RANDOM SAMPLING OVER LCA GROUPS AND INVERSION OF THE RADON TRANSFORM

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Abstract: We consider the problem of reconstructing a measurable function over G from a countable subset of samples, taken accordingly to a Poisson random point process, when G is a locally compact abelian group. This results are applied to the problem of approximating the inverse Radon Transform of a function.

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1 INTRODUCTION

Let G be an LCA group, i. e. a locally compact abelian group with Hausdorff topology, whose group operation is written additively. Let U be a topological Hausdorff space and $\mathbf{B}(U)$ the Borel σ-algebra of U. The symbol $\mathbf{1}_S$ stands for the indicator function of the set S. Denote by Γ the dual group of G and by $\langle \gamma, x \rangle$ the value of $\gamma \in \Gamma$ at $x \in G$. Let m_G denote the Haar measure on G, i.e. the unique invariant measure with respect to the group operation '$+$'. The respective Lebesgue spaces of functions, for $p \in [1, \infty]$, are denoted by $L^p(G)$. If $f \in L^1(G)$, its Fourier Transform at $\gamma \in \Gamma$ is defined by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_G f(x)\overline{\langle \gamma, x \rangle} dm_G(x)\,.$$

In a similar manner one can define a Haar measure m_Γ over $(\Gamma, \mathbf{B}(\Gamma))$, and then the inverse Fourier transform for $g \in L^1(\Gamma)$ at $x \in G$ is given by: $\mathcal{F}^{-1}g(x) = g^\vee(x) = \int_\Gamma g(\gamma)\langle \gamma, x \rangle dm_\Gamma(\gamma)$. A classical example is $G = \mathbb{R}^d$ with its usual addition operation $+$, and its dual $\Gamma = \mathbb{R}^d$. In this case $\langle \gamma, x \rangle = e^{i\gamma \cdot x}$ with m_G and m_Γ the ordinary Lebesgue measure on $\mathbb{R}^d$. Another very known example is the torus $G = \mathbb{T}$ and its dual $\Gamma = \mathbb{Z}$. In this case, $\langle \gamma, x \rangle = e^{i\gamma \cdot x}$ with m_G again the Lebesgue measure and m_Γ the counting measure. With these basic definitions in hand, the most relevant results for these classical cases, such as Plancherel's formula, can be extended with no difficulty to the general abstract setting of an LCA group G [4]. A key result in Harmonic Analysis and Signal Processing for $G = \mathbb{R}^d$ is the so called *Whittaker-Kotel'nikov-Shannon* (WKS) sampling theorem which gives conditions to reconstruct (interpolate) a band limited $L^2(\mathbb{R}^d)$ function from its discrete samples taken at a uniform and appropriate rate. The WKS theorem was extended to $L^2(G)$, with G an LCA group, by Kluvánek [5]. In addtion to its elegance, Kluvanek's result provides an example of a unified theory which gives a positive answer to several similar problems that may seem different at first glance. The WKS theorem and its abstract version by Kluvanek, are examples of uniform sampling theorems. In practice, this means that if f represents a signal its samples are taken at equidistant points, in the abstract setting this traduces in evaluating f over a discrete subgroup $H \subset G$. However, in some applications, non uniform sampling may be preferable to the uniform sampling approach. An interesting case arises when the sample points are chosen randomly. Several authors studied the case of sampling points given by a Poisson point process over $\mathbb{R}^d$, see e.g.[8]. Poisson point processes or related *Random measures* can be defined over general σ-finite measure spaces [6]. In particular one can consider the measure space $(G, \mathbf{B}(G), m_G)$. Here, we shall study some results for this abstract sampling setting and then we will apply these to the particular case of $G = \mathbb{R} \times \mathbb{T}$ arising in the context of the Radon transform.

2 Some preliminaries

We begin by recalling some definitions. Let G be a LCA group. Analogously to the $\mathbb{R}^d$ case, for a measurable $S \subset \Gamma$, one can define the Paley-Wiener spaces of S-band limited functions as: $PW_S = \{f \in L^2(G) : \operatorname{supp}(\hat{f}) \subset S\}$. If $m_\Gamma(S) < \infty$, we call $k_S := \mathcal{F}^{-1}\mathbf{1}_S$. Then the orthogonal projection over PW_S is the operator $T_S : L^2(G) \longrightarrow L^2(G)$ defined as

$$T_S f(x) = k_S * f(x) = \int\limits_G f(t)k_S(x-t)dm_G(t) = \int\limits_S \hat{f}(\gamma)\langle\gamma, x\rangle dm_\Gamma(\gamma)\,.$$

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and X a random variable defined on it. If φ is any Borel measurable real function, we denote $\mathbf{E}(\varphi(X))$ the expectation of $\varphi(X)$. If $\mathbf{B}_0(G) = \{A \in \mathbf{B}(G) : m_G(A) < \infty\}$ and $\lambda > 0$, we can define a Poisson random measure [6] $N : \mathbf{B}_0(G) \longrightarrow L^2(\Omega, \mathcal{G}, \mathbf{P})$, where $\mathcal{G}$ is the sub σ-algebra of $\mathcal{A}$ generated by the famility of random variables $\{N(A),\ A \in \mathbf{B}_0(G)\}$ and such that $\mathbf{E}(N(A)) = \lambda m_G(A)$ and $Var(N(A)) = \lambda m_G(A)$. We aim to define an stochastic integral for some suitable random integrands. Let $\mathcal{F}$ be a sub σ-algebra of $\mathcal{A}$ and independent of $\mathcal{G}$. If $p > 0$, we define the spaces

$$\mathbf{L}^p(\mathcal{F}) = \{f : \Omega \times G \longrightarrow \mathbb{C} \ \ is\ \mathcal{F} \otimes \mathbf{B}(G)\text{- measurable and } \|f\|_{\mathbf{L}^p} < \infty\}\,,$$

where $\|f\|^p_{\mathbf{L}^p} = \mathbf{E}\,\|f\|^p_{L^p(G)}$. Now, let $\mathcal{R} = \{R = F \times B,\ F \in \mathcal{F},\ B \in \mathbf{B}_0(G)\}$ be the class of measurable rectangles. From N we define a new random measure $M : \mathcal{R} \longrightarrow L^2(\Omega, \sigma(\mathcal{F} \cup \mathcal{G}), \mathbf{P})$, by

$$M(R) = M(F \times B) = N(B)\mathbf{1}_F\,. \tag{1}$$

If f is a $\mathcal{R}$-simple function, $f = \sum_{i=i}^N a_i \mathbf{1}_{R_i}$ with $a_i \in \mathbb{C}$ and the $R_i \in \mathcal{R}$ disjoint, we define the stochastic integral of f with respect to M as the random variable:

$$\int_G f(x)dM(x) := \sum_{i=i}^N a_i M(R_i)\,.$$

The integral $\int_G f(x)dM(x)$ verifies the following properties:

Lemma 1 *Let f be a $\mathcal{R}$- simple function, then:*
i) $\mathbf{E}\left(\int_G f(x)dM(x)\right) = \lambda \int_G \mathbf{E}(f(x))dm_G(x)$.
ii) $\mathbf{E}\left(\int_G f(x)dM(x) - \lambda\int_G \mathbf{E}(f(x))dm_G(x)\right)^2 = \lambda \int_G \mathbf{E}|f(x)|^2 dm_G(x)\,.$

By a limiting process, (ii) allows to extend the definition of $\int_G f(x)dM(x)$ for any $f \in \mathbf{L}^1(\mathcal{F}) \bigcap \mathbf{L}^2(\mathcal{F})$. Moreover, Lemma 1 remains true for $f \in \mathbf{L}^1(\mathcal{F}) \bigcap \mathbf{L}^2(\mathcal{F})$.

3 Random Sampling on G

Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}_0}$ be a sequence of independent sub σ-algebras of $\mathcal{A}$, and let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be the sequence of sub σ-algebras defined as $\mathcal{F}_n = \sigma(\cup_{j=0}^n \mathcal{G}_j)$. Let $\{N_n\}_{n\in\mathbb{N}}$ be a sequence of Poisson random measures of parameter $\lambda > 0$ such that N_n is $\mathcal{G}_n$-measurable, and let M_n be defined as in equation (1), for each n. If $m_\Gamma(S) < \infty$, then for $f \in \mathbf{L}^2(\mathcal{F}_{n-1})$, we define:

$$T_{n\,S} f(x) = \frac{1}{\lambda} \int\limits_G f(t)k_S(x-t)dM_n(t), \tag{2}$$

for each $n \in \mathbb{N}$. We summarize the main facts about $T_{n\,S}$ in the following results:

Theorem 1 *Let $f, g \in \mathbf{L}^2(\mathcal{F}_{n-1})$ then:*
i) $T_{n\,S} f \in \mathbf{L}^2(\mathcal{F}_n)$ and $T_{n\,S} f \in PW_S$ a.s..
ii) If $\gamma > 0$ then:

$$\mathbf{E}\,\|f - \gamma T_{n\,S}\, g\|^2_{L^2(G)} = \mathbf{E}\,\|f - \gamma T_S g\|^2 + \frac{\gamma^2 m_\Gamma(S)}{\lambda}\mathbf{E}\,\|g\|^2_{L^2(G)}\ .$$

Note that for fixed n, if $f = g \in PW_S$ a.s. and $\gamma = 1$ then

$$\mathbf{E}\,\|f - T_{n\,S} f\|^2_{L^2(G)} = \frac{m_\Gamma(S)}{\lambda}\mathbf{E}\,\|f\|^2_{L^2(G)} \;.$$

In this case, note that if $\lambda \longrightarrow \infty$ then $\|f - T_{n\,S} f\|_{L^2(G)} \to 0$, in the mean square sense and in probability. Therefore $T_{n\,S} f$ is an unbiased estimator of f. However, to ensure a good approximation of f the average number of samples determined by λ should be considerably large. Theorem 1 and its consequent derivations lead to an improvement by considering an iterative method. For determinstic $f \in L^2(G)$, define a sequence of functions $\{f_n\}_{n\in\mathbb{N}_0}$, $f_n \in \mathbf{L}^2(\mathcal{F}_n) \cap PW_S$ by: If $n = 0$, take $f_0 = 0$ and if $n \geq 1$:

$$f_{n+1} = f_n + \gamma T_{n+1\,S}(f - f_n)\,. \tag{3}$$

The main result then reads:

Theorem 2 *Let $f \in PW_S$ (deterministic) and f_n be defined by equation (3) then:*

$$\mathbf{E}\,\|f - f_n\|^2_{L^2(G)} \leq \delta(\gamma)^n\,\|f\|^2_{L^2(G)}\;, \tag{4}$$

where $0 < \delta(\gamma) = \left((1-\gamma)^2 + \frac{\gamma^2 m_\Gamma(S)}{\lambda}\right)$. In particular, for fixed $\lambda > 0$, γ can be chosen such that $\delta(\gamma) < 1$ and thus equation (4) tends to 0 exponentially when $n \longrightarrow \infty$.

4 Radon Transform and the group $G = \mathbb{R} \times \mathbb{T}$

Assume that $f \in C_0^\infty(\mathbb{R}^2)$ with compact support. Its Radon Transform $\mathbf{R}f : \mathbb{R} \times [0, 2\pi) \longrightarrow \mathbb{R}$ is defined by:

$$\mathbf{R}f(p,\theta) = \int\limits_{\{x:\,\xi\cdot x = p\}} f(x)\,dx = \int\limits_{\mathbb{R}^2} \delta(p - \xi\cdot x) f(x)\,dx, \;\; \xi(\theta) = (\cos\theta, \sin\theta),\; p \in \mathbb{R}.$$

Briefly, the Radon transform is defined by the integrals of f over the lines $\{x : \xi \cdot x = p\}$. Obviously, $\mathbf{R}$ is a linear transformation and, for fixed p, $\mathbf{R}f(p,\theta)$ can be viewed as a periodic function. At this point, following [2, 7], it is possible to apply the tools developed in the previuos section in case of the group $G = \mathbb{R} \times \mathbb{T}$. The projection-slice [1] theorem can be used to prove the following approximate inversion formula [2]:

Theorem 3 *Let $e \in L^2(\mathbb{R}^2)$ be a radial function such that $|t|^{1/2}\hat{e}(t) \in L^2(\mathbb{R}^2)$ and ψ the even function of one variable given $\hat{e}(t) = (2\pi)^{-1}\psi(|t|)$. Let the associated convolution kernel k be given by $\hat{k}(r) = \frac{1}{2}(2\pi)^{-3/2}|r|\psi(r)$. Then:*

$$e * f(x) = \int\limits_{\mathbb{R}} \int\limits_{[0,2\pi)} k(\xi\cdot x - p)\mathbf{R}f(p,\theta)\,d\theta\,dp.$$

In fact, $e * f$ can be viewed as a filtered or smoothed version of f. If e defines an approximation of the identity then the formula gives an approximation of f. Generally, this filtered version is easier to recover from data $\mathbf{R}f$ than the original function f. If $F = \mathbb{R} \times [0, 2\pi) \longrightarrow \mathbb{R}$ and if $\mathcal{F}_1 F$ denotes the Fourier transform on the first variable, we define the integral operator T_ψ, for $x \in \mathbb{R}^2$ as:

$$T_\psi F(x) = \int\limits_{\mathbb{R}} \int\limits_{[0,2\pi)} |r|\psi(r)\mathcal{F}_1 F(r,\theta)\,drd\theta\,.$$

Indeed, one can prove that:

$$e * f(x) = (T_\psi \mathbf{R}f)(x) \tag{5}$$

Recall that now $G = \mathbb{R} \times \mathbb{T}$. The key property of T_ψ is the following:

Lemma 2 *Let* $F \in L^2(G)$*, then* $|T_\psi F(x)| \leq (2\pi)^{1/2} \, \||r|\psi(r)\|_{L^2(\mathbb{R})} \, \|F\|_{L^2(G)}$ *.*

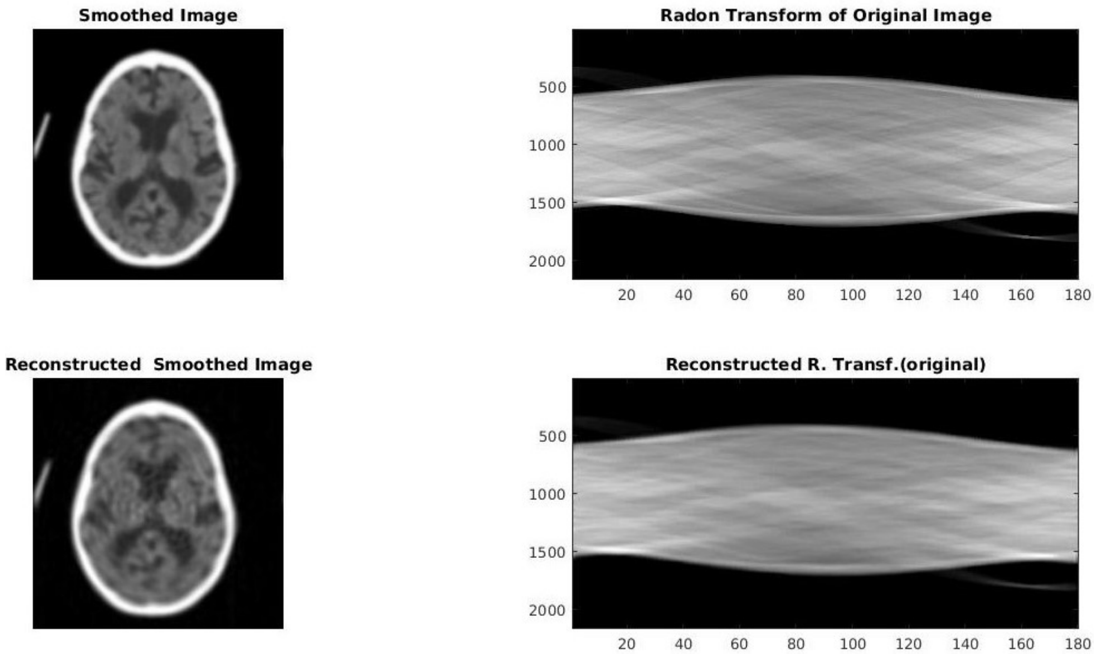
Now, we can combine equation (5) and Lemma 2 with Theorem 2 in the following way. Suppose that $F = \mathbf{R}f \in PW_S$, where $S \subset \Gamma = \mathbb{R} \times \mathbb{Z}$ is such that $m_\Gamma(S) < \infty$. Note that, in this case, m_Γ is the product measure between the Lebesgue measure $m_\mathbb{R}$ and the counting measure c, $m_\Gamma = m_\mathbb{R} \otimes c$. Finally define a sequence $\{F_n\}_n$ as in equation (3), then we can prove:

Theorem 4 *Let* $F = \mathbf{R}f \in PW_S$ *and* F_n *be defined by equation* (3) *then:*

$$\mathbf{E} \, \|e * f - T_\psi F_n\|^2_{L^\infty(\mathbb{R}^2)} \leq \delta(\gamma)^n (2\pi)^{1/2} \, \||r|\psi(r)\|_{L^2(\mathbb{R})} \, \|F\|_{L^2(G)} \, , \tag{6}$$

where $0 < \delta(\gamma)$ *is defined as in Theorem 2.*

To illustrate the proposed method we include some experimental results. On the top we can see a smoothed version of CT image of a brain and the sinogrogram of the original image. Below, on the right side we can see the reconstruction of the original Radon transform after 7 iterations, taking $\lambda = 0.04$ and $\gamma = 0.9$. On the left, the final approximation of the smoothed image.



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