathcal{W}_3$, built with a neighborhood $\mathcal{W}_3$ of $w = 0$ and the union $\mathcal{R}_3$ of two rectangles of the form

$$\mathcal{R}_3 := \{t \mid |\Re t - 1| \leq (\delta_3/2), (\tau_2/2) \leq |\tau| \leq (\tau_4/2)\}$$

so that the spectral radius $R(t,w)$ of the operator $\mathbf{H}_{t,w}$ is strictly less than 1 on the closure of $\overline{\mathcal{T}}_3$.

We may suppose that $\overline{\mathcal{T}}_3 \subset \Gamma(a)$ for $a > 3/4$. Then, on $\overline{\mathcal{T}}_3$, Proposition 6.4 (c) proves that the trace $\operatorname{Tr} \mathbf{E}_{t,w}$ is given by the absolutely convergent series of general term $\operatorname{Tr} \mathbf{H}_{t,w}^k$. Now, in Proposition 6.4 (b), it is proved that each map $(t,w) \mapsto \operatorname{Tr} \mathbf{H}_{t,w}^k$ is an analytic map on $\Gamma(a)$ with $a > 3/4$. Then, on the open domain $\mathcal{T}_3$, the map $(t,w) \mapsto \operatorname{Tr} \mathbf{E}_{t,w}$ is analytic, and it is bounded on $\overline{\mathcal{T}}_3$.

An homothety by a factor of 2 provides a neighborhood $\mathcal{S}_3 := 2\mathcal{R}_3$ so that $\underline{E}(s,w)$ is bounded on $\overline{\mathcal{T}}_3 := \overline{\mathcal{S}}_3 \times \overline{\mathcal{W}}_3$ and analytic on its interior. This neighborhood $\overline{\mathcal{T}}_3$ can be chosen so that $\mathcal{T}_3 \subset \mathcal{S}_0 \times \mathcal{W}_0$ from Proposition 3.2 and then, $E(s,w)$ is also bounded and analytic on $\mathcal{T}_3$.

$\square$

3.7. **Far from the real axis.** We now study the series $E(s,w)$ when w is close to 0, and s in a vertical strip close to the vertical line $\Re s = 2$ but now with large values of $|\Im s|$. We wish to exhibit a polynomial growth of the mapping $s \mapsto E(s,w)$ for large $|\Im s|$. We have already explained in Section 3.1 why the space $\mathcal{A}_\infty(\mathcal{V})$ is not adapted to this task. We thus *do not* use the decomposition (3.10) and directly deal with $E(s,w)$. We will then compare directly the series $E(s,w)$ and iterates $\mathbf{H}_{t,w}^k$ of the transfer operator when acting on $\mathcal{C}^1(\mathcal{I})$, for $t = s/2$ with a real part close to 1.

The behavior of the iterates $\mathbf{H}_t^k$ of the (unweighted) transfer operator $\mathbf{H}_t$ in a vertical strip near $\Re t = 1$ was studied by Dolgopyat for a Markovian dynamical system with a finite number of branches. When such a system satisfies the UNI Property (recalled as Property $(P5)$ in Section 2.2), Dolgopyat exhibits bounds on the iterates $\mathbf{H}_t^k$, which prove that the (plain) quasi-inverse $(I - \mathbf{H}_t)^{-1}$ is analytic with polynomial growth. In [1], Baladi and Vallée extended his work to dynamical systems with an infinite number of branches, and to weighted operators (associated with a cost of moderate growth). Dolgopyat dealt with the Banach space $\mathcal{C}^1(\mathcal{I})$, introduced a family of norms $\| \, \|_{(1,\tau)}$ on the space $\mathcal{C}^1(\mathcal{I})$ indexed by the real parameter $\tau \neq 0$,

$$\|f\|_{(1,\tau)} := \|f\|_0 + \frac{1}{|\tau|}\|f'\|_0 \,,$$

and obtained estimates on the norms $\|\mathbf{H}_{t,w}^k\|_{(1,\tau)}$ for $\tau = \Im t$. The next theorem describes the Dolgopyat-Baladi-Vallée estimates, as they are stated in [1].

Theorem A. [Dolgopyat-Baladi-Vallée estimates]. *There exist a domain $\mathcal{T}_5 := \mathcal{S}_5 \times \mathcal{W}_5$ formed with a neighborhood $\mathcal{W}_5$ of $w = 0$, and a unbounded rectangle $\mathcal{S}_5 := \{s = \sigma + i\tau \mid |\sigma - 1| \le \delta_5, \, |\tau| > \tau_5\}$, an exponent $\xi_5 > 0$, a contraction constant $\gamma_5 < 1$, and a bound M_5, such that, for any $(s,w) \in \mathcal{T}_5$, the norm $(1,\tau)$ of the k-th iterate of the operator $\mathbf{H}_{t,w}$ satisfies, for $\tau = \Im t$,*

$$\|\mathbf{H}_{t,w}^k\|_{(1,\tau)} \le M_5 \cdot |\tau|^{\xi_5} \cdot \gamma_5^k, \qquad \textit{for } k \ge 1 \,. \tag{3.11}$$

For any $w \in \mathcal{W}_5$, the quasi-inverse $t \mapsto (I - \mathbf{H}_{t,w})^{-1}$ is analytic on $\mathcal{S}_5$ and has polynomial growth for $|\Im t| \to \infty$.

We now study the similar bounds for the series $Y_k(t,w)$ that are the components of the series $E(s,w)$, (with $t = s/2$),

$$Y_k(t,w) = \sum_{h \in \mathcal{H}^k} \exp(wc(h))\, |h'(x_h)|^t \,.$$

We precisely relate $Y_k(t,w)$ with the transfer operator $\mathbf{H}^k_{t,w}$, and prove, in the next Theorem 3.5, that $E(s,w)$ is analytic and of polynomial growth in a vertical strip of the form $|\Re s - 2| \le \delta_0$ and $|\Im s| \ge \tau_0$ for some small enough $\delta_0 > 0$ and large enough $\tau_0 > 0$.

Theorem 3.5. *There exist a domain $\mathcal{T}_4 := \mathcal{S}_4 \times \mathcal{W}_4$ formed with a neighborhood $\mathcal{W}_4$ of $w = 0$, and a unbounded rectangle $\mathcal{S}_4 := \{s = \sigma + i\tau \mid |\sigma - 2| \le \delta_4, |\tau| > \tau_4\}$ with $\tau_4 \ge 1$, an exponent $\xi_4 > 0$, a contraction constant $\gamma_4 < 1$, and a bound M_4, such that, for any $w \in \mathcal{W}_4$, $s \mapsto E(s,w)$ is analytic on $\mathcal{S}_4$, and the inequality holds,*

$$|E(s,w)| \le M_4 \cdot |\tau|^{\xi_4} \qquad \text{for any } (s,w) \in \mathcal{T}_4\,.$$

Proof. This proof is an extension of the proof given by Pollicott and Sharp [22]. The authors of [22] use an idea of Ruelle and relate the zeta series to the transfer operator, with two restrictions: they only consider the unweighted case where the dynamical system has a finite number of branches. We here show that their proof extends to a weighted transfer operator with an infinite denumerable set of branches and then relate the weighted zeta series to the weighted transfer operator.

We consider the Dolgopyat domain $\mathcal{T}_5 = \mathcal{S}_5 \times \mathcal{W}_5$ of Theorem A, together with a pair (s,w) with $w \in \mathcal{W}_5$ and s which belongs to the homothetic $2\mathcal{S}_5$. When s belongs to the homothetic $2\mathcal{S}_5$, the complex $t = s/2$ belongs to $\mathcal{S}_5$, and we may apply the Dolgopyat bounds to the iterates $\mathbf{H}^k_{t,w}$.

General strategy. Due to the definition of $E(s,w)$ given in (2.14), we deal with $Y_k(t,w)$ with $t = s/2$. We let $\sigma := \Re t = (1/2)\,\Re s$; $\tau := (1/2)\,\Im s$; $\nu := \Re w$.

We associate with each pair (h,ℓ) with $|h| \le \ell$, the function

$$F_h^{(\ell)} := \mathbf{H}^\ell_{t,w}[\mathbf{1}_{\mathcal{I}_h}] = \sum_{g \in \mathcal{H}^\ell} \mathbf{H}_{[g],t,w}[\mathbf{1}_{\mathcal{I}_h}]\,. \tag{3.12}$$

Denote by m the depth $|h|$ of h. As $\mathbf{H}_{[g],t,w}$ is a component operator which deals with the LFT g, the function $\mathbf{H}_{[g],t,w}[\mathbf{1}_{\mathcal{I}_h}]$ involves the function $\mathbf{1}_{\mathcal{I}_h} \circ g$ and there are two cases according as g begins with h or not:

(i) if there exists u for which $g = h \circ u$, then $\mathbf{H}_{[g],t,w}[\mathbf{1}_{\mathcal{I}_h}] = \mathbf{H}_{[h\circ u],t,w}[\mathbf{1}]$;
(ii) if g is not written as $g = h \circ u$ with $u \in \mathcal{H}^{\ell - m}$, then $\mathbf{H}_{[g],t,w}[\mathbf{1}_{\mathcal{I}_h}] = 0$.

Finally, in all the cases, this leads to another expression for any function $F_h^{(\ell)}$ defined in (3.12) and relative to $h \in \mathcal{H}^m$,

$$F_h^{(\ell)} := \mathbf{H}_{t,w}^{\ell}[\mathbf{1}_{\mathcal{I}_h}] = \sum_{u \in \mathcal{H}^{\ell-m}} \mathbf{H}_{[h\circ u],t,w}[\mathbf{1}] = \mathbf{H}_{t,w}^{\ell-m}[F_h^{(m)}]\,. \tag{3.13}$$

This entails, with distorsion property and Lasota-Yorke bounds (see [1]), the useful inequality

$$\sum_{h \in \mathcal{H}^m} ||F_h^{(\ell)}||_{(1,1)} \le \mathbf{H}_{\sigma,\nu}^{m}[1](-1/4)\,. \tag{3.14}$$

Our object of interest $Y_k(t,w)$ is written in terms of these $F_h^{(k)}$, namely,

$$Y_k(t,w) = \sum_{h \in \mathcal{H}^k} F_h^{(k)}(x_h)\,.$$

The main idea (which extends an idea due to Ruelle) is to write $Y_k(t,w)$ as a sum

(3.15)

$$Y_k(t,w) = \sum_{h \in \mathcal{H}} F_h^{(k)}(x_h) + \sum_{m=2}^{k} \Delta_m, \;\; \Delta_m := \sum_{h \in \mathcal{H}^m} F_h^{(k)}(x_h) - \sum_{h \in \mathcal{H}^{m-1}} F_h^{(k)}(x_h).$$

We first study the first term $T_k(t,w) := \sum_{h\in\mathcal{H}} F_h^{(k)}(x_h)$, then each difference Δ_m contained in the second term.

First term. In this case, $F_h^{(k)}$ is written as $F_h^{(k)} := \mathbf{H}_{t,w}^{k-1}[F_h^{(1)}]$. Then, we have

$$|T_k(t,w)| \le \sum_{h \in \mathcal{H}} ||F_h^{(k)}||_0 \le ||\mathbf{H}_{t,w}^{k-1}||_0 \sum_{h \in \mathcal{H}} ||F_h^{(1)}||_0\,.$$

Using the relation between 0-norm and Dolgopyat norm, together with the Dolgopyat bound (3.11), and Inequality (3.14), we obtain

$$|T_k(t,w)| \le \sum_{h \in \mathcal{H}} ||F_h^{(k)}||_0|| \ll |\tau|^{\xi_5} \gamma_5^{k-1} \mathbf{H}_{\sigma,\nu}[1](-1/4) \ll |\tau|^{\xi_5}\, \gamma_5^{k}\,.$$

Second Term.

We first prove the equality

$$F_{b(h)}^{(k)} = \sum_{h | h = b(h)\circ g} F_h^{(k)} \qquad \text{for any } b(h) \in \mathcal{H}^{m-1}\,. \tag{3.16}$$

The fundamental interval $\mathcal{I}_{b(h)}$ relative to the LFT $b(h)$ is the disjoint union of the fundamental intervals $\mathcal{I}_{b(h)\circ g}$ for $g \in \mathcal{H}$ and, using two times

(3.13), the sequence of equalities hold:

$$\begin{aligned}
F^{(k)}_{b(h)} &= \mathbf{H}^k_{t,w}[\mathbf{1}_{\mathcal{I}_h}] = \sum_{g\in\mathcal{H}} \mathbf{H}^k_{t,w}[\mathbf{1}_{\mathcal{I}_{b(h)\circ g}}] = \sum_{g\in\mathcal{H}} \sum_{u\in\mathcal{H}^{k-m}} \mathbf{H}_{[b(h)\circ g\circ u],t,w}[\mathbf{1}] \\
&= \sum_{u\in\mathcal{H}^{k-m}} \sum_{g\in\mathcal{H}} \mathbf{H}_{[b(h)\circ g\circ u],t,w}[\mathbf{1}] = \sum_{u\in\mathcal{H}^{k-m}} \sum_{h\mid h=b(h)\circ g} \mathbf{H}_{[h\circ u],t,w}[\mathbf{1}] \\
&= \sum_{h\mid h=b(h)\circ g} \sum_{u\in\mathcal{H}^{k-m}} \mathbf{H}_{[h\circ u],t,w}[\mathbf{1}] = \sum_{h\mid h=b(h)\circ g} \mathbf{H}^k_{t,w}[\mathbf{1}_{\mathcal{I}_h}] = \sum_{h\mid h=b(h)\circ g} F^{(k)}_h .
\end{aligned}$$

The equality (3.16) is proven. Now, as any $h \in \mathcal{H}^{m-1}$ is the beginning $b(h)$ of some $h \in \mathcal{H}^m$, it easily entails the equality

(3.17)
$$\Delta_m = \sum_{h\in\mathcal{H}^m} F^{(k)}_h(x_h) - \sum_{h\in\mathcal{H}^{m-1}} F^{(k)}_h(x_h) = \sum_{h\in\mathcal{H}^m} \left[F^{(k)}_h(x_h) - F^{(k)}_h(x_{b(h)})\right] .$$

As the two points x_h and $x_{b(h)}$ belong to the same fundamental interval $\mathcal{I}_{b(h)}$ of depth $m-1$, whose length is less than ρ^{m-1}, the difference Δ_m admits the bound

$$|\Delta_m| \le \rho^{m-1} \cdot \left[\sum_{h\in\mathcal{H}^m} ||F^{(k)}_h||_1\right] .$$

Using the expression of the function $F^{(k)}_h$ obtained in (3.13), we are then led to evaluate the norm $(1,1)$ of each function, for $h \in \mathcal{H}^m$

$$F^{(k)}_h = \mathbf{H}^{k-m}_{t,w} \circ \mathbf{H}_{[h],t,w}[\mathbf{1}] .$$

Using the relation[7] between the norm $(1,1)$ and the norm $(1,\tau)$, together with the Dolgopyat bound (3.11) and Inequality (3.14), this entails the following bound for Δ_m, for any $\gamma_4 \ge \gamma_5$,

$$\begin{aligned}
|\Delta_m| &\ll |\tau| \cdot ||\mathbf{H}^{k-m}_{t,w}||_{(1,\tau)} \cdot \mathbf{H}^m_{\sigma,\nu}[1](-1/4) \cdot \rho^m \\
&\ll |\tau|^{1+\xi_5} \cdot \gamma_4^{k-m} \cdot \left[\rho^m \mathbf{H}^m_{\sigma,\nu}[1](-1/4)\right] \\
&\ll |\tau|^{1+\xi_5}\, \gamma_4^k \cdot \left[\rho^m \cdot \mathbf{H}^m_{\sigma,\nu}[1](-1/4) \cdot \gamma_4^{-m}\right] .
\end{aligned}$$

The bound (3.8) leads to the inequality

$$\rho^m \cdot \mathbf{H}^m_{\sigma,\nu}[1](-1/4) \ll \rho^m \cdot \rho^{m(\sigma-1-d|\nu|)} = \rho^{m(\sigma-d|\nu|)} .$$

The exponent satisfies $\sigma - d|\nu| \ge 1 - \delta_5 - d\nu_5$. Then, choosing $\gamma_4 < 1$ and $\gamma_4 \ge \gamma_5$ entails the final bound

$$\sum_{m=2}^{k} |\Delta_m| \ll k\, |\tau|^{\xi_4} \cdot \gamma_4^k , \qquad \text{with} \qquad \xi_4 = 1 + \xi_5 .$$

[7] In the paper [22], the authors seem to have forgotten the factor $|\tau|$....

Now, we use the bound $|Y_k(t,w)| \le |T_k(t,w)| + \sum_{m=2}^{k} |\Delta_m|$, we change τ into 2τ, and the series $Y_k(t,w)$ itself satisfies, for some constant $\widetilde{M}_4$,

$$|Y_k(t,w)| \ll \widetilde{M}_4 \cdot k \cdot |\tau|^{\xi_4} \cdot \gamma_4^k \,.$$

This entails the final bound with $M_4 := \frac{\widetilde{M}_4}{\gamma_4} \left(\frac{1}{1-\gamma_4}\right)^2$. □

3.8. **Conclusion of Section 3. main properties of the series** $P(s,w)$**.** We now adjust various neighborhoods and bounds from Proposition 2.5 in Section 2.6, Proposition 3.3 in Section 3.5, Proposition 3.4 in Section 3.6 and finally Theorem 3.5 in Section 3.7.

Each result provides a domain $\mathcal{T}_i := \mathcal{S}_i \times \mathcal{W}_i$: The index i equals 0 for Proposition 2.5, it equals $i = 2$ for Proposition 3.3, it equals $i = 3$ for Proposition 3.4, and it equals $i = 4$ for Theorem 3.5. Each $\mathcal{W}_i$ is a neighborhood of 0 and we denote by $\mathcal{W}'$ the intersection of the four neighborhoods. Each $\mathcal{S}_i$ is a vertical strip or a part of a vertical strip of the form $\{s \mid |\Re s - 2| < \delta_i\}$. We thus let $\delta := \min(\delta_0, \delta_2, \delta_3, \delta_4)$ and consider the strip $\mathcal{S} := \{s \mid |\Re s - 2| < \delta\}$. Then, the horizontal lines $|\tau| = \tau_2$ from Proposition 3.3, and $|\tau| = \tau_4$ from Theorem 3.5 separate the strip $\mathcal{S}$ in three regions, where the three results of Proposition 2.5, Proposition 3.4, Theorem 3.5 hold when w belongs to $\mathcal{W}'$.

We need a last adjustment of the neighborhood $\mathcal{W}'$ to better deal with the rectangle "near the real axis", namely $\{s \mid |\Re s - 2| < \delta, |\tau| \le \tau_2\}$; we want indeed the set of poles $\{s_w \mid w \in \mathcal{W}'\}$ to be not too close to the left vertical line $\Re s = 2 - \delta$. As the mapping σ is analytic on $\mathcal{W}'$, the derivative $|\sigma'(w)|$ is bounded on $\overline{\mathcal{W}'}$ (say by some $B > 0$). Then, with the new (and final) neighborhood $\mathcal{W}$, defined as

$$\mathcal{W} := \mathcal{W}' \cap \left\{ |w| \le \frac{\delta}{4B} \right\} ,$$

the following holds, for $w \in \mathcal{W}$,

$$|\Re s_w - 2| = 2|\Re\sigma(w) - 1| \le 2|w|B \le (1/2)\,\delta, \qquad |\Re s_w - (2-\delta)| \ge (1/2)\,\delta\,.$$

Then we let $\tau_0 := \tau_4 > 1$ and $\xi := \xi_4$. We then obtain (after adjustment of the various constants M_i) the final result of this section, which is clearer when using the following notations.

Notations. A width $\delta > 0$ and two horizontal lines $|\tau| = \tau_0$ (with $\tau_0 > 1$) divide the strip $\{s \mid |\Re s - 2| < \delta\}$ into three domains: a bounded rectangle $\mathcal{R} := \{s \mid |\Re s - 2| < \delta, |\tau| \le \tau_0\}$ and the union $\mathcal{U} := \{s \mid |\Re s - 2| < \delta,\ |\tau| \ge \tau_0\}$ of two unbounded rectangles.

Theorem 3.6. *Consider the bivariate generating series $P(s, w)$ associated with a digit-cost c of moderate growth and defined in* (2.9)*. There exists a neighborhood $\mathcal{W}$ of $w = 0$, a width $\delta > 0$, two horizontal lines $|\tau| = \tau_0$ (with $\tau_0 > 1$), an exponent $\xi > 0$ and a bound M, for which the following holds:*

- (a) *For $w \in \mathcal{W}$, the mapping $s \mapsto P(s, w)$ is meromorphic on the half plane $\{s \mid \Re s > 2 - \delta\}$.*
- (b) *It has a unique (simple) pole, located inside the rectangle $\mathcal{R}$, at the point $s_w = 2\sigma(w)$ defined by the equation $\lambda(\sigma(w), w) = 1$, $\sigma(0) = 1$.*
  The residue $v(w)$ of $s \mapsto P(s, w)$ at $s = s_w$ equals $-1/\lambda'_t(\sigma(w), w)$.
  The distance between s_w and the vertical line $\Re s = 2 - \delta$ is at least $(1/2)\delta$.
  The function $v(s, w) := (1/s)(s - s_w)P(s, w)$ satisfies $|v(s, w)| \leq M$ on $\overline{\mathcal{R}} \times \overline{\mathcal{W}}$.
- (c) *The inequality $|P(s, w)| \leq M|\Im s|^{\xi}$ holds on $\overline{\mathcal{U}} \times \overline{\mathcal{W}}$;*
- (d) *The inequality $|P(s, w)| \leq M$ holds on $\{s \mid \Re s \geq 2 + \delta\} \times \overline{\mathcal{W}}$.*

4. Extraction of coefficients and obtention of the Gaussian law.

The previous section provides a precise description of the analytic properties of the Dirichlet bivariate generating function $P(s, w)$. We now return to our probabilistic setting. Section 4.1 recalls the role played by the moment generating function $M_N(w)$ of cost C on the subset $\mathcal{P}_N$. Then, Section 4.2 describes asymptotic estimates of this moment generating function that are obtained by "extracting" the coefficients of the generating function $P(s, w)$: applying a uniform version of the Landau Theorem leads to uniform estimates when w belongs to the neighborhood $\mathcal{W}$ defined in Theorem 3.6. With this uniform estimate of $M_N(w)$ at hands, Section 4.3 applies the Quasi-Powers Theorem, which "transfers" this (uniform) asymptotic estimate into an asymptotic Gaussian law, with the speed of convergence stated in Theorem 1.1. The precise study of the constants of interest, performed in Section 4.4, gives rise to Theorem 1.2.

4.1. **Using the moment generating function.** In this paper, we study (when $N \to \infty$) the asymptotic distribution of the cost C [restricted to $\mathcal{P}_N$] via the sequence of *moment generating functions* $M_N(w) :=$

$\mathbb{E}_N[\exp(wC)]$ where w is a complex number close to 0. From its definition, the moment generating function $M_N(w)$ is written as a quotient, namely,

$$M_N(w) := \mathbb{E}_N[\exp(wC)] = \frac{S_w^{[C]}(N)}{S_0^{[C]}(N)} \tag{4.1}$$

and involves the *cumulative sum* $S_w^{[C]}(N)$ of the cost $\exp(wC)$ over $\mathcal{P}_N$,

$$S_w^{[C]}(N) := \sum_{x|\epsilon(x)\le N} \exp(wC(x)) \quad \text{with} \quad S_0^{[C]}(N) = |\mathcal{P}_N|\,. \tag{4.2}$$

4.2. **Landau Theorem.** The Landau Theorem [15] was proved by Landau in 1924. This is a strong tool which gives estimates on the sum of coefficients of a Dirichlet series, provided the series satisfies (not too) strong analytical properties. This theorem is not so well-known, and many works (as [1] or [22]) that use it, do not deal with a *strong* version (that yet exists in the original work of Landau [15], but is perhaps a little bit hidden). These authors begin with a weak version and prove the strong version by hand[8]. What we mean by *weak* or *strong* is related to the exponent ξ: the weak version only deals with $\xi < 1$, and very often (as it is the case here) one would need a *strong* version which would deal with an exponent ξ that may be larger than 1. A proof of the *strong* version is precisely given in Roux's thesis [23] and also available in [2]. We here need a "uniform" version (with respect to parameter w) of this strong version:

Theorem B. [Landau Theorem – uniform version] *If there is a complex neighborhood $\mathcal{W}$ of 0 where the series $P(s,w)$ satisfies the "uniform" conclusions of Theorem 3.6, with an exponent $\xi > 0$, and a width δ, the following holds for any $N \ge 1$*

$$S_w^{[c]}(N) = \frac{v(w)}{2\sigma(w)}\, N^{2\sigma(w)}\, \left[1 + O(N^{-\beta})\right], \qquad \beta := \frac{\delta}{2(\lfloor \xi \rfloor + 3)}\,,$$

where the constant in the O–term is uniform for $w \in \mathcal{W}$.

This leads to a first result, easily deduced from Theorem 3.6 and Theorem B:

Proposition 4.1. *Consider the set $\mathcal{P}$ of rqi numbers, and a cost C on the set $\mathcal{P}$ associated with a digit-cost c of moderate growth. Denote by C_N the restriction of cost C to the set $\mathcal{P}_N$ which gathers the rqi numbers x with $\epsilon(x) \le N$. Then, there exists a neighborhood $\mathcal{W}$ of 0, on which the*

[8] In [1], the authors use the *weak* version of the Landau Theorem, and they have to introduce the so-called *smoothed* probabilistic model. With this *strong* version at hands, the part of their work may be changed and shortened.

moment generating function $M_N(w)$ admits for any $N \geq 1$ a "uniform quasi-powers" form

$$M_N(w) = \frac{v(w)}{\sigma(w) \cdot v(0)} N^{2(\sigma(w)-1)} \left[1 + O(N^{-\beta})\right], \tag{4.3}$$

which involves the exponent β defined in Theorem B, two analytic functions, the function $w \mapsto \sigma(w)$ defined implicitly by $\lambda(\sigma(w), w) = 1$, with $\sigma(0) = 1$, and the residue function $w \mapsto v(w)$, defined in Theorem 3.6. Moreover, the constant in the O–term is uniform for $w \in \mathcal{W}$.

4.3. **Quasi-Powers Theorem.** The following result, known as the Quasi-Powers Theorem, due to Hwang [12], shows that a "uniform quasi-powers" expression for the moment generating functions $M_N(w) := E_N[\exp(wC_N)]$ entails an asymptotic Gaussian law for cost C_N.

Theorem C. [Quasi-Powers Theorem] *Consider a sequence C_N of variables defined on probability spaces $(\mathcal{P}_N, \mathbb{P}_N)$, and their moment generating functions $M_N(w) := \mathbb{E}_N[\exp(wC_N)]$. Suppose that the functions $M_N(w)$ are analytic on a complex neighborhood $\mathcal{W}$ of zero, and each one satisfies there*

$$M_N(w) = \exp[\beta_N U(w) + V(w)] \left(1 + O(\kappa_N^{-1})\right),$$

where the O–term is uniform on $\mathcal{W}$. Moreover, the two sequences β_N and κ_N tend to ∞ as $N \to \infty$, and $U(w)$, $V(w)$ are analytic on $\mathcal{W}$. Then:

(i) The mean and the variance satisfy

$$\mathbb{E}_N[C_N] = \beta_N U'(0) + V'(0) + O(\kappa_N^{-1}),$$
$$\mathbb{V}_N[C_N] = \beta_N U''(0) + V''(0) + O(\kappa_N^{-1}).$$

(ii) Moreover, if $U''(0) \neq 0$, the distribution of C_N on $\mathcal{P}_N$ is asymptotically Gaussian, with speed of convergence $O(\alpha_N)$ with $\alpha_N = (\kappa_N^{-1} + \beta_N^{-1/2})$,

$$\mathbb{P}_N\left[x \mid \frac{C_N(x) - U'(0)\log N}{\sqrt{U''(0)\log N}} \leq t\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du + O(\alpha_N).$$

Proposition 4.1 shows that Theorem C may be applied to our setting with

$$\beta_N = \log N, \quad \kappa_N = N^{\beta}, \qquad \text{and thus} \quad \alpha_N = (\log N)^{-1/2},$$

together with functions U, V defined as

$$U(w) = 2(\sigma(w) - 1), \quad \text{and} \quad V(w) = \log\left(\frac{v(w)}{\sigma(w) \cdot v(0)}\right), \tag{4.4}$$

that involve the two mappings, the function $w \mapsto \sigma(w)$ and the function $w \mapsto v(w)$ defined in Proposition 4.1. The next section performs a deeper study of these functions near $w = 0$, and proves that the two functions U, V will be analytic on $\mathcal{W}$ with $U''(0) \neq 0$. This finishes the proof of our main Theorem 1.1.

4.4. **Constants.** The function U exactly coincides with its analog which intervenes in the Gaussian law of the cost C on rational trajectories and was studied in [1]. One has

$$U(0) = 0, \qquad U'(0) = 2\sigma'(0) = -2\frac{\lambda'_w(1,0)}{\lambda'_t(1,0)}\,. \tag{4.5}$$

and Baladi and Vallée prove $U''(0) \neq 0$.

With the expression of $v(w)$ given in Proposition 4.1, one obtains

$$V(w) = \log\left(\frac{\lambda'_t(1,0)}{\lambda'_t(\sigma(w), w)}\right) - \log\sigma(w)\,.$$

The first two derivatives $V'(0), V''(0)$ then involve the first three derivatives of the mapping $(t, w) \mapsto \lambda(t, w)$ at $(t, w) = (1, 0)$.

The present function V *does not coincide* with its analog V_r that is associated with rational trajectories and described in [1]. The function V_r is expressed in terms of the analog of the residual function v_r as

$$V_r(w) = \log\left(\frac{v_r(w)}{\sigma(w)\cdot v_r(0)}\right),$$

$$v_r(w) := \frac{-1}{\lambda'_t(\sigma(w), w)}\frac{1}{\sigma(w)}\mathbf{F}_{\sigma(w),w} \circ \mathbf{P}_{\sigma(w),w}[1](0).$$

This expression involves the projector $\mathbf{P}_{t,w}$ on the dominant eigensubspace together with the "final" transfer operator $\mathbf{F}_{t,w}$ which only deals with digits $m \geq 2$, and is defined as

$$\mathbf{F}_{t,w}[f](x) := \mathbf{H}_{t,w}[f](x) - \frac{e^{wc(1)}}{(1+x)^{2t}} f\left(\frac{1}{1+x}\right).$$

The equality $\mathbf{F}_{t,w} \circ \mathbf{P}_{t,w}[1](0) = f_{t,w}(0) - e^{wc(1)} f_{t,w}(1)$ holds and involves the dominant eigenfunction $f_{t,w}$. Finally,

$$v_r(w) = \frac{-1}{\sigma(w)\,\lambda'_t(\sigma(w), w)}\left(f_{\sigma(w),w}(0) - e^{wc(1)} f_{\sigma(w),w}(1)\right).$$

Then Theorems 1.1 and 1.2 are now proven, and this ends the main part of our study.

5. Possible extensions and open problems.

This section describes possible (easy) extensions of the present work. First, Section 5.1 studies the probabilistic behavior of the "cost" $\log \epsilon(x)$. Then, the following two sections (5.2 and 5.3) are devoted to the "constrained" cases, where the numbers have bounded digits (in their continued fraction expansion). Finally, Sections 5.4 and 5.5 discuss possible obtention of local limit theorems, and study the speed of convergence towards the local limit law

5.1. The cost $C(x) := \log \epsilon(x)$.

Even though the cost $x \mapsto \log \epsilon(x)$ is *not* an additive cost, it can be easily studied via our methods. The bivariate generating functions (primitive or general) involve the general term

$$e^{w \log \epsilon(x)} \cdot \epsilon(x)^{-s} = \epsilon(x)^{-(s-w)}$$

and exactly coincide with the (univariate) generating functions $P(s-w), Z(s-w)$. The pole function σ is defined as $\sigma(w) := 1 + w$ and the residue function v associates with w the real $-1/\lambda'(1+w)$. Then $U'(w) = 2$ for any w, and the second derivative $U''(0)$ equals 0. Then, Assertion (i) of Theorem C applies and shows that the variance is very small, of order $O(1)$, whereas the mean is of order $\log N$. This means that the cost $C := \log \epsilon$ is very concentrated around its mean. Assertion (ii) of Theorem C does not apply, and there does not exist a Gaussian limit law. Always with Assertion (i) of Theorem C, we obtain precise estimates of the expectation and the variance of the cost $x \mapsto \log \epsilon(x)$ on the set $\mathcal{P}_N$ which gathers the rqi numbers with size $\log \epsilon(x) \le N$. These estimates involve the first derivatives of the dominant eigenvalue $t \mapsto \lambda(t)$ at $t = 1$,

$$\mathbb{E}_N[C] = 2 \log N - \frac{\lambda''(1)}{\lambda'(1)} + O(N^{-\beta}),$$

$$\mathbb{V}_N[C] = \frac{\lambda'''(1)\lambda'(1) - \lambda''(1)^2}{\lambda'(1)^2} + O(N^{-\beta}) \ .$$

The dominant term of the mean value is well-known, but the other terms of the previous estimates appear to be new, notably the important fact that the variance is of order $O(1)$.

5.2. Quadratic irrational numbers with bounded digits (I).

Here, a given bound $M \ge 2$ is fixed and possibly equal to ∞. This section is devoted to study the "constrained" set $\mathcal{P}[M]$ which gathers rqi numbers whose continued fraction expansion only contains digits at most equal to M. The analog set $\mathcal{R}[M]$ of reals whose continued fraction expansion only contains digits at most equal to M is extensively studied, and there are many works that deal with its Hausdorff dimension σ_M. The constrained (unweighted) operator defined as

$$\mathbf{H}_{M,t}[f](x) := \sum_{m \le M} \frac{1}{(m+x)^{2t}}\, f\left(\frac{1}{m+x}\right) \tag{5.1}$$

directly appears here, together with its dominant eigenvalue $t \mapsto \lambda_M(t)$, as σ_M coincides with the value t for which $\lambda_M(t) = 1$. The strict inequality $\sigma_M < 1$ holds as soon as $M < \infty$.

It is natural to study the rational numbers and the rqi numbers whose continued fractions expansion follows the same constraints, namely the sets $\mathcal{Q}[M] := \mathcal{Q} \cap \mathcal{R}[M]$ and $\mathcal{P}[M] := \mathcal{P} \cap \mathcal{R}[M]$. The rational case was studied by Cusick and Hensley, then Vallée in [27] provided an unified framework based on analytic combinatorics, where she analyses, for a fixed bound M, the probability $\pi(N, M)$ that a rational number of denominator at most N (or a quadratic number x with $\epsilon(x) \le N$) has all its digits at most equal to M. In each case, the estimate

$$\pi(N, M) = N^{2(\sigma_M - 1)}\,[1 + \eta_M(N)] \qquad \text{where } \eta_M(N) \to 0 \text{ when } N \to \infty\,,$$

is proven to hold. Moreover, she also studies the mean value of the depth p (in the rational case) or of the period length p (in the quadratic irrational case), and proves the asymptotic estimates, for any fixed M,

$$\mathbb{E}_N[p] = \mu_M \cdot \log N\,[1 + \eta_M(N)], \text{ with } \mu_M := \frac{-2}{\lambda'_M(\sigma_M)},\ \eta_M(N) \to 0\,.$$

We note that the constant μ_M is the same in the two cases.

She deals in the two cases with the "constrained" analogs $Q_M(s)$ and $P_M(s)$ of univariate "unconstrained" generating functions $Q(s)$ and $P(s)$, namely,

$$Q_M(s) := \sum_{x \in \mathcal{Q}[M]} q(x)^{-s}, \qquad P_M(s) := \sum_{x \in \mathcal{P}[M]} \epsilon(x)^{-s}$$

that she relates to the constrained (plain) transfer operator defined in (5.1). Her results are based on spectral properties of this constrained operator on the half-plane $\Re s > \sigma_M$, that are further transfered to the coefficients of univariate series $P_M(s), Q_M(s)$. As she was (only) interested in average-case results, and did not know (at the moment) the existence of bounds à la Dolgopyat, she dealt with Tauberian Theorems, which do not provide estimates for the remainder term $\eta_M(N)$ for $N \to \infty$.

We now mix her previous approach with our present methods, and exhibit a general "constrained" framework which gathers the present rqi framework and the rational framework of [1]. We introduce the bivariate constrained generating functions associated with some additive cost C, namely

$$Q_M(s,w) := \sum_{x\in\mathcal{Q}[M]} e^{wC(x)}\cdot q(x)^{-s}, \qquad P_M(s,w) := \sum_{x\in\mathcal{P}[M]} e^{wC(x)}\cdot \epsilon(x)^{-s},$$

together with the weighted constrained operator

$$\mathbf{H}_{M,t,w}[f](x) = \sum_{m\le M} \frac{e^{wc(m)}}{(m+x)^{2t}}\, f\left(\frac{1}{m+x}\right).$$

We obtain (easily) the following result:

Theorem 5.1. *With a bound $M \ge 2$, and a cost of moderate growth, associate the additive cost C on each set $\mathcal{Q}[M]$ or $\mathcal{P}[M]$. Then, as soon as $M \ge M_0$ (for some $M_0 \ge 3$), and on each of the two sets,*

- *the set $\mathcal{Q}_N[M]$ of rationals $x \in \mathcal{Q}[M]$ with denominator $q(x) \le N$*
- *or the set $\mathcal{P}_N[M]$ of quadratic irrationals $x \in \mathcal{P}[M]$ with $\epsilon(x) \le N$,*

the distribution of C is asymptotically Gaussian (for $N \to \infty$), and there are two constants $\mu_M(c), \nu_M(c)$ for which

$$\mathbb{P}_{N,M}\left[x \mid \frac{C(x)-\mu_M(c)\log N}{\sqrt{\nu_M(c)\log N}} \le v\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{v} e^{-\frac{u^2}{2}}du + O\left(\frac{1}{\sqrt{\log N}}\right).$$

Sketch of the proof. We adapt here the general description given in the introduction. Now the Dirichlet series $Q_M(s,w), P_M(s,w)$ have both a pole on the curve $s = 2\sigma_M(w)$ where $\sigma_M(w)$ is implicitly defined by the equation

$$\lambda_M(\sigma_M(w),w) = 1, \quad \sigma_M(0) = \sigma_M.$$

Moreover, if M is large enough (say $M \ge M_0$), the real number σ_M is close enough to 1, and the vertical line $\Re s = \sigma_M$ is contained in the Dolgopyat strip of Theorem A. Then, bounds à la Dolgopyat are available for the constrained transfer operator (which can be viewed as a perturbation of the unconstrained one, as in [4]). This entails the polynomial growth of the quasi-inverse $(I - \mathbf{H}_{M,t,w})^{-1}$, and thus the polynomial growth of the two series $Q_M(s,w), P_M(s,w)$ for $|\Im s| \to \infty$ and w close to 0. The constants $\mu_M(c)$ and $\nu_M(c)$ are equal to the first and second derivatives of the function $w \mapsto 2\sigma_M(w)$ at $w = 0$. □

5.3. **Quadratic irrational numbers with bounded digits (II).** Now, the bound M is no longer fixed, and we wish to describe the behavior of the probability $\pi(N,M)$ (both in the rational *and* in the rqi case) when

N and M both tend to ∞. First, the behavior of the Hausdorff dimension σ_M of the set $\mathcal{R}[M]$ has been studied by Hensley in [11], who exhibits an asymptotic estimate for the speed of convergence of σ_M towards 1,

$$2(\sigma_M - 1) = -\frac{2}{\zeta(2)}\frac{1}{M} - \frac{4}{\zeta(2)^2}\frac{\log M}{M^2} + O\left(\frac{1}{M^2}\right) \qquad (M \to \infty)\,.$$

Then, the authors of the present paper studied in [4] the probability $\pi(N, M)$ in the rational case. Besides some precisions and extensions of a first result, previously obtained by Hensley in [10], they specially develop a methodology (based on principles of analytic combinatorics) which can be easily transfered from the rational case to the rqi case and implies the following general result:

Theorem 5.2. *There are an integer $M_0 = M_0 \geq 3$, and a real β, with $0 < \beta < 1/2$ so that, for any $N \geq 1, M \geq M_0$, the two probabilities $\pi(N, M)$ that*

- *a reduced quadratic irrational x with a size $\epsilon(x) \leq N$*
- *or a rational number x with a denominator $q(x) \leq N$*

has all its digits less than M, satisfy

$$\pi(N, M) = N^{2(\sigma_M - 1)}\left[1 + O(N^{-\alpha})\right] \cdot \left[1 + O\left(\frac{\log M}{M}\right)\right]\,.$$

Here, σ_M is the Hausdorff dimension of the set $\mathcal{R}[M]$, and β is the width of the Dolgopyat strip.

This result exhibits a threshold phenomenon, already obtained by Hensley and precisely described in [4] in the rational case, depending on the relative order of $\sigma_M - 1$ (of order $O(1/M)$) with respect to $n := \log N$,

(a) If $M/n \to \infty$, then, almost any number of size at most N has all its digits less than M.

(b) If $M/n \to 0$, then, almost any number of size at most N has at least one of its digits greater than M.

5.4. **Local limit laws - case of a lattice cost.** For any additive cost C associated with a digit-cost c of moderate growth, Baladi and Vallée also obtained in [1] a local limit theorem on rational trajectories. Moreover, in the case when the cost c is lattice, they prove an optimal speed of convergence. We recall that a cost is *lattice* if it is non zero and there exists $L \in \mathbb{R}^+$ so that c/L is integer-valued. The largest such L is then called the span of c. For instance, the three costs of interest are integer valued and thus lattice (with span 1).

Local limit theorems also deal with the moment generating function $M_N(w) := \mathbb{E}_N[\exp(wC)]$, but now with a complex number w on the vertical line $\Re w = 0$. When the cost c is lattice with span L, it is enough to study the generating function $P(s, w)$ or its analog $Z(s, w)$ for a pair (s, w) when s belongs to a vertical strip near $\Re s = 2$ and $w \in]-i\pi/L, +i\pi/L[$. It is straightforward to adapt the proof of [1] to the present framework and derive the following result which is expressed with the following scale,

$$Q(y, N) := \mu(c) \log N + y \ \sqrt{\nu(c) \log N}, \qquad (y \in \mathbb{R}) . \tag{5.2}$$

Theorem 5.3. [Local Limit Theorem for lattice costs.] *For any lattice digit-cost c of moderate growth and of span L, letting $\mu(c) > 0$ and $\nu(c) > 0$ be the constants from Theorems 1.1 and 1.2, the following holds for the cost C associated with c on the set $\mathcal{P}_N$,*

$$\mathbb{P}_N \left[x \in \mathcal{P} \mid |C(x) - Q(y, N)| \leq L/2 \right] = \frac{1}{\sqrt{\nu(c) \log N}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} + O\left(\frac{1}{\log N} \right)$$

with a O uniform for $y \in \mathbb{R}$.

5.5. **Local limit laws - case of a non lattice cost.** In [1], the authors did not succeed to estimate the speed of convergence when the cost is non lattice: this problem is known to be difficult, even in the simpler case of dynamical system with affine branches. The problem was later considered by Baladi and Hachemi, then by Vallée in [30] (always for rational trajectories) who obtained a local limit theorem with a speed of convergence, that now depends on arithmetical properties of the cost c. More precisely, when the cost satisfies a diophantine condition, then the speed depends on the irrationality exponent that intervenes in the diophantine condition, as we now explain.

We first recall that a real number y is diophantine of exponent μ if there exists $C > 0$ such that the inequality $|x - (p/q)| > Cq^{-(2+\mu)}$ is satisfied for any pair $(p, q) \in \mathbb{N}^2$. We now state the diophantine condition on the cost c.

Definition 5.4. For any pair $(h, k) \in \mathcal{H}^+$ let $c(h, k) := |h|c(k) - |k|c(h)$. The cost c is diophantine of exponent μ if there exists a triple $(h, k, \ell) \in (\mathcal{H}^+)^3$ for which

(i) $c(h, k)$ and $c(h, \ell)$ are non zero,

(iii) the ratio $c(h, k)/c(h, \ell)$ is diophantine of exponent μ.

Then, Vallée in [30] proves that the cost C associated with a digit cost c (diophantine of exponent μ) and restricted to the rational numbers of denominator at most N asymptotically follows a local limit law, where the speed of convergence depends of the exponent μ.

Theorem D. *Consider a cost c of moderate growth, diophantine of exponent μ, and let $\chi := 6(\mu+1)$. Then, the following holds for the cost C associated with c on the set $\mathcal{P}_N$: for any ϵ with $\epsilon < 1/\chi$, for any compact interval $J \subset \mathbb{R}$, there exists a constant M_J so that, for every $y \in \mathbb{R}$ and all integers $N \geq 1$,*

$$\left| \sqrt{\log N}\, \mathbb{P}_N[C - Q(y,N) \in J] - |J| \frac{e^{-y^2/2}}{\sqrt{2\nu(c)\pi}} \right| \leq \frac{M_J}{\log^\epsilon N} .$$

The proof of [30] is based on fine diophantine arguments, and is closely related to the generating function $Q(s,w)$ of the rational case: it is not clear if this result may be adapted to the generating function $P(s,w)$ of the rqi case.

6. Annex: Nuclearity and traces of quasi-inverses.

This annex is devoted to provide detailed proofs of Propositions 3.2 and 3.3. The main object here is the trace of various transfer operators. In which sense the usual notion of trace of matrices may be extended to operators that act on Banach spaces was mostly clarified by Grothendieck, for instance, in [9]. Later on, Ruelle and Mayer applied Grothendieck's theory to the study of periodic points in dynamical systems as it is explained in Mayer's survey [18].

Most of the classical studies in the dynamical system context indeed deal with the Fredholm determinant $F(u)$ associated with a family of operators $\mathbf{G}_t$, which satisfies the trace formula,

$$F(u,t) := \det(I - u\mathbf{G}_t) = \exp\left[-\sum_{k\geq 1} \frac{u^k}{k} \operatorname{Tr} \mathbf{G}_t^k \right] ,$$

and its logarithmic derivative $t \mapsto \eta(t)$,

$$\eta(t) := \frac{\partial}{\partial u} \log F(u,t)\big|_{u=1} = \sum_{k\geq 1} \operatorname{Tr} \mathbf{G}_t^k .$$

Here, in an approach which appears to be new, we directly deal with this object, which is proven to exactly coincide with the trace of the quasi-inverse $(I - \mathbf{G}_t)^{-1}$, without studying the Fredholm determinant.

6.1. **Basic facts.**

Definition 6.1. [Grothendieck] Consider a Banach space $\mathcal{B}$ that admits a Schauder basis. A linear operator $\mathbf{H} : \mathcal{B} \to \mathcal{B}$ is *nuclear* if it admits a representation

$$\mathbf{H}[f] = \sum_{n \geq 1} \lambda_n \, f_n^\star[f] \, f_n \tag{6.1}$$

where $\{f_n\}$ and $\{f_n^\star\}$ are families, respectively in $\mathcal{B}$ and $\mathcal{B}^\star$, with $\|f_n\| \leq 1$ and $\|f_n^\star\| \leq 1$, and λ_n is a sequence of complex numbers that belongs to ℓ^1.
The sum $\sum_{n\geq 1} \lambda_n f_n^\star[f_n]$ does not depend on the choice of the representation given in (6.1), and defines the *functional trace* of the operator.
If $q := \inf\{p \mid (\lambda_n) \in \ell^p\}$ the operator $\mathbf{H}$ is said to be nuclear of order q.

Remark that such a nuclear operator belongs to $\mathcal{L}(\mathcal{B})$ and is also compact. Its *spectral trace* is defined as the sum of its eigenvalues counted according to their algebraic multiplicities, if it exists. The following result obtained by Grothendieck in [9] and used in Mayer's [18] and Momeni-Venkov's works [19] exhibits important properties of the nuclear operators of order $q \leq 2/3$:

Theorem B. [Grothendieck] *The functional trace of a nuclear operator of order $q \leq 2/3$ coincides with its spectral trace.*

6.2. **Two useful lemmas.**

The following two lemmas describe (sufficient) conditions for dealing with the trace of a series of operators.

Lemma 6.2. *Consider a Banach space $(\mathcal{B}, \| \ \|)$ and a sequence of operators $\mathbf{H}_{[j]} \in \mathcal{L}(\mathcal{B})$ for $j \in \mathcal{J}$. Assume there exists $q_0 > 0$ for which the two following conditions hold:*

- $(H1)$ *Each operator $\mathbf{H}_{[j]}$ is nuclear of order $q < q_0$, associated with a sequence of coefficients $m \mapsto \lambda_{m,[j]}$;*
- $(H2)$ *The series of general term $\sigma_j := \sum_{m=0}^{\infty} |\lambda_{m,[j]}|^{q_0}$ is convergent.*

Then, the following holds for the operator $\mathbf{H} := \sum_{j\in\mathcal{J}} \mathbf{H}_{[j]}$

(i) *It is nuclear of order $q \leq q_0$;*

(ii) *If $q_0 \leq 2/3$, the equality holds between the (spectral) traces,* $\operatorname{Tr} \mathbf{H} = \sum_{j\in\mathcal{J}} \operatorname{Tr} \mathbf{H}_{[j]}$.

Proof. Each operator $\mathbf{H}_{[j]}$ admits a decomposition of type (6.1) which involves a sequence $m \mapsto \lambda_{m,[j]}$ that belongs to ℓ^{q_0} and the inequalities

$||f_{m,[j]}|| \le 1$, $||f^\star_{m,[j]}|| \le 1$ hold. Hypothesis $(H2)$ entails that the sequence $(m,j) \mapsto \lambda_{m,[j]}$ belongs to ℓ^{q_0}. Then, the decomposition

$$\mathbf{H}[f] = \sum_{(m,j)\in\mathbb{N}\times\mathcal{J}} \lambda_{m,[j]}\, f_{m,[j]}\, f^\star_{m,[j]}[f]$$

(with $||f_{m,[j]}|| \le 1$ and $||f^\star_{m,[j]}|| \le 1$) entails that the operator $\mathbf{H}$ is nuclear of order $q \le q_0$. Moreover, for $q_0 \le 2/3$, the spectral traces satisfy

$$\operatorname{Tr}\mathbf{H} = \sum_{m,j} \lambda_{m,[j]}\, f^\star_{m,[j]}[f_{m,[j]}] = \sum_{j\in\mathcal{J}}\sum_{m=0}^{\infty} \lambda_{m,[j]}\, f^\star_{m,[j]}[f_{m,[j]}] = \sum_{j\in\mathcal{J}} \operatorname{Tr}\mathbf{H}_{[j]}\,.$$

□

The next lemma is useful for dealing with operators which involve quasi-inverses.

Lemma 6.3. *Consider an operator* $\mathbf{T} \in \mathcal{L}(\mathcal{B})$ *and a nuclear operator* $\mathbf{A}$ *of order* $q \le 2/3$ *together with its sequence* (σ_n) *of coefficients of* ℓ^1 *norm equal to* σ. *Then:*

(i) *The operator* $\mathbf{AT}$ *is nuclear of order* $q \le 2/3$ *and the inequality holds:*

$$|\operatorname{Tr}\mathbf{AT}| \le \sigma \cdot \|\mathbf{T}\|\,. \tag{6.2}$$

(ii) *Assume that the spectral radius of* T *is strictly less than 1. Then, the operator* $(I-\mathbf{T})^{-1}$ *exists in* $\mathcal{L}(\mathcal{B})$, *the operator* $\mathbf{A}(I-\mathbf{T})^{-1}$ *is nuclear of order* $q \le 2/3$ *and the following equality holds on the traces,*

$$\operatorname{Tr}\mathbf{A}(I-\mathbf{T})^{-1} = \operatorname{Tr}\sum_{k=1}^{\infty}\mathbf{AT}^k = \sum_{k=1}^{\infty}\operatorname{Tr}\mathbf{AT}^k\,.$$

Proof. Assertion (i) is clear. For (ii): The operator $\mathbf{A}(I-\mathbf{T})^{-1}$ is nuclear of order $q \le 2/3$. Now, for two operators $\mathbf{V}$ and $\mathbf{U}$, Eq. (6.2) entails the bound

$$|\operatorname{Tr}\mathbf{AV} - \operatorname{Tr}\mathbf{AU}| = |\operatorname{Tr}\mathbf{A}(\mathbf{V}-\mathbf{U})| \le \sigma\|\mathbf{V}-\mathbf{U}\|\ .$$

Applying the previous relation, with $\mathbf{V}_K := \sum_{k=1}^{K}\mathbf{T}^k$, $\mathbf{U} := (I-\mathbf{T})^{-1}$, together with the bound $||\mathbf{V}_K - \mathbf{U}|| \ll ||\mathbf{T}||^K$, ends the proof. □

6.3. **Nuclearity and trace properties of the transfer operator $\mathbf{H}_{t,w}$ acting on $\mathcal{A}_\infty(\mathcal{V})$.** We now return to the present framework, where the Banach space $\mathcal{A}_\infty(\mathcal{V})$, defined in (3.4), admits the Taylor basis $\{(z-1)^m,\ m \in \mathbb{Z}_{\ge 0}\}$ as a Schauder basis.

The next result summarizes the main properties of the various operators of interest, relative to their nuclearity and their traces.

Proposition 6.4. (*a*) *For any* $h \in \mathcal{H}^+$, *the weighted transfer operator* $\mathbf{H}_{[h],t,w}$ *is nuclear of order 0. Its trace satisfies*

$$\operatorname{Tr} \mathbf{H}_{[h],t,w} = \alpha(h)^{2t} \frac{\exp(wc(h))}{1-(-1)^{|h|}\alpha(h)^2} . \tag{6.3}$$

(Recall that $|h|$ *is the depth of* h*.)*

(*b*) *Consider a pair* (s, w) *in the domain* $\Gamma(a)$ *defined in* (2.15) *with* $a > 3/4$. *Then, the complete operator* $\mathbf{H}_{t,w}$ *and its iterates* $\mathbf{H}^k_{t,w}$ *(for* $k \geq 1$*) are nuclear operators of order* $q \leq 2/3$. *One has, for* $(s, w) \in \Gamma(a)$,

$$\operatorname{Tr} \mathbf{H}^k_{t,w} = \sum_{h \in \mathcal{H}^k} \operatorname{Tr} \mathbf{H}_{[h],t,w} = \sum_{h \in \mathcal{H}^k} \alpha(h)^{2t} \frac{\exp(wc(h))}{1-(-1)^k\alpha(h)^2}, \qquad \text{for } k \geq 1 . \tag{6.4}$$

For any $k \geq 1$, *the mapping* $(t, w) \mapsto \operatorname{Tr} \mathbf{H}^k_{t,w}$ *is analytic on* $\Gamma(a)$.

(*c*) *Consider a pair* (t, w) *in the domain* $\Gamma(a)$ *with* $a > 1$. *Suppose, in addition, that the spectral radius of* $\mathbf{H}_{t,w}$ *is less than 1. Then, the quasi-inverse* $\mathbf{H}_{t,w}(I - \mathbf{H}_{t,w})^{-1}$ *and the even quasi-inverse* $\mathbf{E}_{t,w} := \mathbf{H}^2_{t,w}(I - \mathbf{H}^2_{t,w})^{-1}$ *are nuclear of order* $q \leq 2/3$. *For these pairs* (t, w), *the following holds,*

$$\operatorname{Tr} \mathbf{H}_{t,w}(I - \mathbf{H}_{t,w})^{-1} = \sum_{k \geq 1} \operatorname{Tr} \mathbf{H}^k_{t,w} \qquad \text{and} \qquad \operatorname{Tr} \mathbf{E}_{t,w} = \sum_{\substack{k \text{ even} \\ k \geq 2}} \operatorname{Tr} \mathbf{H}^k_{t,w} . \tag{6.5}$$

Moreover, the mappings $(t, w) \mapsto \operatorname{Tr} \mathbf{H}_{t,w}(I - \mathbf{H}_{t,w})^{-1}$ *and* $(t, w) \mapsto \operatorname{Tr} \mathbf{E}_{t,w}$ *are analytic on* $\Gamma(a)$ *with* $a > 1$, *and the following equalities hold*

$$\operatorname{Tr} \mathbf{H}_{t,w}(I - \mathbf{H}_{t,w})^{-1} = \sum_{h \in \mathcal{H}^+} \alpha(h)^{2t} \frac{\exp(wc(h))}{1-(-1)^{|h|}\alpha(h)^2}, \tag{6.6}$$

$$\operatorname{Tr} \mathbf{E}_{t,w} = \sum_{h \in \mathcal{H}^+,\ |h| \text{ even}} \alpha(h)^{2t} \frac{\exp(wc(h))}{1-\alpha(h)^2}.$$

Proof. (*a*) Mayer and Momeni-Venkov have studied the nuclearity of the *unweighted* transfer operator. We follow the approach described in Lemma 2.7 of [19], where the authors use the Schauder basis of $\mathcal{A}_\infty(\mathcal{V})$, formed with the polynomials $(z-1)^m$ with $m \geq 0$. We adapt their method to the case of the *weighted* operator $\mathbf{H}_{[h],t,w}$ for any $h \in \mathcal{H}^+$.

Using the Taylor expansion of f at $z = 1$ in the expression of $\mathbf{H}_{[h],t,w}[f](z)$ provides a new expression

$$\mathbf{H}_{[h],t,w}[f](z) = e^{wc(h)} \left[\sum_{m=0}^{\infty} \frac{f^{(m)}(1)}{m!} \left(h(z) - 1\right)^m \right] \underline{h}^t(z) .$$

that involves the analytic extensions $\underline{h}$ of $|h'|$. We consider here the case $r = 5/4$ in Property $(P3)$, and the related radii $r, \tilde{r}$. For any $h \in \mathcal{H}^+$, and $t = \sigma + i\tau$, we introduce with $(P3)(ii)$, the bound

$$a_h(t) := \sup_{z \in \mathcal{V}} |\underline{h}^t(z)| \le e^{\pi|\tau|} |h'(-1/4)|^\sigma .$$

The following objects defined for $m \ge 0$ and $h \in \mathcal{H}^+$,

$$f_{m,[h]}(z) = \frac{\underline{h}^t(z)}{a_h(t)} \left(\frac{h(z) - 1}{\tilde{r}} \right)^m, \quad f^\star_{m,[h]}[f] = r^m \frac{f^{(m)}(1)}{m!} ,$$

satisfy with $(P3)$ and Cauchy estimates, the two bounds

$$||f_{m,[h]}|| \le 1, \quad ||f^\star_{m,[h]}|| \le 1 ,$$

and the decomposition

$$\mathbf{H}_{[h],t,w}[f] = \sum_{m=0}^{\infty} \lambda_{m,[h]} \, f_{m,[h]} \, f^\star_{m,[h]}[f], \quad \text{with} \quad \lambda_{m,[h]} := \left(\frac{r}{\tilde{r}}\right)^{-m} e^{wc(h)} a_h(t)$$

holds. As the sequence $m \mapsto \lambda_{m,[h]}$ is geometric and thus belongs to ℓ^p for any $0 < p \le 1$, this proves the nuclearity of order 0. Each component operator $\mathbf{H}_{[h],t,w}$ is a composition operator of the form $f \mapsto g \cdot f \circ h$ where $g := \exp(wc(h)) \cdot \underline{h}^t$. With Properties $(P3)$, it acts on $\mathcal{A}_\infty(\mathcal{V})$. Moreover, due to $(P3)(i)$, any branch $h \in \mathcal{H}^+$ maps the domain $\mathcal{V}$ strictly inside itself. According to [18, Lemma 7.10], the spectrum of such a composition operator acting on $\mathcal{A}_\infty(\mathcal{V})$ is the set

$$\{\mu_n := g(x_h) \cdot (h'(x_h))^n, \ | \, n \in \mathbb{Z}_{\ge 0}\} ,$$

that involves the unique fixed point x_h of h. Every eigenvalue is simple. This result applies to $\mathbf{H}_{[h],t,w}$ and provides the expression given in (6.3) for the trace of $\mathbf{H}_{[h],t,w}$ which involves $\alpha(h)$ via (2.4).

(b) For any fixed $h \in \mathcal{H}^+$, any (t, w), and any real $p > 0$, the sequence $m \mapsto \lambda_{m,[h]}$ satisfies, with Lemma 1,

$$\sum_{m \ge 0} |\lambda_{m,[h]}|^p \le \left(\frac{\tilde{r}}{r}\right)^{pm} |e^{\nu p c(h)}| \, a_h(t)^p \ll \left(\frac{\tilde{r}}{r}\right)^{pm} e^{\pi p |\tau|} |h'(-1/4)|^{(\sigma - d|\nu|)p} .$$

Consider now a pair $(t, w) \in \Gamma(a)$ and an exponent p for which $a\,p := (\sigma - d|\nu|)p > 1/2$. Then, using the bound (3.7) we obtain, for any $(t, w) \in \Gamma(a)$

$$\sum_{(m,h) \in \mathbb{N} \times \mathcal{H}^k} |\lambda_{m,[h]}|^p \le \frac{e^{p\pi|\tau|}}{1 - (\tilde{r}/r)^p} \mathbf{H}^k_{ap}[1](-1/4) .$$

Then, for $(t, w) \in \Gamma(a)$, the operator $\mathbf{H}^k_{t,w}$ is nuclear of order $q < 1/(2a)$ and is nuclear of order $q < 2/3$ as soon as $(t, w) \in \Gamma(a)$ with $a > 3/4$.

Then, for $a > 3/4$, we apply Lemma 6.2 with $q_0 = 2/3$, and get

$$\operatorname{Tr} \mathbf{H}_{t,w}^k = \sum_{h \in \mathcal{H}^k} \operatorname{Tr} \mathbf{H}_{[h],t,w} \,. \tag{6.7}$$

The explicit expression given in (6.3) shows that the mapping $(t, w) \mapsto \operatorname{Tr} \mathbf{H}_{t,w}^k$ is analytic on $\Gamma(a)$.

(c) Eq. (6.5) is a direct application of Lemma 6.3 (ii) with $\mathbf{A} = \mathbf{T} = \mathbf{H}_{t,w}$ or $\mathbf{A} = \mathbf{T} = \mathbf{H}_{t,w}^2$. When $(t, w) \in \Gamma(a)$ with $a > 1$ the spectral radii of $\mathbf{H}_{t,w}$ and of $\mathbf{H}_{t,w}^2$ are less than 1 (see Proposition 3.1), hence Relation (6.5) holds. On this domain $\Gamma(a)$, the expression of $\operatorname{Tr} \mathbf{H}_{t,w}^k$ given in (6.4) is the general term of a series which is absolutely convergent. This thus entails the relations given in (6.6).

□

6.4. **Spectral dominant properties of the transfer operator $\mathbf{H}_{t,w}$ acting on $\mathcal{A}_\infty(\mathcal{V})$.** For real pairs $(t, w) \in \Gamma(a)$ (with $a > 1/2$), the operator $\mathbf{H}_{t,w}$ satisfies strong positive properties that entail the existence of dominant spectral objects, in the same vein as the Perron-Frobenius properties. Also, for $(t, w) \in \Gamma(a)$, (with $a > 1/2$), the map $(t, w) \mapsto \mathbf{H}_{t,w}$ is analytic in the sense of Kato [13, Chapter 3]. As $\mathbf{H}_{t,w}$ is compact, the dominant spectral objects defined for real pairs may be extended (with analytic perturbation of the dominant part of the spectrum) when the pair (t, w) is close to a real pair.

We are mainly interested in a neighborhood of the real pair $(1, 0)$, and we begin with a complex neighborhood $\mathcal{T} = \mathcal{S} \times \mathcal{W}$ of $(1, 0)$. For $(t, w) \in \mathcal{T}$, a spectral decomposition for the operator $\mathbf{H}_{t,w}$ holds and there exist operators $\mathbf{P}_{t,w}$ and $\mathbf{N}_{t,w}$ for which the operator $\mathbf{H}_{t,w}$ decomposes as

$$\mathbf{H}_{t,w} = \lambda(t, w)\mathbf{P}_{t,w} + \mathbf{N}_{t,w} \tag{6.8}$$

with $\mathbf{P}_{t,w} \circ \mathbf{N}_{t,w} = \mathbf{N}_{t,w} \circ \mathbf{P}_{t,w} = 0$. As $\mathbf{H}_{t,w}$ is compact, and due to the equality $\lambda(1, 0) = 1$, the spectral radius $R(t, w)$ of $\mathbf{N}_{t,w}$ is at most equal to $R < 1$. The operator $\mathbf{P}_{t,w}$ is rank one, of the form

$$\mathbf{P}_{t,w}[f](x) = f_{t,w}(x)\, \mu_{t,w}[f] \tag{6.9}$$

and involves the dominant eigenfunction $f_{t,w}$ of $\mathbf{H}_{t,w}$ and the eigenmeasure $\mu_{t,w}$ of the adjoint operator $\mathbf{H}_{t,w}^\star$. Moreover, with analytic perturbation of the dominant part of the spectrum, all the spectral objects depend analytically on $\mathcal{T}$.

The spectral decomposition extends to the iterates of $\mathbf{H}_{t,w}^k$,

$$\mathbf{H}_{t,w}^k = \lambda(t,w)^k \, \mathbf{P}_{t,w} + \mathbf{N}_{t,w}^k, \qquad \text{for any } k \geq 1 \,. \tag{6.10}$$

As the "remainder quasi-inverse" $\mathbf{N}_{t,w}(I-\mathbf{N}_{t,w})^{-1}$ is analytic on $\mathcal{T}$, and the quotient $\lambda(t,w)/(1-\lambda(t,w))$ is meromorphic on $\mathcal{T}$, the decomposition of the quasi-inverse,

$$(I-\mathbf{H}_{t,w})^{-1} = \frac{\lambda(t,w)}{1-\lambda(t,w)}\mathbf{P}_{t,w} + (I-\mathbf{N}_{t,w})^{-1}\,, \tag{6.11}$$

proves that the quasi-inverse $(I-\mathbf{H}_{t,w})^{-1}$ is meromorphic on $\mathcal{T}$. There is an analog proof for the "even quasi-inverse" $\mathbf{E}_{t,w}$ defined in (3.3),

$$\mathbf{E}_{t,w} := \mathbf{H}_{t,w}^2(I-\mathbf{H}_{t,w}^2)^{-1} = \frac{\lambda^2(t,w)}{1-\lambda^2(t,w)}\mathbf{P}_{t,w} + \mathbf{N}_{t,w}^2(I-\mathbf{N}_{t,w}^2)^{-1}\,. \tag{6.12}$$

6.5. **Spectral decomposition and traces.** The projector $\mathbf{P}_{t,w}$ is clearly nuclear of order 0 (as any operator of finite rank), with a trace equal to 1. Then, as $\mathcal{T}$ is a subset of $\Gamma(a)$ for some $a > 3/4$, and due to Proposition 6.4, all the operators involved are nuclear of order $q < 2/3$. Moreover, as the dominant eigenvalue $\lambda(t,w)$ and the traces $\operatorname{Tr}\mathbf{H}_{t,w}^k$ are well-defined and analytic, it is the same for the operators $\mathbf{N}_{t,w}^k$: their traces are well-defined and analytic on $\mathcal{T}$. As the norm of the operator $\mathbf{N}_{t,w}$ is strictly less than 1 on $\mathcal{T}$, Lemma 6.3 applies and entails that the sequences

$$\left[\sum_{k=1}^{K} \operatorname{Tr}\mathbf{H}_{t,w}^{2k}\right] - \sum_{k=1}^{K} \lambda^{2k}(t,w)\,,$$

has a limit (for $K \to \infty$) that is analytic on $\mathcal{T}$. As

$$\lim_{k\to\infty} \sum_{k=1}^{K} \lambda^{2k}(t,w) = \frac{\lambda^2(t,w)}{1-\lambda^2(t,w)}$$

is meromorphic on $\mathcal{T}$, we have shown the equality

$$\operatorname{Tr}\mathbf{H}_{t,w}^2(I-\mathbf{H}_{t,w}^2)^{-1} = \sum_{k=1}^{\infty} \operatorname{Tr}\mathbf{H}_{t,w}^{2k}$$

that defines a meromorphic function on $\mathcal{T}$. In particular,

$$\operatorname{Tr}\mathbf{E}_{t,w} = \frac{\lambda^2(t,w)}{1-\lambda^2(t,w)} + \operatorname{Tr}\mathbf{N}_{t,w}^2(I-\mathbf{N}_{t,w}^2)^{-1}\,. \tag{6.13}$$

defines a meromorphic function on $\mathcal{T}$, whose poles are located on the curve $\{(t,w) \in \mathcal{T} \mid \lambda^2(t,w) = 1\}$ that contains the point $(1,0)$ and thus the curve $\{(t,w) \in \mathcal{T} \mid \lambda(t,w) = 1\}$.

6.6. **Properties near** $(t, w) = (1, 0)$**.** The operator $\mathbf{H}_{1,0}$ is a density transformer, and thus $\lambda(1, 0) = 1$. Furthermore, the dominant eigenfunction $f_{1,0}$ coincides with the Gauss density $\psi(x) := (1/\log 2) \cdot 1/(1 + x)$. There are simple formulae for the two partial derivatives of $\lambda(t, w)$ at $(1, 0)$. They are easy consequences of the spectral equality $\mathbf{H}_{t,w}[f_{t,w}] = \lambda(t, w)f_{t,w}$, the fact that $\mathbf{H}_{1,0}$ is a density transformer, and the explicit forms of the derivatives of the operator. One obtains

$$-\lambda'_t(1, 0) = \int_{\mathcal{I}} \log |T'(x)|\, \psi(x)\, dx = \frac{\pi^2}{6 \log 2}\,, \tag{6.14}$$

$$\lambda'_w(1, 0) = \sum_{h \in \mathcal{H}} c(h) \int_{h(\mathcal{I})} \psi(x)\, dx\,. \tag{6.15}$$

The constant $-\lambda'_t(1, 0) > 0$ is the Kolmogorov entropy (denoted here by $\mathcal{E}$) and $\lambda'_w(1, 0)$ is the average $\mathbb{E}[c]$ of the cost c with respect to the Gauss density ψ. They are both non zero.

Since both derivatives are non zero at $(t, w) = (1, 0)$, there is a complex neighborhood $\mathcal{S}_1$ of $t = 1$, a neighborhood $\mathcal{W}_1$ of 0 and a unique analytic function $\sigma : \mathcal{W}_1 \to \mathbb{C}$ for which

$$\{(t, w) \in \mathcal{S}_1 \times \mathcal{W}_1 \mid \lambda(t, w) = 1\} = \{w \in \mathcal{W}_1 \mid \lambda(\sigma(w), w) = 1,\, \sigma(0) = 1\}\,,$$

$$\sigma'(w) = -\frac{\lambda'_w(\sigma(w), w)}{\lambda'_t(\sigma(w), w)}\,. \tag{6.16}$$

First, we may choose the neighborhood $\mathcal{S}_1$ to be a rectangle of the form $\mathcal{S}_1 = \{s = \sigma + i\tau \mid |\tau| < \tau_1, |\sigma - 1| \leq \delta_1\}$. Second, "perturbating" the inequality $\lambda'_t(1, 0) \neq 0$, and taking a (possibly) smaller neighborhood $\mathcal{T}_1 := \mathcal{S}_1 \times \mathcal{W}_1$, exhibits a constant $A > 0$ for which the ratio $|\lambda(t, w) - 1|/|s - \sigma(w)|$ is at least $A > 0$ on $\overline{\mathcal{T}}_1$ (and its inverse is then bounded).

Then, for each $w \in \mathcal{W}_1$, the map $t \mapsto \operatorname{Tr} \mathbf{E}_{t,w}$ has a unique (simple) pole on $\mathcal{S}_1$, located at $s = \sigma(w)$, with a residue equal to

$$\operatorname{Res}[\operatorname{Tr} \mathbf{E}_{t,w}; s = \sigma(w)] = \frac{1}{2} \left(\frac{-1}{\lambda'_t(\sigma(w), w)} \right)\,. \tag{6.17}$$

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