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Localization of tetravalent modal algebras

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The main aim of this paper is to define the localization of a tetravalent modal algebra A with respect to a topology $\mathcal{F}$ on A. In Sec. 5, we prove that the tetravalent modal algebra of fractions relative to a $\wedge$-closed system (defined in Definition 3.1) is a tetravalent modal algebra of localization.

Keywords: Tetravalent modal algebra; tetravalent modal algebra of fractions; $\wedge$-closed system.

AMS Subject Classification: 06D30, 03G25

1. Introduction

A remarkable construction in ring theory is the *localization ring* $A_{\mathcal{F}}$ associated with a Gabriel topology $\mathcal{F}$ on a ring A (see [18, 19]). In Lambek's book [11], it introduces the notion of complete ring of quotients of a commutative ring, as a particular case of localization ring (relative to the topology of dense ideals).

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Starting from the example of the rings, Schmid introduced in [20, 21] the notion of maximal lattice of quotients for a distributive lattice. The central role in this construction is played by the concept of multipliers defined by Cornish in [5].

Using the model of localization ring, in [10], Georgescu defined the localization lattice $A_{\mathcal{F}}$ for a bounded distributive lattice A with respect to a topology $\mathcal{F}$ on A and prove that the maximal lattice of quotients for a distributive lattice is a lattice of localization (relative to the topology of regular ideals). Analogous results we have for lattices of fractions of bounded distributive lattices relative to $\wedge$-closed systems.

In 1978, Monteiro introduced tetravalent modal algebras as a very interesting generalization of three-valued Lukasiewicz–Moisil algebras. These algebras do really offer a genuine interest, both from the point of view of algebra and from that of logic, and specially from the one of Algebraic Logic (see [8]). An algebraic study of tetravalent modal algebras can be found in [12–15] and [4, 6, 7].

The main aim of this paper is to develop a theory of localization for tetravalent modal algebras. Since three-valued Łukasiewicz–Moisil algebras is a particular case of tetravalent modal algebra (see [1]), the results of this paper generalize a part of the results from [2, 3, 9] (for LM_3-algebras).

2. Preliminaries

In 1978, Monteiro introduced the tetravalent modal algebras (or TM-algebras) as algebras $\langle A, \vee, \wedge, \sim, \nabla, 1\rangle$ of type $(2, 2, 1, 1, 0)$ which verify:

(M1) $x \wedge (x \vee y) = x$,
(M2) $x \wedge (y \vee z) = (z \wedge x) \vee (z \wedge y)$,
(M3) $\sim\sim x = x$,
(M4) $\sim (x \vee y) =\sim x \wedge \sim y$,
(M5) $\nabla x \vee \sim x = 1$,
(M6) $\nabla x \wedge \sim x =\sim x \wedge x$.

We denote by **TM** the category of TM-algebras.

It is easy to see that every TM-algebra satisfies:

(M7) $1 \vee x = 1$.

From M1, M2, M7, M3, M4 it follows that $\langle A, \wedge, \vee, \sim, 1, 0\rangle$ is a De Morgan algebra with greatest element 1 and least element $0 =\sim 1$. Taking into account [16, 17], we have that three-valued Łukasiewicz–Moisil algebras (or LM_3-algebras) are TM-algebras which, moreover, satisfy:

(M6′) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

The results announced here for TM-algebras will be used throughout the paper

(M8) $x \leq \nabla x$,
(M9) $\nabla 0 = 0$,
(M10) $\nabla 1 = 1$,

(M11) $\nabla\nabla x = \nabla x$
(M12) $\nabla(x \vee y) = \nabla x \vee \nabla y$,
(M13) $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$,
(M14) $x \in \nabla(A)$ if and only if $\nabla x = x$,
(M15) ∇x and $\sim \nabla x$ are Boolean complements,
(M16) $\nabla \sim \nabla x = \sim \nabla x$.

From (M8), (M9), (M13) and (M16), we have that ∇ is an existential quantifier in the sense of Halmos.

3. TM-Algebra of Fractions Relative to an $\wedge$-Closed System

Definition 3.1. A nonempty subset S of a TM-algebra A is called $\wedge$-closed system in A if:

(S1) $1 \in S$,
(S2) $x, y \in S$ implies $x \wedge y \in S$.

We denote by $S(A)$ the set of all $\wedge$-closed systems of A.

Lemma 3.1. *Let S be a $\wedge$-closed system of a* TM-*algebra A. Then, the relation θ_S defined by $(x, y) \in \theta_S$ if and only if there is $s \in S \cap \nabla(A)$ such that $x \wedge s = y \wedge s$ is a congruence on A.*

Proof. We need only to prove that θ_S is compatible with $\sim$ and ∇. Let $(x, y) \in \theta_S$. Then there is $s \in S \cap \nabla(A)$ such that (1) $x \wedge s = y \wedge s$. Thus, (2) $\nabla s = s$ by (M14) and $\sim x \vee \sim s = \sim y \vee \sim s$. From this assertion and (M15), we get that $\sim x \wedge \nabla s = \sim y \wedge \nabla s$. Hence, by (2), we obtain that $(\sim x, \sim y) \in \theta_S$. On the other hand, from (1), (2) and (M13), we have that (3) $\nabla x \wedge \nabla s = \nabla y \wedge \nabla s$. Besides, from (2), we deduce that $\nabla s \in S \cap \nabla(A)$. Therefore, from (3), we conclude that $(\nabla x, \nabla y) \in \theta_S$. □

Let $A \in$ **TM**. For $x \in A$, we denote by $[x]_S$ the equivalence class of x relative to θ_S and by $A[S] = A/\theta_S$.

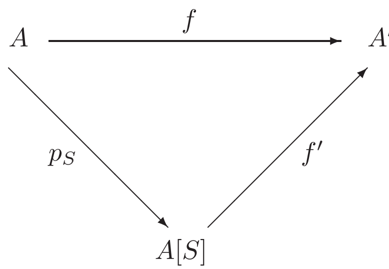
By $p_S : A \to A[S]$, we denote the canonical map defined by $p_S(x) = [x]_S$, for every $x \in A$.

Remark 3.1. Since for every $s \in S \cap \nabla(A)$, $s \wedge s = s \wedge 1$, we deduce that $[s]_S = [1]_S$, hence $p_S(S \cap \nabla(A)) = \{[1]_S\}$.

Proposition 3.1. *If $a \in A$, then $[a]_S \in \nabla(A[S])$ if and only if there exists $s \in S \cap \nabla(A)$ such that $a \wedge s \in \nabla(A)$. So, if $a \in \nabla(A)$, then $[a]_S \in \nabla(A[S])$.*

Proof. For $a \in A$, we have $[a]_S \in \nabla(A[S])$ if and only if $\nabla[a]_S = [a]_S$, that is, $[\nabla a]_S = [a]_S$. So, $(\nabla a, a) \in \theta_S$, which it means that there exists $s \in S \cap \nabla(A)$ such that $\nabla a \wedge s = a \wedge s$, that is, $\nabla(a \wedge s) = \nabla(\nabla a \wedge s) = \nabla a \wedge \nabla s = \nabla a \wedge s = a \wedge s$, hence $a \wedge s \in \nabla(A)$. If $a \in \nabla(A)$, since $1 \in S \cap \nabla(A)$ and $a \wedge 1 = a \in \nabla(A)$, we deduce that $[a]_S \in \nabla(A[S])$. □

Theorem 3.1. *If A is a* TM-*algebra and $f : A \to A'$ is a morphism of* TM-*algebras such that $f(S \cap \nabla(A)) = \{1\}$, then there is an unique morphism of* TM-*algebras $f' : A[S] \to A'$ such that the diagram*



commutes (i.e. $f' \circ p_S = f$).

Remark 3.2. The previous theorem allows us to call $A[S]$ the TM-algebra of fractions relative to the $\wedge$-closed system S.

Example 3.1.

(1) If $S = \{1\}$ or is such that $1 \in S$ and $S \cap (\nabla(A) \setminus \{1\}) = \emptyset$, then for $x, y \in A$, $(x, y) \in \theta_S \Leftrightarrow 1 \wedge x = 1 \wedge y \Leftrightarrow x = y$, hence in this case $A[S] = A$.
(2) If S is an $\wedge$-closed system such that $0 \in S$ (for example $S = A$ or $S = \nabla(A)$), then for every $x, y \in A$, $(x, y) \in S$ (since $x \wedge 0 = y \wedge 0$ and $0 \in S \cap \nabla(A)$), hence in this case $A[S] = \{[0]_S\}$.

4. Topologies on TM-Algebras

Definition 4.1. An ideal of a TM-algebra A is a subset I of A satisfying the following conditions:

(I1) $0 \in I$,
(I2) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$.
(I3) If $x, y \in I$, then $x \vee y \in I$.

We shall denote by $\mathcal{I}(A)$ the lattice of all ideals of A.

Definition 4.2. A nonempty set $\mathcal{F}$ of ideals of A will be called a topology on A if the following properties hold:

(T1) If $I_1 \in \mathcal{F}$, $I_2 \in \mathcal{I}(A)$ and $I_1 \subseteq I_2$, then $I_2 \in \mathcal{F}$ (hence $A \in \mathcal{F}$),
(T2) If $I_1, I_2 \in \mathcal{F}$, then $I_1 \cap I_2 \in \mathcal{F}$.

Clearly, if $\mathcal{F}$ is a topology on A, then $(A, \mathcal{F} \cup \{\emptyset\})$ is a topological space. Any intersection of topologies on A is a topology, hence the set $T(A)$ of all topologies of A is a complete lattice with respect to inclusion. $\mathcal{F}$ is a topology on A if and only if $\mathcal{F}$ is a filter of the lattice of power set of A, for this reason, a topology on A is usually called a Gabriel filter on $\mathcal{I}(A)$.

Example 4.1. $\mathcal{F}_S = \{I \in \mathcal{I}(A) : I \cap S \cap \nabla(A) \neq \emptyset\}$ is a topology on A, for every $S \in S(A)$.

Definition 4.3. The topology $\mathcal{F}_S$ is called the topology associated with the $\wedge$-closed system S.

5. $\mathcal{F}$-Multipliers and Localization of TM-Algebra

Let $\mathcal{F}$ be a topology on A. We consider the relation $\theta_{\mathcal{F}}$ of A

$(x, y) \in \theta_{\mathcal{F}}$ if and only if there exists $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I \cap \nabla(A)$.

Lemma 5.1. *$\theta_{\mathcal{F}}$ is a congruence on A.*

Proof. We need only to prove that $\theta_{\mathcal{F}}$ is compatible with $\sim$ and ∇. Let $(x, y) \in \theta_{\mathcal{F}}$. Then there is $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I \cap \nabla(A)$. Let $e \in I \cap \nabla(A)$, then $e \wedge x = e \wedge y$. From this last assertion and (M15), we deduce that $\sim x \wedge e = (\sim x \wedge \nabla e) \vee (\sim \nabla e \wedge \nabla e) = (\sim x \vee \sim \nabla e) \wedge \nabla e = (\sim y \vee \sim \nabla e) \wedge \nabla e = (\sim y \wedge \nabla e) \vee (\sim \nabla e \wedge \nabla e) = \sim y \wedge \nabla e$. Therefore, $(\sim x, \sim y) \in \theta_{\mathcal{F}}$. On the other hand, from (M13), we have that $\nabla x \wedge e = \nabla x \wedge \nabla e = \nabla(x \wedge \nabla e) = \nabla(x \wedge e) = \nabla(y \wedge e) = \nabla(y \wedge \nabla e) = \nabla y \wedge \nabla e$. Therefore, $(\nabla x, \nabla y) \in \theta_{\mathcal{F}}$. □

We shall denote by $[x]_{\theta_{\mathcal{F}}}$ the congruence class of an element $x \in A$, by $A/\theta_{\mathcal{F}}$ the quotient TM-algebra and by $p_{\mathcal{F}} : A \longrightarrow A/\theta_{\mathcal{F}}$ the canonical morphism of TM-algebras.

Lemma 5.2. *For $a \in A$, $[a]_{\theta_{\mathcal{F}}} \in \nabla(A/\theta_{\mathcal{F}})$ if and only if there exists $I \in \mathcal{F}$ such that $e \wedge \nabla a = e \wedge a$ for every $e \in I \cap \nabla(A)$. So, if $a \in \nabla(A)$, then $[a]_{\theta_{\mathcal{F}}} \in \nabla(A/\theta_{\mathcal{F}})$.*

Proof. For $a \in A$, $[a]_{\theta_{\mathcal{F}}} \in \nabla(A/\theta_{\mathcal{F}})$ if and only if $\nabla[a]_{\theta_{\mathcal{F}}} = [a]_{\theta_{\mathcal{F}}}$ if and only if $[\nabla a]_{\theta_{\mathcal{F}}} = [a]_{\theta_{\mathcal{F}}}$. So, $(\nabla a, a) \in \theta_{\mathcal{F}}$, that is, there exists $I \in \mathcal{F}$ such that $e \wedge \nabla a = e \wedge a$ for every $e \in I \cap \nabla(A)$. So, if $a \in \nabla(A)$, then for every $I \in \mathcal{F}$ and $e \in I \cap \nabla(A)$, $e \wedge \nabla a = e \wedge a$, hence $[a]_{\theta_{\mathcal{F}}} \in \nabla(A/\theta_{\mathcal{F}})$. □

Definition 5.1. Let $\mathcal{F}$ be a topology on A. By an $\mathcal{F}$-multiplier on A, we means a map $f : I \to A/\theta_{\mathcal{F}}$, which verifies the following condition:

$$f(e \wedge x) = [e]_{\theta_{\mathcal{F}}} \wedge f(x), \quad \text{for all } e \in \nabla(A) \quad \text{and} \quad x \in I.$$

Example 5.1. The maps $\mathbf{0}, \mathbf{1} : A \longrightarrow A/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = [0]_{\theta_{\mathcal{F}}}$ and $\mathbf{1}(x) = [x]_{\theta_{\mathcal{F}}}$ for every $x \in A$ are $\mathcal{F}$-multipliers. Also, for $a \in \nabla(A)$ and $I \in \mathcal{F}$, $f_a : I \longrightarrow A/\theta_{\mathcal{F}}$ defined by $f_a(x) = [a]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}}$ is an $\mathcal{F}$-multiplier.

Lemma 5.3. *For each $\mathcal{F}$-multiplier $f : I \to A/\theta_{\mathcal{F}}$, the following properties hold*:

(1) $f(x) \leq [x]_{\theta_{\mathcal{F}}}$ *for all* $x \in I$,
(2) $f(x \wedge y) = f(x) \wedge f(y)$,
(3) $[x]_{\theta_F} \wedge f(y) = [y]_{\theta_{\mathcal{F}}} \wedge f(x)$.

Proof. It is routine. □

We shall denote by $M(I, A/\theta_{\mathcal{F}})$ the set of all the $\mathcal{F}$-multipliers having the domain $I \in \mathcal{F}$ and

$$M(A/\theta_{\mathcal{F}}) = \bigcup_{I \in \mathcal{F}} M(I, A/\theta_{\mathcal{F}}).$$

If $I_1, I_2 \in \mathcal{F}$, $I_1 \subseteq I_2$, we have a canonical mapping $\varphi_{I_1,I_2} : M(I_2, A/\theta_{\mathcal{F}}) \to M(I_1, A/\theta_{\mathcal{F}})$ defined by $\varphi_{I_1,I_2}(f) = f_{|I_1}$ for $f \in M(I_2, A/\theta_{\mathcal{F}})$.

Let us consider the directed system of sets

$$\langle \{M(I, A/\theta_{\mathcal{F}})\}_{I \in \mathcal{F}}, \{\varphi_{I_1,I_2}\}_{I_1,I_2 \in \mathcal{F}, I_1 \subseteq I_2} \rangle$$

and denote by $A_{\mathcal{F}}$ the direct limit (in the category of sets):

$$A_{\mathcal{F}} = \lim_{\overrightarrow{I \in \mathcal{F}}} M(I, A/\theta_{\mathcal{F}}).$$

For any $\mathcal{F}$-multiplier $f : I \to A/\theta_{\mathcal{F}}$, we shall denote by $\widehat{(I, f)}$ the equivalence class of f in $A_{\mathcal{F}}$.

Remark 5.1. We recall that if $f_i : I_i \to A/\theta_{\mathcal{F}}$, $i = 1, 2$, are $\mathcal{F}$-multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $A_{\mathcal{F}}$) if and only if there exists $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.

Definition 5.2. If $I_1, I_2 \in \mathcal{I}(A)$ and $f_i \in M(I_i, A/\theta_{\mathcal{F}})$, $i = 1, 2$, we define

$$f_1 \wedge f_2, f_1 \vee f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$

by

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$$
$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$

for every $x \in I_1 \cap I_2$.

Let $\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2)$ and $\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2)$.

Definition 5.3. Let $I \in \mathcal{I}(A)$ and $f \in M(I, A/\theta_{\mathcal{F}})$, we define $f^* : I \to A/\theta_{\mathcal{F}}$ by

$$f^*(x) = [x]_{\theta_{\mathcal{F}}} \wedge \sim f(\nabla x)$$

for any $x \in I$.

Let $\widehat{(I,f)}^{*} = \widehat{(I,f^*)}$.

Lemma 5.4. *If* $I_1, I_2 \in \mathcal{I}(A)$ *and* $f_i \in M(I_i, A/\theta_{\mathcal{F}})$, $i = 1, 2$, *then* $f_1 \wedge f_2, f_1 \vee f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. It is routine. □

Remark 5.2. For $x \in A$, we have $\mathbf{0}^*(x) = [x]_{\theta_{\mathcal{F}}} \wedge \sim [0]_{\theta_{\mathcal{F}}} = [x]_{\theta_{\mathcal{F}}} \wedge [1]_{\theta_{\mathcal{F}}} = [x]_{\theta_{\mathcal{F}}}$, that is, $\mathbf{0}^* = \mathbf{1}$, and similarly $\mathbf{1}^* = \mathbf{0}$.

Lemma 5.5. *If* $I \in \mathcal{I}(A)$ *and* $f \in M(I, A/\theta_{\mathcal{F}})$, *then* $f^* \in M(I, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I$ and $e \in \nabla(A)$, then

$$\begin{aligned} f^*(e \wedge x) &= [e \wedge x]_{\theta_{\mathcal{F}}} \wedge \sim f(\nabla(e \wedge x)) \\ &= [e \wedge x]_{\theta_{\mathcal{F}}} \wedge \sim f(\nabla e \wedge \nabla x) \\ &= [e]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}} \wedge \sim (\nabla [e]_{\theta_{\mathcal{F}}} \wedge f(\nabla x)) \\ &= [e]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}} \wedge (\sim \nabla [e]_{\theta_{\mathcal{F}}} \vee \sim f(\nabla x)) \\ &= ([e]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}} \wedge \sim \nabla [e]_{\theta_{\mathcal{F}}}) \vee ([e]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}} \wedge \sim f(\nabla x)) \\ &= [0]_{\theta_{\mathcal{F}}} \vee ([e]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}} \wedge \sim f(\nabla x)) \\ &= [e]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}} \wedge \sim f(\nabla x) \\ &= [e]_{\theta_{\mathcal{F}}} \wedge f^*(x). \end{aligned}$$

□

Definition 5.4. For $I \in \mathcal{I}(A)$, we define $\widetilde{\nabla} : M(I, A/\theta_{\mathcal{F}}) \to M(I, A/\theta_{\mathcal{F}})$, by

$$\widetilde{\nabla}(f)(x) = [x]_{\theta_{\mathcal{F}}} \wedge \nabla f(\nabla x)$$

for every $f \in M(I, A/\theta_{\mathcal{F}})$ and $x \in I$.

Lemma 5.6. *If* $I \in \mathcal{I}(A)$, $f \in M(I, A/\theta_{\mathcal{F}})$, *then* $\widetilde{\nabla}(f) \in M(I, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I$ and $e \in \nabla(A)$, then we have

$$\begin{aligned} \widetilde{\nabla}(f)(e \wedge x) &= [e \wedge x]_{\theta_{\mathcal{F}}} \wedge \nabla f(\nabla(e \wedge x)) \\ &= [e \wedge x]_{\theta_{\mathcal{F}}} \wedge \nabla f(\nabla e \wedge \nabla x)) \\ &= [e]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}} \wedge \nabla(\nabla [e]_{\theta_{\mathcal{F}}} \wedge f(\nabla x)) \\ &= [e]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}} \wedge \nabla [e]_{\theta_{\mathcal{F}}} \wedge \nabla f(\nabla x) \\ &= [e]_{\theta_{\mathcal{F}}} \wedge [x]_{\theta_{\mathcal{F}}} \wedge \nabla f(\nabla x) \\ &= [e]_{\theta_{\mathcal{F}}} \wedge \widetilde{\nabla}(f)(x). \end{aligned}$$

□

Let $\nabla^{\mathcal{F}} : A_{\mathcal{F}} \to A_{\mathcal{F}}$ defined by $\nabla^{\mathcal{F}}(\widehat{(I,f)}) = (\widehat{I, \widetilde{\nabla}(f)})$.

Proposition 5.1. $\langle A_{\mathcal{F}}, \wedge, \vee, *, \nabla^{\mathcal{F}}, \mathbf{0}, \mathbf{1} \rangle$ *is a* TM-*algebra.*

Proof. We verify the axioms of TM-algebras. In the following, we work with $f \in M(I, A/\theta_{\mathcal{F}})$, where $I \in \mathcal{I}(A)$. It is easy to verify that $\langle A_{\mathcal{F}}, \wedge, \vee, \mathbf{0}, \mathbf{1}\rangle$ is a bounded distributive lattice.

$$\begin{aligned}
\text{(M3)} \quad (f^*)^*(x) &= [x]_{\theta_{\mathcal{F}}} \wedge \sim f^*(\nabla x), \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sim ([\nabla x]_{\theta_{\mathcal{F}}} \wedge \sim f(\nabla\nabla x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge (\sim [\nabla x]_{\theta_{\mathcal{F}}} \vee f(\nabla x)) \\
&= ([x]_{\theta_{\mathcal{F}}} \wedge \sim \nabla [x]_{\theta_{\mathcal{F}}}) \vee ([x]_{\theta_{\mathcal{F}}} \wedge f(\nabla x)) \\
&= [0]_{\theta_{\mathcal{F}}} \vee ([x]_{\theta_{\mathcal{F}}} \wedge f(\nabla x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge f(\nabla x) \\
&= f(x \wedge \nabla x) \\
&= f(x).
\end{aligned}$$

$$\begin{aligned}
\text{(M4)} \quad (f_1 \vee f_2)^*(x) &= [x]_{\theta_{\mathcal{F}}} \wedge \sim (f_1 \vee f_2)(\nabla x) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sim (f_1(\nabla x) \vee f_2(\nabla x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge \sim f_1(\nabla x) \wedge \sim f_2(\nabla x) \\
&= f_1^*(x) \wedge f_2^*(x) \\
&= (f_1^* \wedge f_2^*)(x).
\end{aligned}$$

For $x \in I$, we have

$$\begin{aligned}
\text{(M5)} \quad \widetilde{\nabla}(f)(x) \vee f(x) &= ([x]_{\theta_{\mathcal{F}}} \wedge \nabla f(\nabla x)) \vee ([x]_{\theta_{\mathcal{F}}} \wedge \sim f(\nabla x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge (\nabla f(\nabla x) \vee \sim f(\nabla x)) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge [1]_{\theta_{\mathcal{F}}} \\
&= [x]_{\theta_{\mathcal{F}}},
\end{aligned}$$

hence $\widetilde{\nabla}(f) \vee f = \mathbf{1}$, that is, $\nabla^{\mathcal{F}}\widehat{(I,f)} \vee \widehat{(I,f)} = \widehat{(A,\mathbf{1})}$.

For $x \in I$, then

$$\begin{aligned}
\text{(M6)} \quad \widetilde{\nabla}(f)(x) \wedge f^*(x) &= [x]_{\theta_{\mathcal{F}}} \wedge \nabla f(\nabla x) \wedge [x]_{\theta_{\mathcal{F}}} \wedge \sim f(\nabla x) \\
&= [x]_{\theta_{\mathcal{F}}} \wedge f(\nabla x) \wedge [x]_{\theta_{\mathcal{F}}} \wedge \sim f(\nabla x) \\
&= f(x \wedge \nabla x) \wedge f^*(x) \\
&= f^*(x) \wedge f(x),
\end{aligned}$$

hence $\nabla^{\mathcal{F}}(f) \wedge f^* = f^* \wedge f$, that is, $\nabla^{\mathcal{F}}\widehat{(I,f)} \wedge \widehat{(I,f)}^* = \widehat{(I,f)}^* \wedge \widehat{(I,f)}$. □

Definition 5.5. The TM-algebra $A_{\mathcal{F}}$ will be called the localization TM-algebra of A with respect to the topology $\mathcal{F}$.

Lemma 5.7. *If $\mathcal{F}_S$ is the topology associated with the $\wedge$-closed system $S \subseteq A$, then $\theta_{\mathcal{F}_S} = \theta_S$.*

Proof. Let $x, y \in A$. If $(x, y) \in \theta_{\mathcal{F}_S}$, then there exists $I \in \mathcal{F}_S$ such that $x \wedge e = y \wedge e$ for any $e \in I \cap \nabla(A)$. Since $I \cap S \cap \nabla(A) \neq \emptyset$ there exists $e_o \in I \cap S \cap \nabla(A)$ such that $x \wedge e_o = y \wedge e_o$, that is, $(x, y) \in \theta_S$. So, $\theta_{\mathcal{F}_S} \subseteq \theta_S$. If $(x, y) \in \theta_S$, there exists $e_o \in S \cap \nabla(A)$ such that $x \wedge e_o = y \wedge e_o$. If we set $I_o = \{x \in A : x \leq e_o\}$, then $I_o \in \mathcal{I}(A)$. Since $e_o \in I_o$, we have that $e_o \in I_o \cap S \cap \nabla(A)$, hence $I_o \cap S \cap \nabla(A) \neq \emptyset$, that is, $I_o \in \mathcal{F}_S$. For every $e \in I_o$, $e \leq e_o$, then $e = e \wedge e_o$, so $x \wedge e = x \wedge (e \wedge e_o) = (x \wedge e_o) \wedge e = (y \wedge e_o) \wedge e = y \wedge (e \wedge e_o) = y \wedge e$, hence $(x, y) \in \theta_{\mathcal{F}_S}$, that is, $\theta_S \subseteq \theta_{\mathcal{F}_S}$. Therefore, $\theta_S = \theta_{\mathcal{F}_S}$. □

Thus, $A/\theta_{\mathcal{F}_S} = A[S]$, hence an $\mathcal{F}_S$-multiplier can be considered in this case as a mapping $f : I \longrightarrow A[S]$ $(I \in \mathcal{F}_S)$ having the property

$$f(e \wedge x) = [e]_S \wedge f(x)$$

for every $x \in I$ and $e \in \nabla(A)$.

Theorem 5.1. *If $\mathcal{F}_S$ is the topology associated with the $\wedge$-closed system $S \subseteq A$, then the* TM-*algebra $A_{\mathcal{F}_S}$ is isomorphic in* **TM** *with $A[S]$.*

Proof. If $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in A_{\mathcal{F}_S} = \lim_{I \vec{\in} \mathcal{F}} M(I, A[S])$ and $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ then there exists $I \in \mathcal{F}_S$ such that $I \subseteq I_1 \cap I_2$ and $f_{1|I} = f_{2|I}$. Since $I, I_1, I_2 \in \mathcal{F}_S$, there exists $e \in I \cap S \cap \nabla(A)$, $e_1 \in I_1 \cap S \cap \nabla(A)$ and $e_2 \in I_2 \cap S \cap \nabla(A)$. We shall prove that $f_1(e_1) = f_2(e_2)$. If we denote $e' = e \wedge e_1 \wedge e_2$, then $e' \in I \cap S \cap \nabla(A)$ and $e' \leq e_1, e_2$. Since $e_1 \wedge e' = e_2 \wedge e' \in I$ then $f_1(e_1 \wedge e') = f_2(e_2 \wedge e')$, hence $f_1(e_1) \wedge [e']_S = f_2(e_2) \wedge [e']_S$, so $f_1(e_1) \wedge [1]_S = f_2(e_2) \wedge [1]_S$, that is, $f_1(e_1) = f_2(e_2)$. In a similar way, we can show that $f_1(e_1) = f_2(e_2)$ for any $e_1, e_2 \in I \cap S \cap \nabla(A)$. In accordance with these considerations, we can define the mapping:

$$\alpha : A_{\mathcal{F}_S} \rightarrow A[S]$$

by putting

$$\alpha(\widehat{(I, f)}) = f(s),$$

where $s \in I \cap S \cap \nabla(A)$.

We have $\alpha(\mathbf{1}) = \alpha(\widehat{(A, \mathbf{1})}) = \mathbf{1}(s) = [s]_S = \mathbf{1}$ for every $s \in S \cap \nabla(A)$.

Also, for every $\widehat{(I_i, f_i)} \in A_{\mathcal{F}_S}$, $i = 1, 2$, we have

$$\begin{aligned} \alpha(\widehat{(I_1, f_1)} \wedge \widehat{(I_2, f_2)}) &= \alpha(\widehat{(I_1 \cap I_2, f_1 \wedge f_2)}) \\ &= (f_1 \wedge f_2)(s) = f_1(s) \wedge s_2(s) \\ &= \alpha(\widehat{(I_1, f_1)}) \wedge \alpha(\widehat{(I_2, f_2)}), \end{aligned}$$

and

$$\begin{aligned} \alpha(\widehat{(I_1, f_1)} \vee \widehat{(I_2, f_2)}) &= \alpha(\widehat{(I_1 \cap I_2, f_1 \vee f_2)}) \\ &= (f_1 \vee f_2)(s) = f_1(s) \vee s_2(s) \\ &= \alpha(\widehat{(I_1, f_1)}) \vee \alpha(\widehat{(I_2, f_2)}), \end{aligned}$$

with $s \in I_1 \cap I_2 \cap \nabla(A)$.

If $\widehat{(I,f)} \in A_{\mathcal{F}_S}$, we have

$$\begin{aligned}\alpha(\widehat{(I,f)}^*) &= \alpha(\widehat{(I,f^*)}),\\ &= f^*(s)\\ &= [s]_S \wedge \sim f(s),\\ &= [1]_S \wedge \sim f(s),\\ &= \sim f(s)\\ &= \sim \alpha(\widehat{(I,f)}),\end{aligned}$$

where $s \in I \cap S \cap \nabla(A)$.

If $\widehat{(I,f)} \in A_{\mathcal{F}_S}$ and $s \in I \cap S \cap \nabla(A)$, we have

$$\begin{aligned}\alpha(\nabla^{\mathcal{F}}(\widehat{(I,f)})) &= \alpha(\widehat{(I,\widetilde{\nabla} f)})\\ &= \widetilde{\nabla}(f)(s)\\ &= [s]_S \wedge \nabla f(s)\\ &= [1]_S \wedge \nabla f(s)\\ &= \nabla f(s)\\ &= \nabla \alpha(\widehat{(I,f)}).\end{aligned}$$

Therefore, this mapping is a morphism of TM-algebras.

We shall prove that α is injective and surjective. To prove injectivity of α, let $\widehat{(I_1,f_1)}, \widehat{(I_2,f_2)} \in A_{\mathcal{F}}$ such that $\alpha(\widehat{(I_1,f_1)}) = \alpha(\widehat{(I_2,f_2)})$. Then for any $s_1 \in I_1 \cap S \cap \nabla(A)$, $e_2 \in I_2 \cap S \cap \nabla(A)$ we have $f_1(e_1) = f_2(e_2)$. If $f_1(e_1) = [x]_S$ and $f_2(e_2) = [y]_S$ with $x, y \in A$, since $[x]_S = [y]_S$, there exists $e \in S \cap \nabla(A)$ such that $x \wedge e = y \wedge e$. If we consider $e' = e \wedge e_1 \wedge e_2 \in I_1 \cap I_2 \cap S \cap \nabla(A)$, we have $x \wedge e' = y \wedge e'$ and $e' \leq e_1, e_2$. If follows that $f_1(e') = f_1(e' \wedge e_1) = f_1(e_1) \wedge [e']_S = [x]_S \wedge [1]_S = [x]_S = [y]_S = [y]_S \wedge [1]_S = f_2(e_2) \wedge [e']_S = f_2(e_2 \wedge e') = f_2(e')$. If we denote $I = \{x \in A : x \leq e'\}$ (since $e' \in \nabla(A)$), then we obtain that $I \in \mathcal{F}_S$, $I \subseteq I_1 \cap I_2$ and $f_{1|I} = f_{2|I}$, hence $\widehat{(I_1,f_1)} = \widehat{(I_2,f_2)}$, that is, α is injective. To prove the surjectivity of α, let $[a]_S \in A[S]$ (hence there exists $e_o \in S \cap \nabla(A)$ such that $a \wedge e_o \in \nabla(A)$). We consider $I_o = (e_o] = \{x \in A : x \leq e_o\}$ (since $e_o \in I_o \cap S \cap \nabla(A)$, then $I_o \in \mathcal{F}_S$) and define $f_a : I_o \to A[S]$ by putting $f_a(x) = [a]_S \wedge [x]_S = [a \wedge x]_S$ for every $x \in I_o$. It is easy to see that f_a is an $\mathcal{F}_S$-multiplier and $\alpha(\widehat{(I_o,f_a)}) = f_a(s) = [a \wedge s]_S = [a]_S \wedge [s]_S = [a]_S \wedge [1]_S = [a]_S$, where $s \in S \cap \nabla(A)$. So α is surjective. Therefore, α is an isomorphism of TM-algebras. □

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