# Semi-Intuitionistic Logic with Strong Negation

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**Abstract**

Motivated by the definition of semi-Nelson algebras, a propositional calculus called semi-intuitionistic logic with strong negation is introduced and proved to be complete with respect to that class of algebras. An axiomatic extension is proved to have as algebraic semantics the class of Nelson algebras.

## 1 Introduction

There is a well known interplay between the study of varieties of algebras and propositional calculus of various logics. Prime examples of this are boolean algebras and classical logic, and Heyting algebras and intuitionistic logic [Fit69]. After the class of Heyting algebras was generalized to the semi-Heyting algebras by H. Sankappanavar in [San85] and [San08], its logic counterpart was developed in [Cor11] and further studied in [CV15].

Nelson algebras, or N-lattices were defined by H. Rasiowa [Ras58] to provide an algebraic semantics to the constructive logic with strong negation proposed by Nelson in [Nel49]. D. Vakarelov in [Vak77] presented a construction of Nelson algebras starting from Heyting ones. Applying this construction to semi-Heyting algebras, we introduced in [CV16] the variety of semi-Nelson algebras as a natural generalization of Nelson algebras. In this variety, the lattice of congruences of an algebra is determined through some of its deductive systems. Furthermore, the class of semi-Nelson algebras is arithmetical, has equationally definable principal congruences and has the congruence extension property.

The purpose of this article is to present a Hilbert-style propositional calculus which is complete with respect to the variety of semi-Nelson algebras. Naming this logic *semi-intuitionistic logic with strong negation* was a natural choice. We believe that this logic will be of interest from the point of view of Many-Valued Logic, since its algebraic semantics show that it can provide many different interpretations for the implication connective. For example, on a chain with five elements, ten different semi-Nelson algebras may be defined, by changing the implication operation.

We present first the algebraic motivation, defining Nelson, semi-Nelson, Heyting and semi-Heyting algebras. We also show some properties of these algebras which are relevant for the proof of completeness. In the next section we introduce the axioms and inference rule for the semi-intuitionistic logic with strong negation, together with some of their consequences. The proofs of lemmas 3.1 and 3.3 have been omitted here and posted online [CV17] due to their length. The final section deals with completeness of the logic with respect to the class of semi-Nelson algebras, and presents an axiomatic extension that has the variety of Nelson algebras as algebraic semantics.

## 2 Nelson and semi-Nelson algebras

In this section, we present the algebraic background that motivated our definition of semi-intuitionistic logic with strong negation, proving along the way some useful results.

**Definition 2.1** *(see [MM96]) Nelson algebras are algebras* $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim, \top \rangle$ *that satisfy the conditions:*

(N1) $x \wedge (x \vee y) \approx x,$

(N2) $x \wedge (y \vee z) \approx (z \wedge x) \vee (y \wedge x),$

(N3) $\sim\sim x \approx x,$

(N4) $\sim (x \wedge y) \approx \sim x \vee \sim y,$

(N5) $x \wedge \sim x \approx (x \wedge \sim x) \wedge (y \vee \sim y),$

(N6) $x \to x \approx \top,$

(N7) $x \to (y \to z) \approx (x \wedge y) \to z,$

(N8) $x \wedge (x \to y) \approx x \wedge (\sim x \vee y).$

*We denote by* $\mathbf{N}$ *the variety of Nelson algebras.*

Let us recall that if $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim, \top \rangle \in \mathbf{N}$ then $\langle A; \wedge, \vee \rangle$ is a distributive lattice with bottom element $\sim \top$ and top element $\top$.

Semi-Heyting algebras are a generalization of Heyting algebras introduced by H. Sankappanavar in [San85], and they share with Heyting algebras the properties of being pseudo-complemented, distributive and having their congruences determined by filters. An algebra $\mathbf{A} = \langle A; \wedge, \vee, \to, \perp, \top \rangle$ is said to be a *semi-Heyting algebra* if $\langle A; \wedge, \vee, \perp, \top \rangle$ is a bounded lattice, and it satisfies the identities: $x \wedge (x \to y) \approx x \wedge y$, $x \wedge (y \to z) \approx x \wedge ((x \wedge y) \to (x \wedge z))$, and $x \to x \approx \top$. To define Heyting algebras, all we need to do is to replace the last identity by $(x \wedge y) \to y \approx \top$.

In [Vak77], D. Vakarelov presented a construction of Nelson algebras from Heyting algebras. This construction has proven fruitful for the study of Nelson algebras, and in [CV16] we applied it to semi-Heyting algebras, which motivated the definition of semi-Nelson algebras.

**Definition 2.2** *An algebra* $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim, \top \rangle$ *of type* $(2, 2, 2, 1, 0)$ *is a semi-Nelson algebra* [CV16] *if the following conditions are satisfied:*

(SN1) $x \wedge (x \vee y) \approx x,$

(SN2) $x \wedge (y \vee z) \approx (z \wedge x) \vee (y \wedge x),$

(SN3) $\sim\sim x \approx x,$

(SN4) $\sim (x \wedge y) \approx \sim x \vee \sim y,$

(SN5) $x \wedge \sim x \approx (x \wedge \sim x) \wedge (y \vee \sim y),$

(SN6) $x \wedge (x \to_N y) \approx x \wedge (\sim x \vee y),$

(SN7) $x \to_N (y \to_N z) \approx (x \wedge y) \to_N z$,

(SN8) $(x \to_N y) \to_N [(y \to_N x) \to_N [(x \to z) \to_N (y \to z)]] \approx \top$,

(SN9) $(x \to_N y) \to_N [(y \to_N x) \to_N [(z \to x) \to_N (z \to y)]] \approx \top$,

(SN10) $(\sim (x \to y)) \to_N (x \wedge \sim y) \approx \top$,

(SN11) $(x \wedge \sim y) \to_N (\sim (x \to y)) \approx \top$,

*where $x \to_N y$ stands for the term $x \to (x \wedge y)$.*

Identities (SN1) to (SN5) are the same as (N1) to (N5), and they define *Kleene algebras.* We denote with **SN** the variety of semi Nelson algebras.

The term $x \to_N y = x \to (x \wedge y)$ is quite important for the study of semi-Nelson algebras. There can be many semi-Nelson implication operations defined over the same underlying distributive lattice, but all of them will yield the same implication when the term defining $\to_N$ is calculated. Furthermore, the algebra with this implication turns out to be the only Nelson algebra definable over the lattice. In fact, we have the following theorem:

**Theorem 2.3** *[CV16, Theorem 2.8] The variety* **N** *of Nelson algebras is the subvariety of* **SN** *defined by the identity $x \to y \approx x \to_N y$.*

**Lemma 2.4** *Let $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim, \top \rangle$ be a semi-Nelson algebra and let $a, b, c, d \in A$. Then the following equations hold:*

(a) $(a \wedge \sim a) \to_N b = \top$ ,

(b) $a \leq b$ *if and only if* $a \to_N b = \sim b \to_N \sim a = \top$,

(c) $(a \vee b) \to_N c = (a \to_N c) \wedge (b \to_N c)$,

(d) $(a \to_N b) \to_N ((a \to_N c) \to_N (a \to_N (b \wedge c))) = \top$,

(e) $(a \to_N c) \to_N ((b \to_N c) \to_N ((a \vee b) \to_N c)) = \top$,

(f) $a \to_N b = a \to_N (a \wedge b)$,

(g) $(a \wedge \sim a \wedge b) \to_N c = \top$,

(h) $(b \wedge (\sim b \vee c) \wedge d) \to_N c = \top$,

(i) $(\sim c \to_N \sim a) \to_N [(\sim c \to_N \sim b) \to_N [\sim c \to_N \sim (a \vee b)]] = \top$,

(j) $(a \to_N b) \to_N ((b \to_N c) \to_N (a \to_N c)) = \top$.

**Proof** As a direct consequence of Theorem 2.3, the identities (N1)-(N8) are valid in semi-Nelson algebras when written for the implication $\to_N$. Similarly, the proofs for (a) through (e) that appear in [Vig99] can be adapted to this context. Part (f), was proved in [CV16, Lemma 2.7 (c)]. For the remaining equalities we have:

(g)

$$\begin{array}{rcll} (a \wedge \sim a \wedge b) \to_N c & = & (a \wedge \sim a) \to_N (b \to_N c) & \text{by (N7)}, \\ & = & \top & \text{by (a)}. \end{array}$$

(h)

$$\begin{array}{rcll}(b \wedge (\sim b \vee c) \wedge d) \rightarrow_N c & = & ((b \wedge d \wedge \sim b) \vee (b \wedge d \wedge c)) \rightarrow_N c & \\ & = & [(b \wedge d \wedge \sim b) \rightarrow_N c] \wedge [(b \wedge d \wedge c) \rightarrow_N c] & \text{by part (c),} \\ & = & \top \wedge [(b \wedge d \wedge c) \rightarrow_N c] & \text{by (g),} \\ & = & (b \wedge d \wedge c) \rightarrow_N c & \\ & = & \top & \text{by part (b).}\end{array}$$

(i) $(\sim c \rightarrow_N \sim a) \rightarrow_N [(\sim c \rightarrow_N \sim b) \rightarrow_N [\sim c \rightarrow_N \sim (a \vee b)]] =$

$$\begin{array}{cll} = & [(\sim c) \wedge (\sim c \rightarrow_N \sim a) \wedge (\sim c \rightarrow_N \sim b)] \rightarrow_N [\sim (a \vee b)] & \text{by (N7),} \\ = & [(\sim c) \wedge (\sim\sim c \vee \sim a) \wedge (\sim\sim c \vee \sim b)] \rightarrow_N [\sim (a \vee b)] & \text{by (N8),} \\ = & [(\sim c) \wedge (\sim\sim c \vee (\sim a \wedge \sim b)] \rightarrow_N [\sim (a \vee b)] & \\ = & [(\sim c) \wedge (\sim\sim c \vee (\sim (a \vee b))] \rightarrow_N [\sim (a \vee b)] & \\ = & \top & \text{by (h).}\end{array}$$

(j) $(a \rightarrow_N b) \rightarrow_N ((b \rightarrow_N c) \rightarrow_N (a \rightarrow_N c)) =$

$$\begin{array}{cll} = & [(a \rightarrow_N b) \wedge (b \rightarrow_N c)] \rightarrow_N (a \rightarrow_N c) & \text{by (N7),} \\ = & [(a \rightarrow_N b) \wedge (b \rightarrow_N c) \wedge a] \rightarrow_N c & \text{by (N7),} \\ = & [(\sim a \vee b) \wedge (b \rightarrow_N c) \wedge a] \rightarrow_N c & \text{by (N8),} \\ = & [((\sim a \wedge a) \vee (b \wedge a)) \wedge (b \rightarrow_N c)] \rightarrow_N c & \\ = & [[(\sim a \wedge a) \wedge (b \rightarrow_N c)] \vee [(b \wedge a) \wedge (b \rightarrow_N c)]] \rightarrow_N c & \\ = & [[(\sim a \wedge a) \wedge (b \rightarrow_N c)] \rightarrow_N c] \wedge [[(b \wedge a) \wedge (b \rightarrow_N c)] \rightarrow_N c] & \text{by (c),} \\ = & [(b \wedge a) \wedge (b \rightarrow_N c)] \rightarrow_N c & \text{by part (g),} \\ = & [b \wedge a \wedge (\sim b \vee c)] \rightarrow_N c & \text{by (N8),} \\ = & [(b \wedge a \wedge \sim b) \vee (b \wedge a \wedge c)] \rightarrow_N c & \\ = & [(b \wedge a \wedge \sim b) \rightarrow_N c] \wedge [(b \wedge a \wedge c)] \rightarrow_N c & \text{by part (c),} \\ = & (b \wedge a \wedge c) \rightarrow_N c & \text{by part (g),} \\ = & \top & \text{by (b).}\end{array}$$

□

# 3 Semi-intuitionistic logic with strong negation

This section is devoted to define a logic whose algebraic semantics is the variety of semi-Nelson algebras, and to prove the properties of the propositional calculus that allow us to prove the completeness result.

A *logical language* $\mathbf{L}$, as defined in [FJP03], is a set of connectives, each with a fixed arity $n \geq 0$. For a countably infinite set $Var$ of propositional variables, the *formulas* of the logical language $\mathbf{L}$ are inductively defined as usual.

A *logic*, in the language $\mathbf{L}$, is a pair $\mathcal{L} = \langle Fm_{\mathbf{L}}, \vdash_{\mathcal{L}} \rangle$ where $Fm_{\mathbf{L}}$ is the set of formulas and $\vdash_{\mathcal{L}}$ is a substitution-invariant consequence relation on $Fm_{\mathbf{L}}$. As usual, the set $Fm_{\mathbf{L}}$ may also be endowed with an algebraic structure, just by regarding the connectives of the language as operation symbols. The resulting algebra is the *algebra of formulas*, denoted by $\mathsf{Fm}_{\mathbf{L}}$. The finitary logic is presented by means of their "Hilbert style" sets of axioms and inferences rules.

We define *semi Intuitionistic logic with strong negation* $\mathcal{SN}$ over the language $\mathbf{L} = \{\top, \sim, \wedge, \vee, \rightarrow\}$ in terms of the following set of axiom schemata, in which we use the following definitions:

- $\alpha \to_N \beta := \alpha \to (\alpha \wedge \beta)$,
- $\alpha \Rightarrow \beta := (\alpha \to_N \beta) \wedge ({\sim} \beta \to_N {\sim} \alpha)$.

$(A1)$ $(\alpha \to_N \beta) \to_N ((\beta \to_N \gamma) \to_N (\alpha \to_N \gamma))$,

$(A2)$ $(\alpha \to_N \beta) \to_N ((\alpha \to_N \gamma) \to_N (\alpha \to_N (\beta \wedge \gamma)))$,

$(A3)$ $(\alpha \wedge \beta) \to_N \alpha$,

$(A4)$ $(\alpha \wedge \beta) \to_N \beta$,

$(A5)$ $\alpha \to_N (\alpha \vee \beta)$,

$(A6)$ $\beta \to_N (\alpha \vee \beta)$,

$(A7)$ ${\sim} (\alpha \vee \beta) \to_N {\sim} \alpha$,

$(A8)$ ${\sim} (\alpha \vee \beta) \to_N {\sim} \beta$,

$(A9)$ $(\alpha \to_N \gamma) \to_N ((\beta \to_N \gamma) \to_N ((\alpha \vee \beta) \to_N \gamma))$,

$(A10)$ $({\sim} \alpha \to_N {\sim} \beta) \to_N (({\sim} \alpha \to_N {\sim} \gamma) \to_N ({\sim} \alpha \to_N {\sim} (\beta \vee \gamma)))$,

$(A11)$ $\alpha \Rightarrow ({\sim\sim} \alpha)$,

$(A12)$ $({\sim\sim} \alpha) \Rightarrow \alpha$,

$(A13)$ $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\alpha \to \gamma) \to_N (\beta \to \gamma)]]$,

$(A14)$ $(\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\gamma \to \alpha) \to_N (\gamma \to \beta)]]$,

$(A15)$ $[(\alpha \wedge \beta) \to_N \gamma] \Rightarrow [\alpha \to_N (\beta \to_N \gamma)]$,

$(A16)$ $({\sim} (\alpha \wedge \beta)) \Rightarrow ({\sim} \alpha \vee {\sim} \beta)$,

$(A17)$ $({\sim} \alpha \vee {\sim} \beta) \Rightarrow ({\sim} (\alpha \wedge \beta))$,

$(A18)$ $(\alpha \wedge ({\sim} \alpha \vee \beta)) \Rightarrow (\alpha \wedge (\alpha \to_N \beta))$,

$(A19)$ $(\alpha \to_N (\beta \to_N \gamma)) \Rightarrow ((\alpha \wedge \beta) \to_N \gamma)$,

$(A20)$ $({\sim} (\alpha \to \beta)) \to_N (\alpha \wedge {\sim} \beta)$,

$(A21)$ $(\alpha \wedge {\sim} \beta) \to_N ({\sim} (\alpha \to \beta))$,

$(A22)$ $[{\sim} (\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha)))] \to_N [{\sim} (\alpha \wedge (\beta \vee \gamma))]$,

$(A23)$ $\top$.

The only inference rule is Modus Ponens for the implication $\to_N$, which we denominate $\mathcal{N}$*-Modus Ponens* ($\mathcal{N}$-MP): $\Gamma \vdash_{\mathcal{SN}} \phi$ and $\Gamma \vdash_{\mathcal{SN}} \phi \to_N \gamma$ yield $\Gamma \vdash_{\mathcal{SN}} \gamma$.

**Lemma 3.1** *Let* $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. *In* $\mathcal{SN}$ *the following properties hold:*

(a) *If* $\Gamma \vdash \alpha$ *then* $\Gamma \vdash \beta \to_N \alpha$,

(b) $\Gamma \vdash \alpha \to_N \alpha$,

(c) *If* $\Gamma \vdash \alpha \Rightarrow \beta$ *then* $\Gamma \vdash \alpha \to_N \beta$ *and* $\Gamma \vdash \sim \beta \to_N \sim \alpha$,

(d) $\Gamma \vdash \sim \alpha \to_N \sim (\alpha \wedge \beta)$,

(e) $\Gamma \vdash \sim \beta \to_N \sim (\alpha \wedge \beta)$,

(f) $\Gamma, \alpha, \alpha \Rightarrow \beta \vdash \beta$,

(g) *If* $\Gamma \vdash \alpha \Rightarrow \beta$ *and* $\Gamma \vdash \alpha$ *then* $\Gamma \vdash \beta$,

(h) *If* $\Gamma \vdash \alpha$ *and* $\Gamma \vdash \beta$ *then* $\Gamma \vdash \alpha \wedge \beta$,

(i) $\Gamma \vdash \alpha \wedge \beta \Rightarrow \alpha$ *and* $\Gamma \vdash \alpha \wedge \beta \Rightarrow \beta$,

(j) $\Gamma \vdash \alpha \Rightarrow \alpha \vee \beta$ *and* $\Gamma \vdash \alpha \Rightarrow \beta \vee \alpha$,

(k) $\Gamma \vdash \alpha \Rightarrow \alpha$,

(l) *If* $\Gamma \vdash \alpha \Rightarrow \beta$ *and* $\Gamma \vdash \beta \Rightarrow \gamma$ *then* $\Gamma \vdash \alpha \Rightarrow \gamma$,

(m) $\Gamma, \alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vdash \alpha \Rightarrow \gamma$,

(n) $\Gamma \vdash \alpha \to_N \beta$ *then* $\Gamma \vdash (\gamma \wedge \alpha) \to_N (\gamma \wedge \beta)$ *and* $\Gamma \vdash (\alpha \wedge \gamma) \to_N (\beta \wedge \gamma)$,

(o) $\Gamma \vdash \alpha \to_N \beta$ *then* $\Gamma \vdash (\gamma \vee \alpha) \to_N (\gamma \vee \beta)$ *and* $\Gamma \vdash (\alpha \vee \gamma) \to_N (\beta \vee \gamma)$,

(p) $\Gamma \vdash (\alpha \vee \beta) \to_N (\beta \vee \alpha)$,

(q) $\Gamma \vdash (\alpha \wedge \beta) \to_N (\beta \wedge \alpha)$,

(r) $\Gamma, \alpha \Rightarrow \beta \vdash (\alpha \vee \gamma) \Rightarrow (\beta \vee \gamma)$,

(s) $\Gamma, \alpha \Rightarrow \beta \vdash (\gamma \vee \alpha) \Rightarrow (\gamma \vee \beta)$,

(t) $\Gamma, \alpha \Rightarrow \beta, \gamma \Rightarrow t \vdash (\alpha \vee \gamma) \Rightarrow (\beta \vee t)$,

(u) $\Gamma, \beta \Rightarrow \alpha \vdash (\sim \alpha) \Rightarrow (\sim \beta)$,

(v) $\Gamma \vdash (\sim (\alpha \to \beta)) \to_N (\sim (\alpha \to_N \beta))$.

**Theorem 3.2** *(Deduction Theorem) Let* $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. *Then*

$$\Gamma \vdash \alpha \to_N \beta \text{ if and only if } \Gamma, \alpha \vdash \beta$$

**Proof** For one implication we have:

1. $\Gamma \vdash \alpha \to_N \beta$ by hypothesis.
2. $\Gamma, \alpha \vdash \alpha \to_N \beta$.
3. $\Gamma, \alpha \vdash \alpha$.
4. $\Gamma, \alpha \vdash \beta$ by ($\mathcal{N}$-MP) applied to 2 and 3.

For the other one, assume that $\Gamma, \alpha \vdash \beta$. We prove the result by induction on the lenght of the proof of $\Gamma, \alpha \vdash \beta$.

- If $\vdash \beta$ or $\beta \in \Gamma$ then $\Gamma \vdash \beta$ . By Lemma 3.1 (a) we have that $\vdash \alpha \to_N \beta$. Consequently, $\Gamma \vdash \alpha \to_N \beta$.
- If $\beta = \alpha$, using Lemma 3.1 (b), $\Gamma \vdash \alpha \to_N \beta$.
- If $\beta$ comes from applying the inference rule then there exist $\gamma \in Fm_{\mathbf{L}}$ such that $\Gamma, \alpha \vdash \gamma$ and $\Gamma, \alpha \vdash \gamma \to_N \beta$. Then

  1. $\Gamma \vdash \alpha \to_N (\gamma \to_N \beta)$ by inductive hypothesis.
  2. $\Gamma \vdash \alpha \to_N \gamma$ by inductive hypothesis.
  3. $\Gamma \vdash [\alpha \to_N (\gamma \to_N \beta)] \Rightarrow [(\alpha \wedge \gamma) \to_N \beta]$ by axiom $(A19)$.
  4. $\Gamma \vdash [[\alpha \to_N (\gamma \to_N \beta)] \Rightarrow [(\alpha \wedge \gamma) \to_N \beta]] \to_N [[\alpha \to_N (\gamma \to_N \beta)] \to_N [(\alpha \wedge \gamma) \to_N \beta]]$ by axiom $(A3)$.
  5. $\Gamma \vdash [\alpha \to_N (\gamma \to_N \beta)] \to_N [(\alpha \wedge \gamma) \to_N \beta]$ by ($\mathcal{N}$-MP) applied to 3 and 4.
  6. $\Gamma \vdash (\alpha \wedge \gamma) \to_N \beta$ by ($\mathcal{N}$-MP) applied to 1 and 5.
  7. $\Gamma \vdash \alpha \to_N \alpha$ by Lemma 3.1 (b).
  8. $\Gamma \vdash (\alpha \to_N \alpha) \to_N [(\alpha \to_N \gamma) \to_N (\alpha \to_N (\alpha \wedge \gamma))]$ by axiom $(A2)$.
  9. $\Gamma \vdash (\alpha \to_N \gamma) \to_N (\alpha \to_N (\alpha \wedge \gamma))$ by ($\mathcal{N}$-MP) applied to 7 and 8.
  10. $\Gamma \vdash \alpha \to_N (\alpha \wedge \gamma)$ by ($\mathcal{N}$-MP) applied to 2 and 9.
  11. $\Gamma \vdash (\alpha \to_N (\alpha \wedge \gamma)) \to_N [[(\alpha \wedge \gamma) \to_N \beta] \to_N [\alpha \to_N \beta]]$ by axiom $(A1)$.
  12. $\Gamma \vdash [(\alpha \wedge \gamma) \to_N \beta] \to_N [\alpha \to_N \beta]$ by ($\mathcal{N}$-MP) applied to 10 and 11.
  13. $\Gamma \vdash \alpha \to_N \beta$ by ($\mathcal{N}$-MP) applied to 6 and 12.

$\square$

We use $\alpha \leftrightarrow_N \beta$ as an abbreviation for the formula $(\alpha \to_N \beta) \wedge (\beta \to_N \alpha)$.

**Lemma 3.3** *Let $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. In $\mathcal{SN}$ the following properties hold:*

(a) $\Gamma \vdash (\sim \alpha \wedge \sim \beta) \leftrightarrow_N \sim (\alpha \vee \beta)$,

(b) $\Gamma \vdash (\alpha \wedge (\alpha \to_N \beta)) \Rightarrow (\alpha \wedge (\sim \alpha \vee \beta))$,

(c) *If* $\Gamma \vdash \alpha \leftrightarrow_N \beta$ *then* $\Gamma \vdash \beta \leftrightarrow_N \alpha$,

(d) $\Gamma \vdash \alpha \Rightarrow (\alpha \wedge (\alpha \vee \beta))$,

(e) $\Gamma \vdash [\alpha \wedge [(\gamma \wedge \alpha) \vee (\beta \wedge \alpha)]] \Rightarrow [\alpha \wedge (\beta \vee \gamma)]$,

(f) $\Gamma \vdash (\alpha \wedge (\beta \vee \gamma)) \Rightarrow [\alpha \wedge ((\gamma \wedge \alpha) \vee (\beta \wedge \alpha))]$,

(g) $\Gamma \vdash (\sim \beta \to_N \sim \alpha) \to_N ((\sim \gamma \to_N \sim \alpha) \to_N (\sim (\beta \wedge \gamma) \to_N \sim \alpha))$,

(h) *If* $\Gamma \vdash \alpha \Rightarrow \beta$ *and* $\Gamma \vdash \alpha \Rightarrow \gamma$ *then* $\Gamma \vdash \alpha \Rightarrow \beta \wedge \gamma$ *and* $\Gamma \vdash \alpha \Rightarrow \gamma \wedge \beta$,

(i) *If* $\Gamma \vdash \alpha \Rightarrow \beta$ *then* $\Gamma \vdash \alpha \Rightarrow (\beta \vee \gamma)$ *and* $\Gamma \vdash \alpha \Rightarrow (\gamma \vee \beta)$,

(j) *If* $\Gamma \vdash \alpha \Rightarrow \beta$ *then* $\Gamma \vdash \alpha \wedge \gamma \Rightarrow \beta$ *and* $\Gamma \vdash \gamma \wedge \alpha \Rightarrow \beta$,

(k) $\Gamma, \alpha \Rightarrow \beta \vdash (\alpha \wedge \gamma) \Rightarrow (\beta \wedge \gamma)$,

(l) $\Gamma, \alpha \Rightarrow \beta \vdash (\gamma \wedge \alpha) \Rightarrow (\gamma \wedge \beta)$,

(m) $\Gamma, \alpha \Rightarrow \beta, \gamma \Rightarrow t \vdash (\alpha \wedge \gamma) \Rightarrow (\beta \wedge t)$,

(n) $\Gamma \vdash [\sim (\alpha \to_N \beta)] \to_N (\alpha \wedge \sim \beta)$,

(o) $\Gamma \vdash (\alpha \wedge \sim \alpha) \to_N \beta$,

(p) $\Gamma \vdash (\alpha \wedge \sim \alpha) \Rightarrow (\beta \vee \sim \beta)$,

(q) $\Gamma \vdash (\alpha \to_N \beta) \to_N [(\beta \to_N \alpha) \to_N [(\sim (\beta \to \gamma)) \to_N (\sim (\alpha \to \gamma))]]$,

(r) $\Gamma \vdash (\sim \alpha \to_N \sim \beta) \to_N [(\sim \beta \to_N \sim \alpha) \to_N [(\sim (\gamma \to \alpha)) \to_N (\sim (\gamma \to \beta))]]$,

(s) $\Gamma \vdash \alpha \Rightarrow \beta, \beta \Rightarrow \alpha, \gamma \Rightarrow t, t \Rightarrow \gamma \vdash (\alpha \to \gamma) \Rightarrow (\beta \to t)$,

(t) $\Gamma, \alpha \vdash \beta \Rightarrow \alpha$,

(u) $\Gamma \vdash (\alpha \to \beta) \to_N (\alpha \to_N \beta)$.

# 4 Completeness

In this section we prove the completeness of the semi-intuitionistic logic with strong negation with respect to semi-Nelson algebras. Then we show an axiomatic extension equivalent to the intuitionistic logic with strong negation $\mathcal{N}$ associated to Nelson algebras. We prove the completeness result by showing that $\mathcal{SN}$ is an implicative logic and then proving that the corresponding class of algebras $\mathsf{Alg}^*\mathcal{SN}$ coincides with $\mathbf{SN}$.

**Definition 4.1** *[Ras74] Let $\mathcal{L}$ be a logic in a language with a binary connective $\to$, either primitive or defined by a term in exactly two variables. Then $\mathcal{L}$ is called an implicative logic with respect to the binary connective $\to$ if the following conditions are satisfied:*

(IL1) $\vdash_{\mathcal{L}} \alpha \to \alpha$.

(IL2) $\alpha \to \beta, \beta \to \gamma \vdash_{\mathcal{L}} \alpha \to \gamma$.

(IL3) *For each connective $f$ in the language of arity $n > 0$,*
$$\left\{ \begin{array}{c} \alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n \\ \beta_1 \to \alpha_1, \ldots, \beta_n \to \alpha_n \end{array} \right\} \vdash_{\mathcal{L}} f(\alpha_1, \ldots, \alpha_n) \to f(\beta_1, \ldots, \beta_n).$$

(IL4) $\alpha, \alpha \to \beta \vdash_{\mathcal{L}} \beta$.

(IL5) $\alpha \vdash_{\mathcal{L}} \beta \to \alpha$.

**Theorem 4.2** *$\mathcal{SN}$ is implicative with respect to the connective $\Rightarrow$.*

**Proof** Condition (IL1) follows from Lemma 3.1 (k). By Lemma 3.1 (m), (IL2) holds. To see that condition (IL3) is satisfied, we use Lemma 3.3 (m) for the connective $\wedge$, Lemma 3.1 (t) for $\vee$, Lemma 3.3 (s) for the implication $\to$, and Lemma 3.1 (u) for $\sim$. Condition (IL4) holds by Lemma (3.1) (f). Finally (IL5) obtains from Lemma 3.3 (t). $\square$

**Definition 4.3** *[Ras74, Definition 6, page 181] Let $\mathcal{L}$ be an implicative logic on the language $\mathbf{L}$. An $\mathcal{L}$-algebra is an algebra $\mathbf{A}$ of similarity type $\mathbf{L}$ that has an element $\top$ with the following properties:*

(LALG1) *For all* $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}$ *and all* $h \in Hom(Fm_{\mathbf{L}}, \mathbf{A})$, *if* $\Gamma \vdash_{\mathcal{L}} \phi$ *and* $h\Gamma \subseteq \{\top\}$ *then* $h\phi = \top$, *where* $h\Gamma = \{h\gamma : \gamma \in \Gamma\}$.

(LALG2) *For all* $a, b \in A$, *if* $a \to b = \top$ *and* $b \to a = \top$ *then* $a = b$.

*The class of* $\mathcal{L}$*-algebras is denoted by* $\mathsf{Alg}^*\mathcal{L}$.

Since $\mathcal{SN}$ is an implicative logic with respect to the binary connective $\Rightarrow$, we have the next result using [Ras74, Theorem 7.1, pag 222].

**Theorem 4.4** *The logic* $\mathcal{SN}$ *is complete with respect to the class* $\mathsf{Alg}^*\mathcal{SN}$. *In other words, for all* $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}$,

$$\Gamma \vdash_{\mathcal{SN}} \phi \text{ if and only if } h\Gamma \subseteq \{\top\} \text{ implies } h\phi = \top,$$

*for all* $h \in Hom(Fm_{\mathbf{L}}, \mathbf{A})$ *and all* $\mathbf{A} \in \mathsf{Alg}^*\mathcal{SN}$.

**Theorem 4.5** $\mathsf{Alg}^*\mathcal{SN} \subseteq \mathbf{SN}$.

**Proof** Let $\mathbf{A} \in \mathsf{Alg}^*\mathcal{SN}$. By Lemma 3.1 (i), we have that $\Gamma \vdash (\alpha \wedge (\alpha \vee \beta)) \Rightarrow \alpha$ whenever $\Gamma \subseteq Fm_{\mathbf{L}}$. So, from Lemma 3.3 parts (d), (e) and (f), $\mathbf{A}$ satisfies the identities (SN1) and (SN2).

By the axioms $(A11)$ - $(A12)$ and $(A16)$ - $(A17)$, the identities (SN3) and (SN4) hold as well.

To prove that the identity (SN5) is valid in $\mathbf{A}$ we use the following deduction:

1. $\Gamma \vdash (\alpha \wedge \sim \alpha) \Rightarrow (\alpha \wedge \sim \alpha)$ by Lemma 3.1 (k).
2. $\Gamma \vdash (\alpha \wedge \sim \alpha) \Rightarrow (\beta \vee \sim \beta)$ by Lemma 3.3 (p).
3. $\Gamma \vdash (\alpha \wedge \sim \alpha) \Rightarrow [(\alpha \wedge \sim \alpha) \wedge (\beta \vee \sim \beta)]$ by Lemma 3.3 (h).
4. $\Gamma \vdash [(\alpha \wedge \sim \alpha) \wedge (\beta \vee \sim \beta)] \Rightarrow (\alpha \wedge \sim \alpha)$ by Lemma 3.1 (i).

Then applying (LALG2) to 3 and 4 above, the identity follows.

From 3.3 (b) and axioms $(A18)$, $(A15)$ and $(A19)$, respectively, identities (SN6) and (SN7) hold.

Finally, the identities (SN8), (SN9), (SN10) and (SN11) hold as a consequence of axioms $(A13)$, $(A14)$, $(A20)$ and $(A21)$. $\square$

**Theorem 4.6** *Let* $\mathbf{A}$ *be a semi-Nelson algebra. Then* $\mathbf{A}$ *satisfies the conditions (LALG1) and (LALG2) with respect to the implication* $\Rightarrow$.

**Proof** Let $a, b \in A$ be such that $a \Rightarrow b = b \Rightarrow a = \top$. By Lemma 2.4 (b), $a = b$, so $\mathbf{A}$ satisfies (LALG2).

Let $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}$ and $h \in Hom(Fm_{\mathbf{L}}, \mathbf{A})$ be such that $\Gamma \vdash_{\mathcal{L}} \phi$ and $h\Gamma \subseteq \{\top\}$. We shall prove that $h(\phi) = \top$ by induction on the proof of $\Gamma \vdash_{\mathcal{L}} \phi$. We consider the following cases:

- If $\phi \in \Gamma$, then $h(\phi) = \top$ by hypothesis.
- If there exists $\alpha \in Fm_{\mathbf{L}}$ such that $\Gamma \vdash_{\mathcal{L}} \alpha$ and $\Gamma \vdash_{\mathcal{L}} \alpha \to_N \phi$ then, by inductive hypothesis, $h(\alpha) = h(\alpha \to_N \phi) = \top$. Therefore, $\top = h(\alpha) \wedge h(\alpha \to_N \phi) = h(\alpha \wedge (\alpha \to_N \phi)) = h(\alpha \wedge (\sim \alpha \vee \phi)) = h(\alpha) \wedge [(\sim h(\alpha)) \vee h(\phi)] = \top \wedge [(\sim \top) \vee h(\phi)] = h(\phi)$, using (SN6).

- Assume that $\phi$ is an axiom of $\mathcal{SN}$.

  If $\phi$ is $(A1)$ or $(A2)$, then by Lemma 2.4 (j) or Lemma 2.4 (d), respectively, the result holds.

  If $\phi$ is one of the axioms $(A3)$ to $(A8)$ then clearly $h(\phi) = \top$ by Lemma 2.4 (b).

  Lemmas 2.4 (e) and 2.4 (i) prove the result for axioms $(A9)$ and $(A10)$.

  In any of the other cases, it is enough to consider the corresponding identity from the definition of the class **SN** and Lemma 2.4 (b).

□

Now we can conclude the main result:

**Theorem 4.7** *The logic $\mathcal{SN}$ is complete with respect to the class* **SN**.

**Proof** From theorems 4.2 and 4.4 it follows that $\mathsf{Alg}^*\mathcal{SN} = \mathbf{SN}$. Therefore, by Theorem 4.4, $\mathcal{SN}$ is complete with respect to the class of semi-Nelson algebras. □

Since adding the identity $x \to y = x \to_N y$ to the definition of semi-Nelson algebras yields a characterization of Nelson algebras, we can carry this result to the logic $\mathcal{SN}$.

**Theorem 4.8** *The logic $\mathcal{N}$, which is $\mathcal{SN}$ together with the axioms:*

$(A24)$ $(\alpha \to_N \beta) \to_N (\alpha \to \beta)$,

$(A25)$ $\sim (\alpha \to_N \beta) \to_N \sim (\alpha \to \beta)$,

*has the variety of Nelson algebras as its algebraic semantis.*

**Proof** By Theorem 4.7, it will be enough to show that both $(\alpha \to \beta) \Rightarrow (\alpha \to_N \beta)$ and $(\alpha \to_N \beta) \Rightarrow (\alpha \to \beta)$ are theorems of $\mathcal{N}$. The first one follows from Lemma 3.3 (u) and $(A25)$, while the second one comes from $(A24)$ and Lemma 3.1 (v).

Thus $\mathsf{Alg}^*\mathcal{N}$ is a subvariety of $\mathsf{Alg}^*\mathcal{SN} = \mathbf{SN}$ in which $x \to y \approx x \to_N y$ holds. Therefore, by Theorem 2.3, $\mathsf{Alg}^*\mathcal{N} = \mathbf{N}$. □

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