A Topological Duality for Tense LM_n-Algebras and Applications[1]

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Abstract

In 2007, tense n-valued Łukasiewicz–Moisil algebras (or tense LM_n-algebras) were introduced by Diaconescu and Georgescu as an algebraic counterpart of the tense n-valued Moisil logic. In this article we continue the study of tense LM_n-algebras initiated by Figallo and Pelaitay (2014, *Log. J. IGPL*, **22**, 255–267). More precisely, we determine a topological duality for these algebras. This duality enables us not only to describe the tense LM_n-congruences on a tense LM_n-algebra, but also to characterize the simple and subdirectly irreducible tense LM_n-algebras. Furthermore, by means of the aforementioned duality, a representation theorem for tense LM_n-algebras is proved, which was formulated and proved by a different method by Georgescu and Diaconescu (2007, *Fund. Inform.*, **81**, 379–408).

Keywords: Łukasiewicz–Moisil algebras, tense Łukasiewicz–Moisil algebras, topological duality.

1 Introduction

In 1940, Gr. C. Moisil [22] introduced n-valued Łukasiewicz algebras (now these algebras are known as n-valued Łukasiewicz–Moisil algebras or LM_n-algebras for short). From that moment on, many articles have been published about this class of algebras. Many of the results obtained have been reproduced in the important book by Boicescu *et al*. [1] which can be consulted by any reader interested in broadening their knowledge on these algebras.

Propositional logics usually do not incorporate the dimension of time; consequently, in order to obtain a tense logic, a propositional logic is enriched by the addition of new unary operators (or connectives) which are usually denoted by G, H, F and P. We can define F and P by means of G and H as follows: $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$, where $\neg x$ denotes negation of the proposition x. Tense algebras (or tense Boolean algebras) are algebraic structures corresponding to the propositional tense logic (see [4, 21]). An algebra $\langle A, \vee, \wedge, \neg, G, H, 0, 1\rangle$ is a tense algebra if $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and G, H are unary operators on A which satisfy the following axioms for all $x, y \in A$:

$$G(1) = 1,\ H(1) = 1,$$

$$G(x \wedge y) = G(x) \wedge G(y),\ H(x \wedge y) = H(x) \wedge H(y),$$

$$x \leq GP(x),\ x \leq HF(x),$$

where $P(x) = \neg H(\neg x)$ and $F(x) = \neg G(\neg x)$.

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Taking into account that tense algebras constitute the algebraic basis for the bivalent tense logic, Diaconescu and Georgescu introduced in [13] the tense MV-algebras and the tense Łukasiewicz–Moisil algebras (or tense n-valued Łukasiewicz–Moisil algebras) as algebraic structures for some many-valued tense logics. In recent years, these two classes of algebras have become very interesting for several authors (see [2, 6–10, 17, 19, 20]). In particular, in [8, 9], Chiriţă introduced tense θ-valued Łukasiewicz–Moisil algebras and proved an important representation theorem which made it possible to show the completeness of the tense θ-valued Moisil logic (see [8]). In [13], the authors formulated an open problem about representation of tense MV-algebras, this problem was solved in [3, 23] for semi-simple tense MV-algebras. Also, in [2], tense basic algebras which are an interesting generalization of tense MV-algebras were studied.

The main purpose of this article is to give a topological duality for tense n-valued Łukasiewicz–Moisil algebras. In order to achieve this we will extend the topological duality given in [16], for n-valued Łukasiewicz–Moisil algebras. In [14], another duality for Łukasiewicz–Moisil algebras was developed, starting from Boolean spaces and adding a family of open sets.

The article is organized as follows: In Section 2, we briefly summarize the main definitions and results needed throughout this article. In Section 3, we developed a topological duality for tense n-valued Łukasiewicz–Moisil algebras, extending the one obtained in [16] for n-valued Łukasiewicz–Moisil algebras. In Section 4, the results of Section 3 are applied. First, we characterize congruences on tense n-valued Łukasiewicz–Moisil algebras by certain closed and increasing subsets of the space associated with them. This enables us to describe the subdirectly irreducible tense n-valued Łukasiewicz–Moisil algebras and the simple tense n-valued Łukasiewicz–Moisil algebras. In Section 5, which is the core of this article, a representation theorem for tense n-valued Łukasiewicz–Moisil algebras is proved using the duality obtained for these algebras. The proof of this result could be of interest for people working in duality theory. Finally, in Section 6, we describe the simple and subdirectly irreducible complete tense n-valued Łukasiewicz–Moisil algebras. We also provide a further characterization of the simple and subdirectly irreducible finite algebras.

2 Preliminaries

2.1 Tense De Morgan algebras

In [18], Figallo and Pelaitay introduced the variety of algebras, which they call tense De Morgan algebras, and they also developed a representation theory for this class of algebras.

First, recall that an algebra $\langle A,\vee,\wedge,\sim,0,1\rangle$ is a De Morgan algebra if $\langle A,\vee,\wedge,0,1\rangle$ is a bounded distributive lattice and $\sim$ is a unary operation on A satisfying the following identities for all $x,y\in A$:

1. $\sim(x\vee y)=\sim x\wedge\sim y$,
2. $\sim\sim x=x$,
3. $\sim 0=1$.

In what follows a De Morgan algebra $\langle A,\vee,\wedge,\sim,0,1\rangle$ will be denoted briefly by $(A,\sim)$.

Definition 2.1
An algebra $(A,\sim,G,H)$ is a tense De Morgan algebra if $(A,\sim)$ is a De Morgan algebra and G and H are two unary operations on A such that for any $x,y\in A$:

1. $G(1)=1$ and $H(1)=1$,
2. $G(x\wedge y)=G(x)\wedge G(y)$ and $H(x\wedge y)=H(x)\wedge H(y)$,

3. $x \leq GP(x)$ and $x \leq HF(x)$, where $F(x) = \sim G(\sim x)$ and $P(x) = \sim H(\sim x)$,
4. $G(x \vee y) \leq G(x) \vee F(y)$ and $H(x \vee y) \leq H(x) \vee P(y)$.

In [18], a duality for tense De Morgan algebras is described taking into account the results established by Cornish and Fowler in [12]. To this purpose, the topological category ***tmPS*** of *tmP*-spaces and *tmP*-functions was considered, which we indicate below:

Definition 2.2
A tense De Morgan space (or *tmP*-space) is a system (X, g, R, R^{-1}), where

(i) (X, g) is an *mP*-space ([12]). More precisely,

(mP1) X is a Priestley space (or P-space),
(mP2) $g: X \longrightarrow X$ is an involutive homeomorphism and an anti-isomorphism,

(ii) R is a binary relation on X and R^{-1} is the converse of R such that:

(tS1) For each $U \in D(X)$ it holds that $G_R(U)$, $H_{R^{-1}}(U) \in D(X)$, where G_R and $H_{R^{-1}}$ are two operators on $\mathcal{P}(X)$ defined for any $U \subseteq X$ as follows:

$$G_R(U) = \{x \in X \mid R(x) \subseteq U\}, \tag{2.1}$$

$$H_{R^{-1}}(U) = \{x \in X \mid R^{-1}(x) \subseteq U\}, \tag{2.2}$$

and $D(X)$ is the set of all increasing and clopen subsets of X,
(tS2) $(x, y) \in R$ implies $(g(x), g(y)) \in R$ for any $x, y \in X$,
(tS3) for each $x \in X$, $R(x)$ is a closed set in X,
(tS4) for each $x \in X$, $R(x) = \downarrow R(x) \cap \uparrow R(x)$, where $\downarrow Y$ ($\uparrow Y$) denotes the set of all $x \in X$ such that $x \leq y$ ($y \leq x$) for some $y \in Y \subseteq X$.

Definition 2.3
A *tmP*-function from a *tmP*-space $(X_1, g_1, R_1, R_1^{-1})$ into another one, $(X_2, g_2, R_2, R_2^{-1})$, is a continuous and increasing function (P-function) $f: X_1 \longrightarrow X_2$, which satisfies the following conditions:

(mf) $f \circ g_1 = g_2 \circ f$ (*mP*-function [12]),
(tf1) $(x, y) \in R_1$ implies $(f(x), f(y)) \in R_2$ for any $x, y \in X_1$,
(tf2) if $(f(x), y) \in R_2$, then there is an element $z \in X_1$ such that $(x, z) \in R_1$ and $f(z) \leq y$,
(tf3) if $(y, f(x)) \in R_2$, then there is an element $z \in X_1$ such that $(z, x) \in R_1$ and $f(z) \leq y$.

Next, Figallo and Pelaitay (see [18, Section 5]) showed that the category ***tmPS*** is dually equivalent to the category ***TDMA*** of tense De Morgan algebras and tense De Morgan homomorphisms. The following results are used to show the dual equivalence:

- Let (X, g, R, R^{-1}) be a *tmP*-space. Then, $(D(X), \sim_g, G_R, H_{R^{-1}})$ is a tense De Morgan algebra, where for all $U \in D(X)$, $\sim_g U$ is defined by

$$\sim_g U = X \setminus g(U), \tag{2.3}$$

and $G_R(U)$ and $H_{R^{-1}}(U)$ are defined as in (2.1) and (2.2), respectively.
- Let $(A, \sim, G, H)$ be a tense De Morgan algebra and $\mathfrak{X}(A)$ be the Priestley space associated with A, i.e. $\mathfrak{X}(A)$ is the set of all prime filters of A, ordered by inclusion and with the topology having as a sub-basis the following subsets of $\mathfrak{X}(A)$:

$$\sigma_A(a) = \{S \in \mathfrak{X}(A) : a \in S\} \text{ for each } a \in A, \tag{2.4}$$

and

$$\mathfrak{X}(A)\setminus\sigma_A(a) \text{ for each } a\in A.$$

Then, $(\mathfrak{X}(A),g_A,R_G^A,R_H^A)$ is a *tmP*-space, where $g_A(S)$ is defined by

$$g_A(S)=\{x\in A:\sim x\notin S\}, \text{ for all } S\in\mathfrak{X}(A), \tag{2.5}$$

and the relations R_G^A and R_H^A are defined for all $S,T\in\mathfrak{X}(A)$ as follows:

$$(S,T)\in R_G^A \Longleftrightarrow G^{-1}(S)\subseteq T\subseteq F^{-1}(S), \tag{2.6}$$

$$(S,T)\in R_H^A \Longleftrightarrow H^{-1}(S)\subseteq T\subseteq P^{-1}(S). \tag{2.7}$$

- Let $(A,\sim,G,H)$ be a tense De Morgan algebra; then, the function $\sigma_A:A\longrightarrow D(\mathfrak{X}(A))$ is a tense De Morgan isomorphism, where σ_A is defined as in (2.4).
- Let (X,g,R,R^{-1}) be a *tmP*-space; then, $\varepsilon_X:X\longrightarrow\mathfrak{X}(D(X))$ is an isomorphism of *tmP*-spaces, where ε_X is defined by

$$\varepsilon_X(x)=\{U\in D(X):x\in U\}, \text{ for all } x\in X. \tag{2.8}$$

- Let $h:(A_1,\sim_1,G_1,H_1)\longrightarrow(A_2,\sim_2,G_2,H_2)$ be a tense De Morgan morphism. Then, the map $\Phi(h):X(A_2)\longrightarrow X(A_1)$ is a morphism of *tmP*-spaces, where

$$\Phi(h)(S)=h^{-1}(S), \text{ for all } S\in X(A_2). \tag{2.9}$$

- Let $f:(X_1,g_1,R_1,R_1^{-1})\longrightarrow(X_2,g_2,R_2,R_2^{-1})$ be a morphism of *tmP*-spaces. Then, $\Psi(f):D(X_2)\longrightarrow D(X_1)$ is a tense De Morgan morphism, where

$$\Psi(f)(U)=f^{-1}(U), \text{ for all } U\in D(X_2). \tag{2.10}$$

In [18], the duality described above was used to characterize the congruence lattice $Con_{tM}(A)$ of a tense De Morgan algebra $(A,\sim,G,H)$. First the following notion was introduced:

Definition 2.4
Let $(X,\leq,g,R,R^{-1})$ be a *tmP*-space. An involutive closed subset Y (i.e. $Y=g(Y)$ [12]) of X is a *tmP*-subset if it satisfies the following conditions for $u,v\in X$:

(ts1) if $(v,u)\in R$ and $u\in Y$, then there exists, $w\in Y$ such that $(w,u)\in R$ and $w\leq v$.
(ts2) if $(u,v)\in R$ and $u\in Y$, then there exists, $z\in Y$ such that $(u,z)\in R$ and $z\leq v$.

The lattice of all *tmP*-subsets of the *tmP*-space associated with a tense De Morgan algebra was taken into account to characterize the congruence lattice of this algebra as it is indicated in the following theorem:

Theorem 2.5 ([18, Theorem 6.4])
Let $(A,\sim,G,H)$ be a tense De Morgan algebra and $(\mathfrak{X}(A),\subseteq,g_A,R_G^A,R_H^A)$ be the *tmP*-space associated with A. Then, the lattice $\mathcal{C}_T(\mathfrak{X}(A))$ of all *tmP*-subsets of $\mathfrak{X}(A)$ is anti-isomorphic to the lattice $Con_{tM}(A)$ of the tense De Morgan congruences on A, and the anti-isomorphism is the function Θ_T defined by the prescription:

$$\Theta_T(Y)=\{(a,b)\in A\times A:\sigma_A(a)\cap Y=\sigma_A(b)\cap Y\}, \text{ for all } Y\in\mathcal{C}_T(\mathfrak{X}(A)). \tag{2.11}$$

2.2 n-valued Łukasiewicz–Moisil algebras

In the sequel n is an integer number and we use the notation $[n]:=\{1,\ldots,n\}$.

In [11] (see also [1]), Cignoli defined the n-valued Łukasiewicz–Moisil algebras (or LM_n-algebras) in the following way:

Definition 2.6
An algebra $\langle A,\vee,\wedge,\sim,\{\varphi_i\}_{i\in[n-1]},0,1\rangle$ is an n-valued Łukasiewicz–Moisil algebra (or LM_n-algebra), where $n\geq 2$ is an integer number, if

(i) $\langle A,\vee,\wedge,\sim,0,1\rangle$ is a De Morgan algebra,
(ii) φ_i, $i\in[n-1]$, are unary operations on A which satisfy the following conditions for any $i,j\in[n-1]$ and $x,y\in A$:

(L1) $\varphi_i(x\vee y)=\varphi_i(x)\vee\varphi_i(y)$,
(L2) $\varphi_i(x)\vee\sim\varphi_i(x)=1$,
(L3) $\varphi_i(\varphi_j(x))=\varphi_j(x)$,
(L4) $\varphi_i(\sim x)=\sim\varphi_{n-i}(x)$,
(L5) $i\leq j$ implies $\varphi_i(x)\leq\varphi_j(y)$,
(L6) $\varphi_i(x)=\varphi_i(y)$ for all $i\in[n-1]$, implies $x=y$.

The operators $\varphi_i:A\longrightarrow A$, $i\in[n-1]$, are known as chrysippian endomorphisms and the axiom (L6) is known as Moisil's determination principle.

An LM_n-algebra $\langle A,\vee,\wedge,\sim,\{\varphi_i\}_{i\in[n-1]},0,1\rangle$ will be denoted in the rest of this article by its universe A or by $(A,\sim,\{\varphi_i\}_{i\in[n-1]})$.

In Lemma 2.7, we will summarize some properties of these algebras.

Lemma 2.7 ([1])
Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]})$ be an LM_n-algebra. Then the following properties are satisfied, for all $x,y\in A$:

(L7) $\varphi_i(x\wedge y)=\varphi_i(x)\wedge\varphi_i(y)$ for any $i\in[n-1]$
(L8) $x\leq y$ if and only if $\varphi_i(x)\leq\varphi_i(y)$ for any $i\in[n-1]$,
(L9) $\varphi_1(x)\leq x$,
(L10) $x\leq\varphi_{n-1}(x)$,
(L11) $\varphi_i(1)=1$, $\varphi_i(0)=0$ for any $i\in[n-1]$,
(L12) any chrysippian endomorphism φ_i preserves arbitrary suprema and infima, whenever they exist.

Definition 2.8
Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]})$ and $(A',\sim',\{\varphi'_i\}_{i\in[n-1]})$ be two LM_n-algebras. A morphism of LM_n-algebras is a map $f:A\longrightarrow A'$, which satisfies the conditions:

(Lf1) $f(0)=0, f(1)=1, f\circ\sim=\sim'\circ f$,
(Lf2) $f(x\wedge y)=f(x)\wedge f(y), f(x\vee y)=f(x)\vee f(y)$, for any $x,y\in A$,
(Lf3) $f\circ\varphi_i=\varphi'_i\circ f$, for any $i\in[n-1]$.

Lemma 2.9 ([1])
Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]})$ be an LM_n-algebra and $\mathcal{B}(A)$ be the set of all complemented elements of A. Then, the following conditions are equivalent for all $a\in A$:

1. $a\in\mathcal{B}(A)$
2. $a\vee\sim a=1$ and $a\wedge\sim a=0$,

3. there are $b\in A$, $i\in[n-1]$, such that $a=\varphi_i(b)$,
4. there is $i\in[n-1]$ such that $a=\varphi_i(a)$,
5. for all $i\in[n-1]$, $a=\varphi_i(a)$,
6. for all $i,j\in[n-1]$, $\varphi_i(a)=\varphi_j(a)$.

EXAMPLE 2.10
An example of an LM_n-algebra is the chain of n rational fractions $L_n=\left\{\frac{j}{n-1}:0\leq j\leq n-1\right\}$, in which $n\geq 2$ is an integer number, endowed with the natural lattice structure and the unary operations $\sim$ and φ_i, defined as follows:
$\sim(\frac{j}{n-1})=1-\frac{j}{n-1}$ and $\varphi_i(\frac{j}{n-1})=0$ if $i+j<n$ or $\varphi_i(\frac{j}{n-1})=1$ if $i+j\geq n$.

The importance of Example 2.10 is seen in [11]:

THEOREM 2.11
Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]})$ be a non-trivial LM_n-algebra. Then, the following conditions are equivalent:

(i) A is a subdirectly irreducible LM_n-algebra,
(ii) A is a simple LM_n-algebra,
(iii) A is isomorphic to a LM_n-subalgebra of L_n.

The following result was obtained as a consequence of this last theorem.

COROLLARY 2.12
Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]})$ be a non-trivial LM_n-algebra. Then, the following conditions are equivalent:

(i) A is a simple LM_n-algebra,
(ii) $\mathcal{B}(A)=\{0,1\}$.

Another example of an LM_n-algebra is the following one:

EXAMPLE 2.13
Let $\langle B,\wedge,\vee,-,0,1\rangle$ be a Boolean algebra. Let us consider the following set $D(B)=\{(x_1,\ldots,x_{n-1})\in B^{n-1}|x_1\leq\cdots\leq x_{n-1}\}$. We define the following unary operations on $D(B)$, for all $(x_1,\ldots,x_{n-1})\in D(B)$:

$N:D(B)\longrightarrow D(B), N(x_1,\ldots,x_{n-1})=(-x_{n-1},\ldots,-x_1)$,
$\varphi_i:D(B)\longrightarrow D(B), \varphi_i(x_1,\ldots,x_{n-1})=(x_i,\ldots,x_i)$ for each $i\in[n-1]$.

Then, $\langle D(B),\wedge,\vee,N,\{\varphi_i\}_{i\in[n-1]},(0,\ldots,0),(1,\ldots,1)\rangle$ is an LM_n-algebra.

The following theorem reduces the calculus in an arbitrary LM_n-algebra A to the calculus in L_n:

THEOREM 2.14 (Moisil's representation theorem)
For any LM_n-algebra A, there exists a non-empty set X and an injective morphism of LM_n-algebras $\Omega:A\longrightarrow L_n^X$.

In [16], Figallo, *et al*. determined a topological duality for LM_n-algebras. To this aim, these authors considered the topological category $\boldsymbol{LM_nP}$ of LM_n-spaces and LM_n-functions. Specifically:

Definition 2.15
A system $(X, g, \{f_i\}_{i\in[n-1]})$ is an n-valued Łukaziewicz–Moisil space (or LM_n-space) if the following properties are fulfilled:

(LP1) (X, g) is an mP-space ([12]),
(LP2) $f_i : X \longrightarrow X$ is a continuous function,
(LP3) $x \leq y$ implies $f_i(x) = f_i(y)$ for all $i \in [n-1]$,
(LP4) $i \leq j$ implies $f_i(x) \leq f_j(x)$,
(LP5) $f_i \circ f_j = f_i$,
(LP6) $f_i \circ g = f_i$,
(LP7) $g \circ f_i = f_{n-i}$,
(LP8) $X = \bigcup\limits_{i=1}^{n-1} f_i(X)$,
for any $i, j \in [n-1]$ and for any $x, y \in X$.

Definition 2.16
If $(X, g, \{f_i\}_{i\in[n-1]})$ and $(X', g', \{f'_i\}_{i\in[n-1]})$ are two LM_n-spaces, then an LM_n-function f from X to X' is a continuous and increasing function (P-function), which satisfies the following conditions:

(mPf) $f \circ g = g' \circ f$ (mP-function [12]),
(LPf) $f'_i \circ f = f \circ f_i$ for all $i \in [n-1]$.

It is routine to prove that the condition (LP8) in Definition 2.15 is equivalent to any of these conditions:

(LP9) for each $x \in X$ there is an index $i \in [n-1]$, such that $x = f_i(x)$,
(LP10) if Y, Z are subsets of X and $f_i^{-1}(Y) = f_i^{-1}(Z)$ for all $i \in [n-1]$, then $Y = Z$.

It is worth mentioning the following properties of LM_n-spaces because they are useful to describe these spaces:

(LP11) Every LM_n-space $(X, g, \{f_i\}_{i\in[n-1]})$ is the cardinal sum of a family of chains, each of which has at most $n-1$ elements.
(LP12) If $(X, g, \{f_i\}_{i\in[n-1]})$ is an LM_n-space, $x \in X$ and C_x denotes the unique maximal chain containing x, then $C_x = \{f_i(x) : i \in [n-1]\}$.

In addition, in [16], the following results were established:

- If $(X, g, \{f_i\}_{i\in[n-1]})$ is an LM_n-space. Then, $(D(X), \sim_g, \{\varphi_i^X\}_{i\in[n-1]})$ is an LM_n-algebra, where for every $U \in D(X)$, $\sim_g U$ is defined as in Equation (2.3) and

$$\varphi_i^X(U) = f_i^{-1}(U) \text{ for all } i \in [n-1]. \tag{2.12}$$

- If $(A, \sim, \{\varphi_i\}_{i\in[n-1]})$ is an LM_n-algebra and $\mathfrak{X}(A)$ is the Priestley space associated with A, then $(\mathfrak{X}(A), g_A, \{f_i^A\}_{i\in[n-1]})$ is an LM_n-space, where for every $S \in \mathfrak{X}(A)$, $g_A(S)$ is defined as Equation (2.5) and

$$f_i^A(S) = \varphi_i^{-1}(S) \text{ for all } i \in [n-1]. \tag{2.13}$$

- $(A,\sim,\{\varphi_i\}_{i\in[n-1]})\cong(D(\mathfrak{X}(A)),\sim,\{\varphi_i^{\mathfrak{X}(A)}\}_{i\in[n-1]})$ and
- $(X,g,\{f_i\}_{i\in[n-1]})\cong(X(D(X)),g_{D(X)},\{f_i^{D(X)}\}_{i\in[n-1]})$, via the natural isomorphisms denoted by σ_A and ε_X respectively, which are defined as in Equations (2.4) and (2.8), respectively.
- The correspondences between the morphisms of both categories are defined in the usual way as in Equations (2.9) and (2.10).

Then, from these results it was concluded that the category $\boldsymbol{LM_nP}$ is dually equivalent to the category $\boldsymbol{LM_nA}$ of LM_n-algebras and LM_n-homomorphisms. Moreover, this duality was taken into account to characterize the congruence lattice on an LM_n-algebra as is indicated in Theorem 2.18. In order to obtain this characterization the modal subsets of the LM_n-spaces were taken into account, which we mention below:

Definition 2.17
Let $(X,g,\{f_i\}_{i\in[n-1]})$ be an LM_n-space. A subset Y of X is modal if $Y=f_i^{-1}(Y)$ for all $i\in[n-1]$.

Theorem 2.18
[[16, Theorem 3.1]] Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]})$ be an LM_n-algebra and $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]})$ be the LM_n-space associated with A. Then, the lattice $\mathcal{C}_M(\mathfrak{X}(A))$ of all modal and closed subsets of $\mathfrak{X}(A)$ is anti-isomorphic to the lattice $Con_{LM_n}(A)$ of LM_n-congruences on A, and the anti-isomorphism is the function $\Theta_M:\mathcal{C}_M(\mathfrak{X}(A))\longrightarrow Con_{LM_n}(A)$ defined by the same prescription in (2.11).

2.3 Tense n-valued Łukasiewicz–Moisil algebras

In [13], Diaconescu and Georgescu introduce the following notion:

Definition 2.19
An algebra $\langle A,\vee,\wedge,\sim,\{\varphi_i\}_{i\in[n-1]},G,H,0,1\rangle$ is a tense n-valued Łukasiewicz–Moisil algebra (or tense LM_n-algebra) if $\langle A,\vee,\wedge,\sim,\{\varphi_i\}_{i\in[n-1]},0,1\rangle$, is an LM_n-algebra and G, H are two unary operators on A which satisfy the following properties:

(T1) $G(1)=1$ and $H(1)=1$,
(T2) $G(x\wedge y)=G(x)\wedge G(y)$ and $H(x\wedge y)=H(x)\wedge H(y)$,
(T3) $G\varphi_i(x)=\varphi_iG(x)$ and $H\varphi_i(x)=\varphi_iH(x)$,
(T4) $x\leq GP(x)$ and $x\leq HF(x)$, where $P(x)=\sim H(\sim x)$ and $F(x)=\sim G(\sim x)$,
for any $x,y\in X$ and $i\in[n-1]$.

A tense LM_n-algebra $\langle A,\vee,\wedge,\sim,\{\varphi_i\}_{i\in[n-1]},G,H,0,1\rangle$ will be denoted in the rest of this paper by (A,G,H) or by $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$.

The following lemma contains properties of tense LM_n-algebras that are useful in what follows.

Lemma 2.20 ([13, 20])
The following properties hold in every tense LM_n-algebra (A,G,H):

(T5) $x\leq y$ implies $G(x)\leq G(y)$ and $H(x)\leq H(y)$,
(T6) $x\leq y$ implies $F(x)\leq F(y)$ and $P(x)\leq P(y)$,
(T7) $F(0)=0$ and $P(0)=0$,
(T8) $F(x\vee y)=F(x)\vee F(y)$ and $P(x\vee y)=P(x)\vee P(y)$,
(T9) $PG(x)\leq x$ and $FH(x)\leq x$,
(T10) $GP(x)\wedge F(y)\leq F(P(x)\wedge y)$ and $HF(x)\wedge P(y)\leq P(F(x)\wedge y)$,
(T11) $G(x)\wedge F(y)\leq F(x\wedge y)$ and $H(x)\wedge P(y)\leq P(x\wedge y)$,

(T12) $G(x\vee y)\leq G(x)\vee F(y)$ and $H(x\vee y)\leq H(x)\vee P(y)$,
for any $x,y\in X$.

Definition 2.21 ([13])
If (A,G,H) and (A',G',H') are two tense LM_n-algebras, then a morphism of tense LM_n-algebras $f:(A,G,H)\longrightarrow(A',G',H')$ is a morphism of LM_n-algebras such that

(tf) $f(G(a))=G'(f(a))$ and $f(H(a))=H'(f(a))$, for any $a\in A$.

In [13], the following example was given:

Example 2.22
Let (X,R) be a frame (i.e. X is a non-empty set and R is a binary relation on X) and $G^*,H^*:L_n^X\longrightarrow L_n^X$ be defined as follows:

$$G^*(p)(x)=\bigwedge\{p(y)|y\in X,xRy\},\ H^*(p)(x)=\bigwedge\{p(y)|y\in X,yRx\},$$

for all $p\in L_n^X$ and $x\in X$. Then, (L_n^X,G^*,H^*) is a tense LM_n-algebra, where the operations of the LM_n-algebra L_n^X are defined pointwise.

Also, Diaconescu and Georgescu proved the following important result in [13]. We will offer an alternative proof of this result in Section 5.

Theorem 2.23
For any tense LM_n-algebra (A,G,H), there exists a frame (X,R) and an injective morphism of tense LM_n-algebras from (A,G,H) into (L_n^X,G^*,H^*).

3 Topological duality for tense LM_n-algebras

In this section, we will develop a topological duality for tense n-valued Łukasiewicz–Moisil algebras, taking into account the results established by Figallo *et al*. in [16] and the results obtained by A.V. Figallo and G. Pelaitay in [18]. In order to determine this duality, we introduce a topological category whose objects and their corresponding morphisms are described below.

Definition 3.1
A system $(X,g,\{f_i\}_{i\in[n-1]},R)$ is a tense LM_n-space if the following conditions are satisfied:

(i) $(X,g,\{f_i\}_{i\in[n-1]})$ is an LM_n-space (Definition 2.15),
(ii) R is a binary relation on X and R^{-1} is the converse of R such that:

(tS1) $(x,y)\in R$ implies $(g(x),g(y))\in R$,
(tS2) for each $x\in X$, $R(x)$ and $R^{-1}(x)$ are closed subsets of X,
(tS3) for each $x\in X$, $R(x)={\downarrow}R(x)\cap{\uparrow}R(x)$,
(tS4) $(x,y)\in R$ implies $(f_i(x),f_i(y))\in R$ for any $i\in[n-1]$,
(tS5) $(f_i(x),y)\in R$, $i\in[n-1]$, implies that there exists $z\in X$ such that $(x,z)\in R$ and $f_i(z)\leq y$,
(tS6) $(y,f_i(x))\in R$, $i\in[n-1]$, implies that there exists $z\in X$ such that $(z,x)\in R$ and $f_i(z)\leq y$,
(tS7) for each $U\in D(X)$, $G_R(U),H_{R^{-1}}(U)\in D(X)$, where G_R and $H_{R^{-1}}$ are operators on $\mathcal{P}(X)$ defined as in (2.1) and (2.2), respectively.

Remark 3.2

(i) Note that if $(X,g,\{f_i\}_{i\in[n-1]},R)$ is a tense LM_n-space, then $(X,g,\{f_i\}_{i\in[n-1]},R,R^{-1})$ is a tense LM_n-frame (see [20, Definition 3.3]).

(ii) It should be mentioned that if $(X,g,\{f_i\}_{i\in[n-1]})$ is an LM_n-space and $R=\emptyset\subset X\times X$, then $R^{-1}=\emptyset$ and it immediately follows that the conditions (tS1), (tS2), (tS3), (tS4), (tS5) and (tS6) hold. Besides, for all $U\in D(X)$, $G_R(U)=X$ and $H_{R^{-1}}(U)=X$, from which we obtain that the relations R and R^{-1} satisfies the condition (tS7). Therefore, $(X,g,\{f_i\}_{i\in[n-1]},R)$ is a tense LM_n-space.

Definition 3.3
A tense LM_n-function f from a tense LM_n-space $(X_1,g_1,\{f_i^1\}_{i\in[n-1]},R_1)$ into another one, $(X_2,g_2,\{f_i^2\}_{i\in[n-1]},R_2)$, is a function $f:X_1\longrightarrow X_2$ such that:

(i) $f:X_1\longrightarrow X_2$ is an LM_n-function (Definition 2.16),
(ii) $f:X_1\longrightarrow X_2$ satisfies the following conditions, for all $x\in X_1$:

(tf1) $f(R_1(x))\subseteq R_2(f(x))$ and $f(R_1^{-1}(x))\subseteq R_2^{-1}(f(x))$,
(tf2) $R_2(f(x))\subseteq\uparrow f(R_1(x))$,
(tf3) $R_2{}^{-1}(f(x))\subseteq\uparrow f(R_1{}^{-1}(x))$.

The category that has tense LM_n-spaces as objects and tense LM_n-functions as morphisms will be denoted by $\boldsymbol{tLM_nS}$, and $\boldsymbol{tLM_nA}$ will denote the category of tense LM_n-algebras and tense LM_n-homomorphisms. Our next task will be to determine that the category $\boldsymbol{tLM_nS}$ is naturally equivalent to the dual category of $\boldsymbol{tLM_nA}$.

Now we will show a characterization of tense LM_n-functions which will be useful later.

Lemma 3.4
Let $(X_1,g_1,\{f_i^1\}_{i\in[n-1]},R_1)$ and $(X_2,g_2,\{f_i^2\}_{i\in[n-1]},R_2)$ be two tense LM_n-spaces and $f:X_1\longrightarrow X_2$ be a tense LM_n-function. Then, f satisfies the following conditions:

(tf4) $\uparrow f(R_1(x))=\uparrow R_2(f(x))$,
(tf5) $\uparrow f(R_1{}^{-1}(x))=\uparrow R_2{}^{-1}(f(x))$, for any $x\in X$.

Proof.
(tf4): From (tf1), we obtain that $\uparrow f(R_1(x))\subseteq\uparrow R_2(f(x))$ for any $x\in X$. On the other hand, from (tf2) we infer that $\uparrow R_2(f(x))\subseteq\uparrow f(R_1(x))$, and therefore the proof is complete.

(tf5): It follows from (tf1) and (tf3). ■

Lemma 3.5
Let $(X,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space. Then for all $x,y\in X$ such that $(x,y)\notin R$, the following conditions are satisfied:

(i) There is $U\in D(X)$ such that $y\notin U$ and $x\in G_R(U)$ or $y\in U$ and $x\notin F_R(U)$, where $F_R(U):=\{x\in X:R(x)\cap U\neq\emptyset\}$.
(ii) There is $V\in D(X)$ such that $y\notin V$ and $x\in H_{R^{-1}}(V)$ or $y\in V$ and $x\notin P_{R^{-1}}(V)$, where $P_{R^{-1}}(V):=\{x\in X:R^{-1}(x)\cap V\neq\emptyset\}$.

Proof.
(i): Let $x,y\in X$ such that $y\notin R(x)$. Then, from property (tS3) we have that $y\notin\uparrow R(x)$ or $y\notin\downarrow R(x)$. Suppose that $y\notin\uparrow R(x)$. From property (tS2), $R(x)$ is compact. From this last fact, we infer that there is $U\in D(X)$ such that $y\notin U$ and $R(x)\subseteq U$. Therefore, $x\in G_R(U)$. Suppose now that $y\notin\downarrow R(x)$.

Then, taking into account that $R(x)$ is compact, we infer that there is $V\in D(X)$ such that $y\in V$ and $R(x)\cap V=\emptyset$ and so $x\notin F_R(V)$.

(ii): It can be proved in a similar way. ∎

Lemma 3.6
Let $(X_1,g_1,\{f_i^1\}_{i\in[n-1]},R_1)$ and $(X_2,g_2,\{f_i^2\}_{i\in[n-1]},R_2)$ be two tense LM_n-spaces. Then, the following conditions are equivalent:

(i) $f:X_1\longrightarrow X_2$ is a tense LM_n-function,
(ii) $f:X_1\longrightarrow X_2$ is an LM_n-function such that, for any $U\in D(X_2)$:

(tf6) $f^{-1}(G_{R_2}(U))=G_{R_1}(f^{-1}(U))$,
(tf7) $f^{-1}(H_{R_2^{-1}}(U))=H_{R_1^{-1}}(f^{-1}(U))$.

Proof.
(i)$\Rightarrow$(ii):
(tf6): Let $x\in f^{-1}(G_{R_2}(U))$. Hence, $R_2(f(x))\subseteq U$. Since U is increasing, from (tf2), we have that $\uparrow f(R_1(x))\subseteq U$ and so, $f(R_1(x))\subseteq U$. Taking into account that $R_1(x)\subseteq f^{-1}(f(R_1(x)))$, we obtain that $R_1(x)\subseteq f^{-1}(U)$. Thus, $x\in G_{R_1}(f^{-1}(U))$. On the other hand, suppose that $x\in G_{R_1}(f^{-1}(U))$. Then, $R_1(x)\subseteq f^{-1}(U)$. Since $f(f^{-1}(U))\subseteq U$ and U is increasing, we obtain $\uparrow f(R_1(x))\subseteq U$. From the last assertion and (tf2), we have that $R_2(f(x))\subseteq U$. Therefore, $x\in f^{-1}(G_{R_2}(U))$.

(tf7): It can be proved in a similar way.

(ii)$\Rightarrow$(i): First, taking into account that f is an LM_n-function which satisfies (tf6) and (tf7) we can see that the following conditions are verified:

(tf8) $f^{-1}(F_{R_2}(U))=F_{R_1}(f^{-1}(U))$ for any $U\in D(X_2)$,
(tf9) $f^{-1}(P_{R_2^{-1}}(U))=P_{R_1^{-1}}(f^{-1}(U))$ for any $U\in D(X_2)$.

Indeed, let $U\in D(X_2)$, then it follows that $f^{-1}(F_{R_2}(U))=f^{-1}(\sim_{g_2}(G_{R_2}(\sim_{g_2}U))=$
$\sim_{g_1}(f^{-1}(G_{R_2}(\sim_{g_2}U)))=\sim_{g_1}(G_{R_1}(f^{-1}(\sim_{g_2}U)))=\sim_{g_1}(G_{R_1}(\sim_{g_1}(f^{-1}(U)))=$
$F_{R_1}(f^{-1}(U))$ and so (tf8) holds. Property (tf9) can be proved in a similar way.

(tf1): Let $x,y\in X_1$ such that (1) $(x,y)\in R_1$. Suppose that $(f(x),f(y))\notin R_2$. Then, from Lemma 3.5 it follows that there is $U\in D(X_2)$ such that (2) $f(y)\notin U$ and $f(x)\in G_{R_2}(U)$ or there is $V\in D(X_2)$ such that (3) $f(y)\in V$ and $f(x)\notin F_{R_2}(V)$. If (2) holds, then (4) $y\notin f^{-1}(U)$ and $x\in f^{-1}(G_{R_2}(U))$. From this last statement and (tf6) we obtain that $x\in G_{R_1}(f^{-1}(U))$, and so from (1) we infer that $y\in f^{-1}(U)$, which contradicts (4). If (3) holds, then (5) $y\in f^{-1}(V)$ and $x\notin f^{-1}(F_{R_2}(V))$, from which it follows by (tf8) that $x\notin F_{R_1}(f^{-1}(V))$. Therefore, $R_1(x)\cap f^{-1}(V)=\emptyset$. From this last assertion and (1) we infer that $y\notin f^{-1}(V)$, which contradicts (5). Thus, we conclude that $(f(x),f(y))\in R_2$.

(tf2): Let (1) $y\in R_2(f(x))$. Suppose that (2) $y\notin\uparrow f(R_1(x))$. Taking into account that X_1 is a compact space, property (tS2) of LM_n-spaces and the fact that f is a continuous function it follows that $f(R_1(x))$ is compact in X_2. Consequently, from the last statement and (2) we infer that there exists $U\in D(X_2)$ such that (3) $f(R_1(x))\subseteq U$ and $y\notin U$. Hence, from this assertion and (1), we obtain that $x\notin f^{-1}(G_{R_2}(U))$ and so by (tf4), we get that $x\notin G_{R_1}(f^{-1}(U))$. This statement contradicts (3). Therefore, $R_2(f(x))\subseteq\uparrow f(R_1(x))$.

(tf3): It can be proved in a similar way. ∎

Next, we will describe some properties of tense LM_n-spaces which will be quite useful for determining the duality for tense LM_n-algebras that we are interested in. Here and subsequently, $\max Y$ denotes the set of maximal elements of Y.

Lemma 3.7 ([16])
Let $(X,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space. Then, the following conditions are verified:

(tS8) X is the cardinal sum of a family of chains, each of which has at most $n-1$ elements.
(tS9) If $x\in X$ and C_x denotes the unique maximum chain containing x, then $C_x=\{f_i(x): i\in[n-1]\}$.
(tS10) $y\in \max X$ if and only if $y=f_{n-1}(y)$.

Proof. It is a consequence of the fact that every LM_n-space satisfies properties (tS8), (tS9) and (tS10). ∎

Lemma 3.8
Let $(X,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space. Then, the following conditions are satisfied for any $x,y,\in X$ and $i\in[n-1]$:

(tS11) $R(g(x))=g(R(x))$, $R^{-1}(g(x))=g(R^{-1}(x))$,
(tS12) $R(f_i(x))\subseteq \bigcup\limits_{y\in R(f_i(x))} \uparrow f_i(y)$,
(tS13) $R^{-1}(f_i(x))\subseteq \bigcup\limits_{y\in R^{-1}(f_i(x))} \uparrow f_i(y)$,
(tS14) $\uparrow f_i(R_1(x))=\uparrow R_2(f(x))$,
(tS15) $\uparrow f_i({R_1}^{-1}(x))=\uparrow {R_2}^{-1}(f(x))$,
(tS16) $f_i^{-1}(G_R(U))=G_R(f_i^{-1}(U))$,
(tS17) $f_i^{-1}(H_{R^{-1}}(U))=H_{R^{-1}}(f_i^{-1}(U))$,
(tS18) $f_i^{-1}(\sim_g(U))=\sim_g(f_{n-i}^{-1}(U))$,
(tS19) $f_i^{-1}(F_R(U))=F_R(f_i^{-1}(U))$,
(tS20) $f_i^{-1}(P_{R^{-1}}(U))=P_{R^{-1}}(f_i^{-1}(U))$.

Proof.
(tS11): It is a consequence of (tS1) and the fact that g is involutive.

(tS12): Let $x,y\in X$ such that $(f_i(x),y)\in R$, then from (tS5) there exists $z\in X$ such that $(x,z)\in R$ and $f_i(z)\leq y$. From this last assertion and properties (LP3) and (LP5) it follows that $f_i(z)=f_i(y)$ and so $f_i(y)\leq y$.

(tS13): It can be proved using a similar technique to that used in the proof of (tS12).

From (tS4), (tS5) and (tS6) it follows that properties (tS14) and (tS15) hold.

(tS16): Taking into account (tS14), the fact that the functions f_i, $i\in[n-1]$, are continuous and the proof of Lemma 3.6, we obtain that (tS16) holds.

(tS17): It can be proved in a similar way taking into account (tS15).

(tS18): $f_i^{-1}(\sim_g U)=f_i^{-1}(X\setminus g^{-1}(U))=X\setminus f_i^{-1}(g^{-1}(U))=X\setminus(g\circ f_i)^{-1}(U)=X\setminus f_{n-i}^{-1}(U)=X\setminus(f_{n-i}\circ g)^{-1}(U)=X\setminus g^{-1}(f_{n-i}^{-1}(U))=X\setminus g(f_{n-i}^{-1}(U))=\sim_g f_{n-i}^{-1}(U)$.
(tS19): $F_R(f_i^{-1}(U))=\sim_g(G_R(\sim_g(f_i^{-1}(U))))=\sim_g(G_R(\sim_g(f_i^{-1}(U))))=\sim_g G_R(f_{n-i}^{-1}(\sim_g U))=\sim_g(f_{n-i}^{-1}(G_R(\sim_g U)))=f_i^{-1}(\sim_g G_R(\sim_g U))=f_i^{-1}(F_R(U))$.

(tS20): It can be proved using a similar technique to that used in the proof of property (tS19). ∎

Corollary 3.9
Let $(X,g,\{f_i\}_{i\in[n-1]})$ be a tense LM_n-space. Then, the conditions (tS4), (tS5) and (tS6) can be replaced by the following conditions:

(tS16) $f_i^{-1}(G_R(U))=G_R(f_i^{-1}(U))$ for any $U\in D(X)$,
(tS17) $f_i^{-1}(H_{R^{-1}}(U))=H_{R^{-1}}(f_i^{-1}(U))$ for any $U\in D(X)$.

PROOF.
(⇒): It follows from Lemma 3.8.

(⇐): It can be proved using a similar technique to that used in the proof of Lemma 3.6. ∎

Next, we will define a contravariant functor from ***tLMnS*** to ***tLMnA***.

LEMMA 3.10
Let $(X, g, \{f_i\}_{i\in[n-1]}, R)$ be a tense LM_n-space. Then,

$$\Psi(X) = \langle D(X), \sim_g, \{\varphi_i^X\}_{i\in[n-1]}, G_R, H_{R^{-1}}, \emptyset, X\rangle$$

is a tense LM_n-algebra, where for all $U \in D(X)$, $\sim_g U$, $\varphi_i^X(U)$, $i \in [n-1]$, $G_R(U)$ and $H_{R^{-1}}(U)$ are defined as in Equations (2.3), (2.12), (2.1) and (2.2), respectively.

PROOF. From [16, Lemma 2.1] and [18, Lemma 4.3] it follows that $\langle D(X), \sim_g, \{\varphi_i^X\}_{i\in[n-1]}, \emptyset, X\rangle$ is an LM_n-algebra and $\langle D(X), \sim_g, G_R, H_{R^{-1}}, \emptyset, X\rangle$ is a tense De Morgan algebra, respectively. Therefore, the properties (T1), (T2) and (T4) of tense LM_n-algebras (Definition 2.19) hold. In addition, since any $U \in D(X)$ satisfies properties (tS16) and (tS17) in Lemma 3.8, then we can assert that property (T3) holds too, and so the proof is complete. ∎

LEMMA 3.11
Let $f : (X_1, g_1, \{f_i^1\}_{i\in[n-1]}) \longrightarrow (X_2, g_2, \{f_i^2\}_{i\in[n-1]})$ be a morphism of tense LM_n-spaces. Then, the map $\Psi(f) : D(X_2) \longrightarrow D(X_1)$ defined by $\Psi(f)(U) = f^{-1}(U)$ for all $U \in D(X_2)$, is a tense LM_n-homomorphism.

PROOF. It follows from the results established in [16, Lemma 2.3] and Lemma 3.6. ∎

The previous two lemmas show that Ψ is a contravariant functor from $\boldsymbol{tLM_nS}$ to $\boldsymbol{tLM_nA}$. To achieve our goal we need to define a contravariant functor from $\boldsymbol{tLM_nA}$ to $\boldsymbol{tLM_nS}$.

LEMMA 3.12
([20, Lemma 3.8]) Let (A, G, H) be a tense LM_n-algebra and let $S, T \in \mathfrak{X}(A)$. Then the following conditions are equivalent:

(i) $G^{-1}(S) \subseteq T \subseteq F^{-1}(S)$,
(ii) $H^{-1}(T) \subseteq S \subseteq P^{-1}(T)$.

DEFINITION 3.13
Let (A, G, H) be a tense LM_n-algebra and let R^A be the relation defined on $\mathfrak{X}(A)$ by the prescription:

$$(S, T) \in R^A \Longleftrightarrow G^{-1}(S) \subseteq T \subseteq F^{-1}(S). \tag{3.1}$$

REMARK 3.14
Lemma 3.12 means that we have two ways to define the relation R^A, either by using G and F, or by using H and P.

The following lemma, whose proof can be found in [20, Lemma 3.11], will be essential for the proof of Lemma 3.16.

LEMMA 3.15
Let (A, G, H) be a tense LM_n-algebra and let $S \in \mathfrak{X}(A)$ and $a \in A$. Then,

(i) $G(a) \notin S$ if and only if there exists $T \in \mathfrak{X}(A)$ such that $(S, T) \in R^A$ and $a \notin T$,

(ii) $H(a) \notin S$ if and only if there exists $T \in \mathfrak{X}(A)$ such that $(S,T) \in R^{A^{-1}}$ and $a \notin T$.

LEMMA 3.16
Let (A,G,H) be an LM_n-algebra and $\mathfrak{X}(A)$ be the Priestley space associated with A. Then, $\Phi(A) = (\mathfrak{X}(A), g_A, \{f_i^A\}_{i\in[n-1]}, R^A)$ is a tense LM_n-space, where for every $S \in \mathfrak{X}(A)$, $g_A(S)$ and $f_i^A(S)$ are defined as in (2.5) and (2.13), respectively and R^A is the relation defined on $\mathfrak{X}(A)$ as in (3.1). Besides, $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$, defined by the prescription (2.4), is a tense LM_n-isomorphism.

PROOF. From [16, Lemma 2.2] and [18, Lemma 5.6] it follows that $(\mathfrak{X}(A), g_A, \{f_i^A\}_{i\in[n-1]})$ is an LM_n-space and $(\mathfrak{X}(A), g_A, R^A, R^{A^{-1}})$ is a tense mP-space, and so properties (tS1), (tS2), (tS3) and (tS7) of tense LM_n-spaces hold (Definition 3.1). Also, from Corollary 3.9 we have that the conditions (tS4), (tS5) and (tS6) are satisfied. Therefore, we have that $(\mathfrak{X}(A), g_A, \{f_i^A\}_{i\in[n-1]}, R^A)$ is a tense LM_n-space. In addition, from [16, Lemma 2.2], we have that σ_A is an LM_n-isomorphism. Also for all $a \in A$, $G_{R^A}(\sigma_A(a)) = \sigma_A(G(a))$ and $H_{R^{A^{-1}}}(\sigma_A(a)) = \sigma_A(H(a))$. Indeed, let us take a prime filter S such that $G(a) \notin S$. By Lemma 3.15, there exists $T \in \mathfrak{X}(A)$ such that $(S,T) \in R^A$ and $a \notin T$. Then, $R^A(S) \not\subseteq \sigma_A(a)$. So, $S \notin G_{R^A}(\sigma_A(a))$ and, therefore, $G_{R^A}(\sigma(a)) \subseteq \sigma_A(G(a))$. Moreover, it is immediate that $\sigma_A(G(a)) \subseteq G_{R_A}(\sigma_A(a))$. Similarly we obtain that $H_{R^{A^{-1}}}(\sigma_A(a)) = \sigma_A(H(a))$ and so σ_A is a tense LM_n-isomorphism. ∎

LEMMA 3.17
Let (A_1, G_1, H_1) and (A_2, G_2, H_2) be two LM_n-algebras and $h : A_1 \longrightarrow A_2$ be a tense LM_n-homomorphism. Then, the map $\Phi(h) : \mathfrak{X}(A_2) \longrightarrow \mathfrak{X}(A_1)$, defined by $\Phi(h)(S) = h^{-1}(S)$ for all $S \in \mathfrak{X}(A_2)$, is a tense LM_n-function.

PROOF. It follows from the results established in [16, Lemma 2.4] and [18, Lemma 5.7]. ∎

Lemmas 3.16 and 3.17 show that Φ is a contravariant functor from ***tLM***$_n$***A*** to ***tLM***$_n$***S***.

The following characterization of isomorphisms in the category ***tLM***$_n$***S*** will be used to determine the duality that we set out to prove.

PROPOSITION 3.18
Let $(X_1, g_1, \{f_i^1\}_{i\in[n-1]}, R_1)$ and $(X_2, g_2, \{f_i^2\}_{i\in[n-1]}, R_2)$ be two tense LM_n-spaces. Then, the following conditions are equivalent, for every function $f : X_1 \longrightarrow X_2$:

(i) f is an isomorphism in the category ***tLM***$_n$***S***,
(ii) f is a bijective LM_n-function such that for all $x, y \in X_1$:
 (itf) $(x,y) \in R_1 \iff (f(x), f(y)) \in R_2$.

PROOF. It is routine. ∎

The map $\varepsilon_X : X \longrightarrow \mathfrak{X}(D(X))$, defined as in Equation (2.8), leads to another characterization of tense LM_n-spaces, which also allow us to assert that this map is an isomorphism in the category ***tLMnS***, as we will describe below:

LEMMA 3.19
Let $(X, \leq, g, \{f_i\}_{i\in[n-1]}, R)$ be a tense LM_n-space, $\varepsilon_X : X \longrightarrow \mathfrak{X}(D(X))$ be the map defined by the prescription (2.8) and let $R^{D(X)}$ be the relation defined on $\mathfrak{X}(D(X))$ by means of the operators G_R and F_R as follows:

$$(\varepsilon_X(x), \varepsilon_X(y)) \in R^{D(X)} \iff G_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_X(y) \subseteq F_R^{-1}(\varepsilon_X(x)). \tag{3.2}$$

Then, the following property holds:

(tS5) $(x,y)\in R$ implies $(\varepsilon_X(x),\varepsilon_X(y))\in R^{D(X)}$.

Proof. It is routine. ∎

Proposition 3.20
Let $(X,\leq,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space, $\varepsilon_X: X\longrightarrow \mathfrak{X}(D(X))$ be the function defined by the prescription (2.8) and let $R^{D(X)}$ be the relation defined on $\mathfrak{X}(D(X))$ by the prescription (3.2). Then, the condition (tS3) can be replaced by the following one:

(tS18) $(\varepsilon_X(x),\varepsilon_X(y))\in R^{D(X)} \Longleftrightarrow (x,y)\in R$.

Proof. It can be proved in a similar way to [18, Proposition 5.5]. ∎

Corollary 3.21
Let $(X,\leq,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space. Then, the map $\varepsilon_X: X\longrightarrow \mathfrak{X}(D(X))$ is an isomorphism in the category ***tLMnS***.

Proof. It follows from the results established in [16, Lemma 3.19, Propositions 3.18 and 3.20]. ∎

Then, from the above results and using the usual procedures we can prove that the functors $\Phi\circ\Psi$ and $\Psi\circ\Phi$ are naturally equivalent to the identity functors on $\boldsymbol{tLM_nS}$ and $\boldsymbol{tLM_nA}$, respectively, from which we conclude:

Theorem 3.22
The category $\boldsymbol{tLM_nS}$ is naturally equivalent to the dual of the category $\boldsymbol{tLM_nA}$.

4 Simple and subdirectly irreducible tense LM_n-algebras

In this section, our first objective is the characterization of the congruence lattice on a tense LM_n-algebra by means of certain closed and modal subsets of its associated tense LM_n-space. Later, this result will be taken into account to characterize simple and subdirectly irreducible tense LM_n-algebras. With this purpose, we will start by introducing the following notion.

Definition 4.1
Let $(X,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space. A subset Y of X is a tense subset if it satisfies the following conditions for all $y,z\in X$:

(ts1) if $y\in Y$ and $z\in R(y)$, then there is $w\in Y$ such that $w\in R(y)\cap\downarrow z$,
(ts2) if $y\in Y$ and $z\in R^{-1}(y)$, then there is $v\in Y$ such that $v\in R^{-1}(y)\cap\downarrow z$.

In [16], the following characterizations of a modal subset of an LM_n-space were obtained.

Proposition 4.2
[[16, Proposition 3.5]] Let $(X,g,\{f_i\}_{i\in[n-1]})$ be an LM_n-space and Y be a non-empty subset of X. Then, the following conditions are equivalent:

(i) Y is modal,
(ii) Y is involutive and increasing,
(iii) Y is a cardinal sum of maximal chains in X.

Corollary 4.3
Let $(X,g,\{f_i\}_{i\in[n-1]})$ be an LM_n-space. If $\{Y_i\}_{i\in I}$ is a family of modal subsets of X, then $\bigcap_{i\in I} Y_i$ is a modal subset of X.

PROOF. It is a direct consequence of Proposition 4.2. ∎

The notion of a modal and tense subset of a tense LM_n-space has several equivalent formulations, which will be useful later:

PROPOSITION 4.4
Let $(X,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space. If Y is a modal subset of X, then the following conditions are equivalent:

(i) Y is a tense subset,
(ii) for all $y\in Y$, the following conditions are satisfied:
  (ts3) $R(y)\subseteq Y$,
  (ts4) $R^{-1}(y)\subseteq Y$,
(iii) $Y=G_R(Y)\cap Y\cap H_{R^{-1}}(Y)$, where $G_R(Y):=\{x\in X:R(x)\subseteq Y\}$ and $H_{R^{-1}}(Y):=\{x\in X:R^{-1}(x)\subseteq Y\}$.

PROOF.
(i) $\Rightarrow$ (ii): Let $y\in Y$ and $z\in R(y)$, then by (i) and (ts1), there is $w\in Y$ such that $w\in R(y)$ and $w\leq z$. Since Y is modal, from Proposition 4.2 it follows that $z\in Y$ and therefore $R(y)\subseteq Y$. The proof that $R^{-1}(y)\subseteq Y$ is similar.

(ii) $\Rightarrow$ (i): It is immediate.

(ii) $\Leftrightarrow$ (iii): It is immediate. ∎

The closed, modal and tense subsets of the tense LM_n-space associated with a tense LM_n-algebra perform a fundamental role in the characterization of the tense LM_n-congruences on these algebras as we will show next.

THEOREM 4.5
Let (A,G,H) be a tense LM_n-algebra, and $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A. Then, the lattice $\mathcal{C}_{MT}(\mathfrak{X}(A))$ of all closed, modal and tense subsets of $\mathfrak{X}(A)$ is anti-isomorphic to the lattice $Con_{tLM_n}(A)$ of tense LM_n-congruences on A, and the anti-isomorphism is the function Θ_{MT} defined by the same prescription as in Equation (2.11).

PROOF. It immediately follows from Theorems 2.5 and 2.18 and the fact that $\mathcal{C}_{MT}(\mathfrak{X}(A))=\mathcal{C}_M(\mathfrak{X}(A))\cap\mathcal{C}_T(\mathfrak{X}(A))$ and for all $\varphi\subseteq A\times A$, $\varphi\in Con_{tLM_n}(A)$ iff φ is both an LM_n-congruence on A and a tense De Morgan congruence on A. ∎

Next, we will use the results already obtained in order to determine the simple and subdirectly irreducible tense LM_n-algebras.

COROLLARY 4.6
Let (A,G,H) be a tense LM_n-algebra, and $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A. Then, the following conditions are equivalent:

(i) (A,G,H) is a simple tense LM_n-algebra,
(ii) $\mathcal{C}_{MT}(\mathfrak{X}(A))=\{\emptyset,\mathfrak{X}(A)\}$.

PROOF. It is a direct consequence of Theorem 4.5. ∎

Corollary 4.7
Let (A,G,H) be a tense LM_n-algebra, and $(\mathfrak{X}(A),\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A. Then, the following conditions are equivalent:

(i) (A,G,H) is a subdirectly irreducible tense LM_n-algebra,
(ii) there is $Y\in\mathcal{C}_{MT}(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$ such that $Z\subseteq Y$ for all $Z\in\mathcal{C}_{MT}(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$.

Proof. It is a direct consequence of Theorem 4.5. ■

Proposition 4.8
Let $(X,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space. If Y is a modal subset of X, then $G_R(Y)$ and $H_{R^{-1}}(Y)$ are also modal.

Proof. Let Y be a modal subset of X and $z\in G_R(Y)$. Therefore, (1) $R(z)\subseteq Y$. Let (2) $w\in C_z$, where C_z is the unique maximal chain containing z, and so from property (tS9), $C_z=\{f_i(z):i\in[n-1]\}$. Then, there are $n_0,n_1\in[n-1]$ such that (3) $z=f_{n_0}(z)$ and (4) $w=f_{n_1}(z)$. Let (5) $t\in R(w)$, then by (4) we have that $(f_{n_1}(z),t)\in R$ and therefore, from (tS4) we infer that $(f_{n_0}(f_{n_1}(z)),f_{n_0}(t))\in R$. From this last fact, property (LP5) and (3) it follows that $(z,f_{n_0}(t))\in R$. Consequently by (1), we get that $f_{n_0}(t)\in Y$. Since Y is modal, from Proposition 4.2 and (tS9), we obtain that $C_t=\{f_i(t):i\in[n-1]\}\subseteq Y$, and so from (LP9), we conclude that $t\in Y$, which allows us to assert from (5) that $R(w)\subseteq Y$ and thus $w\in G_R(Y)$. Finally, we can say from (2) that $C_z\subseteq G_R(Y)$ for all $z\in G_R(Y)$, and hence from Proposition 4.2, we conclude that $G_R(Y)$ is modal. The proof that $H_{R^{-1}}(Y)$ is modal is similar. ■

The characterization of modal and tense subsets of a tense LM_n-space, given in Proposition 4.4, prompts us to introduce the following definition:

Definition 4.9
Let $(X,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space and let $d_X:\mathcal{P}(X)\longrightarrow\mathcal{P}(X)$ defined by:

$$d_X(Z)=G_R(Z)\cap Z\cap H_{R^{-1}}(Z),\ \text{for all}\ Z\in\mathcal{P}(X). \tag{4.1}$$

For each $n\in\omega$, let $d_X^n:\mathcal{P}(X)\longrightarrow\mathcal{P}(X)$, defined by:

$$d_X^0(Z)=Z,\ d_X^{n+1}(Z)=d_X(d_X^n(Z)),\ \text{for all}\ Z\in\mathcal{P}(X). \tag{4.2}$$

By using the above functions d_X, d_X^n, $n\in\omega$, we obtain another equivalent formulation of the notion of modal and tense subset of a tense LM_n-space.

Lemma 4.10
Let $(X,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space. If Y is modal subset of X, then the following conditions are equivalent:

(i) Y is a tense subset,
(ii) $Y=d_X^n(Y)$ for all $n\in\omega$,
(iii) $Y=\bigcap_{n\in\omega}d_X^n(Y)$.

Proof. It is an immediate consequence of Proposition 4.4 and Definition 4.9. ■

Proposition 4.11
Let $(X,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space and $(D(X),G_R,H_{R^{-1}})$ be the tense LM_n-algebra associated with X. Then, for all $n\in\omega$, for all $U,V\in D(X)$ and for all $i\in[n-1]$, the following conditions

are satisfied:

(d0) $d_X^n(U) \in D(X)$,
(d1) $d_X^n(X) = X$ and $d_X^n(\emptyset) = \emptyset$,
(d2) $d_X^{n+1}(U) \subseteq d_X^n(U)$,
(d3) $d_X^n(U \cap V) = d_X^n(U) \cap d_X^n(V)$,
(d4) $U \subseteq V$ implies $d_X^n(U) \subseteq d_X^n(V)$,
(d5) $d_X^n(U) \subseteq U$,
(d6) $d_X^{n+1}(U) \subseteq G_R(d_X^n(U))$ and $d_X^{n+1}(U) \subseteq H_{R^{-1}}(d_X^n(U))$,
(d7) $d_X^n(f_i^{-1}(U)) = f_i^{-1}(d_X^n(U))$ for any $n \in \omega$ and $i \in [n-1]$,
(d8) if U is modal, then $d_X^n(U)$ is modal,
(d9) $\bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))$ is a closed, modal and tense subset of X and therefore
$d_X(\bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))) = \bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))$.

Proof. From Definition 4.9, Lemma 3.16 and the fact that G_R, $H_{R^{-1}}$ and d_X^n, $n \in \omega$, are monotonic operations it immediately follows that properties (d0), (d1), (d2), (d3), (d4), (d5) and (d6) hold.

(d7): Let $U \in D(X)$ and $i \in [n-1]$, then $d_X(f_i^{-1}(U)) = f_i^{-1}(U) \cap G_R(f_i^{-1}(U)) \cap H_{R^{-1}}(f_i^{-1}(U))$. From the last assertion and properties (tS17) and (tS18) in Lemma 3.8, we infer that (1) $d_X(f_i^{-1}(U)) = f_i^{-1}(U \cap G_R(U) \cap H_{R^{-1}}(U)) = f_i^{-1}(d_X(U))$ for any $U \in D(X)$ and $i \in [n-1]$ Suppose that $d_X^{n-1}(f_i^{-1}(U)) = f_i^{-1}(d_X^{n-1}(U))$, for any $n \in \omega$ and $i \in [n-1]$, then (2) $d_X^n(f_i^{-1}(U)) = d_X (d_X^{n-1}(f_i^{-1}(U))) = d_X(f_i^{-1}(d_X^{n-1}(U)))$. Taking into account that $d_X^{n-1}(U) \in D(X)$ and (1), we get that $d_X(f_i^{-1}(d_X^{n-1}(U))) = f_i^{-1}(d_X(d_X^{n-1}(U))) = f_i^{-1}(d_X^n(U))$, and so from (2) the proof is complete.

(d8): It is a direct consequence of Corollary 4.3 and Proposition 4.8.

(d9): Let $U \in D(X)$. Then, from Lemma 3.16 and the prescription (2.12), we have that $f_i^{-1}(U) \in D(X)$. Also, from (LP5), $f_i^{-1}(U)$ is a modal subset of X for all $i \in [n-1]$, from which it follows by (d7) that for $n \in \omega$ and $i \in [n-1]$, $d_X^n(f_i^{-1}(U))$ is a modal and closed subset of X, and so by Corollary 4.3 and the fact that the arbitrary intersection of closed subsets of X is closed, we get that $\bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))$ is a modal and closed subset of X. If $\bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U)) = \emptyset$, then it is verified that $\bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))$ is a closed, modal and tense subset of X. Suppose now that there exists $y \in \bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))$. Since, $f_i^{-1}(U) \in D(X)$ for any $i \in [n-1]$, then from (d6) it follows that $y \in G_R(d_X^{n-1}(f_i^{-1}(U)))$ and $y \in H_{R^{-1}}(d_X^{n-1}(f_i^{-1}(U)))$ for all $n \in \omega$. Therefore, $R(y) \subseteq d_X^{n-1}(f_i^{-1}(U))$ and $R^{-1}(y) \subseteq d_X^{n-1}(f_i^{-1}(U))$ for all $n \in \omega$ and consequently $R(y) \subseteq \bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))$ and $R^{-1}(y) \subseteq \bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))$ for all $i \in [n-1]$. From these last assertions, the fact that $\bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))$ is a modal and closed subset of X and Proposition 4.11, we have that $\bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))$ is a tense subset, from which we conclude, by Lemma 4.10, that $d_X(\bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))) = \bigcap\limits_{n \in \omega} d_X^n(f_i^{-1}(U))$. ∎

As consequences of Proposition 4.11 and the above duality for tense LM_n-algebras (Lemma 3.16) we obtain the following corollaries.

Corollary 4.12
Let $(A, \sim, \{\varphi_i\}_{i \in [n-1]}, G, H)$ be a tense LM_n-algebra and consider the function $d : A \longrightarrow A$, defined by $d(a) = G(a) \wedge a \wedge H(a)$, for all $a \in A$. For all $n \in \omega$, let $d^n : A \longrightarrow A$ be a function, defined by $d^0(a) = a$

and $d^{n+1}(a)=d(d^n(a))$, for all $a\in A$. Then, for all $n\in\omega$ and $a,b\in A$, the following conditions are satisfied:

(d1) $d^n(1)=1$ and $d^n(0)=0$,
(d2) $d^{n+1}(a)\leq d^n(a)$,
(d3) $d^n(a\wedge b)=d^n(a)\wedge d^n(b)$,
(d4) $a\leq b$ implies $d^n(a)\leq d^n(b)$,
(d5) $d^n(a)\leq a$,
(d6) $d^{n+1}(a)\leq G(d^n(a))$ and $d^{n+1}(a)\leq H(d^n(a))$,
(d7) for all $i\in[n-1]$ and $n\in\omega$, $d^n(\varphi_i(a))=\varphi_i(d^n(a))$.

Corollary 4.13
Let (A,G,H) be a tense LM_n-algebra, $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A and let $\sigma_A:A\longrightarrow D(\mathfrak{X}(A))$ be the map defined by the prescription (2.4). Then, $\sigma_A(d^n(a))=d^n_{\mathfrak{X}(A)}(\sigma_A(a))$ for all $a\in A$ and $n\in\omega$.

Proof. It is a direct consequence of Lemma 3.16. ∎

It seems worth mentioning that the operator d defined in Corollary 4.12 was previously defined in [21] for tense algebras, in [13] for tense MV-algebras, and in [8, 9] for tense θ-valued Łukasiewicz–Moisil algebras, respectively.

Lemma 4.14
Let (A,G,H) be a tense LM_n-algebra. If $\bigwedge\limits_{i\in I}a_i$ exists, then the following conditions hold:

(i) $\bigwedge\limits_{i\in I}G(a_i)$ exists and $\bigwedge\limits_{i\in I}G(a_i)=G(\bigwedge\limits_{i\in I}a_i)$,
(ii) $\bigwedge\limits_{i\in I}H(a_i)$ exists and $\bigwedge\limits_{i\in I}H(a_i)=H(\bigwedge\limits_{i\in I}a_i)$,
(iii) $\bigwedge\limits_{i\in I}d(a_i)$ exists and $\bigwedge\limits_{i\in I}d^n(a_i)=d^n(\bigwedge\limits_{i\in I}a_i)$ for all $n\in\omega$.

Proof. (i): Assume that $a_i\in A$ for all $i\in I$ and $\bigwedge\limits_{i\in I}a_i$ exists. Since $\bigwedge\limits_{i\in I}a_i\leq a_i$, we have by (T2) that $G(\bigwedge\limits_{i\in I}a_i)\leq G(a_i)$ for each $i\in I$. Thus, $G(\bigwedge\limits_{i\in I}a_i)$ is a lower bound of the set $\{G(a_i):i\in I\}$. Assume now that b is a lower bound of the set $\{G(a_i):i\in I\}$. From (T5) and (T6) we have that $P(b)\leq PG(a_i)\leq a_i$ for each $i\in I$. So, $P(b)\leq\bigwedge\limits_{i\in I}a_i$. Besides, the pair (G,P) is a Galois connection, this means that $x\leq G(y)\Longleftrightarrow P(x)\leq y$, for all $x,y\in A$. So, we can infer that $b\leq G(\bigwedge\limits_{i\in I}a_i)$. This proves that $\bigwedge\limits_{i\in I}G(a_i)$ exists and $\bigwedge\limits_{i\in I}G(a_i)=G(\bigwedge\limits_{i\in I}a_i)$.

(ii): The proof for the operator H is analogous to the proof for G.

(iii): It is a direct consequence of (i) and (ii). ∎

For invariance properties we have:

Lemma 4.15
Let $(X,g,\{f_i\}_{i\in[n-1]},R)$ be a tense LM_n-space and $(D(X),G_R,H_{R^{-1}})$ be the tense LM_n-algebra associated with X. Then, for all $U,V,W\in D(X)$ such that $U=d_X(U)$, $V=d_X(V)$ and for some $i_0\in[n-1]$,

$d_X(f_{i_0}^{-1}(W)) = f_{i_0}^{-1}(W)$, the following properties are satisfied:

(i) $U \cap V = d_X(U \cap V)$,
(ii) $U \cup V = d_X(U \cup V)$,
(iii) $\sim_g U = d_X(\sim_g U)$,
(iv) $d_X(f_i^{-1}(W)) = f_i^{-1}(W)$ for all $i \in [n-1]$.

PROOF.
(i): It immediately follows from the definition of the function d_X and property (T2) of tense LM_n-algebras.

(ii): Taking into account that $U = d_X(U)$ and $V = d_X(V)$ and the fact that the operations G_R and $H_{R^{-1}}$ are increasing, we infer that $U \cup V \subseteq G_R(U \cup V)$ and $U \cup V \subseteq H_{R^{-1}}(U \cup V)$, which imply that $U \cup V = d_X(U \cup V)$,

(iii): Let $U \in D(X)$ such that (1) $U = d_X(U)$. Then, it is verified that $\sim_g U \subseteq G_R(\sim_g U)$. Indeed, let $x \in \sim_g U$ and (2) $y \in R(x)$. Then, $x \in X \setminus g(U)$ and hence (3) $x \notin g(U)$. Suppose that $y \in g(U)$, then there is $z \in U$ such that $y = g(z)$, and by (tS11) in Lemma 3.8, we get that $R^{-1}(y) = R^{-1}(g(z)) = g(R^{-1}(z))$. Since $z \in U$, from (1) it follows that $R^{-1}(z) \subseteq U$ and so $g(R^{-1}(z))) \subseteq g(U)$. Thus, $R^{-1}(y) \subseteq g(U)$. From the last statement and (2), we infer that $x \in g(U)$, which contradicts (3). Consequently, $y \in \sim_g U$, which allows us to assert that $R(x) \subseteq \sim_g U$ and therefore $\sim_g U \subseteq G_R(\sim U)$. In a similar way, we can prove that $\sim_g U \subseteq H_{R^{-1}}(\sim_g U)$. From the two last assertions we conclude that $\sim_g U = d_X(\sim_g U)$.

(iv): If $W \in D(X)$ and $d_X(f_{i_0}^{-1}(W)) = f_{i_0}^{-1}(W)$ for some $i_0 \in [n-1]$, then from (d7) it follows that $f_{i_0}^{-1}(d_X(W)) = f_{i_0}^{-1}(W)$. From the last assertion and (LP5) we infer that $f_i^{-1}(d_X(W)) = f_i^{-1}(W)$ for all $i \in [n-1]$, and so from (d7), we get that $d_X(f_i^{-1}(W)) = f_i^{-1}(W)$ for all $i \in [n-1]$. ∎

COROLLARY 4.16
Let $(A, \sim, \{\varphi_i\}_{i \in [n-1]}, G, H)$ be a tense LM_n-algebra. Then, for all $a, b, c \in A$, such that $a = d(a)$, $b = d(b)$ and $\varphi_{i_0}(c) = d(\varphi_{i_0}(c))$ for some $i_0 \in [n-1]$, the following properties are satisfied:

(i) $d(a \wedge b) = a \wedge b$,
(ii) $d(a \vee b) = a \vee b$,
(iii) $d(\sim a) = \sim a$,
(iv) $\varphi_i(c) = d(\varphi_i(c))$ for all all $i \in [n-1]$.

PROOF. It is a direct consequence of Lemmas 3.16 and 4.15. ∎

LEMMA 4.17
Let (A, G, H) be a tense LM_n-algebra. Then, for all $a \in A$, the following conditions are equivalent:

(i) $a = d(a)$,
(ii) $a = d^n(a)$ for all $n \in \omega$.

PROOF. It immediately follows from Corollary 4.12. ∎

LEMMA 4.18
Let $(A, \sim, \{\varphi_i\}_{i \in [n-1]}, G, H)$ be a tense LM_n-algebra and $\mathcal{C}(A) := \{a \in A : d(a) = a\}$. Then, $\langle \mathcal{C}(A), \vee, \wedge, \sim, \{\varphi_i\}_{i \in [n-1]}, 0, 1\rangle$ is an LM_n-algebra.

PROOF. From Corollary 4.16 and property (d1) in Corollary 4.12, we have that $\langle \mathcal{C}(A), \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra. Taking into account that $a = d(a)$ for all $a \in \mathcal{C}(A)$, and the property (iv) in Corollary 4.16 it follows that $\varphi_i(a) = \varphi_i(d(a)) = d(\varphi_i(a))$ for all $a \in \mathcal{C}(A)$

and $i \in [n-1]$. Therefore, $\varphi_i(a) \in \mathcal{C}(A)$ for all $a \in \mathcal{C}(A)$ and $i \in [n-1]$, from which we conclude that $\langle \mathcal{C}(A), \vee, \wedge, \sim, \{\varphi_i\}_{i\in[n-1]}, 0, 1\rangle$ is an LM_n-algebra. ∎

COROLLARY 4.19
Let (A, G, H) be a tense LM_n-algebra. Then, $(\mathcal{B}(\mathcal{C}(A)), G, H)$ is a tense Boolean algebra, where $\mathcal{B}(\mathcal{C}(A))$ is the Boolean algebra of all complemented elements of $\mathcal{C}(A)$.

PROOF. It is a direct consequence of Lemmas 2.9 and 4.18 and property (iv) in Corollary 4.16. ∎

REMARK 4.20
Let us recall that under the Priestley duality, the lattice of all filters of a bounded distributive lattice is dually isomorphic to the lattice of all increasing closed subsets of the dual space. Under that isomorphism, any filter T of a bounded distributive lattice A corresponds to the increasing closed set

$$Y_T = \{S \in \mathfrak{X}(A) : T \subseteq S\} = \bigcap \{\sigma_A(a) : a \in T\} \tag{4.3}$$

and $\Theta_C(Y_T) = \Theta(T)$, where Θ_C is defined as in (2.11) and $\Theta(T)$ is the lattice congruence associated with T.

Conversely any increasing closed subset Y of $\mathfrak{X}(A)$ corresponds to the filter

$$T_Y = \{a \in A : Y \subseteq \sigma_A(a)\}, \tag{4.4}$$

and $\Theta(T_Y) = \Theta_C(Y)$, where Θ_C is defined as in (2.11), and $\Theta(T_Y)$ is the lattice congruence associated with T_Y.

Taking into account these last remarks on Priestley duality, Theorem 4.5 and Proposition 4.2, we can say that the congruences on a tense LM_n-algebra are the lattice congruences associated with certain filters of this algebra. So our next goal is to determine the conditions that a filter of a tense LM_n-algebra must fulfil for the associated lattice congruence to be a tense LM_n-congruence.

THEOREM 4.21
Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ be a tense LM_n-algebra. If S is a filter of A, then, the following conditions are equivalent:

(i) $\Theta(S) \in Con_{tLM_n}(A)$,
(ii) $d(\varphi_i(a)) \in S$ for any $a \in S$ and $i \in [n-1]$,
(iii) $d^n(\varphi_i(a)) \in S$ for any $a \in S$, $n \in \omega$ and $i \in [n-1]$.

PROOF.
(i) $\Rightarrow$ (ii): Let S be a filter of A such that $\Theta(S) \in Con_{tLM_n}(A)$. Then, from Priestley duality and Theorem 4.5 it follows that $\Theta(S) = \Theta_{MT}(Y_S)$, where $\Theta(S)$ is the lattice congruence associated with S, and $Y_S = \{x \in \mathfrak{X}(A) : S \subseteq x\} = \bigcap\limits_{a\in S} \sigma_A(a)$ is a closed, modal and tense subset of the tense LM_n-space $\mathfrak{X}(A)$, associated with A. Since Y_S is modal and σ_A is an LM_n-isomorphism, then $Y_S = f_i^{A^{-1}}(Y_S) = f_i^{A^{-1}}\Big(\bigcap\limits_{a\in S} \sigma_A(a)\Big) = \bigcap\limits_{a\in S} \sigma_A(\varphi_i(a))$ for any $i \in [n-1]$. From the last assertion, and taking into account that Y is a tense subset, Lemmas 4.10 and 4.12, Corollary 4.13 and the fact that the function $d_{\mathfrak{X}(A)} : \mathfrak{X}(A) \longrightarrow \mathfrak{X}(A)$ is monotone, we infer that $Y_S = d_{\mathfrak{X}(A)}\Big(\bigcap\limits_{a\in S} \sigma_A(\varphi_i(a))\Big) \subseteq \bigcap\limits_{a\in S} d_{\mathfrak{X}(A)}(\sigma_A(\varphi_i(a))) = \bigcap\limits_{a\in S} \sigma_A(d(\varphi_i(a))) \subseteq \bigcap\limits_{a\in S} \sigma_A(\varphi_i(a)) = Y_S$, for any $i \in [n-1]$. Hence $Y_S = \bigcap\limits_{a\in S} \sigma_A(d(\varphi_i(a)))$ for any $i \in [n-1]$, from which we conclude that $d(\varphi_i(a)) \in S$ for any $a \in S$ and $i \in [n-1]$. Indeed, assume that $a \in S$,

then $a \in x$ for all $x \in Y_S$, from which it follows that $x \in \bigcap_{a \in S} \sigma_A(d(\varphi_i(a)))$ for any $i \in [n-1]$, and thus $d(\varphi_i(a)) \in x$ for all $x \in Y_S$ and $i \in [n-1]$. Therefore, $d(\varphi_i(a)) \in \bigcap_{x \in Y_S} x$ for any $i \in [n-1]$, and taking into account that $S = \bigcap_{x \in Y_S} x$, we obtain that $d(\varphi_i(a)) \in S$ for any $i \in [n-1]$.

(ii) $\Rightarrow$ (i): From Priestley duality and Equation (4.3), we have that $\bigcap_{a \in S} \sigma_A(a) = Y_S = \{x \in \mathfrak{X}(A) : S \subseteq x\}$ is an increasing and closed subset of $\mathfrak{X}(A)$ and $\Theta(S) = \Theta(Y_S)$. By Theorem 4.5, it remains to show that Y_S is a modal and tense subset of $\mathfrak{X}(A)$. From the hypothesis (ii), it follows that for all $a \in S$, $i \in [n-1]$ and $x \in Y_S$, $d(\varphi_i(a)) \in x$. Therefore, from this last fact and Corollary 4.16, it results that $\varphi_i(d(a)) \in x$ for all $i \in [n-1]$ and all $x \in Y_S$, and hence (1) $Y_S \subseteq \bigcap_{a \in S} \sigma_A(\varphi_i(d(a)))$ for all $i \in [n-1]$. Consequently, by Corollary 4.12, $Y_S \subseteq \bigcap_{a \in S} \sigma_A(\varphi_i(a))$ for all $i \in [n-1]$, and from this assertion it follows that $Y_S \subseteq \bigcap_{a \in S} \sigma_A(\varphi_1(a)) \subseteq \bigcap_{a \in S} \sigma_A(a) = Y_s$. Since σ_A is an LM_n-isomorphism, then we get that $\cdot$(2) $Y_s = \bigcap_{a \in S} \sigma_A(\varphi_1(a)) = \bigcap_{a \in S} f_1^{A^{-1}}(\sigma_A(a)) = f_1^{A^{-1}}\left(\bigcap_{a \in S} \sigma_A(a)\right) = f_1^{A^{-1}}(Y_S)$. Therefore from the last statement and (LP5) we conclude that $Y_S = f_i^A(Y_S)$ for all $i \in [n-1]$ and so Y_S is modal. In addition, from (1), (2) and Corollary 4.12 we infer that $Y_S \subseteq \bigcap_{a \in S} \sigma_A(d(\varphi_1(a))) \subseteq \bigcap_{a \in S} \sigma_A(\varphi_1(a)) = Y_s$ and hence, $Y_S = \bigcap_{a \in S} \sigma_A(d(\varphi_1(a)))$. Then, taking into account Corollary 4.13 and that $\bigcap_{a \in S} d_{\mathfrak{X}(A)}(\sigma_A(\varphi_1(a))) = d_{\mathfrak{X}(A)}\left(\bigcap_{a \in S} \sigma_A(\varphi_1(a))\right)$, we obtain that $Y_S = d_{\mathfrak{X}(A)}(Y_S)$, and thus, from Lemma 4.10 and the fact that Y_S is modal, we infer that Y_S is a tense subset of $\mathfrak{X}(A)$. Finally, since Y_S is a closed, modal and tense subset of $\mathfrak{X}(A)$ and $\Theta(S) = \Theta_{MT}(Y_S)$, we conclude, from Theorem 4.5, that $\Theta(S) \in Con_{tLM_n}(A)$.

(ii) $\Leftrightarrow$ (iii): It is trivial. ∎

Theorem 4.21 leads us to introduce the following definition:

Definition 4.22
Let (A, G, H) be a tense LM_n-algebra. A filter S of A is a tense filter iff

(tf) $d(a) \in S$ for all $a \in S$ or equivalently $d^n(a) \in S$ for all $a \in S$ and $n \in \omega$.

Now, we remember the notion of Stone filter of an LM_n-algebra.

Definition 4.23
Let $(A, \sim, \{\varphi_i\}_{i \in [n-1]})$ be an LM_n-algebra. A filter S of A is a Stone filter iff

(sf) $\varphi_i(a) \in S$ for all $a \in S$ and $i \in [n-1]$, or equivalently $\varphi_1(a) \in S$ for all $a \in S$.

Lemma 4.24
Let $(A, \sim, \{\varphi_i\}_{i \in [n-1]}, G, H)$ be a tense LM_n-algebra. If S is a Stone filter of A, then the following conditions are equivalent:

(i) S is a tense filter of A,
(ii) $d^n(\varphi_i(a)) \in S$ for all $a \in S$, $n \in \omega$ and $i \in [n-1]$.

Proof.
(i) $\Rightarrow$ (ii): Let S be a Stone filter of A, $a \in S$, $n \in \omega$ and $i \in [n-1]$. Since S is an Stone filter of A, we have that $\varphi_i(a) \in S$. From this last assertion and the fact that S is a tense filter we conclude that $d^n(\varphi_i(a)) \in S$.

(ii) $\Rightarrow$ (i): Let $a\in S$. Then, from the hypothesis (ii) we obtain that $d^n(\varphi_1(a))\in S$. From the last assertion, properties (L9) and (d5) and the fact that S is a filter of A we infer that $d^n(a)\in S$ for all $n\in\omega$, and therefore S is a tense filter of A. ∎

We will denote by $\mathcal{F}_{TS}(A)$ the set of all tense Stone filters of a tense LM_n-algebra (A,G,H).

Proposition 4.25
Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ be a tense LM_n-algebra. Then, the following conditions are equivalent for all $\theta\subseteq A\times A$:

(i) $\theta\in Con_{tLM_n}(A)$,
(ii) there is $S\in\mathcal{F}_{TS}(A)$ such that $\theta=\Theta(S)$, where $\Theta(S)$ is the lattice congruence associated with the filter S.

Proof.
(i) $\Rightarrow$ (ii): From (i) and Theorem 4.5, it follows that there exists $Y\in\mathcal{C}_{MT}(\mathfrak{X}(A))$ such that (1) $\Theta_{MT}(Y)=\theta$. Then, from Remark 4.20, we infer that $T_Y=\{a\in A: Y\subseteq\sigma_A(a)\}$ is a filter on A and (2) $\Theta(T_Y)=\Theta(Y)=\Theta_{MT}(Y)$. Therefore $\Theta(T_Y)\in Con_{tLM_n}(A)$, and so from Theorem 4.21, we obtain that $Y\in\mathcal{F}_{TS}(A)$. This last assertion, (1) and (2) enable us to conclude the proof.

(ii) $\Rightarrow$ (i): It immediatly follows from Theorem 4.21. ∎

Corollary 4.26
Let (A,G,H) be a tense LM_n-algebra. Then,

(i) (A,G,H) is a simple tense LM_n-algebra if and only if $\mathcal{F}_{TS}(A)=\{A,\{1\}\}$.
(ii) (A,G,H) is a subdirectly irreducible tense LM_n-algebra if and only if there is $T\in\mathcal{F}_{TS}(A)$, $T\neq\{1\}$ such that $T\subseteq S\cdot$ for all $S\in\mathcal{F}_{TS}(A)$, $S\neq\{1\}$.

Proof. It is a direct consequence of Corollaries 4.6 and 4.7, Remark 4.20 and Proposition 4.25. ∎

Finally, we will describe the simple and subdirectly irreducible tense LM_n-algebras.

In the proof of the following proposition we will use the finite intersection property of compact spaces, which establishes that if X is a compact topological space, then for each family $\{M_i\}_{i\in I}$ of closed subsets of X satisfying $\bigcap\limits_{i\in I}M_i=\emptyset$, there is a finite subfamily $\{M_{i_1},\ldots,M_{i_n}\}$ such that $\bigcap\limits_{j=1}^{n}M_{i_j}=\emptyset$.

Proposition 4.27
Let (A,G,H) be a tense LM_n-algebra and $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A. Then, the following conditions are equivalent:

(i) (A,G,H) is a simple tense LM_n-algebra,
(ii) for every $U\in D(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$ and for every $i\in[n-1]$ such that $f_i^{A^{-1}}(U)\neq\mathfrak{X}(A)$, $\bigcap\limits_{n\in\omega}d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U))=\emptyset$,
(iii) for every $U\in D(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$ and for every $i\in[n-1]$ such that $f_i^{A^{-1}}(U)\neq\mathfrak{X}(A)$, $d^{n_i^U}_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U))=\emptyset$ for some $n_i^U\in\omega$,
(iv) for every $U\in\mathcal{B}(D(\mathfrak{X}(A)))\setminus\{\mathfrak{X}(A)\}$, there is $n_U\in\omega$ such that $d^{n_U}_{\mathfrak{X}(A)}(U)=\emptyset$,
(v) $\mathcal{F}_{TS}(D(\mathfrak{X}(A)))=\{D(\mathfrak{X}(A)),\{\mathfrak{X}(A)\}\}$.

Proof.
(i) $\Rightarrow$ (ii): Let $U \in D(\mathfrak{X}(A)) \setminus \{\mathfrak{X}(A)\}$. Then, from Lemma 3.15 and property (LP10) of LM_n-spaces, we infer that there is at least $i_0 \in [n-1]$ such that $f_{i_0}^{A^{-1}}(U) \neq \mathfrak{X}(A)$. Now, let $i \in [n-1]$ such that $f_i^{A^{-1}}(U) \neq \mathfrak{X}(A)$, then from (d5) in Proposition 4.11 we have that $d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U)) \neq \mathfrak{X}(A)$. From this last assertion and (d9) in Proposition 4.11, we obtain that $\bigcap\limits_{n\in\omega} d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U)) \in \mathcal{C}_{MT}(\mathfrak{X}(A)) \setminus \{\mathfrak{X}(A)\}$. From this last statement, the hypothesis (i) and Corollary 4.6, we conclude that $\bigcap\limits_{n\in\omega} d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U)) = \emptyset$.

(ii) $\Rightarrow$ (iii): Let $U \in D(\mathfrak{X}(A)) \setminus \{\mathfrak{X}(A)\}$ and $i \in [n-1]$ such that $f_i^{A^{-1}}(U) \neq \mathfrak{X}(A)$. Then, from the hypothesis (ii), we have that (1) $\bigcap\limits_{n\in\omega} d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U)) = \emptyset$. Besides, for all $n \in \omega$, $d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U))$ is a closed subset of $\mathfrak{X}(A)$ and $d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U)) = \bigcap\limits_{j=1}^{n} d^j_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U))$. Then, from (1), the last statement, the fact that $\mathfrak{X}(A)$ is compact and the finite intersection property of compact spaces, we conclude that there is $n_i^U \in \omega$ such that $d^{n_i^U}_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U)) = \emptyset$.

(iii) $\Rightarrow$ (iv): From Lemma 2.9, we have that $U \in \mathcal{B}(D(\mathfrak{X}(A)))$ if and only if $U = f_i^{A^{-1}}(U)$ for all $i \in [n-1]$, and so from property (LP10) of LM_n-spaces, we infer that $U \in \mathcal{B}(D(\mathfrak{X}(A))) \setminus \{\mathfrak{X}(A)\}$ iff $f_i^{A^{-1}}(U) \neq \mathfrak{X}(A)$ for all $i \in [n-1]$. Therefore, from the previous assertion and the hypothesis (iii), we obtain that for each $U \in \mathcal{B}(D(\mathfrak{X}(A)))$ and each $i \in [n-1]$, there is $n_i^U \in \omega$ such that $d^{n_i^U}_{\mathfrak{X}(A)}(U) = \emptyset$. In addittion, from (1) it follows that for all $i, j \in [n-1]$, $n_i^U = n_j^U = n_U$, and so the proof is complete.

(iv) $\Rightarrow$ (v): Assume that $S \in \mathcal{F}_{TS}(D(\mathfrak{X}(A)))$, $S \neq \{\mathfrak{X}(A)\}$. Then there is (1) $U \in S$, $U \neq \mathfrak{X}(A)$ and so from property (LP10) of LM_n-spaces, we infer that there is $i \in [n-1]$ such that $f_i^{A^{-1}}(U) \neq \mathfrak{X}(A)$. Considering (2) $V = f_i^{A^{-1}}(U)$, then from Lemma 2.9, we obtain that $V \in \mathcal{B}(D(\mathfrak{X}(A)))$, $V \neq \mathfrak{X}(A)$. Hence, from the hypothesis (iv), we can assert that there is $n_V \in \omega$ such that $d^{n_V}_{\mathfrak{X}(A)}(V) = \emptyset$. From (1), (2), the preceding assertion and Definitions 4.22 and 4.23, we deduce that $\emptyset \in S$, which implies that $S = D(\mathfrak{X}(A))$.

(v) $\Rightarrow$ (i): It immediately follows from Corollary 4.26 and the fact that (A, G, H) is isomorphic to the tense LM_n-algebra $(D(\mathfrak{X}(A)), G_{R^A}, H_{R^{A^{-1}}})$. ∎

Corollary 4.28
Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ be a tense LM_n-algebra. Then, the following conditions are equivalent:

(i) $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ is a simple tense LM_n-algebra,
(ii) for every $a \in A \setminus \{1\}$ and for every $i \in [n-1]$ such that $\varphi_i(a) \neq 1$, $d^{n_i^a}(\varphi_i(a)) = 0$ for some $n_i^a \in \omega$,
(iii) for each $a \in \mathcal{B}(A) \setminus \{1\}$, there is $n_a \in \omega$ such that $d^{n_a}(a) = 0$,
(iv) $\mathcal{F}_{TS}(A) = \{A, \{1\}\}$.

Proof. It is a direct consequence of Proposition 4.27 and the fact that $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$ is a tense LM_n-isomorphism. ∎

Corollary 4.29
If $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ is a simple tense LM_n-algebra, then $\mathcal{B}(\mathcal{C}(A)) = \{0, 1\}$ and therefore $(\mathcal{C}(A), \sim, \{\varphi_i\}_{i\in[n-1]})$ is a simple LM_n-algebra.

Proof. From Lemmas 2.9 and 4.17, property (iv) in Corollary 4.16 and property (ii) in Corollary 4.28 it follows that $\mathcal{B}(\mathcal{C}(A))=\{0,1\}$. From this last assertion, Corollary 2.12 and Lemma 4.18 the proof is complete. ∎

Next, we will recall two concepts which will play a fundamental role in this article. Let Y be a topological space and $y_0\in Y$. A net in a space Y is a map $\varphi:\mathcal{D}\longrightarrow Y$ of some directed set $(\mathcal{D},\prec)$ (i.e. $\mathcal{D}\neq\emptyset$ and $\prec$ is a preorder on $\mathcal{D}$ and for all $d_1,d_2\in\mathcal{D}$ there is $d_3\in\mathcal{D}$ such that $d_1\prec d_3$ and $d_2\prec d_3$). Besides, we say that φ converges to y_0 (written $\varphi\to y_0$) if for all neighborhoods $U(y_0)$ of y_0 there is $d_0\in\mathcal{D}$ such that for all $d\in\mathcal{D}$, $d_0\prec d$, $\varphi(d)\in U(y_0)$. We also say that φ accumulates at y_0 (written $\varphi\succ y_0$) if for all neighborhoods $U(y_0)$ of y_0 and for all $d\in\mathcal{D}$, there is $d_c\in\mathcal{D}$ such that $d\prec d_c$ and $\varphi(d_c)\in U(y_0)$. If $\varphi:\mathcal{D}\longrightarrow Y$ is a net and $y_d=\varphi(d)$ for all $d\in\mathcal{D}$, then the net φ it will be denoted by $(y_d)_{d\in\mathcal{D}}$. If $\varphi\to y_0$, it will be denoted by $(y_d)\underset{d\in\mathcal{D}}{\longrightarrow}y_0$. If $\varphi\succ y_0$, it will be denoted by $(y_d)_{d\in\mathcal{D}}\succ y_0$.

Proposition 4.30
Let (A,G,H) be a tense LM_n-algebra and $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A. Then, the following conditions are equivalent:

(i) (A,G,H) is a subdirectly irreducible tense LM_n-algebra,
(ii) there is $V\in\mathcal{B}(D(\mathfrak{X}(A)))$, $V\neq\mathfrak{X}(A)$, such that for each $U\in D(\mathfrak{X}(A))$, $U\neq\mathfrak{X}(A)$ and for each $i\in[n-1]$ such that $f_i^{A^{-1}}(U)\neq\mathfrak{X}(A)$, $\bigcap\limits_{n\in\omega}d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U))\subseteq V$,
(iii) there is $V\in\mathcal{B}(D(\mathfrak{X}(A)))$, $V\neq\mathfrak{X}(A)$, such that for each $U\in D(\mathfrak{X}(A))$, $U\neq\mathfrak{X}(A)$ and for each $i\in[n-1]$ such that $f_i^{A^{-1}}(U)\neq\mathfrak{X}(A)$, $d^{n_i^U}_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U))\subseteq V$ for some $n_i^U\in\omega$,
(iv) there is $V\in\mathcal{B}(D(\mathfrak{X}(A)))$, $V\neq\mathfrak{X}(A)$, such that for all $U\in\mathcal{B}(D(\mathfrak{X}(A)))$, $U\neq\mathfrak{X}(A)$, $d^{n^U}_{\mathfrak{X}(A)}(U)\subseteq V$, for some $n^U\in\omega$,
(v) there is $T\in\mathcal{F}_{TS}(D(\mathfrak{X}(A)))$, $T\neq\{\mathfrak{X}(A)\}$, such that $T\subseteq S$ for all $S\in\mathcal{F}_{TS}(D(\mathfrak{X}(A)))$, $S\neq\{\mathfrak{X}(A)\}$.

Proof.
(i) $\Rightarrow$ (ii): From (i) and Corollary 4.7 we infer that there exists $Y\in\mathcal{C}_{MT}(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$ such that (1) $Z\subseteq Y$ for all $Z\in\mathcal{C}_{MT}(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$. Since Y is modal, then by Proposition 4.2, there is (2) $x\in\max\mathfrak{X}(A)\setminus Y$. Taking into account that Y is a closed subset of $\mathfrak{X}(A)$ and hence it is compact, we can assert that there is $W\in D(\mathfrak{X}(A))$, such that (3) $Y\subseteq W$ and (4) $x\notin W$. In addition from (2) and (tS10) in Lemma 3.7, we have that $x=f^A_{n-1}(x)$ and so by (4) we infer that $x\notin f^{A^{-1}}_{n-1}(W)$. If $V=f^{A^{-1}}_{n-1}(W)$, then $V\in\mathcal{B}(D(\mathfrak{X}(A)))\setminus\{\mathfrak{X}(A)\}$. Besides, from (3) and the fact that $Y=f^{A^{-1}}_{n-1}(Y)$, we get that (5) $Y\subseteq f^{A^{-1}}_{n-1}(W)=V$. On the other hand, if $U\in D(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$, then from Lemma 3.16 and property (LP10) of LM_n-spaces, we infer that there is at least $i_0\in[n-1]$ such that $f^{A^{-1}}_{i_0}(U)\neq\mathfrak{X}(A)$. Now, let $i\in[n-1]$ such that $f_i^{A^{-1}}(U)\neq\mathfrak{X}(A)$, then from Proposition 4.11 we obtain that $\bigcap\limits_{n\in\omega}d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U))\in\mathcal{C}_{MT}(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$, from which we conclude, by the assertions (1) and (5), that $\bigcap\limits_{n\in\omega}d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U))\subseteq V$.

(ii) $\Rightarrow$ (iii): From the hypothesis (ii), we have that there is $V\in\mathcal{B}(D(\mathfrak{X}(A)))\setminus\{\mathfrak{X}(A)\}$, such that (1) $\bigcap\limits_{n\in\omega}d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U))\subseteq V$ for each $U\in D(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$ and each $i\in[n-1]$ such that $f_i^{A^{-1}}(U)\neq\mathfrak{X}(A)$. Suppose that there is $U\in D(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$ and there is $i_0\in[n-1]$, which satisfy (1) and $d^n_{\mathfrak{X}(A)}(f_{i_0}^{A^{-1}}(U))\nsubseteq V$ for all $n\in\omega$. Then for each $n\in\omega$, there exists (2) $x_n\in d^n_{\mathfrak{X}(A)}(f_{i_0}^{A^{-1}}(U))$ and $x_n\notin V$. Hence $(x_n)_{n\in\omega}$ is a sequence in $\mathfrak{X}(A)\setminus V$ and since $\mathfrak{X}(A)\setminus V$ is compact, we can assert that there exists (3) $x\in\mathfrak{X}(A)\setminus V$ such that $(x_n)_{n\in\omega}$ accumulates at x. In addition, by (1) and (3), we have that

$x \notin \bigcap_{n\in\omega} d^n_{\mathfrak{X}(A)}(f_{i_0}^{A^{-1}}(U))$, and thus $x \in \mathfrak{X}(A) \setminus d^{n_0}_{\mathfrak{X}(A)}(f_{i_0}^{A^{-1}}(U))$ for some $n_0 \in \omega$. Since x is an accumulation point of $(x_n)_{n\in\omega}$, then the preceding assertion and the fact that $\mathfrak{X}(A) \setminus d^{n_0}_{\mathfrak{X}(A)}(f_{i_0}^{A^{-1}}(U))$ is an open subset of $\mathfrak{X}(A)$ allows us to infer that for all $n \in \omega$ there is $m_n \in \omega$ such that $n \leq m_n$ and $x_{m_n} \in \mathfrak{X}(A) \setminus d^{n_0}_{\mathfrak{X}(A)}(f_{i_0}^{A^{-1}}(U))$. Thus $x_{m_{n_0}} \in \mathfrak{X}(A) \setminus d^{n_0}_{\mathfrak{X}(A)}(f_{i_0}^{A^{-1}}(U))$ and $n_0 \leq m_{n_0}$. As a consequence of Proposition 4.11 we have that $\mathfrak{X}(A) \setminus d^{n_0}_{\mathfrak{X}(A)}(f_{i_0}^{A^{-1}}(U)) \subseteq \mathfrak{X}(A) \setminus d^{m_{n_0}}_{\mathfrak{X}(A)}(f_{i_0}^{A^{-1}}(U))$ and so $x_{m_{n_0}} \notin d^{m_{n_0}}_{\mathfrak{X}(A)}(f_{i_0}^{A^{-1}}(U)))$, which contradicts (2). Therefore, for every $U \in D(\mathfrak{X}(A)) \setminus \{\mathfrak{X}(A)\}$ and $i \in [n-1]$ such that $f_i^{A^{-1}}(U) \neq \mathfrak{X}(A)$, $d^{n_i^U}_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U)) \subseteq V$ for some $n_i^U \in \omega$.

(iii) $\Rightarrow$ (iv): From Lemma 2.9 and the property (LP10) of LM_n-spaces, we infer that for all $U \in \mathcal{B}(D(\mathfrak{X}(A)))$, $U \neq \mathfrak{X}(A)$ if and only if $f_i^{A^{-1}}(U) \neq \mathfrak{X}(A)$ for all $i \in [n-1]$. Therefore, from the last statement and the hypothesis (iii), we obtain that for each $U \in \mathcal{B}(D(\mathfrak{X}(A)))$, $U \neq \mathfrak{X}(A)$ and each $i \in [n-1]$, there is $n_i^U \in \omega$ such that $d^{n_i^U}_{\mathfrak{X}(A)}(U) \subseteq V$. Then, considering $n_U = \max\{n_i^U : i \in [n-1]\}$, from (d2) in Proposition 4.11 we conclude that $d^{n_U}_{\mathfrak{X}(A)}(U) \subseteq V$.

(iv) $\Rightarrow$ (v): Let $S \in \mathcal{F}_{TS}(D(\mathfrak{X}(A)))$, $S \neq \{\mathfrak{X}(A)\}$. Then there exists (1) $U \in S \setminus \{\mathfrak{X}(A)\}$ and so from property (LP10) we infer that there is $i \in [n-1]$ such that $f_i^{A^{-1}}(U) \neq \mathfrak{X}(A)$. Let (2) $W = f_i^{A^{-1}}(U)$. Then, from Lemma 2.9 we have that $W \in \mathcal{B}(D(\mathfrak{X}(A)))$, $W \neq \mathfrak{X}(A)$ and thus by the hypothesis (iv), we can assert that there is $n_W \in \omega$ such that (3) $d^{n_W}_{\mathfrak{X}(A)}(W) \subseteq V$. Besides, from the assertions (1) and (2) and Lemma 4.24, we obtain that $d^{n_W}_{\mathfrak{X}(A)}(W) \in S$. From the last statement, (3) and the fact that S is a filter of $D(\mathfrak{X}(A))$, we get that $V \in S$, and so $V \in \bigcap_{\substack{S\in\mathcal{F}_{TS}(D(\mathfrak{X}(A))) \\ S\neq\{\mathfrak{X}(A)\}}} S$. Therefore, considering $T = \bigcap_{\substack{S\in\mathcal{F}_{TS}(D(\mathfrak{X}(A))) \\ S\neq\{\mathfrak{X}(A)\}}} S$ and taking into account that $V \neq \mathfrak{X}(A)$, we conclude that $T \in \mathcal{F}_{TS}(D(\mathfrak{X}(A)))$, $T \neq \{\mathfrak{X}(A)\}$ and $T \subseteq S$, for all $S \in \mathcal{F}_{TS}(D(\mathfrak{X}(A)))$, $S \neq \{\mathfrak{X}(A)\}$.

(v) $\Rightarrow$ (i): It follows from Corollary 4.26 and the fact that (A, G, H) is isomorphic to the tense LM_n-algebra $(D(\mathfrak{X}(A)), G_{R^A}, H_{R^{A^{-1}}})$. ∎

Corollary 4.31
Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ be a tense LM_n-algebra. Then, the following conditions are equivalent:

(i) $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ is a subdirectly irreducible tense LM_n-algebra,
(ii) there is $b \in \mathcal{B}(A) \setminus \{1\}$ such that for every $a \in A \setminus \{1\}$ and for every $i \in [n-1]$ such that $\varphi_i(a) \neq 1$, $d^{n_i^a}(\varphi_i(a)) \leq b$ for some $n_i^a \in \omega$,
(iii) there is $b \in \mathcal{B}(A) \setminus \{1\}$ such that for every $a \in \mathcal{B}(A) \setminus \{1\}$, there is $n_a \in \omega$ such that $d^{n_a}(a) \leq b$,
(iv) there is $T \in \mathcal{F}_{TS}(A)$, $T \neq \{1\}$ such that $T \subseteq S$ for all $S \in \mathcal{F}_{TS}(A)$, $S \neq \{1\}$.

Proof. It is a direct consequence of Proposition 4.30 and the fact that $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$ is a tense LM_n-isomorphism. ∎

Corollary 4.32
Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ be a subdirectly irreducible tense LM_n-algebra such that for every $a \in \mathcal{B}(A) \setminus \{1\}$, $d^n(a) = d^{n_a}(a)$ for some $n_a \in \omega$ and for all $n \in \omega$, $n_a \leq n$. Then, $(\mathcal{C}(A) \sim, \{\varphi_i\}_{i\in[n-1]})$ is a simple LM_n-algebra.

Proof. From Corollary 4.31, we can assert that there exists $b \in \mathcal{B}(A) \setminus \{1\}$ such that (1) for every $a \in \mathcal{B}(A) \setminus \{1\}$, $d^{n'_a}(a) \leq b$ for some $n'_a \in \omega$. Also, from hypothesis we have that there is $n_b \in \omega$ such that $d^n(b) = d^{n_b}(b)$ for all $n \in \omega$, $n_b \leq n$. Considering $u = d^{n_b}(b)$, then from the last assertion, properties (d5) and (d7) in Corollary 4.12 and the fact that $b \in \mathcal{B}(A) \setminus \{1\}$, we obtain that $u \in \mathcal{B}(\mathcal{C}(A))$, $u \neq 1$. In addition, let $c \in \mathcal{B}(\mathcal{C}(A))$, $c \neq 1$, then by Lemma 4.17, $c = d^n(c)$ for all $n \in \omega$, and thus from (1) we get that $c = d^{n'_c}(c) \leq b$. Then from property (d4) in Corollary 4.12, we infer that $c = d^{n_b}(c) \leq d^{n_b}(b) = u$. Consequently, from Corollary 4.19, $\mathcal{B}(\mathcal{C}(A))$ is a totally ordered Boolean algebra and so $\mathcal{B}(\mathcal{C}(A)) = \{0,1\}$. Therefore, from Corollary 2.12 and Lemma 4.18, we conclude that $\big(\mathcal{C}(A){\sim}, \{\varphi_i\}_{i\in[n-1]}\big)$ is a simple LM_n-algebra. ∎

5 A representation theorem for tense LM_n-algebras

In this section, as an application of the categorical equivalence obtained in Section 3, we prove a representation theorem for tense LM_n-algebras, which was formulated and proved by a different method by Diaconescu and Georgescu in [13].

First, we analyse the restriction of the relation R^A, defined on the tense LM_n-space $\mathfrak{X}(A)$ associated with a tense LM_n-algebra $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$, to the set $\max\mathfrak{X}(A)$ of the maximal elements of $\mathfrak{X}(A)$. By virtue of property (tS10) of tense LM_n-spaces and the prescription 2.13 we can assert that $M \in \max\mathfrak{X}(A)$ if and only if $M = \varphi_{n-1}^{-1}(M)$.

Lemma 5.1
Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ be a tense LM_n-algebra and let $(\mathfrak{X}(A), \subseteq, g_A, \{f_i^A\}_{i\in[n-1]}, R^A)$ be the tense LM_n-space associated with A. Then, for $M_1, M_2 \in \max\mathfrak{X}(A)$, the following conditions are equivalent:

(i) $M_1 \subseteq P^{-1}(M_2)$,
(ii) $G^{-1}(M_1) \subseteq M_2$,
(iii) $M_2 \subseteq F^{-1}(M_1)$,
(iv) $H^{-1}(M_2) \subseteq M_1$.

Proof.
(i) $\Rightarrow$ (ii): Let $a \in A$ such that $G(a) \in M_1$. Then, by the hypothesis (i), we infer that $P(G(a)) \in M_2$. Besides, from property (T9) of tense LM_n-algebras, we have that $P(G(a)) \leq a$. Then, taking into account the fact that M_2 is a filter of A, it follows that $a \in M_2$ and so $G^{-1}(M_1) \subseteq M_2$.

(ii) $\Rightarrow$ (iii): Let $a \in M_2$. Then, from property (L10) of LM_n-algebras it follows that $\varphi_{n-1}(a) \in M_2$ and so, from the property (L2) it results that ${\sim}\varphi_{n-1}(a) \notin M_2$. From the last statement and the hypothesis (ii), we have that (1) $G({\sim}\varphi_{n-1}(a)) \notin M_1$. Besides, from properties (T3) and (L4), we obtain that (2) $G({\sim}\varphi_{n-1}(a)) = \varphi_1(G({\sim}a))$. Then, from (1), (2) and property (L2), we infer that ${\sim}\varphi_1(G({\sim}a)) \in M_1$, and so from the fact that $M_1 = \varphi_{n-1}^{-1}(M_1)$, we get that $\varphi_{n-1}({\sim}\varphi_1(G({\sim}a))) \in M_1$. The last statement and properties (L3) and (L4) imply that $\varphi_{n-1}({\sim}G({\sim}a)) \in M_1$, and taking into account that $M_1 = \varphi_{n-1}^{-1}(M_1)$, it follows that ${\sim}G({\sim}a) \in M_1$ or equivalently $F(a) \in M_1$, from which we conclude that $M_2 \subseteq F^{-1}(M_1)$.

(iii) $\Rightarrow$ (iv): Let $a \in A$ such that $H(a) \in M_2$. Then, from (iii) we infer that $F(H(a)) \in M_1$. Also, from property (T9) of tense LM_n-algebras, we have that $F(H(a)) \leq a$. Then, taking into account the fact that M_1 is a filter of A, we conclude that $a \in M_1$ and therefore $H^{-1}(M_2) \subseteq M_1$.

(iv) $\Rightarrow$ (i): Let $a \in M_1$. Then, from property (L10) it follows that $\varphi_{n-1}(a) \in M_1$, and so, from property (L2), we have that ${\sim}\varphi_{n-1}(a) \notin M_1$. From the last assertion and the hypothesis (iv), we obtain that (1) $H({\sim}\varphi_{n-1}(a)) \notin M_2$. Besides, from properties (T3) and (L4), we get that (2) $H({\sim}\varphi_{n-1}(a)) = \varphi_1(H({\sim}a))$. Hence, from (1), (2) and property (L2), we infer that ${\sim}\varphi_1(H({\sim}a)) \in M_2$, and so the fact that $M_2 = \varphi_{n-1}^{-1}(M_2)$ enables us to say that $\varphi_{n-1}({\sim}\varphi_1(H({\sim}a))) \in M_2$. The last assertion and properties (L3) and (L4) imply that $\varphi_{n-1}({\sim}H({\sim}a)) \in M_2$. Then, taking into account that $M_2 = \varphi_{n-1}^{-1}(M_2)$, we can assert that ${\sim}H({\sim}a) \in M_2$ or equivalently $P(a) \in M_2$, from which we conclude that $M_1 \subseteq P^{-1}(M_2)$. ∎

Lemma 5.2

Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ be a tense LM_n-algebra and $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A. If $X=\max\mathfrak{X}(A)$ and $R=R^A|_X$, then for all $M\in X$,

$$R(M)=\{T\in X: G^{-1}(M)\subseteq T\}=\{T\in X: T\subseteq F^{-1}(M)\}$$

$$=\{T\in X: H^{-1}(T)\subseteq M\}=\{T\in X: M\subseteq P^{-1}(T)\}.$$

Proof. It immediately follows from Lemmas 3.12 and 5.1. ∎

Lemma 5.3

Let $(A,\sim,\{\varphi_i^A\}_{i\in[n-1]},G,H)$ be a tense LM_n-algebra, $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A, $\max\mathfrak{X}(A)=X$, $R^A|_{\max\mathfrak{X}(A)}=R$, the frame (X,R) and $(L_n,\sim,\{\varphi_i\}_{i\in[n-1]})$ be the LM_n-algebra given in Example 2.10. Then, the algebra $(L_n^X,\sim,\{\varphi_i^{L_n^X}\}_{i\in[n-1]},G^*,H^*)$ is a tense LM_n-algebra, where the operations of the LM_n-algebra $(L_n^X,\sim,\{\varphi_i^{L_n^X}\}_{i\in[n-1]})$ are defined pointwise and the unary operations G^* and H^* are defined for each $p\in L_n^X$ by the prescriptions:

$$\text{(i) } G^*(p)(M)=\bigwedge\{p(T)\,|\,T\in X, G^{-1}(M)\subseteq T\}$$

$$=\bigwedge\{p(T)\,|\,T\in X, T\subseteq F^{-1}(M)\},$$

$$\text{(ii) } H^*(p)(M)=\bigwedge\{p(S)\,|\,S\in X, H^{-1}(M)\subseteq S\}$$

$$=\bigwedge\{p(S)\,|\,S\in X, S\subseteq P^{-1}(M)\},$$

for all $M\in X$.

Proof. From Lemma 5.2 and Example 2.22, the proof is complete. ∎

Corollary 5.4

Let $(A,\sim,\{\varphi_i^A\}_{i\in[n-1]},G,H)$ be a tense LM_n-algebra such that $(G,H)=(1_A,1_A)$, where $1_A:A\longrightarrow A$ is defined by $1_A(a)=1$ for any $a\in A$. Let $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A, $\max\mathfrak{X}(A)=X$, $R^A|_{\max\mathfrak{X}(A)}=R$, the frame (X,R) and $(L_n,\sim,\{\varphi_i\}_{i\in[n-1]})$ be the LM_n-algebra given in Example 2.10. Then, the algebra $(L_n^X,\sim,\{\varphi_i^{L_n^X}\}_{i\in[n-1]},G^*,H^*)$ is a tense LM_n-algebra, where the operations of the LM_n-algebra $(L_n^X,\sim,\{\varphi_i^{L_n^X}\}_{i\in[n-1]})$, are defined pointwise and $(G^*,H^*)=\left(1_{L_n^X},1_{L_n^X}\right)$, where $1_{L_n^X}:L_n^X\longrightarrow L_n^X$ such that for all $p\in L_n^X$, $1_{L_n^X}(p)$ is the greatest element of L_n^X (i.e. $1_{L_n^X}(p)(M)=1$ for all $M\in X$).

Proof. From Lemma 5.3, $(L_n^X,\sim,\{\varphi_i^{L_n^X}\}_{i\in[n-1]},G^*,H^*)$ is a tense LM_n-algebra. Since $(G,H)=(1_A,1_A)$ it follows that $G^{-1}(M)=A$ and $H^{-1}(M)=A$ for any $M\in X$. Therefore, from Lemma 5.2, we infer that $R(M)=\emptyset$ for any $M\in X$ and so, from Lemmas 5.2 and 5.3, we get that for each $p\in L_n^X$, $G^*(p)(M)=\bigwedge\{p(T)\,|\,T\in R(M)\}=1$, $H^*(p)(M)=\bigwedge\{p(T)\,|\,M\in R(T)\}=1$, for any $M\in X$. From these last statements, we conclude that $(G^*,H^*)=\left(1_{L_n^X},1_{L_n^X}\right)$, where $1_{L_n^X}:L_n^X\longrightarrow L_n^X$ is defined by $1_{L_n^X}(p)(M)=1$, for any $p\in L_n^X$ and $M\in X$. ∎

In the sequel, we will show some results which will be useful later on.

Lemma 5.5
Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ be a tense LM_n-algebra, $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A and $R^A\mid_{\max\mathfrak{X}(A)}=R$. Then, for all $T\in\max\mathfrak{X}(A)$ such that $G^{-1}(T)$ is a proper filter of A,

$$G^{-1}(T)=\bigcap\{M\in\max\mathfrak{X}(A):G^{-1}(T)\subseteq M\}=\bigcap\{M\in\max\mathfrak{X}(A):M\in R(T)\}.$$

Proof. Let $T\in\max\mathfrak{X}(A)$ such that $G^{-1}(T)$ is a proper filter of A. Then there exists $M\in\max\mathfrak{X}(A)$ such that $G^{-1}(T)\subseteq M$ and therefore,

$$(1)\, G^{-1}(T)\subseteq\bigcap\{M\in\max\mathfrak{X}(A):G^{-1}(T)\subseteq M\}.$$

Suppose that there is $a\in A$ such that

$$(2)\, a\in\bigcap\{M\in\max\mathfrak{X}(A):G^{-1}(T)\subseteq M\}\,\text{and}\,(3)\, G(a)\notin T.$$

Since $T\in\max\mathfrak{X}(A)$ and $G^{-1}(T)$ is a proper filter of A, it follows that $T\cap\mathcal{B}(A)\in\mathfrak{X}(\mathcal{B}(A))$ and $G^{-1}(T)\cap\mathcal{B}(A)$ is a proper filter of $\mathcal{B}(A)$. Taking into account that every prime filter of the Boolean algebra $\mathcal{B}(A)$ is an ultrafilter, the fact that every proper filter of $\mathcal{B}(A)$ is the intersection of all ultrafilters of $\mathcal{B}(A)$ that contain it, and the property that $M'\in\mathfrak{X}(\mathcal{B}(A))$ iff there is $M\in\max\mathfrak{X}(A)$ such that $M'=M\cap\mathcal{B}(A)$, we infer that

$$\begin{aligned}(4)\quad G^{-1}(T)\cap\mathcal{B}(A)&=\bigcap\{M'\in\mathfrak{X}(\mathcal{B}(A)):G^{-1}(T)\cap\mathcal{B}(A)\subseteq M'\}\\&=\bigcap\{M\cap\mathcal{B}(A):M\in\max\mathfrak{X}(A),G^{-1}(T)\cap\mathcal{B}(A)\subseteq M\cap\mathcal{B}(A)\}.\end{aligned}$$

From the assertion (3), and the fact that $T=\varphi_{n-1}^{-1}(T)$, we have that $\varphi_{n-1}(G(a))\notin T$ and so, from property (T3) of tense LM_n-algebras, we obtain that $G(\varphi_{n-1}(a))\notin T$. Therefore, $\varphi_{n-1}(a)\notin G^{-1}(T)\cap \mathcal{B}(A)$. From the previous assertion, Lemma 2.9 and (4) we infer that there is (5) $M_0\in\max\mathfrak{X}(A)$ such that (6) $G^{-1}(T)\cap\mathcal{B}(A)\subseteq M_0\cap\mathcal{B}(A)$ and $\varphi_{n-1}(a)\notin M_0$. From the last statement and the fact that $M_0=\varphi_{n-1}^{-1}(M_0)$, we can assert that (7) $a\notin M_0$. On the other hand, we have that (8) $G^{-1}(T)\subseteq M$ for all $M\in\max\mathfrak{X}(A)$, such that (9) $G^{-1}(T)\cap\mathcal{B}(A)\subseteq M\cap\mathcal{B}(A)$. Indeed, let $b\in A$ such that $G(b)\in T$, then $\varphi_{n-1}(G(b))\in T$, and so from property (T3) it follows that $\varphi_{n-1}(b)\in G^{-1}(T)$. In addition, from Lemma 2.9, we have that $\varphi_{n-1}(b)\in\mathcal{B}(A)$ and therefore $\varphi_{n-1}(b)\in G^{-1}(T)\cap\mathcal{B}(A)$. Then, from (9) we obtain that $b\in M$, from which it results that $G^{-1}(T)\subseteq M$. Then, from (5), (6), (8) and (9), we get that $G^{-1}(T)\subseteq M_0$. This last assertion, (5) and (7) contradicts (2). Consequently, (2) and (3) are not true and so the assertion (1) and Lemma 5.2 enable us to conclude the proof. ∎

In the proof of the following lemma we will use the property of the dense subsets of a topological space (X,τ), which establishes that a subset $\mathfrak{D}$ of X is dense (i.e. the closure of $\mathfrak{D}$ is X) if and only if for any base $\mathcal{B}$ of τ and any $B\in\mathcal{B}\setminus\{\emptyset\}$, $B\cap\mathfrak{D}\neq\emptyset$.

Lemma 5.6
Let $(L_n^X,\sim,\{\varphi_i^{L_n^X}\}_{i\in[n-1]},G^*,H^*)$ be the tense LM_n-algebra described in Lemma 5.3, $\mathfrak{X}\left(L_n^X\right)$ be the tense LM_n-space associated with L_n^X and $\mathfrak{X}(L_n)$ be the LM_n-space associated with the LM_n-algebra L_n, given in Example 2.10. If $\mathfrak{D}=\left\{S\times {L_n}^{X\setminus\{x\}}:S\in\mathfrak{X}(L_n),x\in X\right\}$, where for each $x\in X$, ${L_n}^{X\setminus\{x\}}=\{f:X\setminus\{x\}\longrightarrow L_n\}$, then $\mathfrak{D}$ is a dense subset of $\mathfrak{X}\left(L_n^X\right)$.

PROOF. It is easy to check that $\mathfrak{D}\subseteq\mathfrak{X}(L_n^X)$. If X is a finite set, then $\mathfrak{D}=\mathfrak{X}(L_n^X)$. If X is an infinite set, then taking into account that the set $\mathcal{B}=\{\sigma_{L_n^X}(h)\setminus\sigma_{L_n^X}(g): h,g\in L_n^X\}$ is a basis of the topology of $\mathfrak{X}(L_n^X)$ and the fact that $\sigma_{L_n^X}: L_n^X \longrightarrow D(\mathfrak{X}(L_n^X))$ is an order isomorphism, we infer that for each $B\in\mathcal{B}\setminus\{\emptyset\}$, there are $h,g\in L_n^X$ such that $h\not\leq g$ and $B=\sigma_{L_n^X}(h)\setminus\sigma_{L_n^X}(g)$. From this last assertion it follows that $h(x)\not\leq g(x)$ for some $x\in X$, and since $h(x),g(x)\in L_n$, then there are (1) $j,k\in[n-1], j<k$, such that (2) $h(x)=\frac{k}{n-1}$ and (3) $g(x)=\frac{j}{n-1}$. Let (4) $S=\{\frac{l}{n-1}: k\leq l\leq n-1\}=\uparrow\frac{k}{n-1}\subseteq L_n$. Then, $S\in\mathfrak{X}(L_n)$ and hence (5) $S\times {L_n}^{X\setminus\{x\}}\in\mathfrak{D}$. Besides, taking into account (1), (2) and (4), we obtain that $h(x)\in S$, and from (1), (3) and (4), we get that $g(x)\notin S$. Hence $h\in S\times {L_n}^{X\setminus\{x\}}$ and $g\notin S\times {L_n}^{X\setminus\{x\}}$. These last assertions and the fact that $S\times {L_n}^{X\setminus\{x\}}\in\mathfrak{X}(L_n^X)$ enable us to infer that (6) $S\times {L_n}^{X\setminus\{x\}}\in\sigma_{L_n^X}(h)\setminus\sigma_{L_n^X}(g)$. Therefore, from (5) and (6), it results that $(\sigma_{L_n^X}(h)\setminus\sigma_{L_n^X}(g))\cap\mathfrak{D}\neq\emptyset$, from which we conclude that $\mathfrak{D}$ is dense in $\mathfrak{X}(L_n^X)$. ∎

Next, we will recall some characterizations of continuous functions. These characterizations will play fundamental role in the proofs of Proposition 5.7 and Theorem 5.9.

Let (X,τ_X), (Y,τ_Y) be two topological spaces and $f: X\longrightarrow Y$. Then, the following conditions are equivalent:

(i) f is continuous function,
(ii) $f^{-1}(C)$ is closed in (X,τ_X) for any closed C in (Y,τ_Y),
(iii) $f^{-1}(O)$ is open in (X,τ_X) for any open O in (Y,τ_Y),
(iv) $f^{-1}(B)$ is open in (X,τ_X) for any subbasic B in (Y,τ_Y),
(iv) for all $x\in X$ and for every net $(x_d)_{d\in D}$, $(x_d)\xrightarrow[d\in\mathcal{D}]{} x$ implies that $(f(x_d))\xrightarrow[d\in\mathcal{D}]{} f(x)$.

PROPOSITION 5.7
Let $(A,\sim,\{\varphi_i^A\}_{i\in[n-1]},G,H)$ be a tense LM_n-algebra, $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A, $X=\max\mathfrak{X}(A)$, $\mathfrak{D}=\{Q\times {L_n}^{X\setminus\{M\}}: Q\in\mathfrak{X}(L_n),\ M\in X\}$, $(\mathfrak{X}(L_n),g_{L^n},\{f_i^{L^n}\}_{i\in[n-1]})$ be the LM_n-space associated with the LM_n-algebra $(L_n,\sim,\{\varphi_i\}_{i\in[n-1]})$, given in Example 2.10 and $(\mathfrak{X}(L_n^X),g_{L_n^X},\{f_i^{L_n^X}\}_{i\in[n-1]},R^{L_n^X})$ be the tense LM_n-space associated with the tense LM_n-algebra $(L_n^X,\sim,\{\varphi_i^{L_n^X}\}_{i\in[n-1]},G^*,H^*)$ described in Lemma 5.3. Let $f:\mathfrak{D}\longrightarrow\mathfrak{X}(A)$, defined for each $Q\in\mathfrak{X}(L_n)$ and for each $M\in X$, by the prescription:

$$f\left(Q\times {L_n}^{X\setminus\{M\}}\right)={\varphi_i^A}^{-1}(M),\ \text{if}\, Q=\varphi_i^{-1}(Q)\,\text{for some}\, i\in[n-1]. \tag{5.1}$$

Then, f satisfies the following properties:

(i) for each $a\in A, f^{-1}(\sigma_A(a))=\sigma_{L_n^X}(h_a)\cap\mathfrak{D}$, where $h_a: X\longrightarrow L_n$ is defined for all $M\in X$ by the prescription:

$$h_a(M)=\begin{cases} 0 & \text{if } \varphi_i^A(a)\notin M \text{ for all } i\in[n-1], \text{ or equivalently} \\ & \text{if } a\notin M, \\ 1 & \text{if } \varphi_i^A(a)\in M \text{ for all } i\in[n-1], \text{ or equivalently} \\ & \text{if } \varphi_1^A(a)\in M, \\ \frac{i}{n-1},\ i\in[n-2], & \text{if } \varphi_{n-i}^A(a)\in M \text{ and } \varphi_{n-i-1}^A(a)\notin M. \end{cases}$$

(ii) f is continuous, considering $\mathfrak{D}$ as a subspace of $\mathfrak{X}(L_n^X)$.
(iii) If $(T_d)_{d\in\mathcal{D}}\subseteq\mathfrak{D}$ is a net such that $T_d\xrightarrow[d\in\mathcal{D}]{} T$ for some $T\in\mathfrak{X}(L_n^X)\setminus\mathfrak{D}$, then the net $(f(T_d))_{d\in\mathcal{D}}$ converges in $\mathfrak{X}(A)$.

(iv) If the nets $(T_d)_{d\in\mathcal{D}}\subseteq\mathfrak{D}$ and $(S_d)_{d\in\mathcal{D}}\subseteq\mathfrak{D}$ converge to the same element $T\in\mathfrak{X}\left(L_n^X\right)\setminus\mathfrak{D}$, then the nets $(f(T_d))_{d\in\mathcal{D}}$ and $(f(S_d))_{d\in\mathcal{D}}$ converge to the same element in $\mathfrak{X}(A)$.

Proof.
(i): Let $a\in A$. Taking into account that $h_a:X\longrightarrow L_n$, $L_n=\{\frac{j}{n-1}:0\leq j\leq n-1\}$ and for all $i,j\in[n-1]$, $\varphi_i(\frac{j}{n-1})=0$ if $i+j<n$ or $\varphi_i(\frac{j}{n-1})=1$ in other cases, we infer that (1) $\varphi_i(h_a(M))=1$ or $\varphi_i(h_a(M))=0$ for any $M\in X$ and $i\in[n-1]$. Also, we obtain that for any $M\in X$ and $i\in[n-1]$, $\varphi_i(h_a(M))=1$ implies that $h_a(M)=\frac{j}{n-1}$, $j\in[n-1]$ and $i\geq n-j$, from which we get that $\varphi^A_{n-j}(a)\in M$, and so from property (L5) of LM_n-algebras, we conclude that $\varphi^A_i(a)\in M$. Conversely if $\varphi^A_{i_0}(a)\in M$, for some $i_0\in[n-1]$, then we have that $\varphi^A_i(a)\in M$ for all $i\in[n-1]$, or there exists $j\in[n-1]$, $j\leq i_0$, such that $\varphi^A_j(a)\in M$ and $\varphi^A_{j-1}(a)\notin M$. In the first case, we have that $h_a(M)=1$ and so $\varphi_{i_0}(h_a(M))=1$. In the second case, it follows that $h_a(M)=\frac{n-j}{n-1}$, $j\in[n-1]$, $j\leq i_0$. Since $i_0+(n-j)\geq n$, from the last assertion we obtain that $\varphi_{i_0}(h_a(M))=1$. Therefore, (2) for any $M\in X$ and $i\in[n-1]$, $\varphi_i(h_a(M))=1$ if and only if $\varphi^A_i(a)\in M$. Besides, since $h_a\in L_n^X$, then $\sigma_{L_n^X}(h_a)=\left\{S\in\mathfrak{X}\left(L_n^X\right):h_a\in S\right\}$, from which it results that (3)$\sigma_{L_n^X}(h_a)\cap\mathfrak{D}=\{Q\times {L_n}^{X\setminus\{M\}}:Q\in\mathfrak{X}(L_n), M\in X$ and $h_a\in Q\times {L_n}^{X\setminus\{M\}}\}$. It is immediate that (4) $h_a\in Q\times {L_n}^{X\setminus\{M\}}$ iff $h_a(M)\in Q$, for any $Q\in\mathfrak{X}(L_n)$ and any $M\in X$. In addition, (5) $h_a(M)\in Q$ if and only if $Q\times {L_n}^{X\setminus\{M\}}\in f^{-1}(\sigma_A(a))$, for any $M\in X$ and $Q\in\mathfrak{X}(L_n)$. Indeed, let $Q\in\mathfrak{X}(L_n)$. Then, from property (LP9) of LM_n-spaces, we have that $Q=\varphi_i^{-1}(Q)$ for some $i\in[n-1]$. Taking into account (1) and (2), the provisions of (5) is a consequence of the fact that each of the following statements is equivalent to the next one in the sequence:

$h_a(M)\in Q$; $\quad h_a(M)\in\varphi_i^{-1}(Q)$; $\varphi_i(h_a(M))\in Q$; $\varphi_i(h_a(M))=1$; $\varphi^A_i(a)\in M$;
$a\in{\varphi^A_i}^{-1}(M)$; $a\in f\left(Q\times {L_n}^{X\setminus\{M\}}\right)$; $Q\times {L_n}^{X\setminus\{M\}}\in f^{-1}(\sigma_A(a))$.

Finally, from the assertions (3), (4) and (5) we conclude that $\sigma_{L_n^X}(h_a)\cap\mathfrak{D}=f^{-1}(\sigma_A(a))$ for any $a\in A$.

(ii): From (i), we have that for all $a\in A, f^{-1}(\sigma_A(a))=\sigma_{L_n^X}(h_a)\cap\mathfrak{D}$. Therefore, for all $a\in A, f^{-1}(\sigma_A(a))$ is closed and open in $\mathfrak{D}$. From this last assertion and taking into account that $\{\sigma_A(a):a\in A\}\cup\{\mathfrak{X}(A)\setminus\sigma_A(a):a\in A\}$ is a subbase of the topology of $\mathfrak{X}(A)$, we conclude that $f:\mathfrak{D}\longrightarrow\mathfrak{X}(A)$ is a continuous function.

(iii): Let $(T_d)_{d\in\mathcal{D}}\subseteq\mathfrak{D}$ such that (1) $T_d\ \overrightarrow{_{d\in\mathcal{D}}}\ T$ for some $T\in\mathfrak{X}\left(L_n^X\right)\setminus\mathfrak{D}$. Since $\mathfrak{X}(A)$ is compact and $(f(T_d))_{d\in\mathcal{D}}$ is a net in $\mathfrak{X}(A)$, it follows that (2) $(f(T_d))_{d\in\mathcal{D}}$ accumulates at T for some $T\in\mathfrak{X}(A)$. Suppose that there is $S\in\mathfrak{X}(A)$ such that $S\neq T$ and (3) $(f(T_d))_{d\in\mathcal{D}}$ accumulates at S. Hence, $T\not\subseteq S$ or $S\not\subseteq T$. Suppose that $T\not\subseteq S$, then there exists $a\in A$ such that $a\in T$ and $a\notin S$. Therefore, $T\in\sigma_A(a)$ and $S\in\mathfrak{X}(A)\setminus\sigma_A(a)$. Since $\sigma_A(a)\in D(\mathfrak{X}(A))$, then from (2), (3) and these last assertions, we infer that there are two nets $(f(T_{d_c}))_{c\in\mathcal{D}}$ and $(f(T_{d_b}))_{b\in\mathcal{D}}$ such that $(f(T_{d_c}))_{c\in\mathcal{D}}\subseteq(f(T_d))_{d\in\mathcal{D}}$ and $\{f(T_{d_c})\}_{c\in\mathcal{D}}\subseteq\sigma_A(a)$, $(f(T_{d_b}))_{b\in\mathcal{D}}$
$\subseteq(f(T_d))_{d\in\mathcal{D}}$ and $\{f(T_{d_b})\}_{b\in\mathcal{D}}\subseteq\mathfrak{X}(A)\setminus\sigma_A(a)$. Hence, (4) $\{T_{d_c}\}_{c\in\mathcal{D}}\subseteq f^{-1}(\sigma_A(a))$ and (5) $\{T_{d_b}\}_{b\in\mathcal{D}}\subseteq f^{-1}(\mathfrak{X}(A)\setminus\sigma_A(a))$. For each $a\in A$, let the function $h_a:X\longrightarrow L_n$ be defined, for all $M\in X$ by:

$$h_a(M)=\begin{cases} 0 & \text{if } a\notin M,\\ 1 & \text{if } \varphi^A_1(a)\in M,\\ \frac{i}{n-1},\ 1\leq i\leq n-2, & \text{if } \varphi^A_{n-i}(a)\in M \text{ and } \varphi^A_{n-i-1}(a)\notin M.\end{cases}$$

Then, from (i) we have that $f^{-1}(\sigma_A(a))=\sigma_{L_n^X}(h_a)\cap\mathfrak{D}$ and $f^{-1}(\mathfrak{X}(A)\setminus\sigma_A(a))=\mathfrak{D}\cap\left(\mathfrak{X}\left(L_n^X\right)\setminus\sigma_{L_n^X}(h_a)\right)$, and so from (4) and (5) we infer that (6)$\{T_{c_d}\}_{d\in\mathcal{D}}\subseteq\sigma_{L_n^X}(h_a)$ and (7)$\{T_{b_d}\}_{d\in\mathcal{D}}\subseteq X\left(L_n^X\right)\setminus\sigma_{L_n^X}(h_a)$. On the other hand, from (1) we have that $T_{d_c} \overrightarrow{_{c\in\mathcal{D}}} T$ and since $\sigma_{L_n^X}(h_a)$ is closed in $\mathfrak{X}\left(L_n^X\right)$, then from (6) we can assert that (8)$T\in\sigma_{L_n^X}(h_a)$. In addition, from (1) we obtain $T_{d_b} \overrightarrow{_{b\in\mathcal{D}}} T$ and taking into account (7) and the fact that $\mathfrak{X}\left(L_n^X\right)\setminus\sigma_{L_n^X}(h_a)$ is closed in $\mathfrak{X}\left(L_n^X\right)$, we obtain that $T\notin\sigma_{L_n^X}(h_a)$, which contradicts (8). Analogously we reach a contradiction if $S\not\subseteq T$. Therefore $T=S$, from which we conclude that $(f(T_d))_{d\in\mathcal{D}}$ is a net in $\mathfrak{X}(A)$, which has a unique accumulation point and therefore it is convergent in $\mathfrak{X}(A)$.

(iv): Let $(T_d)_{d\in\mathcal{D}}\subseteq\mathfrak{D}$ and $(S_d)_{d\in\mathcal{D}}\subseteq\mathfrak{D}$ such that $T_d \overrightarrow{_{d\in\mathcal{D}}} T$ and $S_d \overrightarrow{_{d\in\mathcal{D}}} T$, $T\in\mathfrak{X}\left(L_n^X\right)\setminus\mathfrak{D}$. Then, from (i), the nets $(f(T_d))_{d\in\mathcal{D}}$ and $(f(S_d))_{d\in\mathcal{D}}$ converge in $\mathfrak{X}(A)$. Suppose that there are $Q,S\in\mathfrak{X}(A)$ such that $Q\neq S$, $f(T_d) \overrightarrow{_{d\in\mathcal{D}}} Q$ and $f(S_d) \overrightarrow{_{d\in\mathcal{D}}} S$, then using a similar technique to that performed in the demonstration of (iii), we arrive at a contradiction and therefore, $Q=S$. ∎

Hereinafter we need to consider the following theorem of extensions of continuous functions:

THEOREM 5.8 ([15])
Let (X,τ_X) be a topological space, $D\subseteq X$ dense in X, (Y,τ_Y) a regular topological space and $f: D\longrightarrow Y$ a continuous function, then f has a continuous extension $F:X\longrightarrow Y$ iff, for every $x\in X$ and all nets $(x_i)_{i\in I}\subseteq D$ which converge to x, the nets $(f(x_i))_{i\in I}$ converge to the same limit in Y. If F exists, then F is the unique continuous extension of f.

The above theorem is an equivalent formulation of the statement in [15] because in the latter, bases of filters are used instead of nets, but for our purposes the nets are most useful.

THEOREM 5.9
Let $(A,\sim,\{\varphi_i^A\}_{i\in[n-1]},G,H)$ be a tense LM_n-algebra, $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]},R^A)$ be the tense LM_n-space associated with A, (X,R) be a frame, where $X=\max\mathfrak{X}(A)$, $R=R^A|_{\max\mathfrak{X}(A)}$, and $(L_n^X,\sim,\{\varphi_i^{L_n^X}\}_{i\in[n-1]},G^*,H^*)$ be the tense LM_n-algebra described in Lemma 5.3. Then, there exists a surjective tense LM_n-function from $\mathfrak{X}\left(L_n^X\right)$ onto $\mathfrak{X}(A)$, where $(\mathfrak{X}\left(L_n^X\right),g_{L_n^X},\{f_i^{L_n^X}\}_{i\in[n-1]},R^{L_n^X})$ is the tense LM_n-space associated with the LM_n-algebra from Lemma 5.3.

PROOF. In order to prove the existence of a surjective LM_n-function $\Phi:\mathfrak{X}\left(L_n^X\right)\longrightarrow\mathfrak{X}(A)$, we will show the statements set in (I) to (XIV), which are listed below:

(I) *There exists a continuous function* $\Phi:\mathfrak{X}\left(L_n^X\right)\longrightarrow\mathfrak{X}(A)$:
Let $\mathfrak{D}=\left\{Q\times {L_n}^{X\setminus\{M\}}: Q\in\mathfrak{X}(L_n),\, M\in X\right\}$, $\left(\mathfrak{X}(L_n),g_{L^n},\{f_i^{L^n}\}_{i\in[n-1]}\right)$ be the LM_n-space associated with the LM_n-algebra $(L_n,\sim,\{\varphi_i\}_{i\in[n-1]})$ and let $f:\mathfrak{D}\longrightarrow\mathfrak{X}(A)$ be defined as in Proposition 5.7. Taking into account that every Priestley space is a regular space, then from Proposition 5.7 and Theorem 5.8, we can assert that f has a unique continuous extension $\Phi:\mathfrak{X}\left(L_n^X\right)\longrightarrow\mathfrak{X}(A)$. Also, from the proof of Theorem 5.8 we have that

- for all $T\in\mathfrak{X}\left(L_n^X\right)\setminus\mathfrak{D}$, $\Phi(T)=S$ if and only if $f(T_d) \overrightarrow{_{d\in\mathcal{D}}} S$ for any net $(T_d)_{d\in\mathcal{D}}\subseteq\mathfrak{D}$ such that $T_d \overrightarrow{_{d\in D}} T$.

(II) Φ *is surjective*:

Let $S\in\mathfrak{X}(A)$, then by property (LP9) of LM_n-spaces, there is $i\in[n-1]$ such that $S=\varphi_i^{A^{-1}}(S)$. From Lemma 3.16, we have that $(\mathfrak{X}(A),g_A,\{f_i^A\}_{i\in[n-1]})$ is a tense LM_n-space. Then, from the properties (tS19) and (tS10) of tense LM_n-spaces, the definition of the functions f_i^A, $i\in[n-1]$, given by the prescription (2.13) and properties (LP4) and (LP5) of LM_n-spaces, we infer that $M=\varphi_{n-1}^{A^{-1}}(S)\in\max\mathfrak{X}(A)$ and $S=\varphi_i^{A^{-1}}(M)$. If we consider $Q\in\mathfrak{X}(L_n)$ such that $Q=\varphi_i^{-1}(Q)$, then from Proposition 5.7, we have that $Q\times {L_n}^{X\setminus\{M\}}\in\mathfrak{D}$ and $f\left(Q\times {L_n}^{X\setminus\{M\}}\right)=S$. Since $\Phi|_{\mathfrak{D}}=f$, we conclude that $\Phi\left(Q\times {L_n}^{X\setminus\{M\}}\right)=S$.

(III) $\Phi^{-1}(\sigma_A(a))=\sigma_{L_n^X}(h_a)$ *for all* $a\in A$, *where the function* h_a *is defined as in Proposition* 5.7:

From Proposition 5.7, we have that (1) $\sigma_{L_n^X}(h_a)\cap\mathfrak{D}=f^{-1}(\sigma_A(a))$ for all $a\in A$.

Let $a\in A$ and $S\in\sigma_{L_n^X}(h_a)\cap\mathfrak{D}$, then from Lemma 5.6, there exists a net (2) $(S_d)_{d\in\mathcal{D}}\subseteq\mathfrak{D}$ such that (3) $S_d \underset{d\in\mathcal{D}}{\longrightarrow} S$. Therefore, there exists $d_o\in\mathcal{D}$ such that $S_d\in\sigma_{L_n^X}(h_a)\cap\mathfrak{D}$ for all $d\in\mathcal{D}$, $d_o\prec d$, and so from (1) it follows that $S_d\in f^{-1}(\sigma_A(a))$ for all $d\in\mathcal{D}$, $d_o\prec d$. Consequently, from (1) and taking into account that $\Phi|_{\mathfrak{D}}=f$, we have that $S_d\in\Phi^{-1}(\sigma_A(a))$ for all $d\in\mathcal{D}$, $d_o\prec d$. Besides, since Φ is a continuous function we have that $\Phi^{-1}(\sigma_A(a))$ is closed in $\mathfrak{X}\left(L_n^X\right)$. Then, from the last statement and (3), we infer that $S\in\Phi^{-1}(\sigma_A(a))$. Conversely, let $T\in\Phi^{-1}(\sigma_A(a))\cap\mathfrak{D}$. Then $\Phi(T)=f(T)$ and therefore, $T\in f^{-1}(\sigma_A(a))$ and so from (1) we obtain that $T\in\sigma_{L_n^X}(h_a)$.

Now, let (4) $S\in\Phi^{-1}(\sigma_A(a))\cap\mathfrak{D}$, then from Lemma 5.6, there is a net $(S_d)_{d\in\mathcal{D}}\subseteq\mathfrak{D}$ such that (5) $S_d \underset{d\in\mathcal{D}}{\longrightarrow} S$. Besides, since Φ is a continuous function, we have that $\Phi^{-1}(\sigma_A(a))$ is an open subset of $\mathfrak{X}\left(L_n^X\right)$, then from (4) and (5), we can assert that there is $d_o\in\mathcal{D}$ such that $S_d\in\Phi^{-1}(\sigma_A(a))$ for all $d\in\mathcal{D}$, $d_o\prec d$ and hence, $S_d\in f^{-1}(\sigma_A(a))\cap\mathfrak{D}$ for all $d\in\mathcal{D}$, $d_o\prec d$. From the last assertion and (1) it results that $S_d\in\sigma_{L_n^X}(h_a)$ for all $d\in\mathcal{D}$, $d_o\prec d$, and consequently, from (5) and the fact that $\sigma_{L_n^X}(h_a)$ is a closed subset of $\mathfrak{X}\left(L_n^X\right)$, it follows that $S\in\varphi_{L_n^X}(h_a)$. And so we conclude that $\Phi^{-1}(\sigma_A(a))=\sigma_{L_n^X}(h_a)$ for any $a\in A$.

(IV) Φ *is isotone*:

Let $T,S\in\mathfrak{X}\left(L_n^X\right)$ such that $T\subseteq S$. Suppose that $\Phi(T)\not\subseteq\Phi(S)$, then there is $a\in A\cdot$ such that $\Phi(T)\in\sigma_A(a)$ and $\Phi(S)\notin\sigma_A(a)$. Consequently, from (III), we obtain that $T\in\sigma_{L_n^X}(h_a)$ and $S\notin\sigma_{L_n^X}(h_a)$. Then, we have that $h_a\in L_n^X$, $h_a\in T$ and $h_a\notin S$, and so $T\not\subseteq S$, which contradicts the hypothesis. Therefore, $\Phi(T)\subseteq\Phi(S)$.

(V) $f\circ f_i^{L_n^X}|_{\mathfrak{D}}=f_i^A\circ f$ *for all* $i\in[n-1]$, *where* $f_i^{L_n^X}|_{\mathfrak{D}}$ *is the restriction of* $f_i^{L_n^X}$ *to* $\mathfrak{D}$:

From the prescription (5.1), we infer that $f\left(Q\times {L_n}^{X\setminus\{M\}}\right)\subseteq M$ for all $Q\in\mathfrak{X}(L_n)$ and $M\in X$. Then, from the previous assertion, properties (LP5) and (LP9) of LM_n-spaces and the prescription (2.13), we obtain that (2)$f_i^A\left(f\left(Q\times {L_n}^{X\setminus\{M\}}\right)\right)=f_i^A(M)$ for all $Q\in\mathfrak{X}(L_n)$ and $M\in X$. Taking into account the prescription (2.13) and the fact that $\varphi_i^{L_n^X}:L_n^X\longrightarrow L_n^X$ is defined pointwise for any $i\in[n-1]$, it immediately follows that $f_i^{L_n^X}|_{\mathfrak{D}}\left(Q\times {L_n}^{X\setminus\{M\}}\right)=f_i^{L_n^X}\left(Q\times {L_n}^{X\setminus\{M\}}\right)=f_i^{L_n}(Q)\times {L_n}^{X\setminus\{M\}}$, for any $Q\in\mathfrak{X}(L_n)$, $M\in X$ and $i\in[n-1]$ and consequently, (3) $\left(f\circ f_i^{L_n^X}|_{\mathfrak{D}}\right)\left(Q\times {L_n}^{X\setminus\{M\}}\right)=f\left(f_i^{L_n}(Q)\times {L_n}^{X\setminus\{M\}}\right)$, for any $Q\in\mathfrak{X}(L_n)$, $M\in X$ and $i\in[n-1]$. Furthermore, from the prescriptions (2.13) and (5.1), we obtain that (4) for any $Q\in\mathfrak{X}(L_n)$, $M\in X$ and $i\in[n-1]$, $f\left(f_i^{L_n}(Q)\times {L_n}^{X\setminus\{M\}}\right)=f\left({\varphi_i}^{-1}(Q)\times {L_n}^{X\setminus\{M\}}\right)=\varphi_i^{A^{-1}}(M)=f_i^A(M)$. Therefore the statements (2), (3) and (4) allow us to say that (V) holds.

(VI) $\Phi\circ f_i^{L_n^X}=f_i^A\circ\Phi$, *for all* $i\in[n-1]$:

Let $T\in\mathfrak{X}(L_n^X)$, then from Lemma 5.6 there are two nets, $(Q_d)_{d\in\mathcal{D}}\subseteq\mathfrak{X}\left(L_n^X\right)$ and $(M_d)_{d\in\mathcal{D}}\subseteq X$ such that (1)$Q_d\times {L_n}^{X\setminus\{M_d\}}\underset{d\in\mathcal{D}}{\longrightarrow}T$. Taking into account that the functions $f_i^{L_n^X}:\mathfrak{X}\left(L_n^X\right)\longrightarrow X\left(L_n^X\right)$, $i\in[n-1]$, and $\Phi:\mathfrak{X}\left(L_n^X\right)\longrightarrow\mathfrak{X}(A)$ are continuous and the assertion (1), we infer that (2)$\Phi\left(f_i^{L_n^X}\left(Q_d\times L_n^{X\setminus\{M_d\}}\right)\right)\underset{d\in\mathcal{D}}{\longrightarrow}\Phi\left(f_i^{L_n^Y}(T)\right)$, for all $i\in[n-1]$. On the other hand, by virtue of that $f_i^{L_n^X}\left(Q_d\times {L_n}^{X\setminus\{M_d\}}\right)\in\mathfrak{D}$ for all $d\in\mathcal{D}$ and $i\in[n-1]$, then $\Phi\left(f_i^{L_n^X}\left(Q_d\times {L_n}^{X\setminus\{M_d\}}\right)\right)=f\left(f_i^{L_n^X}\left(Q_d\times {L_n}^{X\setminus\{M_d\}}\right)\right)=f\left(f_i^{L_n^X}|_{\mathfrak{D}}\left(Q_d\times {L_n}^{X\setminus\{M_d\}}\right)\right)$ for all $i\in[n-1]$ and $d\in\mathcal{D}$, and so from the statement (V), we obtain that (3) $\Phi\left(f_i^{L_n^X}\left(Q_d\times {L_n}^{X\setminus\{M_d\}}\right)\right)=f_i^A\left(f\left(Q_d\times {L_n}^{X\setminus\{M_d\}}\right)\right)$ for all $d\in\mathcal{D}$ and $i\in[n-1]$. From (1) and the definition of Φ given in (I), we have that $f\left(Q_d\times {L_n}^{X\setminus\{M_d\}}\right)\xrightarrow[d\in\mathcal{D}]{}\Phi(T)$. Since $f_i^A:\mathfrak{X}(A)\longrightarrow\mathfrak{X}(A)$ is continuous, for any $i\in[n-1]$, then $f_i^A\left(f\left(Q_d\times {L_n}^{X\setminus\{M_d\}}\right)\right)\xrightarrow[d\in\mathcal{D}]{}f_i^A(\Phi(T))$ for any $i\in[n-1]$. From the last assertion and (3) it follows that (4) $\Phi\left(f_i^{L_n^X}\left(Q_d\times {L_n}^{X\setminus\{M_d\}}\right)\right)\xrightarrow[d\in\mathcal{D}]{}f_i^A(\Phi(T))$ for any $i\in[n-1]$. Finally, from (2), (4) and the fact that $\mathfrak{X}(A)$ is a Hausdorff space, we infer that $\left(\Phi\circ f_i^{L_n^X}\right)(T)=\left(f_i^A\circ\Phi\right)(T)$ for all $T\in\mathfrak{X}\left(L_n^X\right)$ and $i\in[n-1]$. Therefore, $\Phi\circ f_i^{L_n^X}=f_i^A\circ\Phi$ for any $i\in[n-1]$.

(VII) $f\circ g_{L_n^X}|_{\mathfrak{D}}=g_A\circ f$, *where* $g_{L_n^X}|_{\mathfrak{D}}$ *is the restriction of* $g_{L_n^X}$ *to* $\mathfrak{D}$:

Taking into account the prescription (2.5) and the fact that $\sim$ on L_n^X is defined pointwise, it immediately follows that $g_{L_n^X}|_{\mathfrak{D}}\left(Q\times {L_n}^{X\setminus\{M\}}\right)=g_{L_n^X}\left(Q\times {L_n}^{X\setminus\{M\}}\right)=g_{L_n}(Q)\times {L_n}^{X\setminus\{M\}}$ for all $M\in X$ and $Q\in\mathfrak{X}\left(L_n^X\right)$. Then, for all $Q\in\mathfrak{X}\left(L_n^X\right)$ and $M\in X$, (1) $\left(f\circ g_{L_n^X}|_{\mathfrak{D}}\right)\left(Q\times {L_n}^{X\setminus\{M\}}\right)=f\left(g_{L_n}(Q)\times {L_n}^{X\setminus\{M\}}\right)$. Since $Q\in\mathfrak{X}(L_n)$, then from property (LP9) of LM_n-spaces, there is $i\in[n-1]$ such that $Q=\varphi_i^{-1}(Q)$ and so from the prescription (2.13) and property (LP7) of LM_n-spaces, we infer that $g_{L_n}(Q)=\varphi_{n-i}^{-1}(Q)$, from which it follows that $f\left(Q\times {L_n}^{X\setminus\{M\}}\right)=\varphi_i^{-1}(M)$, and $f\left(g_{L_n}(Q)\times {L_n}^{X\setminus\{M\}}\right)=\varphi_{n-i}^{-1}(M)$. On the other hand, taking into account the prescription (2.13) and property (LP7) of LM_n-spaces, we obtain that $\varphi_{n-i}^{A^{-1}}(M)=f_{n-i}^A(M)=g_A(f_i^A(M))=g_A(\varphi_i^{A^{-1}}(M))$. Therefore, $f(g_{L_n}(Q)\times {L_n}^{X\setminus\{M\}})=g_A\left(f\left(Q\times\times {L_n}^{X\setminus\{M\}}\right)\right)$, and so from (1), we obtain that $\left(f\circ g_{L_n^X}|_{\mathfrak{D}}\right)\left(Q\times {L_n}^{X\setminus\{M\}}\right)=(g_A\circ f)\left(Q\times {L_n}^{X\setminus\{M\}}\right)$ for all $Q\in\mathfrak{X}\left(L_n^X\right)$ and for all $M\in X$, from which we conclude that $f\circ g_{L_n^X}|_{\mathfrak{D}}=g_A\circ f$.

(VIII) $\Phi\circ g_{L_n^X}=g_A\circ\Phi$:

Let $T\in\mathfrak{X}\left(L_n^X\right)$, then from Lemma 5.6, there is (1) $(T_d)_{d\in\mathcal{D}}\subseteq\mathfrak{D}$ such that (2) $T_d\xrightarrow[d\in\mathcal{D}]{}T$. Since $g_{L_n^X}$ is a continuous function, then (3) $g_{L_n^X}(T_d)\xrightarrow[d\in\mathcal{D}]{}g_{L_n^X}(T)$. On the other hand, from (1) we have that for each $d\in\mathcal{D}$ there are $Q_d\in\mathfrak{X}(L_n)$ and $M_d\in X$ such that $T_d=Q_d\times {L_n}^{X\setminus\{M_d\}}$, from which it follows that $g_{L_n^X}(T_d)=g_{L_n}(Q_d)\times {L_n}^{X\setminus\{M_d\}}$ and therefore, (4) $g_{L_n^X}(T_d)\in\mathfrak{D}$ for all $d\in\mathcal{D}$. Since Φ is continuous, $\Phi|_{\mathfrak{D}}=f$ and $g_{L_n^X}|_{\mathfrak{D}}(T_d)=g_{L_n^X}(T_d)$, $d\in\mathcal{D}$, then from (3) and (4) we obtain that $\left(f\circ g_{L_n^X}\right)(T_d)\xrightarrow[d\in\mathcal{D}]{}\left(\Phi\circ g_{L_n^X}\right)(T)$, and so from (VII) we can assert that (5) $(g_A\circ f)(T_d)\xrightarrow[d\in\mathcal{D}]{}\left(\Phi\circ g_{L_n^X}\right)(T)$. On the other hand, from (2) and taking into account that Φ and g_A are continuous and $\Phi|_{\mathfrak{D}}=f$, we infer that (6) $(g_A\circ f)(T_d)\xrightarrow[d\in\mathcal{D}]{}(g_A\circ\Phi)(T)$. Since $\mathfrak{X}(A)$ is a Hausdorff space, then from (5) and (6), we conclude that for all $T\in\mathfrak{X}\left(L_n^X\right)$, $\left(\Phi\circ g_{L_n^X}\right)(T)=(g_A\circ\Phi)(T)$, and so, $\Phi\circ g_{L_n^X}=g_A\circ\Phi$.

The statements (I), (II), (IV), (VI) and (VIII) allow us to assert that Φ is a surjective LM_n-function from $\mathfrak{X}\left(L_n^X\right)$ to $\mathfrak{X}(A)$.

(IX) $G^*(h_a)=h_{G(a)}$ *for all* $a\in A$, *where for all* $b\in A$, *the function* $h_b:X\longrightarrow L_n$ *is defined as in Proposition* 5.7:

Since $h_{G(a)}:X\longrightarrow L_n$ is defined by the prescription:

$$h_{G(a)}(M)=\begin{cases}0 & \text{if } G(a)\notin M,\\ 1 & \text{if } \varphi_1^A(G(a))\in M,\\ \frac{i}{n-1},\ 1\leq i\leq n-2, & \text{if } \varphi_{n-i}^A(G(a))\in M \text{ and } \varphi_{n-i-1}^A(G(a))\notin M,\end{cases}$$

then we only have to prove that the statements (A), (B) and (C) hold.

(A) *For each* $a\in A$, $G^*(h_a)(M)=0$ *iff* $h_{G(a)}(M)=0$ *for all* $M\in X$:

Since for each $a\in A$, $h_a\in L_n^X$, then from Lemmas 5.2 and 5.3, we have that for all $M\in X$, (1) $(G^*(h_a))(M)=\bigwedge_{(M,T)\in R} h_a(T)$, from which we infer that for all $M\in X$, (2) $(G^*(h_a))(M)=0$ iff $\bigwedge_{(M,T)\in R} h_a(T)=0$.
Besides, since L_n is finite and $\{h_a(T)\colon T\in X, T\in R\}\subseteq L_n$, then we have that (3) $\bigwedge_{(M,T)\in R} h_a(T)=0$, iff there is $T_0\in R(M)$ such that $a\notin T_0$. In addition, from (1) and (2), we infer that $R(M)\neq\emptyset$, and so by Lemma 5.2, $G^{-1}(M)$ is a proper filter of A. Therefore, from Lemma 5.5, we have that (4) $G^{-1}(M)=\bigcap_{(M,T)\in R} T$. Consequently, from (2), (3) and (4), it follows that $(G^*(h_a))(M)=0$ iff $G(a)\notin M$ for any $M\in X$. From the last statement and the definition of the function $h_{G(a)}$, we obtain that $(G^*(h_a))(M)=0$ iff $h_{G(a)}(M)=0$ for all $M\in X$ and $a\in A$.

(B) *For each $a\in A$, $G^*(h_a)(M)=1$ iff $h_{G(a)}(M)=1$ for all $M\in X$:*

Let $a\in A$ and $M\in X$ such that (1) $G^*(h_a)(M)=1$. Taking into account the definition of $G^*(h_a)(M)$, then (2) $R(M)\neq\emptyset$ or (3) $R(M)=\emptyset$.
Suppose (2) holds. Then, each of the following conditions is equivalent to the next one in the sequence:
(4) $(G^*(h_a))(M)=1$; $\bigwedge_{(M,T)\in R} h_a(T)=1$; $h_a(T)=1$ for all $T\in R(M)$; $\varphi_1^A(a)\in T$ for all $T\in R(M)$; $\varphi_1^A(a)\in\bigcap_{(M,T)\in R} T$.
Besides, from (2) and Lemma 5.2 we have that $G^{-1}(M)$ is a proper filter of A, and so from Lemma 5.3, we can set that (5) $G^{-1}(M)=\bigcap_{(M,T)\in R} T$. Then, from (4) and (5) it results that $(G^*(h_a))(M)=1$ if and only if $G(\varphi_1^A(a))\in M$. From the last statement and property (T3) of tense LM_n-algebras it follows that $(G^*(h_a))(M)=1$ if and only if $\varphi_1^A(G(a))\in M$, and so by virtue of the definition of the function $h_{G(a)}$, we conclude that (6) $(G^*(h_a))(M)=1$ if only if $h_{G(a)}(M)=1$, for all $a\in A$ and $M\in X$ such that $R(M)\neq\emptyset$.
On the other hand, taking into account that $G^{-1}(M)$ is a filter of A and the definition of $R(M)$, given in Lemma 5.2, it follows that (3) holds iff $G^{-1}(M)=A$. Hence for all $a\in A$, $G(\varphi_1^A(a))\in M$ and so from property (T3) of tense LM_n-algebras, we have that $\varphi_1^A(G(a))\in M$ for all $a\in A$, from which we conclude that $h_{G(a)}(M)=1$ for all $a\in A$. Conversely, if $h_{G(a)}(M)=1$ for all $a\in A$, then $\varphi_1^A(G(a))\in M$ for all $a\in A$, from which it follows by property (L9) of LM_n-algebras that $G(a)\in M$ for all $a\in A$, and so $G^{-1}(M)=A$. Consequently, $R(M)=\emptyset$, from which we obtain that $G^*(h_a)(M)=1$. Therefore, (7) $(G^*(h_a))(M)=1$ if only if $h_{G(a)}(M)=1$, for all $a\in A$ and $M\in X$ such that $R(M)=\emptyset$. And so, from the statements (7) and (8), the proof of (B) is complete.

(C) *For each $a\in A$, $i\in[n-2]$, $G^*(h_a)(M)=\frac{i}{n-1}$, iff $h_{G(a)}(M)=\frac{i}{n-1}$, for all $M\in X$:*

For any $a\in A$ and $i\in[n-1]$, each of the following conditions is equivalent to the next one in the sequence, for all $M\in X$, $a\in A$ and $i\in[n-2]$:

(1) $(G^*(h_a))(M)=\frac{i}{n-1}$; $\bigwedge_{(M,T)\in R} h_a(T)=\frac{i}{n-1}$; there is $T_0\in R(M)$, such that $h_a(T_0)=\frac{i}{n-1}$ and $h_a(T_0)\leq h_a(T)$ for all $T\in R(M)$; there is $T_0\in R(M)$ such that $\varphi_{n-i-1}^A(a)\notin T_0$ and $\varphi_{n-i}^A(a)\in T$, for all $T\in R(M)$.

Since $(G^*(h_a))(M) \neq 1$, then $R(M) \neq \emptyset$, and consequently $G^{-1}(M)$ is a proper filter of A and so, from Lemma 5.3, we have that (2) $G^{-1}(M) = \bigcap\limits_{(M,T)\in R} T$.

Then, from (1) and (2), we infer that

(3) $(G^*(h_a))(M) = \frac{i}{n-1}$, $i \in [n-2]$, iff $G(\varphi^A_{n-i}(a)) \in M$ and $G(\varphi^A_{n-i-1}(a)) \notin M$, $i \in [n-2]$.

Besides, from property (T3), we get that

(4) $G(\varphi^A_{n-i}(a)) \in M$ and $G(\varphi^A_{n-i-1}(a)) \notin M$ iff $\varphi^A_{n-i}(G(a)) \in M$ and $\varphi^A_{n-i-1}(G(a)) \notin M$.

Hence, from (3) and (4) we can assert that $G^*(h_a)(M) = \frac{i}{n-1}$, iff $h_{G(a)}(M) = \frac{i}{n-1}$, for all $a \in A$, $M \in X$ and $i \in [n-2]$.

Finally, from (A), (B), (C), we conclude that $G^*(h_a) = h_{G(a)}$ for all $a \in A$.

(X) *For all $a \in A$, $H^*(h_a) = h_{H(a)}$, where for all $b \in A$, $h_b : X \longrightarrow L_n$ is defined as in Proposition* 5.7:

It follows using a similar technique to that used in the proof of (IX).

(XI) $(S,T) \in R^{L^X_n}$ *implies* $(\Phi(S), \Phi(T)) \in R^A$ *for any* $S, T \in \max \mathfrak{X}(L^X_n)$:

Let $S, T \in \max \mathfrak{X}(L^X_n)$ such that (1)$(T,S) \in R^{L^X_n}$, then from Lemma 5.3, we obtain that $G^{*-1}(T) \subseteq S$. Therefore $G^{*-1}(T)$ is a proper filter of L^X_n. Since $T \in \max \mathfrak{X}(L^X_n)$ and L^X_n is a tense LM_n-algebra then, from Lemma 5.5, it follows that (2) $p \in G^{*-1}(T)$ iff $p \in N$, for all $p \in L^X_n$ and $N \in R^{L^X_n}(T) \cap \max \mathfrak{X}(L^X_n)$. Assume now that (3)$(\Phi(S), \Phi(T)) \notin R^A$. Since $S, T \in \max \mathfrak{X}(L^X_n)$, then there are $M_0, M_1 \in X$ such that (5) $\Phi(S) = M_0$ and $\Phi(T) = M_1$, and therefore, from the assertions (3) and (4) and the fact that $R = R^A|_X$ it follows that $(M_0, M_1) \notin R$, and so, from Lemma 5.3 we obtain that $G^{-1}(M_0) \not\subseteq M_1$. Then, there exists $a \in A$ such that (5)$M_0 \in \sigma_A(G(a))$ and $M_1 \notin \sigma_A(a)$, and hence from the statements (4) and (5), we obtain that $T \in \Phi^{-1}(\sigma_A(G(a)))$ and $S \notin \Phi^{-1}(\sigma_A(a))$. In addition, from (III) we have that $\Phi^{-1}(\sigma_A(a)) = \sigma_{L^X_n}(h_a)$, and $\Phi^{-1}(\sigma_A(G(a))) = \sigma_{L^X_n}(h_{G(a)})$. Also, from (IX) we have that $\sigma_{L^X_n}(h_{G(a)}) = \sigma_{L^X_n}(G^*(h_a))$. Therefore, from these last assertions we get that $T \in \sigma_{L^X_n}(G^*(h_a))$ and $S \notin \sigma_{L^X_n}(h_a)$. Consequently, there exists $a \in A$ such that $h_a \in L^X_n$, $h_a \in G^{*-1}(T)$, $h_a \notin S$, $S \in R^{L^X_n}(T) \cap \max \mathfrak{X}(L^X_n)$, which contradicts (2). Therefore, we have that $(\Phi(S), \Phi(T)) \in R$.

(XII) $(S,T) \in R^{L^X_n}$ *implies* $(\Phi(S), \Phi(T)) \in R^A$ *for any* $S, T \in \mathfrak{X}(L^X_n)$:

Let $S, T \in \mathfrak{X}(L^X_n)$, then from property (tS4) of tense LM_n-spaces, we obtain that

for all $i \in [n-1]$, $(S,T) \in R^{L^X_n}$ iff $(f^{L^X_n}_i(S), f^{L^X_n}_i(T)) \in R^{L^X_n}$,

for all $i \in [n-1]$, $(\Phi(S), \Phi(T)) \in R^A$ iff $(f^A_i(\Phi(S)), f^A_i(\Phi(T))) \in R^A$.

Then from these two assertions and property (LP5) of LM_n-spaces, we infer that

(1)$(S,T) \in R^{L^X_n}$ iff $(f^{L^X_n}_{n-1}(S), f^{L^X_n}_{n-1}(T)) \in R^{L^X_n}$,

(2)$(\Phi(S), \Phi(T)) \in R^A$ iff $(f^A_{n-1}(\Phi(S)), f^A_{n-1}(\Phi(T))) \in R^A$.

On the other hand, from property (LP9) we have that $f^{L^X_n}_{n-1}(S) \in \max \mathfrak{X}(L^X_n)$, $f^{L^X_n}_{n-1}(T) \in \max \mathfrak{X}(L^X_n)$, from which it follows from (XI) that

(3) $\left(f^{L^X_n}_{n-1}(S), f^{L^X_n}_{n-1}(T)\right) \in R^{L^X_n}$ implies $\left(\Phi\left(f^{L^X_n}_{n-1}(S)\right), \Phi\left(f^{L^X_n}_{n-1}(T)\right)\right) \in R^A$.

In addition, from (VI) we have that

(4)$\Phi\left(f^{L^X_n}_{n-1}(S)\right) = f^A_{n-1}(\Phi(S))$ and $\Phi\left(f^{L^X_n}_{n-1}(T)\right) = f^A_{n-1}(\Phi(T))$.

Therefore, from (1), (2), (3) and (4), we conclude that (XII) holds.

(XIII) $\Phi^{-1}(G_{R^A}(U)) = G_{R^{L^X_n}}(\Phi^{-1}(U))$ *for any* $U \in D(\mathfrak{X}(A))$:

Let $U \in D(\mathfrak{X}(A))$. Since, by Lemma 3.16, there is $a \in A$ such that $U = \sigma_A(a)$, we infer that (1) $\Phi^{-1}(G_{R^A}(U)) = \Phi^{-1}(G_{R^A}(\sigma_A(a))) = \Phi^{-1}(\sigma_A(G(a)))$. Taking into account (III), (IX) and the fact that $\sigma_{L_n^X}$ is a tense LM_n-isomorphism, we get that (2) $\Phi^{-1}(\sigma_A(G(a))) = \sigma_{L_n^X}(h_{G(a)}) = \sigma_{L_n^X}(G^*(h_a)) = G_{R^{L_n^X}}(\sigma_{L_n^X}(h_a)) = G_{R^{L_n^X}}(\Phi^{-1}(\sigma_A(a))) = G_{R^{L_n^X}}(\Phi^{-1}(U))$. Therefore, from (1) and (2), we conclude that for all $U \in D(\mathfrak{X}(A))$, $\Phi^{-1}(G_{R^A}(U)) = G_{R^{L_n^X}}(\Phi^{-1}(U))$.

(XIV) $\Phi^{-1}(H_{R^{A^{-1}}}(U)) = H_{R^{L_n^{X^{-1}}}}(\Phi^{-1}(U))$ *for any* $U \in D(\mathfrak{X}(A)$:

The proof is similar to that of (XIII), taking into account (X) and the fact that for all $M_1, M_2 \in X$, $(M_1, M_2) \in R$ iff $H^{-1}(M_2) \subseteq M_1$.

Finally, the statements (I) to (XIV) complete the proof. ∎

Remark 5.10
In Theorem 5.9, we can consider $X = \mathfrak{X}(\mathcal{B}(A))$, where $\mathfrak{X}(\mathcal{B}(A))$ is the lattice of all prime filters of the Boolean algebra $\mathcal{B}(A)$ of the complemented elements of A. It is well known that $\max \mathfrak{X}(A)$ is isomorphic to $\mathfrak{X}(\mathcal{B}(A))$, as ordered sets.

Theorem 5.11 (Representation theorem for tense LM_n-algebras)
For any tense LM_n-algebra (A, G, H) there exists a frame (X, R) and an injective morphism of tense LM_n-algebras from (A, G, H) into (L_n^X, G^*, H^*) and therefore, (A, G, H) is isomorphic to a tense LM_n-subalgebra of (L_n^X, G^*, H^*), where (L_n^X, G^*, H^*) is the tense LM_n-algebra described in Lemma 5.3.

Proof. Lemmas 3.6 and 3.16 and Theorem 5.9 allow us to assert that there exist a frame (X, R) and an injective tense LM_n-homomorphism $\Omega : A \longrightarrow L_n^X$ and therefore A is isomorphic to a tense LM_n-subalgebra of L_n^X. ∎

The previous theorem reduces the calculus in an arbitrary tense LM_n-algebra A to the calculus in L_n^X.

Corollary 5.12
Let $(A, \sim, \{\varphi_i^A\}_{i \in [n-1]}, G, H)$ be a tense LM_n-algebra, $I_{n-1} = [n-1]$, $L_2^{[I_{n-1}]}$ be the set of all increasing functions from I_{n-1} to the Boolean algebra L_2 with two elements, and the LM_n-algebra $\langle L_2^{[I_{n-1}]}, \wedge, \vee, \sim, \{\varphi_i\}_{i \in [n-1]}, 0, 1\rangle$, where the operations of the lattice $\langle L_2^{[I_{n-1}]}, \wedge, \vee, 0, 1\rangle$ are defined pointwise and for all $f \in L_2^{[I_{n-1}]}$ and $i, j \in [n-1]$, $\varphi_i(f)(j) = f(i)$ and $(\sim f)(i) = f(n-i)$. Then, there is a frame (X, R) such that A is isomorphic to a tense LM_n-subalgebra of $L_2^{[I_{n-1}]^X}$.

Proof. It is a direct consequence of Theorem 5.11 and the fact that the LM_n-algebras L_n and $L_2^{[I_{n-1}]}$ are isomorphic. ∎

Corollary 5.13
Let $(A, \sim, \{\varphi_i^A\}_{i \in [n-1]}, G, H)$ be a tense LM_n-algebra, L_2 be the Boolean algebra with two elements, $D(L_2) = \{(x_1, \ldots, x_{n-1}) \in L_2^{n-1} | x_1 \leq \ldots \leq x_{n-1}\}$, and

$$(D(L_2), \wedge, \vee, N, \varphi_1, \ldots, \varphi_{n-1}, (0, \ldots, 0), (1, \ldots, 1)),$$

be the LM_n-algebra described in Example 2.13. Then, there is a frame (X, R) such that A is isomorphic to a tense LM_n-subalgebra of $D(L_2)^X$.

Proof. It is a direct consequence of Corollary 5.12 and the fact that the LM_n-algebras $D(L_2)^X$ and $L_2^{[I_{n-1}]}$ are isomorphic. ∎

LEMMA 5.14
The tense LM_n-algebra $(L_n^X, \sim, \{\varphi_i^{L_n^X}\}_{i\in[n-1]}, G^*, H^*)$ is complete.

PROOF. Let $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subseteq L_n^X$, $f : X \longrightarrow L_n$ and $g : X \longrightarrow L_n$, defined for all $x \in X$, by the prescriptions: $f(x) = \bigwedge\limits_{\alpha\in\mathcal{A}} f_\alpha(x)$, $g(x) = \bigvee\limits_{\alpha\in\mathcal{A}} f_\alpha(x)$. Since L_n is finite, then for all $x \in X$, $\bigwedge\limits_{\alpha\in\mathcal{A}} f_\alpha(x)$, $\bigvee\limits_{\alpha\in\mathcal{A}} f_\alpha(x) \in L_n$, and therefore $f, g \in L_n^X$. It is easy to show that $f = \bigwedge\limits_{\alpha\in\mathcal{A}} f_\alpha$ and $g = \bigvee\limits_{\alpha\in\mathcal{A}} f_\alpha$, which allow us to assert that L_n^X is complete. ∎

COROLLARY 5.15
Any tense LM_n-algebra $(A, \sim, \{\varphi_i^A\}_{i\in[n-1]}, G, H)$ is a subalgebra of a complete tense LM_n-algebra.

PROOF. It is an immediate consequence of Theorem 5.11 and Lemma 5.14. ∎

6 Complete and finite simple and subdirectly irreducible tense $\boldsymbol{LM_n}$-algebras

Now, we are interested in the characterization of the simple and subdirectly irreducible complete tense LM_n-algebras whose filters are complete. To this end, we recall that if A is a complete lattice whose prime filters are complete, then for all $S \subseteq A$, $\sigma_A\Big(\bigwedge\limits_{a\in S} a\Big) = \bigcap\limits_{a\in S} \sigma_A(a)$.

PROPOSITION 6.1
Let (A, G, H) be a complete tense LM_n-algebra. Then, the following conditions are equivalent for any $a \in A$:

(i) $a = d(a)$,
(ii) $a = d^n(a)$ for all $n \in \omega$,
(iii) $a = \bigwedge\limits_{n\in\omega} d^n(a)$,
(iv) $a = \bigwedge\limits_{n\in\omega} d^n(b)$ for some $b \in A$.

PROOF. It follows from Lemmas 4.14 and 4.17 and the fact that for any $a \in A$, $\bigwedge\limits_{n\in\omega} d^n(a) \in A$. ∎

THEOREM 6.2
Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ be a complete tense LM_n-algebra whose filters are complete. Then, the following conditions are equivalent:

(i) $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ is a simple tense LM_n-algebra,
(ii) $\mathcal{B}(\mathcal{C}(A)) = \{0, 1\}$,
(iii) $(\mathcal{C}(A), \sim, \{\varphi_i\}_{i\in[n-1]})$ is a simple LM_n-algebra,
(iv) $(\mathcal{C}(A), \sim, \{\varphi_i\}_{i\in[n-1]})$ is isomorphic to an LM_n-subalgebra of L_n.

PROOF.
(i) ⇒ (ii): It is an immediate consequence of Corollary 4.29.

(ii) ⇔ (iii): It immediately follows from Corollary 2.12 and Lemma 4.18.

(iii) ⇔ (iv): It is a direct consequence of Theorem 2.11.

(ii) ⇒ (i): Taking into account that A is a complete tense LM_n-algebra and Proposition 6.1, we have that $\mathcal{C}(A) = \left\{\bigwedge\limits_{n\in\omega} d^n(a) : a \in A\right\}$, and so from the hypothesis (ii) we obtain that (1) $\varphi_i\Big(\bigwedge\limits_{n\in\omega} d^n(a)\Big) = 0$ or

$\varphi_i\Big(\bigwedge\limits_{n\in\omega} d^n(a)\Big)=1$ for every $a\in A$ and every $i\in[n-1]$. Besides, from property (L6) of LM_n-algebras, we have that for all $a\in A\setminus\{1\}$, there is at least $i_0\in[n-1]$ such that $\varphi_{i_0}(a)\neq 1$, and hence from property (d2) in Corollary 4.12, we obtain that $d^n(\varphi_{i_0}(a))\neq 1$ for any $n\in\omega$. Then, from property (L12) of LM_n-algebras and property (d7) in Corollary 4.12, we get that $\varphi_{i_0}\Big(\bigwedge\limits_{n\in\omega} d^n(a)\Big)=\bigwedge\limits_{n\in\omega}\varphi_{i_0}(d^n(a))=\bigwedge\limits_{n\in\omega} d^n(\varphi_{i_0}(a))\neq 1$. Therefore, we can assert that (2) $\varphi_i\Big(\bigwedge\limits_{n\in\omega} d^n(a)\Big)\neq 1$ for every $a\in A\setminus\{1\}$ and every $i\in[n-1]$ such that $\varphi_i(a)\neq 1$. From (1) and (2) we infer that $\varphi_i\Big(\bigwedge\limits_{n\in\omega} d^n(a)\Big)=0$ for every $a\in A\setminus\{1\}$ and every $i\in[n-1]$ such that $\varphi_i(a)\neq 1$. In addition, from the fact that the prime filters of A are complete, it follows that $\sigma_A\Big(\bigwedge\limits_{n\in\omega} d^n(\varphi_i(a))\Big)=\bigcap\limits_{n\in\omega}\sigma_A(d^n(\varphi_i(a)))$. Consequently, for each $a\in A\setminus\{1\}$ and each $i\in[n-1]$ such that $\varphi_i(a)\neq 1$, $\bigcap\limits_{n\in\omega}\sigma_A(d^n(\varphi_i(a)))=\emptyset$ and thus, from Lemma 3.16 and Corollary 4.13, it results that $\bigcap\limits_{n\in\omega} d^n_{\mathfrak{X}(A)}\Big(\varphi_i^{\mathfrak{X}(A)}(\sigma_A(a))\Big)=\emptyset$ for all $a\in A\setminus\{1\}$ and $i\in[n-1]$ such that $\varphi_i^{\mathfrak{X}(A)}(\sigma_A(a))\neq\mathfrak{X}(A)$. Finally, from this last statement, Lemma 3.16 and the definition of $\varphi_i^{\mathfrak{X}(A)}$ on $D(\mathfrak{X}(A))$, $i\in[n-1]$, given by the prescription 2.12, we conclude that for all $U\in D(\mathfrak{X}(A))\setminus\{\mathfrak{X}(A)\}$ and $i\in[n-1]$ such that $f_i^{A^{-1}}(U)\neq\mathfrak{X}(A)$, $\bigcap\limits_{n\in\omega} d^n_{\mathfrak{X}(A)}(f_i^{A^{-1}}(U))=\emptyset$ and so from Proposition 4.27 the proof is complete. ∎

THEOREM 6.3
Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ be a complete tense LM_n-algebra whose filters are complete. Then, the following conditions are equivalent:

(i) $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ is a subdirectly irreducible tense LM_n-algebra,
(ii) $\mathcal{B}(\mathcal{C}(A))=\{0,1\}$,
(iii) $(\mathcal{C}(A),\sim,\{\varphi_i\}_{i\in[n-1]})$ is a simple LM_n-algebra.

PROOF.
(i) $\Rightarrow$ (ii): From the hypothesis (i) and Corollary 4.31, we have that there is (1) $b\in\mathcal{B}(A)\setminus\{1\}$ such that (2) $d^{n_a}(a)\leq b$ for some $n_a\in\omega$, for all $a\in\mathcal{B}(A)\setminus\{1\}$. Then, from (1) and Lemma 2.9, it follows that (3) $b=\varphi_i(b)$ for all $i\in[n-1]$. Besides, since A is a complete tense LM_n-algebra, then from Proposition 6.1, we get that $\bigwedge\limits_{n\in\omega} d^n(b)\in\mathcal{C}(A)$ and so, from (3), property (L12) of LM_n-algebras, Lemma 2.9 and the property (d7) in Corollary 4.12, we deduce that $\bigwedge\limits_{n\in\omega} d^n(b)\in\mathcal{B}(\mathcal{C}(A))$. Furthermore, from (1) and property (d5) in Corollary 4.12, it results that $\bigwedge\limits_{n\in\omega} d^n(b)\neq 1$. Now, let $c\in\mathcal{B}(\mathcal{C}(A))$, $c\neq 1$, then from (2) and the fact that $c=\bigwedge\limits_{n\in\omega} d^n(c)$, we obtain that $c=\bigwedge\limits_{n\in\omega} d^n(c)\leq d^{n_c}(c)\leq b$, and so from property (d4) in Corollary 4.12, we can assert that $c\leq\bigwedge\limits_{n\in\omega} d^n(b)$. Therefore, from Corollary 4.19, we conclude that $(\mathcal{B}(C(A)),\sim)$ is a totally ordered Boolean algebra and consequently, $\mathcal{B}(\mathcal{C}(A))=\{0,1\}$.

(ii) $\Leftrightarrow$ (iii): It immediately follows from Corollary 2.12 and Lemma 4.18.

(ii) $\Rightarrow$ (i): From the hypothesis (ii) and Theorem 6.2 it results that $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ is a simple tense LM_n-algebra and therefore the proof is complete. ∎

COROLLARY 6.4
Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ be a complete tense LM_n-algebra whose filters are complete, and let $(L_n,\sim,\{\varphi_i\}_{i\in[n-1]})$ and $(L_n^X,\sim,\{\varphi_i^{L_n^X}\}_{i\in[n-1]},G^*,H^*)$ be the algebras described in Example 2.10 and Lemma 5.3, respectively. Then, the following conditions are equivalent:

(i) $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ is a simple tense LM_n-algebra,
(ii) $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ is a subdirectly irreducible tense LM_n-algebra,
(iii) $(\mathcal{C}(A),\sim,\{\varphi_i\}_{i\in[n-1]})$ is isomorphic to an LM_n-subalgebra of L_n,
(iv) $\mathcal{B}(\mathcal{C}(A))=\{0,1\}$,
(v) $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ is isomorphic to a complete tense LM_n-subalgebra of L_n^X
1. whose filters are complete.

PROOF. It is a direct consequence of Corollary 5.15 and Theorems 6.3 and 6.2. ∎

COROLLARY 6.5
Let $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ be a finite tense LM_n-algebra, and let $(L_n,\sim,\{\varphi_i\}_{i\in[n-1]})$ and $(L_n^X,\sim,\{\varphi_i^{L_n^X}\}_{i\in[n-1]},G^*,H^*)$ be the algebras described in Example 2.10 and Lemma 5.3, respectively. Then, the following conditions are equivalent:

(i) $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ is a simple tense LM_n-algebra,
(ii) $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ is a subdirectly irreducible LM_n-algebra,
(iii) $(\mathcal{C}(A),\sim,\{\varphi_i\}_{i\in[n-1]})$ is isomorphic to an LM_n-subalgebra of L_n,
(iv) $\mathcal{B}(\mathcal{C}(A))=\{0,1\}$,
(v) $(A,\sim,\{\varphi_i\}_{i\in[n-1]},G,H)$ is isomorphic to a tense LM_n-subalgebra of L_n^X.

PROOF. It is a direct consequence of Corollary 6.4 and the fact that A is finite. It should be mentioned that in this case since A is finite, then from property (d2) in Corollary 4.12, we have that for every $a\in A$, there is $n_a\in\omega$ such that $d^n(a)=d^{n_a}(a)$ for all $n\in\omega$, $n_a\leq n$, and so $\bigwedge_{n\in\omega} d^n(a)=d^{n_a}(a)$. Also, since A is finite, then $X=\max\mathfrak{X}(A)$ is finite and so L_n^X is finite. ∎

7 Conclusion and future research

In this article, we have determined a topological duality for tense n-valued Łukasiewicz–Moisil algebras, extending the one obtained in [16], in which it is used the definition of LM_n-algebras given in [11]. By means of the above duality we have characterized simple and subdirectly irreducible tense n-valued Łukasiewicz–Moisil algebras, specially complete and finite algebras. Also, we have proved a theorem of representation of these algebras. The proof of this theorem has allowed us to identify some topological properties of Priestley space associated with n-valued Łukasiewicz–Moisil algebra L_n^X (Lemma 5.6, Proposition 5.7), highlighting the effectiveness of the topological procedures used, which could be of interest for people working in duality theory.

It seems worth mentioning that in [14], Diaconescu and Leuştean introduced an alternative definition for LM_{n+1}-algebra as we will indicate below:

DEFINITION 7.1
An LM_{n+1}-algebra is a system of the form $\langle A,\vee,\wedge,^*,J_1,\ldots,J_n,0,1\rangle$ such that the structure $\langle A,\vee,\wedge,^*,0,1\rangle$ is a De Morgan algebra and $J_1,\ldots,J_n$ are unary operations on A such that the following hold:

(J1) $\bigvee_{k=n-i+1}^{n} J_k(x\vee y)=\bigvee_{k=n-i+1}^{n}(J_k(x)\vee J_k(y))$,

(J2) $J_i(x) \vee J_i(y)^* = 1$,
(J3) $J_k(J_i(x)) = 0$ and $J_n(J_i(x)) = J_i(x)$,
(J4) $J_k(x^*) = J_{n-k}(x)$ and $J_n(x^*) = \bigwedge_{i=1}^{n} J_i(x)^*$,
(J5) $J_l(x) \leq (J_1(x) \vee \ldots J_{l-1}(x))^*$,
(J6) if $J_i(x) = J_i(y)$, for all $i \in [n]$, then $x = y$,
for any $i,j \in [n]$, $k \in [n-1]$, $1 < l < n$ and $x, y \in L$.

Moreover, these authors proved that Definitions 2.6 and 7.1 are equivalent.

Furthermore, in [14], it is shown that category $\boldsymbol{LM_{n+1}}$ of LM_{n+1}-algebras and LM_{n+1}-homomorphisms is equivalent to a category which has Boolean algebras endowed with a particular set of Boolean ideals as objects and their corresponding homomorphisms as morphisms. To achieve this goal, first the authors defined *an n symmetric sequence of ideals on a Boolean algebra B* as a finite set $\{I_1, \ldots, I_{n-1}\}$ of ideals on B with the property that $I_i = I_{n-i}$, for any $i \in [n-1]$. Then, they considered the category $\boldsymbol{BoolI_{n+1}}$ whose objects are tuples of the form $(B, I_{n-1}, \ldots, I_1)$, where B is a Boolean algebra and $\{I_1, \ldots, I_{n-1}\}$, is an n symmetric sequence of ideals on B, and whose morphisms are Boolean morphisms $g : (B, I_{n-1}, \ldots, I_1) \longrightarrow (B', I'_{n-1}, \ldots, I'_1)$ such that $g(I_i) \subseteq I'_i$, for any $i \in [n-1]$. Finally, they proved that the categories $\boldsymbol{LM_{n+1}}$ and $\boldsymbol{BoolI_{n+1}}$ are equivalent. This categorical equivalence is a powerful tool for working with LM_{n+1}-algebras.

Also, in [14] the authors developed a Stone-type duality for LM_{n+1}-algebras starting from the Stone duality for Boolean algebras. In order to determine this duality, they took into account that the categories $\boldsymbol{LM_{n+1}}$ and $\boldsymbol{BoolI_{n+1}}$ are equivalent and so they constructed a Stone-type duality for the category $\boldsymbol{BoolI_{n+1}}$. To this purpose, they introduced a topological category, denoted by $\boldsymbol{BoolSO_n}$, whose objects are the *Boolean spaces with n symmetric open sets*, which are tuples of the form $(X, O_1, \ldots, O_{n-1})$, where X is a Boolean space (i.e. a Hausdorff and compact space which has a basis of clopen subsets) and $O_1, \ldots, O_{n-1}$ are open sets in X such that $O_i = O_{n-i}$ for any $i \in [n-1]$, and whose morphisms are continuous maps $f : (X, O_1, \ldots, O_{n-1}) \longrightarrow (Y, U_1, \ldots, U_{n-1})$ such that $f^{-1}(U_i) \subseteq O_i$, for any $i \in [n-1]$. Besides, they proved that the categories $\boldsymbol{BoolI_{n+1}}$ and $\boldsymbol{BoolSO_n}$ are dually equivalent.

One of the referees pointed out that it would be interesting to study how this last duality can be extended for tense $(n+1)$-valued Łukasiewicz–Moisil algebras.

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