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Some remarks on non-symmetric polarization [☆]

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ABSTRACT

Let $P : \mathbb{C}^n \to \mathbb{C}$ be an m-homogeneous polynomial given by

$$P(x) = \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_1} \ldots x_{j_m}.$$

Defant and Schlüters defined a non-symmetric associated m-form $L_P : (\mathbb{C}^n)^m \to \mathbb{C}$ by

$$L_P\left(x^{(1)}, \ldots, x^{(m)}\right) = \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x^{(1)}_{j_1} \ldots x^{(m)}_{j_m}.$$

They estimated the norm of L_P on $(\mathbb{C}^n, \|\cdot\|)^m$ by the norm of P on $(\mathbb{C}^n, \|\cdot\|)$ times a $(c \log n)^{m^2}$ factor for every 1-unconditional norm $\|\cdot\|$ on $\mathbb{C}^n$. A symmetrization procedure based on a card-shuffling algorithm which (together with Defant and Schlüters' argument) brings the constant term down to $(cm \log n)^{m-1}$ is provided. Regarding the lower bound, it is shown that the optimal constant is bigger than $(c \log n)^{m/2}$ when $n \gg m$. Finally, the case of ℓ_p-norms $\|\cdot\|_p$ with $1 \leq p < 2$ is addressed.

1. Introduction

Let $P : \mathbb{C}^n \to \mathbb{C}$ be an m-homogeneous polynomial. It is well-known that there is a unique symmetric m-linear form $B : (\mathbb{C}^n)^m \to \mathbb{C}$, such that $B(x, \ldots, x) = P(x)$ for all $x \in \mathbb{C}$. Moreover, the *polarization*

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formula gives an expression for the m-linear form B in terms of P (see e.g. [3, Section 1.1]). In fact, for every $x^{(1)}, \dots, x^{(m)} \in \mathbb{C}$, we have

$$B\left(x^{(1)}, \dots, x^{(m)}\right) = \frac{1}{2^m m!} \sum_{\varepsilon \in \{-1,1\}^m} P\left(\varepsilon_1 x^{(1)} + \dots + \varepsilon_m x^{(m)}\right).$$

It follows from this identity that

$$\sup_{\|x^{(k)}\| \le 1} \left| B\left(x^{(1)}, \dots, x^{(m)}\right) \right| \le e^m \sup_{\|x\| \le 1} |P(x)|, \tag{1}$$

for any norm $\|\cdot\|$ in $\mathbb{C}^n$.

In [2], Defant and Schlüters defined a non-symmetric m-linear form L_P arising from a given m-homogeneous polynomial P. More precisely, for an m-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ defined by

$$P(x) = \sum_{1 \le j_1 \le \dots \le j_m \le n} c_{j_1 \dots j_m} x_{j_1} \dots x_{j_m},$$

its associated m-linear form $L_P : (\mathbb{C}^n)^m \to \mathbb{C}$ is given by

$$L_P\left(x^{(1)}, \dots, x^{(m)}\right) = \sum_{1 \le j_1 \le \dots \le j_m \le n} c_{j_1 \dots j_m} x^{(1)}_{j_1} \dots x^{(m)}_{j_m}.$$

Assuming unconditionality of the norm $\|\cdot\|$ in $\mathbb{C}^n$, Defant and Schlüters proved that a similar estimate as in (1) holds for L_P. Before providing further details we introduce an *ad hoc* definition:

Definition 1.1. For $m, n \in \mathbb{N}$, we define $C(m, n)$ as the infimum of the constants $C > 0$ such that for every m-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ and every 1-unconditional norm $\|\cdot\|$ on $\mathbb{C}^n$ we have

$$\sup_{\|x^{(k)}\| \le 1} \left| L_P\left(x^{(1)}, \dots, x^{(m)}\right) \right| \le C \sup_{\|x\| \le 1} |P(x)|.$$

Similarly, for $1 \le p < 2$, we take $C_p(m, n)$ as the infimum of the constants $C > 0$ such that for every m-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ we have

$$\sup_{\|x^{(k)}\|_p \le 1} \left| L_P\left(x^{(1)}, \dots, x^{(m)}\right) \right| \le C \sup_{\|x\|_p \le 1} |P(x)|.$$

The aforementioned result of [2] can be stated in terms of the previous definition.

Theorem 1.2 *([2, Theorem 1.1]). There exists a universal constant $c_1 \ge 1$ such that*

$$C(m, n) \le (c_1 \log n)^{m^2}.$$

Moreover, for $1 \le p < 2$, there is a constant $c_2 = c_2(p) \ge 1$ for which

$$C_p(m, n) \le c_2^{m^2}.$$

Note that by the uniqueness of the symmetric m-linear form B we have

$$B\left(x^{(1)}, \ldots, x^{(m)}\right) = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} L_P\left(x^{\sigma(1)}, \ldots, x^{\sigma(m)}\right), \tag{2}$$

where Σ_m is the group of permutations of m elements. The proof of Theorem 1.2 consists of bounding the norm of L_P by successive partial symmetrizations starting at L_P and ending at the fully symmetrized B. Finally, applying (1) yields the result. Changing only the way in which this symmetrization is carried out and using the same arguments as in [2], we obtain improved bounds for the constants $C(m,n)$ and $C_p(m,n)$. Additionally, we provide lower bounds for these constants. Our main result is the following.

Theorem 1.3. *There exists a universal constant $c_1 \geq 1$ such that*

$$\left(\frac{\log\left(\frac{2n}{m}\right) - \pi}{\pi}\right)^{m/2} \leq C(m,n) \leq c_1^m m^m (\log n)^{m-1}.$$

Moreover, for $1 \leq p < 2$, there is a constant $c_2 = c_2(p) \geq 1$ for which

$$m^{\frac{m}{p}} \leq C_p(m,n) \leq c_2^m m^m,$$

where the lower bound holds for $n \geq m$.

Remark 1.4. Defant and Schlüters achieved similar upper bounds by refining their original calculations from [2] as it was mentioned during a personal communication.

Remark 1.5. Scrutiny of the theorem's proof suggests that the underlying reason which determines the magnitude of the constants $C(m,n)$ and $C_p(m,n)$ is the behaviour of the operator known as the main triangle projection. Roughly speaking, the main triangle projection is the operator which given a matrix in $\mathbb{C}^{n\times n}$ returns the same matrix with zeroes below the diagonal. Each norm on $\mathbb{C}^n$ induces an operator norm in $\mathbb{C}^{n\times n}$ and again this induces a norm for the main triangle projection. Estimations of the latter norm are the ones that shape the upper and lower bounds of $C(m,n)$ and $C_p(m,n)$ that were obtained.

2. Symmetrization

The following may be deduced from (2).

$$\begin{aligned}
B\left(x^{(1)}, \ldots, x^{(m)}\right) &= \frac{1}{m!} \sum_{\sigma \in \Sigma_m} L_P\left(x^{\sigma(1)}, \ldots, x^{\sigma(m)}\right) \\
&= \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_1}^{\sigma(1)} \ldots x_{j_m}^{\sigma(m)} \\
&= \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_{\sigma^{-1}(1)}}^{(1)} \ldots x_{j_{\sigma^{-1}(m)}}^{(m)} \\
&= \frac{1}{m!} \sum_{\tau \in \Sigma_m} \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_{\tau(1)}}^{(1)} \ldots x_{j_{\tau(m)}}^{(m)}.
\end{aligned}$$

From a probabilistic point of view, this may be restated as

$$B\left(x^{(1)},\dots,x^{(m)}\right)=E\left[\sum_{1\le j_1\le\dots\le j_m\le n} c_{j_1\dots j_m} x^{(1)}_{j_{\sigma(1)}}\dots x^{(m)}_{j_{\sigma(m)}}\right], \tag{3}$$

where expectation is taken over $\sigma\in\Sigma_m$ and Σ_m is endowed with the equiprobability measure. In other words, B is the expected value of L_P when the order of the monomials' subindices is an equidistributed random variable. Thus, a card-shuffling procedure applied to the order of the subindices will yield a symmetrization procedure for L_P by taking expectation. We will use the Fischer–Yates shuffle in its original version which can be found in [4]. It goes as follows. Choose a random card from an ordered deck and leave it on top. Next, choose a random card between the second and the last place and leave it in the second place, and so on. At the last step, choose between the last two cards which one will go in the penultimate place. After applying this procedure, an ordered deck will be completely shuffled, that is, any arrangement will be equally probable.

Remark 2.1. Note that at any given step, the k-th step say, the first $k-1$ cards (which have been previously selected) are completely random, while the last cards remain completely ordered. This special structure will be crucial in the proof of Theorem 1.3.

Next, we introduce the symmetrization procedure arising from the Fischer–Yates shuffle. For every $1\le k\le m-1$ we let $\mathbb{P}_k$ be the probability distribution on Σ_m associated to performing the first k steps of the shuffling algorithm. We define the k-th shuffle S_k of an m-form $L:(\mathbb{C}^n)^m\to\mathbb{C}$ by

$$S_kL\left(x^{(1)},\dots,x^{(m)}\right)=E\left[\sum_{i_1,\dots,i_m=1}^{n} c_{i_1\dots i_m} x^{(1)}_{i_{\sigma(1)}}\dots x^{(m)}_{i_{\sigma(m)}}\right],$$

where $\sigma\sim\mathbb{P}_k$.

In particular, from (3) and the fact that the $(m-1)$-th step of the shuffle achieves equidistribution we have

$$B=S_{m-1}L_P.$$

However, it should be noticed that the intermediate shuffles are not partial symmetrizations since we are symmetrizing the monomials' subindices rather than the variables.

In order to study the structure of S_k, we define the k-th shuffling step T_k of an m-form $L:(\mathbb{C}^n)^m\to\mathbb{C}$ by

$$\begin{aligned}&T_kL\left(x^{(1)},\dots,x^{(m)}\right)\\&\quad=\frac{1}{m-k+1}\sum_{l=k}^{m}L\left(x^{(1)},\dots,x^{(k-1)},x^{(k+1)},\dots,x^{(l)},x^{(k)},x^{(l+1)},\dots,x^{(m)}\right).\end{aligned}$$

Lemma 2.2. *For every $1\le k\le m-1$ we have that $S_k=T_k\dots T_1$.*

Proof. Since T_k and S_k are linear for every $1\le k\le m-1$, it is enough to check that the equality holds for monomials. Fix $1\le i_1,\dots,i_m\le n$, we have to prove that

$$S_k\left(x^{(1)}_{i_1}\dots x^{(m)}_{i_m}\right)=T_k\dots T_1\left(x^{(1)}_{i_1}\dots x^{(m)}_{i_m}\right).$$

We will proceed by induction. If $k=1$, the random permutation σ is a cycle in Σ_m. More precisely, using the cycle notation in Σ_m we have that σ takes the value $(l\ l-1\ \dots\ 1)$ for some $1 \le l \le m$ with probability $1/m$. Therefore, we get

$$\begin{aligned}S_1\left(x_{i_1}^{(1)}\dots x_{i_m}^{(m)}\right) &= E\left[x_{i_{\sigma(1)}}^{(1)}\dots x_{i_{\sigma(m)}}^{(m)}\right] = \frac{1}{m}\sum_{l=1}^{m} x_{i_l}^{(1)}x_{i_1}^{(2)}\dots x_{i_{l-1}}^{(l)}x_{i_{l+1}}^{(l+1)}\dots x_{i_m}^{(m)} \\ &= \frac{1}{m}\sum_{l=1}^{m} x_{i_1}^{(2)}\dots x_{i_{l-1}}^{(l)}x_{i_l}^{(1)}x_{i_{l+1}}^{(l+1)}\dots x_{i_m}^{(m)} = T_1\left(x_{i_1}^{(1)}\dots x_{i_m}^{(m)}\right).\end{aligned}$$

Only the inductive step remains to be proven. Let $2 \le k \le m-1$ and suppose the lemma holds for $k-1$. From the definition of the Fischer–Yates shuffle we may deduce that a random permutation with law $\mathbb{P}_k$ can be written as the composition of two independent random permutations τ and σ where $\sigma \sim \mathbb{P}_{k-1}$ and τ takes the value $\tau_l = (l\ l-1\ \dots\ k)$ for some $k \le l \le m$ with probability $1/(m-k+1)$. For a fixed τ, we may define new indices $j_1,\dots,j_m$ such that $j_k = i_{\tau(k)}$ for every $1 \le k \le m$. So we obtain

$$\begin{aligned}S_k\left(x_{i_1}^{(1)}\dots\ x_{i_m}^{(m)}\right) &= E_{\tau,\sigma}\left[x_{i_{\tau\sigma(1)}}^{(1)}\dots x_{i_{\tau\sigma(m)}}^{(m)}\right] = E_\tau\left[E_\sigma\left[x_{j_{\sigma(1)}}^{(1)}\dots x_{j_{\sigma(m)}}^{(m)}\right]\right] \\ &= E_\tau\left[S_{k-1}\left(x_{j_1}^{(1)}\dots x_{j_m}^{(m)}\right)\right] = E_\tau\left[S_{k-1}\left(x_{i_{\tau(1)}}^{(1)}\dots x_{i_{\tau(m)}}^{(m)}\right)\right] \\ &= \frac{1}{m-k+1}\sum_{l=k}^{m} S_{k-1}\left(x_{i_{\tau_l(1)}}^{(1)}\dots x_{i_{\tau_l(m)}}^{(m)}\right) \\ &= \frac{1}{m-k+1}\sum_{l=k}^{m} S_{k-1}\left(x_{i_1}^{(1)}\dots x_{i_{k-1}}^{(k-1)}x_{i_l}^{(k)}x_{i_k}^{(k+1)}\dots x_{i_{l-1}}^{(l)}x_{i_{l+1}}^{(l+1)}\dots x_{i_m}^{(m)}\right) \\ &= \frac{1}{m-k+1}\sum_{l=k}^{m} S_{k-1}\left(x_{i_1}^{(1)}\dots x_{i_{k-1}}^{(k-1)}x_{i_k}^{(k+1)}\dots x_{i_{l-1}}^{(l)}x_{i_l}^{(k)}x_{i_{l+1}}^{(l+1)}\dots x_{i_m}^{(m)}\right) \\ &= T_kS_{k-1}\left(x_{i_1}^{(1)}\dots x_{i_m}^{(m)}\right),\end{aligned}$$

which completes the proof. □

Following [2], we turn to study how the coefficients of the successive shuffles of L_P change. Let $L:\ (\mathbb{C}^n)^m \to \mathbb{C}$ be an m-linear form given by

$$L\left(x^{(1)},\dots,x^{(m)}\right) = \sum_{i\in\mathcal{I}(m,n)} c_i x_{i_1}^{(1)}\dots x_{i_m}^{(m)},$$

where $\mathcal{I}(m,n) = \{1,\dots,n\}^m$. We will denote its coefficients by $c_i(L) = c_i$.

Lemma 2.3. *For $m,n\in\mathbb{N}$, $1\le k\le m-1$, $i\in\mathcal{I}(m,n)$ and an m-homogeneous polynomial $P:\mathbb{C}^n\to\mathbb{C}$ we have*

$$c_i\left(S_{k-1}L_P\right) = \begin{cases}(m-k+1)\left(1+\sum_{u=1}^{m-k}\delta_{i_k,i_{k+u}}\left(\frac{1}{u+1}-\frac{1}{u}\right)\right)c_i\left(S_kL_P\right) & \text{if } i_k\le i_{k+1} \\ 0 & \text{otherwise}\end{cases},$$

where δ is the Kronecker delta and we take $S_0L_P = L_P$.

Proof. We begin the proof by calculating the coefficients $c_i\left(S_kL_P\right)$ in terms of the coefficients $c_i\left(S_{k-1}L_P\right)$. Observe that for an m-linear form $L:(\mathbb{C}^n)^m\to\mathbb{C}$ we have

$$\begin{aligned}
&T_k L\left(x^{(1)},\dots,x^{(m)}\right)\\
&= \frac{1}{m-k+1}\sum_{l=k}^{m} L\left(x^{(1)},\dots,x^{(k-1)},x^{(k+1)},\dots,x^{(l)},x^{(k)},x^{(l+1)},\dots,x^{(m)}\right)\\
&= \frac{1}{m-k+1}\sum_{l=k}^{m}\sum_{i\in\mathcal{I}(m,n)} c_i(L)x^{(1)}_{i_1}\dots x^{(k-1)}_{i_{k-1}}x^{(k+1)}_{i_k}\dots x^{(l)}_{i_{l-1}}x^{(k)}_{i_l}x^{(l+1)}_{i_{l+1}}\dots x^{(m)}_{i_m}\\
&= \sum_{i\in\mathcal{I}(m,n)}\frac{1}{m-k+1}\sum_{l=k}^{m} c_i(L)x^{(1)}_{i_1}\dots x^{(k-1)}_{i_{k-1}}x^{(k)}_{i_l}x^{(k+1)}_{i_k}\dots x^{(l)}_{i_{l-1}}x^{(l+1)}_{i_{l+1}}\dots x^{(m)}_{i_m}\\
&= \sum_{i\in\mathcal{I}(m,n)}\frac{1}{m-k+1}\sum_{l=k}^{m} c_{(i_1,\dots,i_{k-1},i_{k+1},\dots,i_l,i_k,i_{l+1},\dots,i_m)}(L)x^{(1)}_{i_1}\dots x^{(m)}_{i_m}.
\end{aligned}$$

Therefore, since $S_k = T_k S_{k-1}$, we deduce the formula

$$c_i\left(S_k L_P\right) = \frac{1}{m-k+1}\sum_{l=k}^{m} c_{(i_1,\dots,i_{k-1},i_{k+1},\dots,i_l,i_k,i_{l+1},\dots,i_m)}\left(S_{k-1}L_P\right). \tag{4}$$

By the definition of L_P if a coefficient $c_i\left(L_P\right)$ is not zero, then the index i must satisfy that $1 \le i_1 \le \dots \le i_m \le n$. We will prove inductively that for $0 \le k \le m-1$, if the coefficient $c_i\left(S_k L_P\right)$ is not zero, then the index i must satisfy that $1 \le i_{k+1} \le \dots \le i_m \le n$.

Since $S_0 L_P = L_P$, the case $k = 0$ is already proven. Now assume the assertion holds for some $0 \le k-1 \le m-1$ and fix $i \in \mathcal{I}(m,n)$ such that $i_s > i_{s+1}$ for some $k+1 \le s \le m-1$. Applying the inductive hypothesis we may deduce that

$$c_{(i_1,\dots,i_{k-1},i_{k+1},\dots,i_l,i_k,i_{l+1},\dots,i_m)}\left(S_{k-1}L_P\right) = 0,$$

for every $k \le l \le m$. Hence, using (4) we get that $c_i\left(S_k L_P\right) = 0$ proving the inductive step. In particular, we have shown that $c_i\left(S_{k-1}L_P\right) = 0$ if $i_k > i_{k+1}$ as sought.

Now assume that $i_k \le i_{k+1}$. If for some $k+1 \le s \le m-1$ we have that $i_s > i_{s+1}$, then by the previous argument we may deduce that $c_i\left(S_{k-1}L_P\right) = c_i\left(S_k L_P\right) = 0$ as desired. Therefore, it remains to check the statement when $1 \le i_k \le \dots \le i_m \le n$. Define $s = \sup\{k \le u \le m \ : i_u = i_k\}$ and notice that

$$c_{(i_1,\dots,i_{k-1},i_{k+1},\dots,i_l,i_k,i_{l+1},\dots,i_m)}\left(S_{k-1}L_P\right) = \begin{cases} c_i\left(S_{k-1}L_P\right) & \text{if } k \le l \le s \\ 0 & \text{if } s < l \le m \end{cases}.$$

Thus, we may push (4) further to get

$$\begin{aligned}
c_i\left(S_k L_P\right) &= \frac{1}{m-k+1}\sum_{l=k}^{m} c_{(i_1,\dots,i_{k-1},i_{k+1},\dots,i_l,i_k,i_{l+1},\dots,i_m)}\left(S_{k-1}L_P\right)\\
&= \frac{1}{m-k+1}\sum_{l=k}^{s} c_i\left(S_{k-1}L_P\right) = \frac{s-k+1}{m-k+1}c_i\left(S_{k-1}L_P\right).
\end{aligned}$$

Since $s \ge k$, we have that $s-k+1 \ne 0$. Thus, we get

$$\begin{aligned}
c_i\left(S_{k-1}L_P\right) &= \frac{m-k+1}{s-k+1} c_i\left(S_k L_P\right) \\
&= (m-k+1)\left(1+\sum_{u=1}^{s-k}\left(\frac{1}{u+1}-\frac{1}{u}\right)\right) c_i\left(S_k L_P\right) \\
&= (m-k+1)\left(1+\sum_{u=1}^{m-k}\delta_{i_k,i_{k+u}}\left(\frac{1}{u+1}-\frac{1}{u}\right)\right) c_i\left(S_k L_P\right).
\end{aligned}$$

This concludes the proof. □

As in [2], we will restate the previous lemma using Schur products. For $A, B \in \mathbb{C}^{\mathcal{I}(m,n)}$, the Schur product $A * B$ is given by

$$c_i(A * B) = c_i(A)c_i(B),$$

where $c_i(\cdot)$ denotes de i-th entry of a matrix. By identifying an m-linear form with its coefficients, we may compute the product between a matrix and an m-form. More precisely, for $A \in \mathbb{C}^{\mathcal{I}(m,n)}$ and an m-linear form $L : (\mathbb{C}^n)^m \to \mathbb{C}$ we define $A * L : (\mathbb{C}^n)^m \to \mathbb{C}$ by

$$c_i(A * L) = c_i(A)c_i(L).$$

With this notation Lemma 2.3 proves the formula

$$S_{k-1}L_P = R_k * S_k L_P, \tag{5}$$

where $R_k \in \mathbb{C}^{\mathcal{I}(m,n)}$ is given by

$$c_i\left(R_k\right) = \begin{cases} (m-k+1)\left(1+\sum_{u=1}^{m-k}\delta_{i_k,i_{k+u}}\left(\frac{1}{u+1}-\frac{1}{u}\right)\right) & \text{if } i_k \le i_{k+1} \\ 0 & \text{otherwise} \end{cases}.$$

The matrix $R_k \in \mathbb{C}^{\mathcal{I}(m,n)}$ may be decomposed as sums and products of simpler matrices. For $u, v \in \{1, \ldots, m\}$, let $D^{u,v}, T^{u,v} \in \mathbb{C}^{\mathcal{I}(m,n)}$ be such that for every $i \in \mathcal{I}(m,n)$ we have

$$c_i\left(D^{u,v}\right) = \begin{cases} 1 & \text{if } i_u = i_v \\ 0 & \text{otherwise} \end{cases},$$

$$c_i\left(T^{u,v}\right) = \begin{cases} 1 & \text{if } i_u \le i_v \\ 0 & \text{otherwise} \end{cases}.$$

Keeping Remark 1.5 in mind, we may observe that $T^{u,v}$ bears a close resemblance with the main triangle projection $T : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$. Indeed, note that $c_i(T^{u,v}) = c_{i_u,i_v}(T)$ for every $i \in \mathcal{I}(m,n)$.

Lemma 2.4. *For $1 \le k \le m-1$, we have*

$$R_k = (m-k+1)T^{k,k+1} * \left(1+\sum_{u=1}^{m-k} D^{k,k+u}\left(\frac{1}{u+1}-\frac{1}{u}\right)\right).$$

Proof. For $i \in \mathcal{I}(m,n)$, we deduce that

$$\begin{aligned} c_i\left((m-k+1)\ T^{k,k+1} * \left(1+\sum_{u=1}^{m-k} D^{k,k+u}\left(\frac{1}{u+1}-\frac{1}{u}\right)\right)\right) &= \\ &= (m-k+1)c_i\left(T^{k,k+1}\right)\left(1+\sum_{u=1}^{m-k} c_i\left(D^{k,k+u}\right)\left(\frac{1}{u+1}-\frac{1}{u}\right)\right) \\ &= c_i\left(T^{k,k+1}\right)(m-k+1)\left(1+\sum_{u=1}^{m-k} \delta_{i_k,i_{k+u}}\left(\frac{1}{u+1}-\frac{1}{u}\right)\right) \\ &= c_i\left(R_k\right), \end{aligned}$$

which proves the statement. □

3. Upper bounds

In this section we provide the upper bounds for Theorem 1.3. Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. For $A \in \mathbb{C}^{\mathcal{I}(m,n)}$, we define $\mu_{\|\cdot\|}(A)$ as the infimum of the constants $C > 0$ such that for every m-linear form $L : (\mathbb{C}^n)^m \to \mathbb{C}$ we have

$$\sup_{\|x^{(k)}\|\leq 1}\left|A * L\left(x^{(1)},\ldots,x^{(m)}\right)\right| \leq C \sup_{\|x^{(k)}\|\leq 1}\left|L\left(x^{(1)},\ldots,x^{(m)}\right)\right|.$$

Note that $\left(\mathbb{C}^{\mathcal{I}(m,n)}, \mu_{\|\cdot\|}\right)$ is a Banach algebra.

We will use the following lemma by Defant and Schlüters.

Lemma 3.1 *([2, Lemma 3.2]). For every $n,m \in \mathbb{N}$, every $u,v \in \{1,\ldots,m\}$ and every 1-unconditional norm $\|\cdot\|$ on $\mathbb{C}^n$*

$$\begin{aligned} \mu_{\|\cdot\|}\left(D^{u,v}\right) &= 1, \\ \mu_{\|\cdot\|}\left(T^{u,v}\right) &\leq \log_2(2n). \end{aligned}$$

Moreover, for every $1 \leq p < 2$, there exists a constant $c = c(p)$ so that for every $n,m \in \mathbb{N}$

$$\mu_{\|\cdot\|_p}\left(T^{u,v}\right) \leq c.$$

As mentioned in Remark 1.5, the estimates for $T^{u,v}$ rely on bounds for the norm of the main triangle projection obtained by Kwapień and Pełczyński in [5] and Bennett in [1].

Corollary 3.2. *For every $n,m \in \mathbb{N}$, every $1 \leq k \leq m-1$ and every 1-unconditional norm $\|\cdot\|$ on $\mathbb{C}^n$ we have*

$$\mu_{\|\cdot\|}\left(R_k\right) \leq 2(m-k+1)\mu_{\|\cdot\|}\left(T^{k,k+1}\right).$$

Proof. From the last lemma we know that $\mu_{\|\cdot\|}\left(D^{u,v}\right) = 1$ for every $u,v \in \{1,\ldots,m\}$. Since $\left(\mathbb{C}^{\mathcal{I}(m,n)}, \mu_{\|\cdot\|}\right)$ is a Banach algebra, we may deduce from Lemma 2.4 that

$$\begin{aligned}
\mu_{\|\cdot\|}\left(R_k\right) &= \mu_{\|\cdot\|}\left((m-k+1)T^{k,k+1} * \left(1+\sum_{u=1}^{m-k} D^{k,k+u}\left(\frac{1}{u+1}-\frac{1}{u}\right)\right)\right) \\
&\leq (m-k+1)\mu_{\|\cdot\|}\left(T^{k,k+1}\right)\left(1+\sum_{u=1}^{m-k}\mu_{\|\cdot\|}\left(D^{k,k+u}\right)\left|\frac{1}{u+1}-\frac{1}{u}\right|\right) \\
&\leq (m-k+1)\left(1+\sum_{u=1}^{\infty}\left(\frac{1}{u}-\frac{1}{u+1}\right)\right)\mu_{\|\cdot\|}\left(T^{k,k+1}\right) \\
&= 2(m-k+1)\mu_{\|\cdot\|}\left(T^{k,k+1}\right),
\end{aligned}$$

as required. □

We are ready to prove the upper bounds for Theorem 1.3.

Theorem 3.3. *There exists a universal constant $c_1 \geq 1$ such that*

$$C(m,n) \leq c_1^m m^m (\log n)^{m-1}.$$

Moreover, for $1 \leq p < 2$, there is a constant $c_2 = c_2(p) \geq 1$ for which

$$C_p(m,n) \leq c_2^m m^m.$$

Proof. Using (5), the definition of $\mu_{\|\cdot\|}$ and the previous corollary we get

$$\begin{aligned}
\sup_{\|x^{(k)}\|\leq 1}\left|S_{k-1}L_P\left(x^{(1)},\dots,x^{(m)}\right)\right| &= \sup_{\|x^{(k)}\|\leq 1}\left|R_k * S_k L_P\left(x^{(1)},\dots,x^{(m)}\right)\right| \\
&\leq \mu_{\|\cdot\|}\left(R_k\right)\sup_{\|x^{(k)}\|\leq 1}\left|S_k L_P\left(x^{(1)},\dots,x^{(m)}\right)\right| \\
&\leq 2(m-k+1)\mu_{\|\cdot\|}\left(T^{k,k+1}\right)\sup_{\|x^{(k)}\|\leq 1}\left|S_k L_P\left(x^{(1)},\dots,x^{(m)}\right)\right|,
\end{aligned}$$

for every $1 \leq k \leq m-1$. Taking $\mu = \sup_{1\leq k\leq m-1}\mu_{\|\cdot\|}\left(T^{k,k+1}\right)$ and linking the previous inequalities together, we deduce that

$$\begin{aligned}
\sup_{\|x^{(k)}\|\leq 1}\left|L_P\left(x^{(1)},\dots,x^{(m)}\right)\right| &\leq 2m\mu\sup_{\|x^{(k)}\|\leq 1}\left|S_1 L_P\left(x^{(1)},\dots,x^{(m)}\right)\right| \\
&\leq 2^2 m(m-1)\mu^2\sup_{\|x^{(k)}\|\leq 1}\left|S_2 L_P\left(x^{(1)},\dots,x^{(m)}\right)\right| \\
&\leq \ldots \leq 2^{m-1}m!\mu^{m-1}\sup_{\|x^{(k)}\|\leq 1}\left|S_{m-1} L_P\left(x^{(1)},\dots,x^{(m)}\right)\right|.
\end{aligned}$$

Using the identity $S_{m-1}L_P = B$ and applying (1), we obtain

$$\begin{aligned}
\sup_{\|x^{(k)}\|\leq 1}\left|L_P\left(x^{(1)},\dots,x^{(m)}\right)\right| &\leq 2^{m-1}m!\mu^{m-1}\sup_{\|x^{(k)}\|\leq 1}\left|B\left(x^{(1)},\dots,x^{(m)}\right)\right| \\
&\leq 2^{m-1}e^m m!\mu^{m-1}\sup_{\|x\|\leq 1}|P(x)|.
\end{aligned}$$

The theorem follows by applying Stirling's formula to estimate $m!$ and Lemma 3.1 to estimate μ. □

4. Lower bounds

Firstly, we provide a lower bound for $C_p(m,n)$.

Lemma 4.1. *For every $n \geq m$ and every $1 \leq p < 2$, we have that $C_p(m,n) \geq m^{\frac{m}{p}}$.*

Proof. Let $P : \mathbb{C}^m \to \mathbb{C}$ be the m-homogeneous polynomial defined by

$$P(x) = x_1 \dots x_m.$$

So, its associated m-linear form $L_P : (\mathbb{C}^m)^m \to \mathbb{C}$ is given by

$$L_P\left(x^{(1)}, \dots, x^{(m)}\right) = x_1^{(1)} \dots x_m^{(m)}.$$

Observe that

$$\sup_{\|x^{(k)}\|_p \leq 1} \left| L_P\left(x^{(1)}, \dots, x^{(m)}\right)\right| = \sup_{\|x^{(k)}\|_p \leq 1} \left| x_1^{(1)} \dots x_m^{(m)} \right| = 1, \tag{6}$$

where equality is achieved by taking $x^{(i)}$ to be the i-th canonical vector of ℓ_p^m.

On the other hand, a straightforward computation using Lagrange multipliers gives

$$\sup_{\|x\|_p \leq 1} |P(x)| = \left| P\left(m^{-\frac{1}{p}}(1, \dots, 1)\right)\right| = m^{-\frac{m}{p}}. \tag{7}$$

Applying (6) and (7) together with the definition of $C_p(m,n)$ we get

$$1 = \sup_{\|x^{(k)}\|_p \leq 1} \left| L_P\left(x^{(1)}, \dots, x^{(m)}\right)\right| \leq C_p(m,n) \sup_{\|x\|_p \leq 1} |P(x)| = m^{-\frac{m}{p}} C_p(m,n),$$

as desired. □

Secondly, we estimate $C(m,n)$ from below. In order to do this we will need the following special case of a theorem proved by Pełczyński.

Theorem 4.2 *([8, Theorem 1]). For a finite index set J, let $(a_j)_{j \in J}$ and $(b_j)_{j \in J}$ be sequences of characters on compact abelian groups S and T respectively. Suppose there are constants $c_1, c_2 > 0$ such that*

$$\frac{1}{c_1} \left\| \sum_{j \in J} \alpha_j a_j \right\|_{C(S)} \leq \left\| \sum_{j \in J} \alpha_j b_j \right\|_{C(T)} \leq c_2 \left\| \sum_{j \in J} \alpha_j a_j \right\|_{C(S)}, \tag{8}$$

for every sequence of scalars $(\alpha_j)_{j \in J} \subseteq \mathbb{C}$. Then, for every Banach space E and every sequence of vectors $(v_j)_{j \in J} \subseteq E$ we have

$$\frac{1}{c_1 c_2} \int_S \left\| \sum_{j \in J} v_j a_j(s) \right\|_E ds \leq \int_T \left\| \sum_{j \in J} v_j b_j(t) \right\|_E dt \leq c_1 c_2 \int_S \left\| \sum_{j \in J} v_j a_j(s) \right\|_E ds. \tag{9}$$

We are ready to provide the lower bound for $C(m,n)$ stated in Theorem 1.3.

Lemma 4.3. *For $n, m \in \mathbb{N}$ such that $\log\left(\frac{2n}{m}\right) \geq \pi$, we have*

$$C(m,n) \geq \left(\frac{\log\left(\frac{2n}{m}\right) - \pi}{\pi}\right)^{m/2}.$$

Proof. Consider the norm $\|\cdot\|_\infty$ on $\mathbb{C}^n$. Since $P(x) = L_P(x, \ldots, x)$, we deduce that

$$\sup_{\|x\|_\infty \leq 1} |P(x)| \leq \sup_{\|x^{(k)}\|_\infty \leq 1} \left| L_P\left(x^{(1)}, \ldots, x^{(m)}\right)\right| \leq C(m,n) \sup_{\|x\|_\infty \leq 1} |P(x)|,$$

for every m-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$. Equivalently, by a careful use of the maximum modulus principle we get

$$\sup_{x \in \mathbb{T}^n} |P(x)| \leq \sup_{x^{(k)} \in \mathbb{T}^n} \left| L_P\left(x^{(1)}, \ldots, x^{(m)}\right)\right| \leq C(m,n) \sup_{x \in \mathbb{T}^n} |P(x)|, \tag{10}$$

where $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$.

Thus, the conditions of Pełczyński's theorem are satisfied. Indeed, denote the compact abelian groups $\mathbb{T}^n$ and $(\mathbb{T}^n)^m$ by S and T respectively and consider the index set $J = \{j \in \mathcal{I}(m,n) \ : \ 1 \leq j_1 \leq \ldots \leq j_m \leq n\}$. For every $j \in J$, define the characters $a_j : S \to \mathbb{T}$ and $b_j : T \to \mathbb{T}$ by

$$a_j(x) = x_{j_1} \ldots x_{j_m} \quad \text{and} \quad b_j\left(x^{(1)}, \ldots, x^{(m)}\right) = x^{(1)}_{j_1} \ldots x^{(m)}_{j_m}.$$

If we restate (10) with this notation we get (8), with $c_1 = 1$ and $c_2 = C(m,n)$. Therefore, we deduce from Pełczyński's theorem that

$$\begin{aligned} \frac{1}{C(m,n)} \int_{\mathbb{T}^n} \left\| \sum_{j \in J} v_j x_{j_1} \ldots x_{j_m} \right\|_E dx &\leq \int_{\mathbb{T}^n} \ldots \int_{\mathbb{T}^n} \left\| \sum_{j \in J} v_j x^{(1)}_{j_1} \ldots x^{(m)}_{j_m} \right\|_E dx^{(1)} \ldots dx^{(m)} \\ &\leq C(m,n) \int_{\mathbb{T}^n} \left\| \sum_{j \in J} v_j x_{j_1} \ldots x_{j_m} \right\|_E dx, \end{aligned} \tag{11}$$

for every Banach space E and every sequence of vectors $(v_j)_{j \in J} \subseteq E$. Choosing the space E and the vectors $(v_j)_{j \in J} \subseteq E$ adequately will yield the estimate we seek.

We will build upon an example provided by Bourgain (unpublished) and included in a paper by McConnell and Taqqu [7, Example 4.1] (see also [6, Section 6.9]). Consider the Banach space $F = \mathcal{L}(\ell_2)$. For every $1 \leq i \neq j \leq n$, define vectors $v_{ij} \in F$ by

$$v_{ij} = \frac{1}{i-j} e_i \otimes e_j + \frac{1}{j-i} e_j \otimes e_i.$$

Using complex Steinhaus variables instead of Bernoulli random variables and proceeding as in [6] we get

$$\int_{\mathbb{T}^n} \left\| \sum_{1 \leq i < j \leq n} v_{ij} x_i x_j \right\| dx \leq \pi \qquad \text{and} \tag{12}$$

$$\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \left\| \sum_{1 \leq i < j \leq n} v_{ij} x^{(1)}_i x^{(2)}_j \right\| dx^{(1)} dx^{(2)} \geq \log n - \pi. \tag{13}$$

Note that by the previous estimations we obtain the desired result for $m = 2$ since we have

$$\log n - \pi \leq \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \left\| \sum_{1\leq i<j\leq n} v_{ij} x_i^{(1)} x_j^{(2)} \right\| dx^{(1)} dx^{(2)}$$
$$\leq C(2,n) \int_{\mathbb{T}^n} \left\| \sum_{1\leq i<j\leq n} v_{ij} x_i x_j \right\| dx \leq C(2,n)\pi.$$

Moreover, this together with Theorem 3.3 shows that the asymptotic behaviour of $C(2, n)$ is logarithmic.

To conclude our argument it remains to extend this 2-variable example to m variables. Assume m is even and let $E = \bigotimes_{k=1}^{m/2} F$ be the projective tensor product of $m/2$ copies of F. Consider the m-homogeneous vector-valued polynomial $P : \mathbb{C}^n \to E$ defined by

$$P(x) = \sum_{\substack{\frac{2n}{m}(k-1)<j_{2k-1}<j_{2k}\leq \frac{2n}{m}k \\ 1\leq k\leq \frac{m}{2}}} v_j x_j, \quad \text{where } v_j = v_{j_1 j_2} \otimes v_{j_3 j_4} \otimes \ldots \otimes v_{j_{m-1} j_m}.$$

Notice that

$$P(x) = \otimes_{k=1}^{m/2} \sum_{\frac{2n}{m}(k-1)<j_{2k-1}<j_{2k}\leq \frac{2n}{m}k} v_{j_{2k-1}j_{2k}} x_{j_{2k-1}} x_{j_{2k}}.$$

Applying (12) we get

$$\int_{\mathbb{T}^n} \|P(x)\|\, dx = \int_{\mathbb{T}^n} \prod_{k=1}^{m/2} \left\| \sum_{\frac{2n}{m}(k-1)<j_{2k-1}<j_{2k}\leq \frac{2n}{m}k} v_{j_{2k-1}j_{2k}} x_{j_{2k-1}} x_{j_{2k}} \right\| dx$$
$$= \prod_{k=1}^{m/2} \int_{\mathbb{T}^n} \left\| \sum_{\frac{2n}{m}(k-1)<j_{2k-1}<j_{2k}\leq \frac{2n}{m}k} v_{j_{2k-1}j_{2k}} x_{j_{2k-1}} x_{j_{2k}} \right\| dx$$
$$\leq \prod_{k=1}^{m/2} \pi = \pi^{m/2}.$$

On the other hand, from (13) we deduce

$$\int_{\mathbb{T}^n} \cdots \int_{\mathbb{T}^n} \left\| L_P\left(x^{(1)}, \ldots, x^{(m)}\right) \right\| dx^{(1)} \ldots dx^{(m)} =$$
$$= \int_{\mathbb{T}^n} \cdots \int_{\mathbb{T}^n} \prod_{k=1}^{m/2} \left\| \sum_{\frac{2n}{m}(k-1)<j_{2k-1}<j_{2k}\leq \frac{2n}{m}k} v_{j_{2k-1}j_{2k}} x_{j_{2k-1}}^{(2k-1)} x_{j_{2k}}^{(2k)} \right\| dx^{(1)} \ldots dx^{(m)}$$
$$= \prod_{k=1}^{m/2} \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \left\| \sum_{\frac{2n}{m}(k-1)<j_{2k-1}<j_{2k}\leq \frac{2n}{m}k} v_{j_{2k-1}j_{2k}} x_{j_{2k-1}}^{(2k-1)} x_{j_{2k}}^{(2k)} \right\| dx^{(2k-1)} dx^{(2k)}$$
$$\geq \prod_{k=1}^{m/2} \left(\log\left(\frac{2n}{m}\right) - \pi\right) = \left(\log\left(\frac{2n}{m}\right) - \pi\right)^{m/2}.$$

Finally, using (11) together with these estimates we obtain

$$C(n,m) \geq \left(\frac{\log\left(\frac{2n}{m}\right) - \pi}{\pi} \right)^{m/2},$$

as desired. □

Note that Lemmas 4.1 and 4.3 together with Theorem 3.3 prove Theorem 1.3.

Remark 4.4. Tracing back the argument to obtain (13), we find that Bourgain's example is based on a lower estimate of the main triangle projection's norm on $\mathcal{L}(\ell_2)$. In other words, the lower bound for $C(m,n)$ was obtained by studying the behaviour of the main triangle projection as mentioned in Remark 1.5. Although $C(m,n)$ and $C_p(m,n)$ were not completely characterized, it seems that the main triangle projection plays a crucial role in determining their asymptotic behaviour.

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