SEPARATION PROPERTIES FOR ITERATED FUNCTION SYSTEMS OF BOUNDED DISTORTION

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ABSTRACT. In this paper we study a general separation property for subsystems G, whose attractor K_G is a sub-self-similar set. This is a generalization of the Lau-Ngai weak separation property for the bounded distortion case. For subsystems with positive Hausdorff measure in its similarity dimension, we characterize the subsets of K_G with positive measure where the separation property may fail. We exhibit two examples of fractal sets. One such that do not satisfy the weak separation property and whose existence was questioned by Zerner. The other one such that have positive Hausdorff measure in its similarity dimension an the separation property fails in a subset of positive measure.

1. INTRODUCTION

Let X be a non empty closed subset of $\mathbb{R}^n$ and let $I^1 = \{1, \ldots, \ell\}$ a finite index alphabet, $\ell > 1$. An *iterated function system* (IFS) consists of a family $\{\varphi_i\}_{i \in I^1}$ of contractions on X. That is, for $i \in I^1$ we have $\varphi_i : X \to X$ such that

$$|\varphi_i(x) - \varphi_i(y)| \leq r_i |x - y| \quad \text{for all } x, y \in X,$$

where $0 < r_{\min} \leq r_i \leq r_{\max} < 1$, and $|\cdot|$ represents the euclidean norm in $\mathbb{R}^n$.

An IFS determines a unique non empty compact set K satisfying

$$K = \bigcup_{i \in I^1} \varphi_i(K),$$

which is in general a fractal set [8, 6]. We will denote by I^k the set of all words of length k, $\omega = \omega_1 \omega_2 \ldots \omega_k$, $\omega_j \in I^1$, and by $I^* = \cup_{i=1}^{\infty} I^k$ the set of all finite words, for $\omega \in I^*$ we will denote by $|\omega|$ the length of ω. For $\omega = \omega_1 \ldots \omega_k \in I^k$

we will denote $r_\omega = r_{\omega_1} \ldots r_{\omega_k}$, $\varphi_\omega = \varphi_{\omega_1} \circ \cdots \circ \varphi_{\omega_k}$, and $K_\omega = \varphi_\omega(K)$. We remark that $K_{\lambda\omega} \subseteq K_\lambda$, for all $\omega, \lambda \in I^*$.

We consider the general class of IFS of *bounded distortion* (BD) [6], that is, we assume that there are constants $M_0, M_1 > 0$ such that

(1.1) $$M_0 |K_\omega||x - y| \le |\varphi_\omega(x) - \varphi_\omega(y)| \le M_1 |K_\omega||x - y|,$$

for all $\omega \in I^*$, $x, y \in X$; where $|K_\omega|$ represents the diameter of the compact set K_ω. The BD property (1.1) is satisfied if the φ_i's are conformal maps [14, 10], and in particular for contracting similitudes.

Let us denote by I the space of infinite successions $\omega = \omega_1\omega_2\ldots$, $\omega_j \in I^1$ with the usual metric: $d(\omega, \lambda) = \sum_{j=1}^\infty \frac{|\omega_j - \lambda_j|}{\ell^j}$, for all $\omega, \lambda \in I$. We consider the natural projection map $\Pi : I \to X$ defined by

$$\Pi(\omega) = \bigcap_{n=1}^\infty K_{\omega_1\ldots\omega_n}.$$

It is clear that $d(\omega, \lambda) \le \frac{1}{\ell^n}$ if $\omega_j = \lambda_j$ for all $j = 1 \ldots n$ and, conversely, $d(\omega, \lambda) < \frac{1}{\ell^n}$ implies $\omega_j = \lambda_j$ for all $j = 1 \ldots n$. Thus Π is a continuous map and $\Pi(I) = \cup_{\omega \in I}\Pi(\omega) = K$. For a general subset of successions $G \subset I$, we will denote by G^k the set of words of length k, $\omega \in I^k$ for which there is $\lambda \in I$ such that $\omega\lambda \in G$; also $G^* = \cup_{k=1}^\infty G^k$.

We say that $G \subseteq I$ is a *subsystem* of I if G is compact and shift invariant, that is if $\omega = \omega_1\omega_2\omega_3 \cdots \in G$, then $\omega_2\omega_3 \cdots \in G$. Then the compact subset $K_G = \Pi(G)$ is a *sub-self-similar* set satisfying $K_G \subseteq \cup_{i \in I^1}\varphi_i(K_G)$. Such constructions were studied by Falconer [7] for similitudes.

For a subsystem G let us define the *similarity dimension* of G by the unique solution s_G of the pressure equation $p_G(s_G) = 0$, where $p_G(s)$ is defined by

$$p_G(s) = \lim_{k\to\infty} \frac{1}{k} \log(\textstyle\sum_{\omega \in G^k} |K_\omega|^s).$$

Separation properties are needed to find formulas for the Hausdorff dimensions of K and K_G. If the IFS I satisfy a strong separation property like the Open Set Condition then the Hausdorff dimension of K coincides with the

similarity dimension of I, $\dim K = s_I$ and $\mathcal{H}^{s_I}(K) > 0$ [8]. This result extends for general subsystems as was showed by Falconer [7] and we recall below. On the other hand if I satisfy the weak separation property of Lau and Ngai [9] then $\dim K$ coincides with the growth dimension of I [16], which is in general smaller than s_I.

In this work we study the following separation property. Let G a subsystem of bounded distortion, for a non empty compact set $A \subseteq X$, we will denote by $G(A)$ the family of finite words

$$G(A) = \{\omega = \omega_1 \ldots \omega_k \in G^* : |K_\omega| < |A| \leq |K_{\omega_1 \ldots \omega_{k-1}}|; K_\omega \cap A \neq \emptyset\}.$$

We will say that a subsystem G is *separated* if there exists $M > 0$ such that $\sharp(G(A)) \leq M$ for all compact A, where $\sharp(\cdot)$ denotes the cardinal (number of elements).

For $G = I$ the separate condition is a reformulation of the Bandt-Graft condition [1] which is equivalent to the Open Set Condition. See [15] for contracting similitudes and [13] for conformal IFS. However is easy to see that our separation property does not implies the Bandt-Graft condition for general subsystems.

In the next section, we prove the dimension formula for sub-self-similar sets and some complementary results. The principal result of this section (Theorem 2.2) was proved by Falconer [7] but we give a different proof here. Sections 3 and 4 contains original results and examples. In Section 3 we characterize the weak separation property and the growth dimension of K through a subsystem W. We propose a generalized weak separation property and give an example of a fractal set that satisfy the generalized property but not the weak separation property. The existence of such fractal set was questioned in Zerner [16]. Finally, in Section 4, we study subsystems for which $\mathcal{H}^{s_G}(K_G) > 0$. For $G = I$ this implies separation but we show that this is not the case for general subsystems. The principal result (Theorem 4.3) characterizes the subsets of K_G with positive measure where the separation

property fails. In the example of this section we show a fractal set where we can find such a subset explicitly.

2. Similarity dimension and separate subsystems

Let G a subsystem, $A \subseteq X$ a compact set and $k, s > 0$, we will denote

$$c_{G,k}^s = \sum_{\omega \in G^k} |K_\omega|^s, \quad \text{and} \quad c_{G,k}^s(A) = \sum_{\substack{\omega \in G^k \\ K_\omega \cap A \neq \emptyset}} |K_\omega|^s.$$

Taking into account the bounded distortion and the shift invariance of G is easy to see that

$$(2.1) \qquad c_{G,k+m}^s = \sum_{\substack{\eta\lambda \in G^{k+m} \\ |\eta|=k, |\lambda|=m}} |K_{\eta\lambda}|^s \leq M_1^s \, c_{G,k}^s \, c_{G,m}^s.$$

Thus $\log(c_{G,k}^s)$ is a subadditive succession, the limit $p_G(s) = \lim_{k\to\infty} \frac{1}{k} \log(c_{G,k}^s)$ exists and

$$(2.2) \qquad M_1^{-s} \, e^{kp_G(s)} \leq c_{G,k}^s.$$

Moreover, we know that $p_G(s)$ is a continuous decreasing function of s and there exists a unique $s_G \geq 0$ such that $p_G(s_G) = 0$ which is the similarity dimension of G. The following conditions are trivial consequences, of the definition, see [7, Proposition 3.2].

Proposition 2.1. *Let s_G the similarity dimension of G, then*

$$s_G = \inf\{s \geq 0 : \sum_{k=1}^\infty c_{G,k}^s < \infty\} = \sup\{s \geq 0 : \sum_{k=1}^\infty c_{G,k}^s = \infty\},$$

$$s_G = \inf\{s \geq 0 : \lim_{k\to\infty} c_{G,k}^s = 0\} = \sup\{s \geq 0 : \lim_{k\to\infty} c_{G,k}^s = \infty\}.$$

It is not true, in general, that $c_{G,k+m}^s \geq C \, c_{G,k}^s \, c_{G,m}^s$ for some constant $C > 0$. However, for $G = I$, it is easy to see that $c_{I,k+m}^s \geq M_0^s \, c_{I,k}^s \, c_{I,m}^s$ and

$$(2.3) \qquad M_1^{-s} \, e^{kp_I(s)} \leq c_{I,k}^s \leq M_0^{-s} \, e^{kp_I(s)}.$$

We remark that if H, G are subsystems, $H \subseteq G \subseteq I$, then $c_{H,k}^s \leq c_{G,k}^s \leq c_{I,k}^s$, $p_H(s) \leq p_G(s) \leq p_I(s)$, for all $k, s > 0$, and $s_H \leq s_G \leq s_I$.

The following theorem relates $\dim K_G$ with s_G. The first part is standard whereas second part was proved in [7] for similitudes. Falconer's proof extends to the bounded distortion case but we offer a different proof here to keep this work self–contained and because it contains techniques that we will use repeatedly.

Theorem 2.2. *Let G a subsystem, then $\dim K_G \leq s_G$. If, in addition, G is separated then $\dim K_G = s_G$ and $\mathcal{H}^{s_G}(K_G) > 0$.*

Proof. Let $t > s_G$, then $p_G(t) < 0$. For large k we have $\frac{1}{k} \log c_{G,k}^t \leq -\epsilon < 0$ and $c_{G,k}^t \leq e^{-k\epsilon} < 1$. This implies that the t-dimensional Hausdorff measure of K_G is finite, $\mathcal{H}^t(K_G) < \infty$, for all $t > s_G$, and thus $\dim K_G \leq s_G$.

Assume now that G is separated, we prove that $s_G \leq \dim K_G$. If $\dim K_G \geq s_I$ there is nothing to prove, since we have $s_I \leq \dim K_G \leq s_G \leq s_I$.

Suppose $\dim K_G < s_I$ and let t be such that

$$(2.4) \qquad\qquad \dim K_G < t < s_I \, .$$

We want to prove that there is $B > 0$ such that $c_{G,k}^t \leq B$ for all k. Then $p_G(t) \leq 0$ which implies $s_G \leq t$ and thus, since this is true for all t satisfying (2.4), we have $s_G \leq \dim K_G$.

By (2.4) $\mathcal{H}^t(K_G) = 0$ thus, for all $\epsilon > 0$ and taking into account that K_G is compact, there exists a finite cover $K_G \subset \bigcup U_j$ such that $\sum |U_j|^t < \epsilon$. We set $\epsilon < M_1^{-t} M^{-1}$, where M is the separation constant, and let l_0 such that $r_{\max}^{l_0}|K| < |U_j|$ for all j. We use induction to prove that

$$(2.5) \qquad\qquad c_{G,k}^t \leq M_0^{-t} \, e^{l_0 p_I(t)} \, ,$$

for all k.

For $k \leq l_0$ we have that

$$c_{G,k}^t \leq c_{I,k}^t \leq M_0^{-t} e^{k p_I(t)} \leq M_0^{-t} e^{l_0 p_I(t)} \, ,$$

by (2.3) and since $p_I(t) > 0$ by (2.4). Now, suppose $l > l_0$ and that (2.5) is true for all $k < l$, we evaluate

$$c^t_{G,l}(U_j) = \sum_{\substack{\omega \in G^l \\ K_\omega \cap U_j \neq \emptyset}} |K_\omega|^t.$$

We remark that for each $\omega \in G^l$, $K_\omega \cap U_j \neq \emptyset$, we have $|K_{\omega_1 \ldots \omega_{l-1}}| \leq r_{\omega_1 \ldots \omega_{l-1}}|K| \leq r^{l-1}_{\max}|K| \leq r^{l_0}_{\max}|K| < |U_j|$, then ω has an initial word η, $\omega = \eta\lambda$, such that $\eta \in G(U_j)$. Thus

$$c^t_{G,l}(U_j) \leq \sum_{\eta \in G(U_j)} \sum_{\eta\lambda \in G^l} |K_{\eta\lambda}|^t \leq \sum_{\eta \in G(U_j)} \sum_{\lambda \in G^{l-|\eta|}} |K_{\eta\lambda}|^t$$

$$\leq M^t_1 \sum_{\eta \in G(U_j)} |K_\eta|^t c^t_{G,l-|\eta|}$$

$$\leq M^{-t}_0 e^{l_0 p_I(t)} M^t_1 \sum_{\eta \in G(U_j)} |K_\eta|^t \ , \ \text{by the inductive hypothesis}$$

$$\leq M^{-t}_0 e^{l_0 p_I(t)} M^t_1 M |U_j|^t \ ,$$

since $|K_\eta| < |U_j|$ and $\sharp(G(U_j)) \leq M$ by the separation property. Thus, since $\{U_j\}$ is a finite cover of K_G,

$$c^t_{G,l} \leq \sum_j c^t_{G,l}(U_j)$$

$$\leq M^{-t}_0 e^{l_0 p_I(t)} M^t_1 M \sum_j |U_j|^t$$

$$\leq M^{-t}_0 e^{l_0 p_I(t)} M^t_1 M \epsilon \leq M^{-t}_0 e^{l_0 p_I(t)} \ ,$$

since $\sum_j |U_j|^t < \epsilon < M^{-t}_1 M^{-1}$. This complete the proof of (2.5) and hence $\dim K_G = s_G$. Now, to prove that $\mathcal{H}^{s_G}(K_G) > 0$, we simply remark that the separation property implies the conditions of Falconer [5, Theorem 2] for K_G. $\square$

Recurrent sets and graph-directed sets [2, 11] are standard generalizations of IFS that corresponds to particular types of subsystems [7]. In the context of this work we consider the following definition. We will call *restriction* to a subset of words of length n, $R \subseteq I^n$, and consider the subset $[I|R]$ of all

successions $\omega = \omega_1 \omega_2 \cdots \in I$ such that $\omega_{i+1}\omega_{i+2}\ldots\omega_{i+n} \in R$ for all i. It is easy to see that $[I|R]$ is in fact a subsystem of I and we say that $[I|R]$ is a *recurrent subsystem* defined by the restriction R.

We associate $[I|R]$ with a directed graph. Let $[I|R]^{n-1}$ be the vertex set, and we draw an edge from λ to η if and only if $\lambda\eta \in [I|R]^*$. If this directed graph is strongly connected (i.e. every two vertices can be connected through a directed path) we say that $[I|R]$ is a *connected recurrent subsystem*. The classical theory of IFS extends to connected recurrent subsystems and separation properties were studied in that case [3].

If G is a subsystem we consider the recurrent subsystems $[I|G^k]$. It is clear that $[I|G^k] \supseteq [I|G^{k+1}] \supseteq G$ for all k and $G = \cap_{k=1}^\infty [I|G^k]$ by compacity. Moreover $s_{[I|G^k]}$ tends to s_G as we would expect.

Proposition 2.3. $\lim_{k\to\infty} s_{[I|G^k]} = s_G$.

Proof. It is clear that $s_{[I|G^k]}$ is a non increasing succession $s_{[I|G^k]} \geq s_{[I|G^{k+1}]} \geq s_G$, then $\lim_{k\to\infty} s_{[I|G^k]} = t \geq s_G$. Let $s > s_G$, then

$$c_{[I|G^k],km}^s \leq M_1^{s(m-1)}\left(c_{[I|G^k],k}^s\right)^m = M_1^{-s}\left(M_1^s c_{G,k}^s\right)^m,$$

by (2.1) and taking into account that $[I|G^k]^k = G^k$. Proposition 2.1 implies that $M_1^s c_{G,k}^s < 1$ for k great enough, then $c_{[I|G^k],km}^s \to 0$ when $m \to \infty$ which implies $t \leq s_{[I|G^k]} \leq s$ and thus $t = s_G$. $\qquad\square$

In some cases, we need to approximate G through a family of systems which are not necessarily subsystems of I. More precisely, let $\{J_h^1\}$ a family of finite subsets $J_h^1 \subset G^*$ indexed by a positive real or integer parameter h. We say that $\{J_h^1\}$ *approaches* G if for each h there are positive integer numbers $k_0(h)$ and $k_1(h)$ such that: $k_0(h) \leq |\lambda| \leq k_1(h)$ for all $\lambda \in J_h^1$; $k_0(h) \to \infty$ when $h \to \infty$ and, for all $\omega \in G^{k_1(h)}$, there exists at least one $\lambda \in J_h^1$ such that $\omega = \lambda\eta$.

Each J_h^1 is in fact an alphabet for a system J_h which is not a subsystem of I according to our definition. However, we have the following result which relates the similarity dimensions of I and J_h.

Proposition 2.4. *Let* $\{J_h^1\}$ *a family that approaches* G, *then* $s_G \leq s_{J_h}$ *for all* h *and* $\lim_{h \to \infty} s_{J_h} = s_G$.

Proof. We show first that $s_G \leq s_{J_h}$. For $m \in \mathbb{N}$, we remark that for each $\omega \in G^{k_1(h)m}$ there exists at least one $\alpha \in J_h^*$ such that $\omega = \alpha\beta$ with $|\beta| < k_1(h)$, thus

$$c_{G,k_1(h)m}^s \leq \sum_{\alpha,\beta} |K_{\alpha\beta}|^s \, ,$$

where $|\alpha\beta| = k_1(h)m$, $\alpha \in J_h^*$ and $|\beta| < k_1(h)$. Then if $\alpha \in J_h^n$ it results that $m - 1 < n \leq (k_1(h)/k_0(h))m$. Let m' the integer part of $(k_1(h)/k_0(h))m$ and $M = \max\{c_{G,k}^s : k < k_1(h)\}$, then

$$c_{G,k_1(h)m}^s \leq M_1^s M \sum_{\alpha} |K_\alpha|^s \leq M_1^s M \sum_{n=m}^{m'} c_{J_h,n}^s \, .$$

If $s > s_{J_h}$ then $\sum_{n=m}^{m'} c_{J_h,n}^s \to 0$ when $m \to \infty$ by Proposition 2.1. Thus $\lim_{m \to \infty} c_{G,k_1(h)m}^s = 0$ for all $s > s_{J_h}$ and $s_G \leq s_{J_h}$.

Now let $s > s_G$ and observe that

$$c_{J_h,m} \leq \left(M_1^s\right)^{m-1} \left(\sum_{\lambda \in J_h^1} |K_\lambda|^s\right)^m \leq M_1^{-s} \left(M_1^s \sum_{n=k_0(h)}^{k_1(h)} c_{G,n}^s\right)^m .$$

By Proposition 2.1 $\sum_{n=k_0(h)}^{k_1(h)} c_{G,n}^s < M_1^{-s}$ for all h great enough, then $c_{J_h,m} \to 0$ when $m \to \infty$ and this implies that $s_{J_h} \leq s$. Thus $\lim_{h \to \infty} s_{J_h} = s_G$. $\qquad\square$

3. Weak separation properties

In this section, we will relate the weak separation property (WSP) of Lau and Ngai [9, 16] with the existence of a separate subsystem and propose a generalization for IFS of bounded distortion. First, we make the following general observation whose proof is left to the reader.

Proposition 3.1. *Suppose we have a succession of subsets $G_1 \subseteq I^1$, $G_2 \subseteq I^2, \ldots$ such that, for all word $\omega_1 \omega_2 \ldots \omega_n \in G_n$, we have that $\omega_2 \ldots \omega_n \in G_{n-1}$. Then $G = \{\omega \in I : \omega_1 \omega_2 \ldots \omega_n \in G_n, \text{ for all } n\}$ is a subsystem of I and $G^n \subseteq G_n$.*

Now, we introduce a total order on I^* by setting $\lambda < \omega$ if $|\lambda| < |\omega|$, and the lexicographic order if $|\lambda| = |\omega|$. Let us define a subsystem W by

$$W_n = \{\omega \in I^n : \varphi_\omega \neq \varphi_\lambda \, ; \text{ for all } \lambda < \omega \,\} \,,$$

and $W = \{\omega \in I : \omega_1 \ldots \omega_n \in W_n \,, \text{ for all } n \,\}$. This is a subsystem by the preceding proposition (we will sketch the argument in Theorem 3.3), moreover it is easy to see that and $K_W = K$.

We recall now some definitions of Zerner [16]. In that follows we assume that the φ_i's are similitudes and K is in general position. For $a, b > 0$ and $A, U \subseteq \mathbb{R}^n$ let us define

$$F = \{\varphi_\omega : \omega \in I^*\} = \{\varphi_\omega : \omega \in W^*\} \,,$$

$$F_b = \{\varphi_\omega \in F : r_\omega \in (b \, r_{\min}, b]\} \,, \text{ and}$$

$$F_{a,U,M} = \{\varphi_\omega \in F_{a|U|} : \varphi_\omega(M) \cap U \neq \emptyset\} \,.$$

It is clear that there is a one to one correspondence between F and W^*. The *growth dimension* β_I of I is defined as the exponential growth rate of $\sharp(F_b)$ for $b \to 0$. In our notation, consider the family $\{J_h^1\}$ where $J_h^1 = \{\omega \in W^* : r_\omega \in (\frac{1}{h} r_{\min}, \frac{1}{h}]\}$ for $h > 0$, then $F_{1/h} = \{\varphi_\omega : \omega \in J_h^1\}$ and $\beta_I = \lim_{h \to \infty} s_{J_h}$ [16, Theorem 2]. Setting $a > 0$ and $M \subseteq \mathbb{R}^n$ non empty, we say that I satisfy WSP if and only if $\sharp(F_{a,U,M})$ is bounded for all $U \subseteq \mathbb{R}^n$ [16, Theorem 1]. In that case $\dim K = \beta_I$. With respect to the subsystem W, we have the following result.

Theorem 3.2. *$\beta_I = s_W$ and I satisfy WSP if and only if W is separated. In that case $\dim K = s_W$ and $\mathcal{H}^{s_W}(K) > 0$.*

Proof. First we observe that $\{J_h^1\}$ is a family that approaches W, then $\beta_I = \lim_{h\to\infty} s_{J_h} = s_W$ by Proposition 2.4. Now, we set $a = \frac{1}{|K|}$, and $M = K$, then by the one to one correspondence between F and W^*,

$$\sharp(F_{a,U,M}) = \sharp\{\omega \in W^* : r_{\min}|U| < |K_\omega| \le |U|;\ K_\omega \cap U \ne \emptyset\},$$

since $|K_\omega| = r_\omega|K|$ for similitudes. We denote by W_U the right set, and compare it with $W(U)$. Let $\omega \in W(U)$, then $|U| \le |K_{\omega_1\ldots\omega_{k-1}}|$ and $r_{\min}|U| \le |K_\omega|$. If $r_{\min}|U| < |K_\omega|$ then $\omega \in W_U$, on the other hand if $r_{\min}|U| = |K_\omega|$, then $r_{\min}|U| < |K_{\omega_1\ldots\omega_{k-1}}| = |U|$ and $\omega_1\ldots\omega_{k-1} \in W_U$. Conversely let $\omega \in W_U$. If $|K_\omega| < |U|$ then there is a $h \le k$ such that $|K_{\omega_1\ldots\omega_h}| < |U| \le |K_{\omega_1\ldots\omega_{h-1}}|$ and $\omega_1\ldots\omega_h \in W(U)$, on the other hand if $|K_\omega| = |U|$, then $|K_{\omega j}| < |U|$ and $\omega j \in W(U)$ for some $j \in I$.

Thus $\sharp(F_{a,U,M})$ is bounded if and only if $\sharp(W(U))$ is bounded, i.e. if and only if W is separated. The last assertion follows directly from Theorem 2.2 since $K_W = K$. $\qquad\square$

Now, we move to the bounded distortion case to generalize WSP. The subsystem W was constructed by eliminating words ω such that $K_\omega = K_\lambda$, for some word $\lambda < \omega$. We propose a direct generalization by eliminating words ω such that $K_\omega \subset K_{\lambda(1)} \cup \cdots \cup K_{\lambda(k)}$ for some words $\lambda(1),\ldots,\lambda(k) < \omega$. Specifically, we define:

$$GW_n = \{\omega \in I^n : K_\omega \not\subset K_{\lambda(1)} \cup \cdots \cup K_{\lambda(k)};\ \text{for all } \lambda(1),\ldots,\lambda(k) < \omega\},$$

and $GW = \{\omega \in I : \omega_1\ldots\omega_n \in GW_n,\ \text{for all } n\ \}$.

Theorem 3.3. *GW is a subsystem with $K_{GW} = K$. If GW is separated then* $\dim K = s_{GW}$ *and* $\mathcal{H}^{s_{GW}}(K) > 0$.

Proof. We will prove that GW is a subsystem by using Proposition 3.1. That $K_{GW} = K$ is clear from the definition and then the theorem follows from Theorem 2.2. Let $\omega_1\omega_2\ldots\omega_n \in GW_n$ and suppose that $\omega_2\ldots\omega_n \notin GW_{n-1}$,

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then there are $\lambda(1), \ldots, \lambda(k) < \omega_2 \ldots \omega_n$ such that $K_{\omega_2 \ldots \omega_n} \subset K_{\lambda(1)} \cup \cdots \cup K_{\lambda(k)}$. We then remark that $\omega_1 \omega_2 \ldots \omega_n > \omega_1 \lambda(i)$ and

$$K_{\omega_1 \omega_2 \ldots \omega_n} = \varphi_{\omega_1}(K_{\omega_2 \ldots \omega_n}) \subset \varphi_{\omega_1}(K_{\lambda(1)} \cup \cdots \cup K_{\lambda(k)})$$

$$\subset \varphi_{\omega_1}(K_{\lambda(1)}) \cup \cdots \cup \varphi_{\omega_1}(K_{\lambda(k)}) = K_{\omega_1 \lambda(1)} \cup \cdots \cup K_{\omega_1 \lambda(k)},$$

which is a contradiction. Thus $\omega_2 \ldots \omega_n \in GW_{n-1}$ and GW is a subsystem. $\square$

If GW is separated we say that I satisfy GWSP. The next example shows a system which satisfy GWSP but do not satisfy WSP.

Example 3.4. Let $X = [0,1] \times [0,1]$ the unit square in $\mathbb{R}^2$. Let $I^1 = \{1, 2, 3, 4, 5, 6, 7\}$;

$$\varphi_1(x) = \tfrac{1}{3} x + (0, \tfrac{2}{3}), \ \varphi_2(x) = \tfrac{1}{3} x + (\tfrac{1}{3}, \tfrac{2}{3}), \ \varphi_3(x) = \tfrac{1}{3} x + (\tfrac{2}{3}, \tfrac{2}{3})$$

$$\varphi_4(x) = \tfrac{1}{3} x, \ \varphi_5(x) = \tfrac{1}{3} x + (\tfrac{1}{3}, 0), \ \varphi_6(x) = \tfrac{1}{3} x + (\tfrac{2}{3}, 0), \ \text{and}$$

$$\varphi_7(x) = \tfrac{1}{3} x + (\delta, 0), \ \text{where } 0 < \delta < \tfrac{1}{3} \text{ is an irrational number.}$$

It is easy to see that $K = [0,1] \times C$, where C is the usual Cantor set. Now we consider the subsystems W and GW. First we observe that if $\omega = \omega_1 \ldots \omega_k \in I^*$, then

$$\varphi_\omega(x) = (\tfrac{1}{3})^k x + (q_1 + q_2 \delta, q_3),$$

where q_1, q_2, q_3 are rational numbers, $0 \le q_1, q_3 \le 1$ and $0 \le q_2 \le \tfrac{3}{2}$. Moreover

$$q_2 = 1^{e_1} + \left(\tfrac{1}{3}\right)^{e_2} + \cdots + \left(\tfrac{1}{3}\right)^{(n-1)e_n}, \ \text{where } e_i = \begin{cases} 0 \text{ if } \omega_i \ne 7 \\ 1 \text{ if } \omega_i = 7 \end{cases}.$$

Taking this into account we can see that $\varphi_\omega = \varphi_\lambda$ if and only if $\omega = \lambda$ for all $\omega, \lambda \in I^*$. Indeed, if $\varphi_\omega = \varphi_\lambda$, then $\omega_i = 7$ if and only if $\lambda_i = 7$ and then must be $\omega = \lambda$. Therefore $W = I$ which is not separated, that is I do not satisfy WSP.

On the other hand $GW_1 = GW^1 = \{1, 2, 3, 4, 5, 6\}$ since $K_7 \subseteq K_4 \cup K_5$, then $GW = \{1, 2, 3, 4, 5, 6\}^\infty$ which is separated (satisfy OSC) and I satisfy GWSP. In Figure 1 we schematize the seven transformations over the unit square X
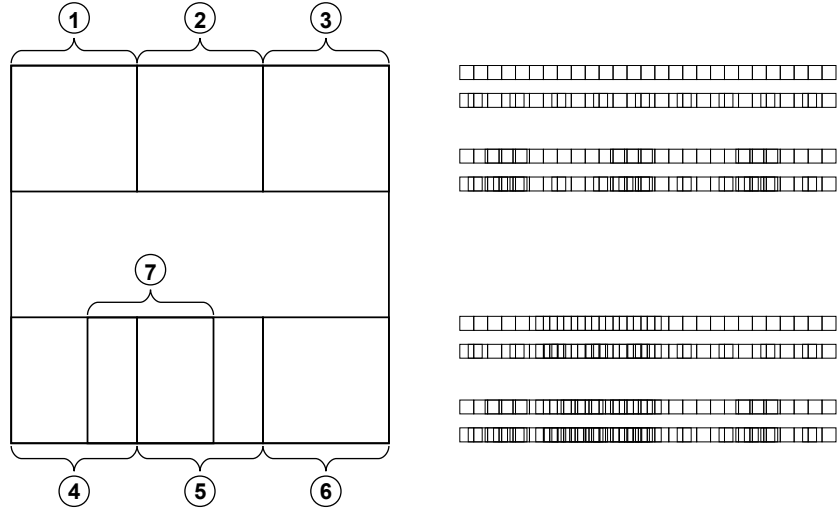
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FIGURE 1. A system satisfying GWSP but not WSP

and the subset $\cup_{\omega \in I^3} \varphi_\omega(X)$ (on the right) which approximates K and shows the overlapping effect.

We remark that K is a self-similar set in general position satisfying $\dim K = s_{GW} = \frac{\log 6}{\log 3} < \frac{\log 7}{\log 3} = s_W < \dim X = 2$, whose existence was questioned in Zerner [16].

4. POSITIVE HAUSDORFF MEASURE

In this section we study subsystems with $\mathcal{H}^{s_G}(K_G) > 0$. We know that $\mathcal{H}^{s_G}(K_G) > 0$ implies that G is separated if $G = I$ (see [15] and [12] for similitudes and [13] for conformal systems), and it is also true if G is a connected recurrent subsystem.

To analyze the general case we must introduce some notation. Let $Z \subset K_G$ the set of points where the separation property fails: $Z = \{z \in K_G :$ for all $N > 0$ there exists a closed set A such that $z \in A$ and $\sharp(G(A)) \geq N\}$. It is easy to see that G is separated if and only if $Z = \emptyset$.

For $\omega \in G$ let us define:

$$\underline{b}(\omega, n) = \liminf_{m \to \infty} \sum_{\substack{\lambda \in G^m \\ \omega_1 \ldots \omega_n \lambda \in G^{n+m}}} |K_\lambda|^{s_G}, \qquad \underline{b}(\omega) = \inf_n \underline{b}(\omega, n)$$

$$\overline{b}(\omega, n) = \limsup_{m \to \infty} \sum_{\substack{\lambda \in G^m \\ \omega_1 \ldots \omega_n \lambda \in G^{n+m}}} |K_\lambda|^{s_G}, \qquad \overline{b}(\omega) = \sup_n \overline{b}(\omega, n)$$

We remark that $0 \leq \underline{b}(\omega) \leq \underline{b}(\omega, n) \leq \overline{b}(\omega, n) \leq \overline{b}(\omega) \leq +\infty$. The functions $\underline{b}(\omega)$ and $\overline{b}(\omega)$ are not continuous in general, but we have the following

Proposition 4.1. *Let $\omega \in G$ and $\epsilon > 0$.*

(1) *There is $\delta > 0$ such that $\underline{b}(\eta) \leq \underline{b}(\omega) + \epsilon$ for all $\eta \in G$ such that $d(\omega, \eta) < \delta$.*

(2) *If $\overline{b}(\omega) > N$, then there is δ such that $\overline{b}(\eta) \geq N - \epsilon$ for all $\eta \in G$ such that $d(\omega, \eta) < \delta$.*

Proof. To prove (1), let n_0 such that $\underline{b}(\omega, n_0) \leq \underline{b}(\omega) + \epsilon$, then if $d(\omega, \eta) < 1/\ell^{n_0}$ we have that $\omega_1 \ldots \omega_{n_0} = \eta_1 \ldots \eta_{n_0}$ and

$$\underline{b}(\eta) \leq \underline{b}(\eta, n_0) = \underline{b}(\omega, n_0) \leq \underline{b}(\omega) + \epsilon.$$

The proof of (2) follows in a similar way. $\qquad\qquad\square$

Next, we will characterize subsets of Z with null and positive s_G-Hausdorff measure using these functions. First we will see, in the following example, that is not true in general that $\mathcal{H}^{s_G}(K_G) > 0$ implies $\mathcal{H}^{s_G}(Z) = 0$.

Example 4.2. We consider the system $I = \{1, 2, 3, 4\}$ and the restrictions $R1 = \{11, 12, 21, 22\}$, $R2 = \{33, 34, 43, 44\}$ and $R = R1 \cup R2 \cup \{13, 14, 23, 24\}$. Let $G1 = [I|R1]$, $G2 = [I|R2]$ and $G = [I|R]$ the corresponding recurrent subsystems, $G1$ and $G2$ are connected whereas G is not. The associated directed graphs are drawn in Figure 2.
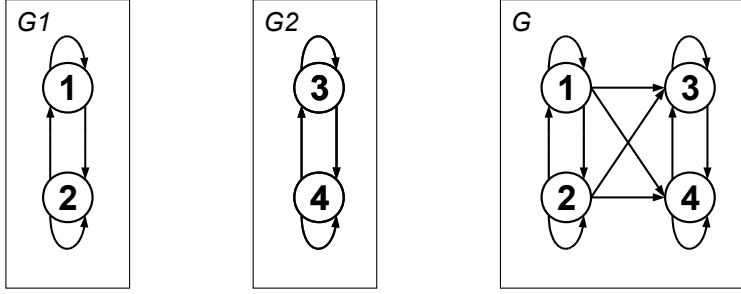
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FIGURE 2. Directed graphs associated with Example 4.2

We assume that $s_{G2} \leq s_{G1}$ and prove that $s_G = s_{G1}$. In fact, $s_{G1} \leq s_G$ since $G1 \subseteq G$. On the other hand

$$c_{G,k}^{s_{G1}} = \sum_{\omega \in G2^k} |K_\omega|^{s_{G1}} + \sum_{h=1}^{k} \sum_{\substack{\alpha \in G1^h \\ \beta \in G2^{k-h}}} |K_{\alpha\beta}|^{s_{G1}}$$

$$\leq c_{G2,k}^{s_{G1}} + M_1^{s_{G1}} \sum_{h=1}^{k} c_{G1,h}^{s_{G1}} c_{G2,k-h}^{s_{G1}}.$$

Since $G1$ and $G2$ are connected recurrent subsystems we know that there exist $D > 0$ such that $c_{Gi,k}^{s_{Gi}} \leq D$ for $i = 1, 2$ and all k. Moreover, taking into account that $s_{G2} \leq s_{G1}$, we can choose D such that $c_{G2,k}^{s_{G1}} \leq D$ for all k (Indeed, if $s_{G2} < s_{G1}$ then $c_{G2,k}^{s_{G1}} \to 0$ when $k \to \infty$). Then

$$c_{G,k}^{s_{G1}} \leq D + k M_1^{s_{G1}} D^2 \leq k(D + M_1^{s_{G1}} D^2) \quad \text{and} \quad \lim_{k \to \infty} \tfrac{1}{k} \log c_{G,k}^{s_{G1}} \leq 0 \,,$$

which implies $s_G \leq s_{G1}$.

Now, we will consider two particular examples in $\mathbb{R}^2$ where $\mathcal{H}^{s_G}(Z) > 0$ and we will study the values of $\bar{b}$ and $\underline{b}$. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ the rotation of angle $\frac{\pi}{4}$ around the origin and let $a, b \in \mathbb{R}$, $0 < a < b < \frac{\sqrt{2}}{2}$. Let

$$\varphi_1(x) = r_1 R(x), \ \varphi_2(x) = r_2 R\big(x - (1,0)\big) + (1,0)$$

$$\varphi_3(x) = r_3 R(x), \ \varphi_4(x) = r_4 R\big(x - (1,0)\big) + (1,0).$$

For the first example we set $r_1 = r_4 = b$ and $r_2 = r_3 = a$, then $s_{G2} = s_{G1} = s_G$. Besides we know that $G1$ is separated, then $\mathcal{H}^{s_G}(K_{G1}) > 0$. On the other
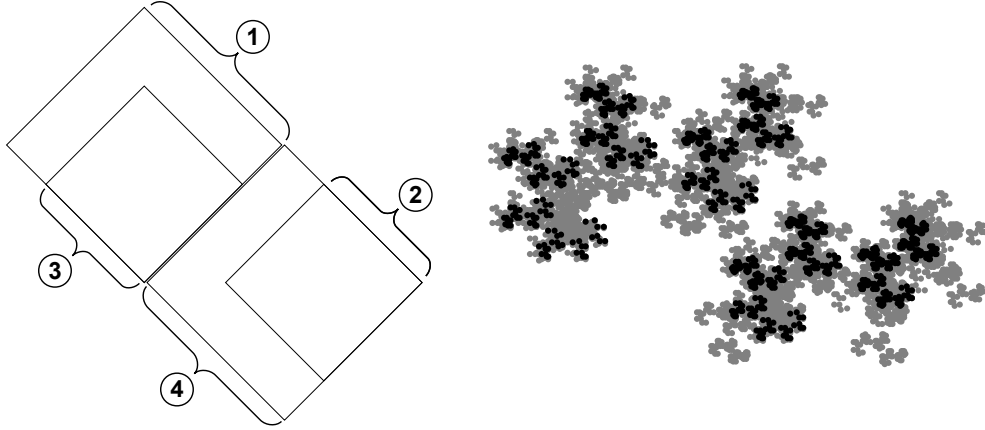
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FIGURE 3. Non separated subsystem: black points are contained in Z and have positive s_G-Hausdorff measure

hand G is not separated: it is easy to see that that $0 \in Z$ and then, since for all $\beta \in G$, $\alpha \in G1$ we have that $\alpha\beta \in G$, it results that $K_{G1} \subseteq Z$. Thus $\mathcal{H}^{s_G}(Z) > 0$. In Figure 3 we show the four transformations applied to the unit square and approximations of K_G in gray, and K_{G1} in black.

By an argument similar to that used above, we can see that there is a constant D' such that

$$c^{s_G}_{G,k} \geq D' + k M^{s_G}_0 D'^2 \,.$$

Thus, if $\omega \in G1$, $\bar{b}(\omega, n) = \limsup_{k \to \infty} c^{s_G}_{G,k} = +\infty$ for all n. Therefore we have that $\bar{b}(\omega) = +\infty$ for all $\omega \in G1$, it is to say $K_{G1} \subseteq \Pi\big(\{\omega \in G : \bar{b}(\omega) = +\infty\}\big)$.

For the second example we set $r_1 = b$ and $r_2 = r_3 = r_4 = a$, then $s_{G2} < s_{G1} = s_G$. Again $G1$ is separated, $\mathcal{H}^{s_G}(K_{G1}) > 0$ and $K_{G1} \subseteq Z$. Now, for $\omega \in G2$, $\underline{b}(\omega, n) = \liminf_{k \to \infty} c^{s_G}_{G2,k} = 0$ for all n. Then $\underline{b}(\omega) = 0$ for all $\omega \in G2$ and, moreover, $\underline{b}(\omega) = 0$ for all $\omega = \alpha\beta$ such that $\beta \in G2$. But every succession $\eta \in G$ may be approximated by ω's such that $\omega = \alpha\beta$, $\beta \in G2$, then $K_{G1} \subseteq K_G = \Pi\big(\overline{\{\omega \in G : \underline{b}(\omega) = 0\}}\big)$.

In these examples we found a subset K' of Z with positive s_G-Hausdorff measure and such that K' is contained in the closure of a subset where $\underline{b}(\omega)$

is arbitrarily small or $\bar{b}(\omega)$ is arbitrarily large. The next Theorem shows that such subset always exist when $\mathcal{H}^{s_G}(Z) > 0$.

Theorem 4.3. *Consider the subset*

$$Z' = \Pi\left(\overline{\cap_{\epsilon>0}\left(\{\omega \in G : \underline{b}(\omega) \leq \epsilon\} \cup \{\omega \in G : \bar{b}(\omega) \geq \tfrac{1}{\epsilon}\}\right)}\right),$$

then $\mathcal{H}^{s_G}(Z - Z') = 0$.

The proof follows from two lemmas. Let us define

$$B'(\epsilon) = \{\omega \in G : \epsilon \leq \underline{b}(\omega), \, \bar{b}(\omega) \leq 1/\epsilon\}; \quad \text{and}$$

$$B(\epsilon) = \{\omega \in B'(\epsilon) : d(\Pi(\omega), K_G - \Pi(B'(\epsilon))) \geq \epsilon\},$$

where d corresponds to the euclidean distance.

Lemma 4.4. *Let $\eta \in G^*$, if there exist $\beta \in G$ such that $\omega = \eta\beta \in B'(\epsilon)$, then*

$$\epsilon/2 \leq \sum_{\substack{\lambda \in G^{k-|\eta|} \\ \eta\lambda \in G^k}} |K_\lambda|^{s_G} \leq 2/\epsilon,$$

for k large enough. In particular, the inequalities follow if η is such that $K_\eta \cap K_G \subseteq \Pi(B'(\epsilon))$.

Proof. From the $B'(\epsilon)$ definition we have that

$$\liminf_{k \to \infty} \sum_{\substack{\lambda \in G^{k-|\eta|} \\ \eta\lambda \in G^k}} |K_\lambda|^{s_G} = \underline{b}(\omega, |\eta|) \geq \underline{b}(\omega) \geq \epsilon$$

$$\limsup_{k \to \infty} \sum_{\substack{\lambda \in G^{k-|\eta|} \\ \eta\lambda \in G^k}} |K_\lambda|^{s_G} = \bar{b}(\omega, |\eta|) \leq \bar{b}(\omega) \leq 1/\epsilon,$$

and the lemma results from limit properties. □

Lemma 4.5. $\mathcal{H}^{s_G}(Z \cap \Pi(B(\epsilon)) = 0$ *for all $\epsilon > 0$.*

Proof. We fix $\epsilon > 0$ and suppose $Z \cap \Pi(B(\epsilon)) \neq \emptyset$. For $N > 0$ let us denote by $\mathcal{A}_N$ the family of sets A such that: $0 < |A| < \epsilon/3$, $A \cap Z \cap \Pi(B(\epsilon)) \neq \emptyset$ and $\sharp(G(A)) \geq N$. It is clear that $\mathcal{A}_N$ is a Vitali family for $Z \cap \Pi(B(\epsilon))$. Now,

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for $A \in \mathcal{A}_N$ we define $U_A = \bigcup_{\eta \in G(A)} K_\eta \cap K_G$, then $\{U_A\}_{A \in \mathcal{A}_N}$ is also a Vitali family for $Z \cap \Pi(B(\epsilon))$. By the Vitali covering theorem [4, Theorem 1.10], we have for all $\epsilon_2 > 0$ that there exists a disjoint finite family $\{U_{A_j}\}$ such that

$$(4.1) \qquad \mathcal{H}^{s_G}(Z \cap \Pi(B(\epsilon)) \leq \sum_j |U_{A_j}|^{s_G} + \epsilon_2 \,.$$

Now, we remark that, for k large enough such that $|K_{\omega_1 \ldots \omega_{k-1}}| < |A_j|$ for all j and all $\omega \in G$, we have

$$c_{G,k}^{s_G}(U_{A_j}) = \sum_{\substack{\omega \in G^k \\ K_\omega \cap U_{A_j} \neq \emptyset}} |K_\omega|^{s_G} \geq \sum_{\eta \in G(A_j)} \sum_{\eta \lambda \in G^k} |K_{\eta\lambda}|^{s_G}$$

$$\geq M_0^{s_G} \sum_{\eta \in G(A_j)} |K_\eta|^{s_G} \left(\sum_{\substack{\lambda \in G^{k-|\eta|} \\ \eta\lambda \in G^k}} |K_\lambda|^{s_G} \right).$$

We observe that

$$(4.2) \qquad d\big(K_\eta, K_G - \Pi(B'(\epsilon))\big) \geq \epsilon/3 \,,$$

for all $\eta \in G(A_j)$, since $K_\eta \cap A_j \neq \emptyset$, $A_j \cap \Pi(B(\epsilon)) \neq \emptyset$ and $|K_\eta| < |A_j| < \epsilon/3$. Thus $K_\eta \cap K_G \subseteq \Pi(B'(\epsilon))$ and using the previous lemma we find that

$$c_{G,k}^{s_G}(U_{A_j}) \geq \frac{M_0^{s_G} \epsilon}{2} \sum_{\eta \in G(A_j)} |K_\eta|^{s_G} \geq C \sum_{\eta \in G(A_j)} |U_{A_j}|^{s_G}$$

$$\geq C \, N \, |U_{A_j}|^{s_G} \,,$$

where $C = \dfrac{\left(M_0^{s_G}\right)^2 \epsilon \, r_{\min}^{s_G}}{2.3^{s_G}}$. Since $|K_\eta| \geq r_{\min} M_0 |A_j| \geq r_{\min} M_0 (|U_{A_j}|/3)$ and $\sharp(G(A_j)) \geq N$. Therefore,

$$(4.3) \qquad \sum_j |U_{A_j}|^{s_G} \leq \frac{1}{CN} \sum_j c_{G,k}^{s_G}(U_{A_j}) \leq \frac{1}{CN} c_{G,k}^{s_G}\big(\Pi(B(\epsilon/3))\big) \,,$$

for k large enough since U_{A_j} are disjoint sets and $U_{A_j} \subset \Pi(B(\epsilon/3))$ which follows from (4.2).

Now, let m such that $|K_\eta| < \epsilon/3$ for all $\eta \in G^m$. If $K_\eta \cap \Pi(B(\epsilon/3)) \neq \emptyset$ then $K_\eta \cap K_G \subset \Pi(B'(\epsilon/3))$ and we have that

$$c_{G,k}^{s_G}\big(\Pi(B(\epsilon/3))\big) \leq M_1^{s_G} \sum_{\substack{\eta \in G^m \\ K_\eta \cap \Pi(B(\epsilon/3)) \neq \emptyset}} |K_\eta|^{s_G} \left(\sum_{\substack{\lambda \in G^{k-m} \\ \eta\lambda \in G^k}} |K_\lambda|^{s_G} \right)$$

$$\leq M_1^{s_G} \frac{\epsilon}{6} \left(\sum_{\substack{\eta \in G^m \\ K_\eta \cap \Pi(B(\epsilon/3)) \neq \emptyset}} |K_\eta|^{s_G} \right),$$

for k large enough. Combining this inequality with (4.1) and (4.3) we obtain

$$\mathcal{H}^{s_G}(Z \cap \Pi(B(\epsilon)) \leq \frac{C'}{N} + \epsilon_2\,,$$

for some constant $C' > 0$ which depends on ϵ but is independent of N and ϵ_2. Thus $\mathcal{H}^{s_G}(Z \cap \Pi(B(\epsilon)) = 0$. $\qquad\square$

Proof of Theorem 4.3. The previous lemma implies that

$$\mathcal{H}^{s_G}\big(Z \cap \cup_{\epsilon>0}\Pi(B(\epsilon))\big) = 0\,.$$

Then we only need to show that $K_G - \cup_{\epsilon>0}\Pi(B(\epsilon)) = Z'$. Let $x \in K_G$, $x = \Pi(\omega)$ such that $x \notin \cup_{\epsilon>0}\Pi(B(\epsilon))$, then either $\omega \notin B'(\epsilon)$ for all ϵ or $\omega \in \Pi(B'(\epsilon))$ but $d(x, K_G - \Pi(B'(\epsilon))) < \epsilon$ for all $\epsilon \leq \epsilon_0$. In the first case, either $\underline{b}(\omega) = 0$ or $\bar{b}(\omega) = \infty$ and then $x \in Z'$. In the second case, let $\delta = d(x, K_G - \Pi(B'(\epsilon)))$. If $0 < \delta < \epsilon$ then $\omega \in B(\delta)$, since $\omega \in B'(\epsilon) \subset B'(\delta)$ and $d(x, K_G - \Pi(B'(\delta)) \geq d(x, K_G - \Pi(B'(\epsilon)) = \delta$, which is a contradiction. Thus must be $\delta = 0$ for all $\epsilon \leq \epsilon_0$ which implies $x \in Z'$. $\qquad\square$

At last, we consider connected subsystems, which generalize connected recurrent subsystems. We say that G is a *connected subsystem* if there exists $T > 0$ such that, for all $\alpha, \beta \in G^*$ there is a $\lambda_\alpha^\beta \in G^*$, $|\lambda_\alpha^\beta| \leq T$, such that $\alpha\lambda_\alpha^\beta\beta \in G^*$.

For $\alpha \in G^*$ we introduce the notation $c(\alpha)_{G,k}^s = \sum_{\substack{\lambda \in G^k \\ \alpha\lambda \in G^*}} |K_\lambda|^s$. Now, we can state the following result whose proof uses standard techniques and inequalities like (2.1) and (2.2).

Proposition 4.6. *If G is a connected subsystem then there are constants C_0, C_1 such that*

$$C_1^{-s} \, e^{kp_G(s)} \leq c(\alpha)_{G,k}^s \leq C_0^{-s} \, e^{kp_G(s)} \,,$$

for all $\alpha \in G^*$.

Theorem 4.7. *Let G a connected subsystem. If $\mathcal{H}^{s_G}(K_G) > 0$ then G is separated.*

Proof. The previous proposition implies that $C_1^{-s_G} \leq c(\alpha)_{G,k}^{s_G} \leq C_0^{-s_G}$ for all $\alpha \in G^*$. Therefore $C_1^{-s_G} \leq \underline{b}(\omega) \leq \bar{b}(\omega) \leq C_0^{-s_G}$ for all $\omega \in G$, $Z' = \emptyset$ and $\mathcal{H}^{s_G}(Z) = 0$ from Theorem 4.3. We want to prove that $Z = \emptyset$. Suppose $Z \neq \emptyset$, let $x \in Z$ and for all $N > 0$ let A_N be a closed set such that $x \in A_N$ and $\sharp(G(A_N)) \geq N$. Let $\alpha \in G^*$, $\Lambda = \{\lambda \in G^* : |\lambda| \leq T\}$ and $M = \sharp(\Lambda)$. As G is connected, we have that for each $\eta \in G(A_N)$ there is a $\lambda \in \Lambda$ such that $\alpha\lambda\eta \in G^*$. If $N > M$ then there exists a $\lambda \in \Lambda$ such that $\sharp(\{\eta \in G(A_N) : \alpha\lambda\eta \in G^*\}) \geq \frac{N}{M}$, thus $\sharp(G(\varphi_{\alpha\lambda}(A_N)) \geq \frac{N}{M}$. We can see in consequence that for all $\alpha \in G^*$ there is a $\omega \in G$ such that $z = \Pi(\alpha\omega) \in Z$, therefore must be $\mathcal{H}^{s_G}(Z) > 0$ which is a contradiction. Then $Z = \emptyset$ and G is separated. $\square$

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