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The holodiagram in a geometrical approach to the calculation of fringes visibility

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Abstract: This work is intended as a help to show, using very simple geometrical models, how visibility is built into the Young's interference patterns produced by incoherent sources. A very simple example, that consisting in only two mutually incoherent point sources, is used. Consonance between independent fringe systems leads straightforwardly to the same results predicted by the Van Cittert-Zernike's Theorem. The homogeneous and non-homogeneous cases are exemplified. Abramson's Holodiagram, both the classic one and the Young's fringes one, concepts are then used to gain further non trivial insight in some simple situations including that of virtual sources. Sensitivity to pinholes displacements and areas of homogeneity are easily observed.

Key words: Holodiagramm – coherence – Van Cittert-Zernike's Theorem

1. Introduction

To adequately describe the partial coherence between two points in the propagation region produced by an extense polychromatic source it is necessary to measure the correlation existing between the field vibrations between these points. The measure of that correlation is intimately related with the interference and diffraction pattern produced when combining the vibrations of both points.

The calculation of the degree of coherence and the mutual intensity (that replaces the mutual coherence function when the finite source is quasi monochromatic) leads to the Van Cittert-Zernike Theorem [1, 2], which is of great utility in coherence theory. This theorem establishes that the complex degree of coherence describing the correlation of the vibrations between a fixed point P_2 and a variable point P_1 in a plane illuminated by a finite, incoherent, quasi monochromatic source equals the value of the normalized complex amplitude in point P_1 of a diffraction pattern. Such pattern is obtained replacing the source by a diffracting aperture identical in size and shape to the source illuminated with a spherical wave converging in P_2, and the amplitude distribution of which, point to point corresponds with the intensity of the source [2].

If the linear dimensions of the source and the distance between P_1 and P_2 are small as compared with the distance between these points to the source, the modulus of the complex degree of coherence equals the normalized Fourier Transform of the intensity distribution of the source.

It could be noticed that the theory of coherence deals with measurable quantities, correlation and intensities and uses the auxiliary concept of visibility of an interference fringe system.

In this sense, we propose in this work an approach to the understanding of the meaning of coherence using paraxial geometrical optics in terms of the visibility of fringes. This idea was already proposed in the original work by Zernike [2].

How is it possible that two (or more) incoherent elements of a source could give rise to nonzero visibility fringes stable in time? Each idealized point element of it produces high contrast fringes, but different elements are not supposed to give rise to stable interference patterns. Then the only possibility for visibility to subsist when both point source elements are present should be that the source points are separated by such a distance that makes the individual fringe systems to coincide.

Then, emphasis is exerted in the fact that the existence of measurable visibility finges is due to superposition in consonance of multiple fringe systems originating in different source elements that are incoherent between them. This idea, originally used to calculate visibility in times before the VCZ Theorem was stated, can be exemplified by using two very small and close pinholes very near to the eyes and observing through them outdoor scenarios. Even if the available light is not monochromatic, fringes can be observed in luminance discontinuities, such as edges, wires or TV poles, images of the Sun in dew drops or cylindrical surfaces also show fringes with high enough visibility to be discerned. It is easy then to figure out that low or zero visibility in extended sources is due to the superposition of shifted fringes systems.

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The case of two quasi monochromatic point sources is taken as example and it is then extended to the case of a continuous space distribution. The non-homogeneous case is motivated with another very simple example consisting in two point sources on a perpendicular line to the plane of Young's apertures.

Then, the concepts of the classical Holodiagram (HD) and its close relative, the Young's Holodiagram (YHD), as introduced by Abramson [3] are used to gain some more insight in these simple cases. The aforementioned examples with two point sources in particular geometric positions are seen as particular cases that can be easily understood in terms of these diagrams and useful information can be gained from their nice properties.

2. Two incoherent point sources (homogeneous case)

In fig. 1, A and A′ in plane P–P′ represent two far quasi monochromatic incoherent point sources separated a distance x'_0.To stress the incoherence idea we can think that both sources are never lit at the same time.

Those sources show the same irradiance and their mean wavelength is λ. The holes P_1 and P_2, separated a distance d have a negligible size, so that the light distribution in plane Q–Q′ does not show appreciable variations due to diffraction in the region of interest. Lens L (not essential for the description), with focal distance f_0, conjugates planes P–P′ and Q–Q′. Besides, distance z' is very much larger than the involved transverse distances, and the image distance $z \cong f_0$.

So, for each point source in P–P′, it corresponds an intensity distribution in plane Q-Q′ as in Young's experiment. These intensity distributions I_A and I_B can be respectively described as

$$I_A(x) = I_0 \cos^2 \left[\frac{2\pi}{\lambda} \frac{d}{z} x\right]$$

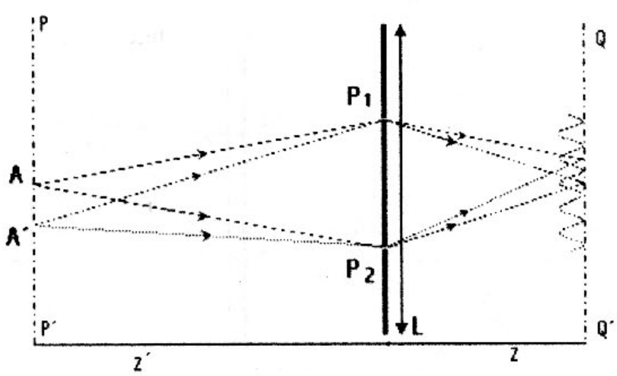

Fig. 1. A two-points incoherent homogeneous source. The lens L is not essential for the description. In all the drawings a small distance between the lens and the aperture is introduced for easiness. Besides, the approximations with respect to the involved distances mentioned in the text are not respected to make the drawings more clear.

and

$$I_{A'}(x) = I_0 \cos^2 \left[\frac{2\pi}{\lambda} \frac{d}{z} (x - x_0)\right]$$

with

$$x_0 = \frac{z}{z'} x'_0 .$$

The position of any interference order (say, for example, the zero order) fully determines the position of the whole fringe system.

As both sources are incoherent, the total intensity is

$$I(x) = I_A(x) + I_{A'}(x)$$

If we call

$$\omega'_x = \frac{2\pi}{\lambda} \frac{d}{z}$$

then it results

$$I(x) = I_0\{1 + \cos [\omega'_x(2x - x_0)] \cos \omega'_x x_0\}$$

and the visibility

$$V \equiv \frac{I_M - I_m}{I_M + I_m} = |\cos \omega'_x x_0| = \left|\cos \frac{2\pi}{\lambda} \frac{x'_0}{z'} d\right| . \qquad (1)$$

That is, the visibility depends on the separation d between P_1 and P_2 (points where correlation is calculated in the Van Cittert-Zernike's Theorem) and also on the ratio

$$\frac{x'_0}{z'} .$$

It is, the relative position of the two fringe systems (one fringe system for A and one for A′) determines the visibility of the composite fringe system when both sources are present.

If, for example, fringes are shifted in phase in integer multiples of 2π (so that they coincide), visibility will be maximum. This result agrees with the intuitive idea that the existence of visibility due to the existence of two sources is due to their consonance.

It is evident from eq. 1 that visibility, and consequently the degree of spatial coherence, shows the same geometry of Young's fringes as predicted by the Van Cittert-Zernike theorem. Notice that it is the relative position of any given interference order of both fringe systems that determines the resulting visibility. So, the location of a pair of homologous points in the fringes systems is enough to determine the visibility value. In what follows we are going to use the location of the point of zero interference order in each system, it is the point where the optical path length difference is zero.

Following the same line of reasoning as before, if there are N discrete point sources with irradiance I_i, located in points x'_i, the intensity distribution in plane Q–Q′ results:

$$I(x) = \sum_{i=1}^{N} I_i \cos^2 (\omega_x(x - x_i))$$

with

$$\omega_x = \omega'_x \, \frac{z}{z'} \, .$$

The case of a continuous intensity distribution source with similar restrictions (quasi monochromatic, incoherent, plane and far away) can be thought as due to a linear distribution of irradiance $I(x')$ in plane P–P', the intensity distribution in plane Q–Q' is

$$I(x) = \int I(x') \cos^2 (\omega_x (x - x)) \, dx'$$
$$= B + \tfrac{1}{2} A \cos (2\omega_x x - \delta)$$

with

$$A \cos \delta = \int I(x') \cos (2\omega_x x') \, dx'$$

and

$$A \sin \delta = \int I(x') \sin (2\omega_x x') \, dx'$$
$$B = \int I(x') \, dx' \, .$$

The integrals are extended to include all points in the source.

Then, the maximum and minimum values of the intensity are respectively

$$I_M = b + \tfrac{1}{2} A$$

and

$$I_m = B - \tfrac{1}{2} A$$

and visibility results to be

$$V \equiv \frac{I_M - I_m}{I_M + I_m} = \frac{A}{2B} \, .$$

Solving for A, B and replacing,

$$V = \frac{\{|\int I(x') \cos (2\omega_x x') \, dx'|^2 + |\int I(x') \sin (2\omega_x x') \, dx'|^2\}^{1/2}}{\int I(x') \, dx'} \, .$$

It is, visibility is represented by the modulus of the normalized Fourier Transform of the intensity distribution of the source and consequently, the complex degree of coherence. This is the same result that could be obtained from the Van Cittert Zernike's Theorem.

Notice that only elementary calculus and trigonometric identities are used and a permanent clear feeling of what is being done is easily kept.

Also, that in this case the visibility depends on the distance d (coordinates difference) between points P_1 and P_2 and not in their actual position in front of the lens.

3. Two sources (non homogeneous case)

Returning to the case of only two point sources, let us assume that we consider the somewhat more general case where the two sources A and A' are not required to be in a plane perpendicular to the optical axis of the lens.

Figure 2 shows two possibilities. In the first (fig. 2a), both sources are aligned with the midpoint between P_1 and P_2 and along the normal direction to the aperture screen. Without further calculation it is evident, because of the symmetry, that both fringe systems are in consonance for any value of d and high visibility is then always obtained.

Let us consider a slightly more general situation where the line joining the two sources is displaced an amount X from the center of the pupil as shown in fig. 2b.

Let us call z_1 and z_2 the distances between the plane containing the holes P_1 and P_2 and each source, respectively. Light going from one source to points P_1 and P_2 will travel different path lengths, and the point of zero path length difference in the fringe system will be located in different positions for one source and for the other.

We start by finding x_0 and x'_0, the location of the zero order fringes for each source. Let us call $\Lambda(A, P_1)$ the optical path from A to P_1 and a similar notation for the other pairs of points. The optical path difference for the light going from A to P_1 and P_2 is

$$\Delta\Lambda(A) = \Lambda(A, P_1) - \Lambda(A, P_2)$$

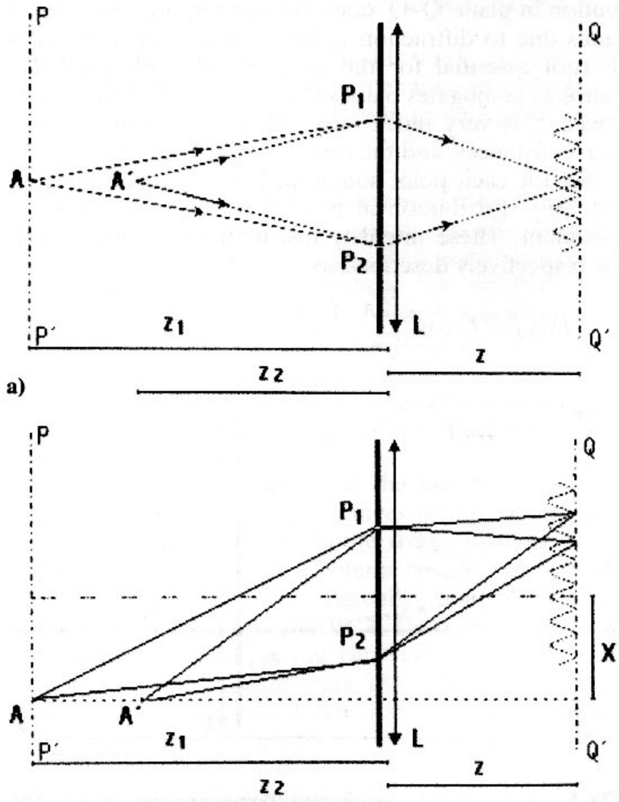

Fig. 2. a) A two-points incoherent source aligned with the midpoint of the apertures. b) A two-points incoherent source on the normal to the apertures plane at a coordinate X with respect to the midpoint between them.

and from A′

$$\Delta\Lambda(A') = \Lambda(A', P_1) - \Lambda(A', P_2)\,.$$

By using the distances defined in fig. 2b, we find that

$$\Lambda(A, P_2) = \sqrt{(X - d/2)^2 + z_1^2}$$
$$\Lambda(A, P_1) = \sqrt{(X + d/2)^2 + z_1^2}$$
$$\Lambda(A', P_1) = \sqrt{(X + d/2)^2 + z_2^2}$$
$$\Lambda(A', P_2) = \sqrt{(X + d/2)^2 + z_2^2}\,.$$

Considering that the sources are assumed to be so far away that the following approximations can be applied:

$$z_1 \gg X \pm d/2$$
$$z_2 \gg X \pm d/2\,.$$

Then

$$\Lambda(A, P_2) \approx z_1\left(1 + \frac{(X - d/2)^2}{2z_1^2}\right)$$
$$\Lambda(A, P_1) \approx z_1\left(1 + \frac{(X + d/2)^2}{2z_1^2}\right)$$
$$\Lambda(A', P_1) \approx z_2\left(1 + \frac{(X + d/2)^2}{2z_2^2}\right)$$
$$\Lambda(A', P_2) \approx z_2\left(1 + \frac{(X - d/2)^2}{2z_2^2}\right)\,.$$

So,

$$\Delta\Lambda(A) \approx \frac{Xd}{z_1}$$
$$\Delta\Lambda(A') \approx \frac{Xd}{z_2}\,,$$

and then

$$\Delta\Lambda \approx Xd\left(\frac{1}{z_2} - \frac{1}{z_1}\right)\,.$$

The X_0 coordinate of the center of the system of fringes produced by A can be located by imposing the zero path difference between the interfering beams. To do this we now calculate the path difference between P_1 and P_2 and an arbitrary point $P(x)$ in the observation screen following the same procedure and the result is:

$$\Lambda(P_1, P(x)) = \sqrt{z^2 + \left(x + \frac{d}{2}\right)^2}$$
$$\Lambda(P_2, P(x)) = \sqrt{z^2 + \left(x - \frac{d}{2}\right)^2}\,.$$

If we restrict our attention to x values so small that we can approximate the square root by the first terms of its series development, then

$$\Lambda(P_1, P(x)) \approx z\left(1 + \frac{\left(x + \frac{d}{2}\right)^2}{z^2}\right)$$
$$\Lambda(P_2, P(x)) \approx z\left(1 + \frac{\left(x - \frac{d}{2}\right)^2}{z^2}\right)\,.$$

The difference in optical path between both is then

$$\Delta(P(x)) \approx \frac{\left(x + \frac{d}{2}\right)^2 - \left(x - \frac{d}{2}\right)^2}{z} = \frac{xd}{z}$$
$$= \left|\cos\frac{2\pi}{\lambda}\,dX\left[\frac{1}{z_1} - \frac{1}{z_2}\right]\right|\,.$$

It can be seen that the case shown in fig. 2a is a particular case ($X = 0$). Visibility is then maximum and does not depend on the value of d.

Besides, by for example shifting the sources by $X = d/2$ to one side the behavior of the visibility drastically changes. It now depends on d^2 in a Fresnel-zone like way.

It is, the visibility of the fringes *depends on the X position of the sources. It was not the case in the former section.* The source is then named *inhomogeneous.*

This behavior can also be predicted from the Van Cittert-Zernike Theorem.

Then, the two double sources (homogeneous and inhomogeneous) considered here give visibility conditions easy to interpret and illustrate on how inhomogeneity (dependence of the visibility on the pinholes coordinates) appears.

The location of the point on the screen of zero path difference, or, conversely, given a point on the screen, the location of openings giving rise to zero path difference, can be facilitated by using the Holodiagram [2]. Given a point P on the screen where Young's fringes are expected and a point light source A, a set of ellipsoids can be drawn with them (P and A) as foci, such that successive ellipsoids differ in constant steps of half wavelength. Then, if the openings are on the same ellipsoid (see fig. 3), the point on the screen has zero path difference. In general, two arbitrary openings will correspond to a path difference given by the order difference of the ellipsoids times half wavelength.

If the location of the openings changes, the path difference in general also changes. The sensitivity of this location variation can be visually estimated very easily using the HD in this way. As an example, if one of the pinholes is displaced along one ellipse of the holodiagram, the interference order at P does not change, while if the displacement is in the direction where the HD fringes are closely packed the change in interference order is the highest. All the previous developments on holographic sensitivity can be applied to this displacement.

By equating $\Delta\Lambda(A)$ with $\Delta\Lambda(P(x))$ we can find the coordinate X_0 of the zero order fringe and the same

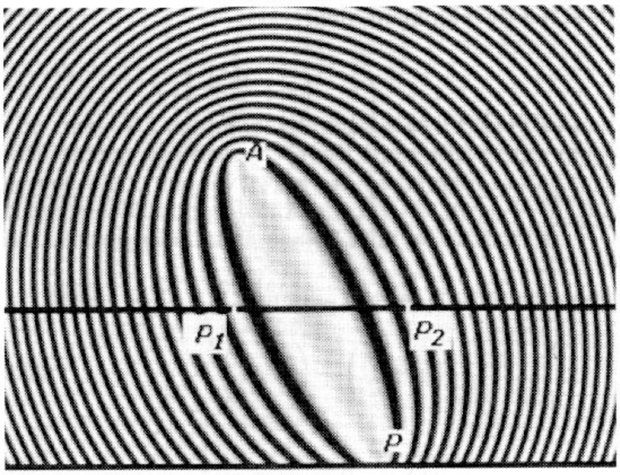

Fig. 3. The Holodiagram used to identify the interference order of a point P.

calculation can be done to find X_0' by equating $\Delta\Lambda(A')$ with $\Delta\Lambda(P(x))$.

$$\Delta\Lambda(A) \approx \frac{d^2}{2z_1} = \frac{X_0 d}{z}$$

$$\Delta\Lambda(A') \approx \frac{d^2}{2z_2} = \frac{X_0' d}{z}.$$

Then

$$X_0 = z\frac{X}{2z_1} \qquad X_0' = z\frac{X}{2z_2}$$

$$X_0 - X_0' = zX\left[\frac{1}{z_1} - \frac{1}{z_2}\right].$$

The intensity distributions I_A and $I_{A'}$ produced by A and A′, can be respectively described as

$$I_A(x) = I_0 \cos^2\left[\frac{2\pi}{\lambda}\frac{d}{z}(x - X_0)\right]$$

and

$$I_{A'}(x) = I_0 \cos^2\left[\frac{2\pi}{\lambda}\frac{d}{z}(x - X_0')\right].$$

As both sources are incoherent, the total intensity is

$$I(x) = I_A(x) + I_{A'}(x),$$

then it results

$$I(x) = I_0\{1 + \cos[\omega_x(2x + X_0' - X_0)]\cos\omega_x(X_0 - X_0')\}$$

and the visibility is

$$V \equiv \frac{I_M - I_m}{I_M + I_m} = |\cos\omega_x(X_0 - X_0')| = \left|\cos\frac{2\pi(X_0 - X_0')}{\lambda}\frac{d}{z}\right|.$$

Notice also that using this approach, the coincidence of the zero path difference points due to sources A and A′ considered above (fig. 2a) when the two sources are aligned with a point midway between the openings is straightforwardly concluded.

Then, the two double sources (homogeneous and inhomogeneous) considered here give visibility conditions easy to interpret and illustrate on how inhomogeneity (dependence of the visibility on the pinholes coordinates) appears.

The behavior of the visibility is identical to that of the interference of two coherent sources. Then, the original pattern used by Young and afterwards by Abramson [2, 4] consisting in a set of revolution hyperboloids (in this case with foci in the incoherent point sources A and A′) can be used to find useful insight on the distribution of coherence as represented by the fringes visibility. This pattern has been named Young's fringes Holodiagram (YHD) [4].

A pattern consisting in a family of hyperboloids with the mentioned foci and differing in constant adequately chosen steps could be thought, painted black and white alternatively (fig. 4). Then, the intersection of this pattern with the plane of the apertures indicates the corresponding visibility value.

It can be understood using a binarized version of the diagram (only for easiness in description). If both aper-

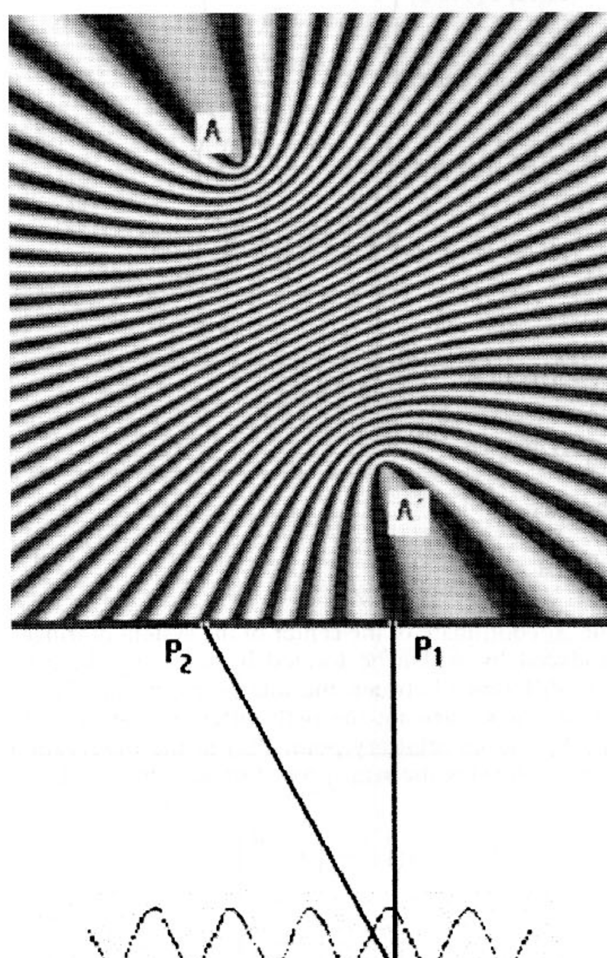

Fig. 4. Young's fringes Holodiagram used to find the visibility of Young's fringes.

tures are on bright (or dark) regions of the YHD then visibility is maximum and if the apertures lie in regions of different brightness value, then it is minimum. The generalization to continuous variation of this binarized version is straightforward.

Again, in this case the sensitivity of the visibility to changes in the position of the pinholes can be inferred from the geometry of the YHD. Displacements of one pinhole along one hyperbola do not change the visibility value while sensitivity to displacements is maximal in the direction of the smallest spacing of the hyperbolae. Besides, a qualitative idea of the homogeneity (or lack of it) can also be gained from the observation of the YHD. In the region where the YHD shows even spacing (the region of classical Young's fringes) coherence is homogeneous, while in regions where the spacing varies the behavior is non-homogeneous.

The HD concept can also be applied to search for the visibility in the case of virtual sources. We are going to call a *virtual source* to the case when one of the light sources is a spherical wave converging to a point behind the plane of the apertures.

The phase difference between the wavelets going through apertures P2 and P1 determines the position of the whole system of fringes. When one of the sources is virtual, (A′ as shown in fig. 5), we can observe that the phase delay between the wavelets coming through P_1 and P_2 for the (virtual) point A′ is the same as in the case of a (real) source A″ located symmetrical with respect to the midpoint O between P_1 and P_2. So, the YHD constructed with A and A″ will describe the coherence corresponding with this case for P_1 and P_2. A different YHD will be required for every position of the apertures if the position of O is modified.

An extreme, 'pathological', case is when one of the sources is in one of the apertures. If the presence of this source does not modify the propagation of the light coming from the other source, then the visibility will be identically $1/2$ for any pair P_1 and P_2. If both sources are in the apertures, then the visibility is zero.

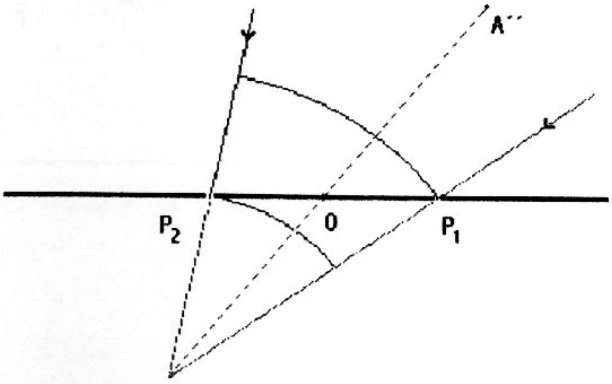

Fig. 5. One virtual source A′ and its symmetrical A″ with respect to the apertures midpoint O.

Next, when both sources are virtual, the YHD can be constructed at either side of the screen with the apertures with identical results.

4. The inverse problem

Which is the set of locations where two incoherent point sources will produce a given (say, maximum) visibility? It is, given that the visibility has a certain value, which is the restriction it imposes on the possible source distributions? This question (given the visibility value, to infer some information about the source) could be named *the inverse problem*. As we have seen that the positions of the zero order fringe of the fringes systems produced by A and A′ separately depend on $\Delta\Lambda$, all points giving rise to the same value of it will produce the same visibility.

The locus of those points is also the Young's fringes Holodiagram (YHD), a plot of hyperboloids that is obtained as in fig. 4, now having as foci the openings in the screen P_1 and P_2 (see fig. 6).

It can be seen that any point sources located on, say, the maxims of this diagram will produce high visibility fringes. Moreover, if a source in the shape of these maxims was used, high visibility should also be obtained, a result that was not at all obvious from the start.

It should be clear now that with such a source, if some regions of it are obscured, no changes in the visibility should occur (only changes the mean intensity), while if new sources were added in the dark regions, visibility in the fringes should diminish.

It is, the use of the YHD adds to the insight of how visibility is built in the fringes, based on very simple

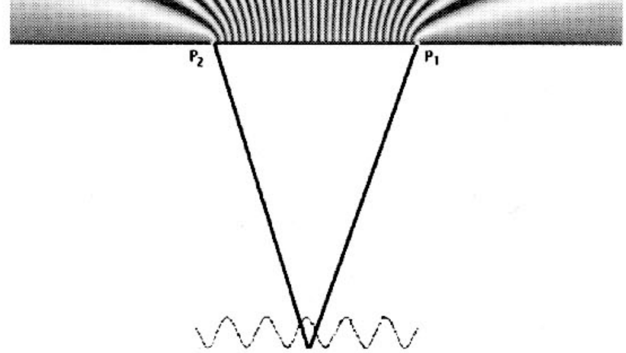

Fig. 6. The inverse problem using Young's fringes Holodiagram.

concepts, easy to handle and not requiring explicitly the use of operations such as ensemble average and correlation, which, of course are present but not required for this geometrical approach.

5. Conclusions

A geometrical approach to the calculation of visibility, stressing on the fact that it is due to the consonance of individual fringe systems between incoherent superposition has been proposed. It was done mostly as an aid to the understanding of how visibility is built into the interference pattern when incoherent sources are present. With this approach, the same results as with the VZ-C Theorem are obtained using only elementary calculus and trigonometric identities. The homogeneous and inhomogeneous cases were illustrated through a very simple example.

The use of the Holodiagram concepts (both the classical and the Young's fringes versions) gives also some insight on the origin of visibility in some idealized situations. Sensitivity to variations in the location of the pinholes can be estimated from previous developments in the HD and the YHD. Regions of homogeneity and non-homogeneity can also be inferred from it. The case of one or both virtual sources was also considered.

A difficult to grasp concept, as is that of spatial coherence is somewhat made more comprehensible using very simple geometrical concepts. Some not obvious conclusions in the inverse problem can be obtained in this way.

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References

[1] Born M, Wolf E: Principles of Optics. Pergamon Press, Oxford 1983
[2] Zernike F: The concept of degree of coherence and its application to optical problems. Physica **V** (no. 8) (1938) 785–795
[3] Abramson N: The making and evaluation of holograms. Academic Press, London 1981
[4] Rabal H: The Holodiagram with virtual sources. Optik **112** (2001) 487–492

Part of this work (the homogeneous case) was published in Anales de la Asociación Física Argentina as 'Enfoque geométrico para obtener la visibilidad. Su relación con el Teorema de Van Cittert-Zernike', H. Rabal and N. Bolognini, Tucumán, october, 1991.