Representation theory of MV-algebras

Eduardo J. Dubuc [a,*], Yuri A. Poveda [b]

[a] Departamento de Matematicas, F.C.E. y N., UBA, Buenos Aires, Argentina

[b] Departamento de Matematicas, Universidad Tecnologica de Pereira, Pereira, Colombia

ARTICLE INFO

ABSTRACT

In this paper we develop a general representation theory for MV-algebras. We furnish the appropriate categorical background to study this problem. Our guide line is the theory of classifying topoi of coherent extensions of universal algebra theories. Our main result corresponds, in the case of MV-algebras and MV-chains, to the representation of commutative rings with unit as rings of global sections of sheaves of local rings. *We prove that any MV-algebra is isomorphic to the MV-algebra of all global sections of a sheaf of MV-chains on a compact topological space.* This result is intimately related to McNaughton's theorem, and we explain why our representation theorem can be viewed as a vast generalization of McNaughton's theorem. In spite of the language used in this abstract, we have written this paper in the hope that it can be read by experts in MV-algebras but not in sheaf theory, and conversely.

Preface

We have written this paper in the hope that it can be read by experts in MV-algebras but not in sheaf theory, and conversely. Our basic reference on MV-algebras is the book 'Algebraic Foundations of Many-valued Reasoning' [1], and we refer to this book and not to the original sources for the known specific results we utilize. Only basic facts of sheaf theory are needed for reading this paper—the reader may consult 'Sheaves in Geometry and Logic' [12]. For general category theory we refer the reader to the classical textbook 'Categories for the working mathematician' [11].

0. Introduction

In this paper we develop a general representation theory for MV-algebras as algebras of global sections of sheaves of MV-chains. Our work generalizes previous results in this direction, like the representation theorem for locally finite MV-algebras proved in [2], and it is based on the second author's thesis [16].

We furnish the appropriate categorical background to study the representation theory of MV-algebras. Our guide line is the theory of classifying topoi first developed in the case of rings by M. Hakim, and thereafter placed in the general context of universal algebra by category theorists, in particular Coste [3]. Our main result corresponds, in the case of MV-algebras and MV-chains, to the representation of commutative rings with unit as rings of global sections of sheaves of local rings (see, e.g., [6]). We prove that *every MV-algebra is isomorphic to the MV-algebra of all global sections of its prime spectrum* (Theorem 3.12). We analyze and develop carefully the various steps that lead to this theorem.

---

* Corresponding author.
E-mail address: edubuc@dm.uba.ar (E.J. Dubuc).

The basic starting construction in the case of rings is the local ring resulting from the *localization* at a prime ideal, while in the case of MV-algebras it is the MV-chain resulting from the *quotient* by a prime ideal. This leads in both cases to the consideration of the set of prime ideals as the set of points of the base space for the spectral sheaf. Some considerations of categorical nature (see (*)) indicated to us that the appropriate topology for this set, in the case of MV-algebras, is not the Zariski topology as in the case of rings, but a topology that we call *co-Zariski*. Its base of open sets is given by the sets $\{P \mid a \in P\}$, letting a range over all elements of A, and not by the sets $\{P \mid a \notin P\}$ as in the Zariski topology. We think that the consideration of the Zariski topology in the representation of MV-algebras as global sections of sheaves of MV-chains has blocked the traditional development of the theory beyond the particular case of hyperarchimedean algebras, where the two topologies coincide.

We construct the prime spectrum sheaf (over the set of prime ideals) of an MV-algebra following standard methods of sheaf theory, and we introduce the notion of *MV-space* and its corresponding category mimicking the algebraic geometry notion of ringed space. Our main result (Theorem 3.12) means that *the category of MV-algebras is the dual of a category of MV-spaces.* This is similar to Grothendieck method in algebraic geometry, which is based on the consideration of the dual of the category of rings as the category of affine schemes. We show that Theorem 3.12 follows from two facts: (i) *The compactness lemma* (Lemma 3.8) stating that the prime spectrum is a compact topological space, and (ii) *The pushout–pullback lemma* (Lemma 3.11). This lemma means that given two elements a_1, a_2 in an MV-algebra A, any two other elements b_1, b_2 such that $b_1 = b_2 \ mod(a_1 \vee a_2)$ can be "glued" into a single element b, unique $mod(a_1 \wedge b_2)$, such that $b = b_1 \ mod(a_1)$, and $b = b_2 \ mod(a_2)$. Not surprisingly, this lemma is deeply linked with McNaughton's theorem.

Section 1 is specially aimed to the non-expert reader. There we recall basic fundamental facts on sheaves and on MV-algebras. In this way we set up a basic dictionary of MV-algebra and sheaf theoretic terms and notation, as well as a place for reference. The knowledgeable reader is probably able to skip most of this section, and jump ahead to Section 2.

In Sections 2 and 3 we develop the basic framework of the representation and duality theory for MV-algebras. We state our main representation Theorem 3.12 and we show how it can be derived from the two lemmas mentioned above. These lemmas are proved in later sections.

Taking into account the equivalence between the categories of MV-algebras and of lattice ordered abelian groups with a distinguished strong unit (see [1]), their respective representation theories should be strongly related. In 3.13 we compare our representation Theorem 3.12 with the representation theorems for l-groups of [10] and [17].

In Section 4 the set of maximal ideals of any MV-algebra is equipped with the co-Zariski topology, while the set of [0, 1]-valued morphisms (introduced in [2]) is equipped with the Zariski topology. These two spectral spaces have the same underlying set, but the co-Zariski topology is always finer than the Zariski topology, and strictly so, unless the algebra is hyperarchimedean. We prove several preliminary results, which will find use in the proof of the pushout–pullback lemma, and also are of help to understand the relationship between this lemma and McNaughton theorem. We are thus naturally led to introduce a concept of *strong semisimplicity* (Definition 4.7). Strongly semisimple MV-algebras form an intermediate class between hyperarchimedean algebras and semisimple algebras. A standard example of strongly semisimple non-hyperarchimedean MV-algebras is given by finitely presented MV-algebras. In 4.10 we relate our concept of strong semisimplicity with the notion of *Yosida frame* of [13].

In Section 5, before attacking the general case, we consider the case of hyperarchimedean algebras, and prove the main representation theorem for these algebras. Here every prime ideal is maximal, the Zariski and co-Zariski topologies coincide, and the prime spectrum is a Hausdorff space. Furthermore, all the fibers in the prime spectrum are subalgebras of the real unit interval [0, 1]. As a consequence, applying the classical methods of [2], we extend to all hyperarchimedean MV-algebras the duality theorem of [2] for locally finite MV-algebras.

In Section 6 we prove and/or recall several results on finitely presented MV-algebras, and prove the pushout–pullback lemma for these algebras. A key result is the gluing Lemma 5.3 of [15], that we adapt and prove in our context in Lemma 6.11.

In Section 7 we prove the general case of the pushout–pullback lemma. This follows by categorical nonsense (finite limits commute with filtered colimits) from the case of finitely presented MV-algebras. We find it convenient and instructive to sketch an explicit proof in the particular case of pullbacks of MV-algebras.

In Section 8 we prove the compactness lemma, that is, we prove that the prime spectrum furnished with the co-Zariski topology is a compact topological space. We develop a construction of its lattice of open sets without constructing the underlying set first. This yields a compact locale. Then the prime spectrum is the space of points of this locale. Such space turns out to be compact, provided the locale has enough points, which follows by a standard application of Zorn's Lemma.

In Section 9 we prove McNaughton's theorem as the special case of our representation theorem for free MV-algebras. We show that a finite open cover of the prime spectrum of the free MV-algebra on n generators yields (by restricting the open sets of the cover to the maximal spectrum), a finite decomposition of the n dimensional cube by convex polyhedra. Once this is understood, an isomorphism between the MV-algebra of McNaughton functions and the MV-algebra of global sections of the prime spectrum of the free algebra becomes evident. These results also furnish a conceptual context for McNaughton's theorem. They show the "local" nature of the concept of McNaughton function, as a particular instance of the usual topological notion of localness. Our representation theorem is a vast generalization of McNaughton's theorem, from free MV-algebras to the totality of MV-algebras.

We relegate to an appendix some general results on posets that we use in this paper. We view posets as $\{0, 1\}$-based categories, and we develop Grothendieck theory of sheaves but dealing directly with posets. In particular,

since our categories are posets, inf-lattices play the role of categories with finite limits, and locales that of Grothendieck topoi.

(*) Zariski versus co-Zariski

Categorical reasons that force the Zariski topology in ring theory:

1. The localization of a ring in a prime ideal is a covariant construction in the sense that if we have two prime ideals $P \subset Q$, we have a morphism of local rings $A_P \to A_Q$.
2. Given an element $a \in A$ and a prime ideal P, there exists a factorization $A\{a^{-1}\} \to A_P$ if and only if $a \notin P$. For a fixed a, the set $\{P \mid \exists\ A\{a^{-1}\} \to A_P\}$ is the *Zariski* open set $D_a = \{P \mid a \notin P\}$.
3. The assignment $a \mapsto A\{a^{-1}\}$ is contravariant (a presheaf) for Zariski opens in the sense that
   $D_a \subset D_b \Rightarrow A\{b^{-1}\} \to A\{a^{-1}\}$.

Categorical reasons that force the co-Zariski topology in MV-algebra theory:

1. The quotient of an MV-algebra by a prime ideal is a contravariant construction, in the sense that if we have two prime ideals $P \subset Q$, we have a morphism of MV-chains in the other direction $A/Q \to A/P$.
2. Given an element $a \in A$ and a prime ideal P, there exists a factorization $A/(a) \to A/P$ if and only if $a \in P$. For a fixed a, the set $\{P \mid \exists\ A/(a) \to A/P, \}$ is the *co-Zariski* open set $W_a = \{P \mid a \in P\}$.
3. The assignment $a \mapsto A/(a)$ is contravariant (a presheaf) for co-Zariski opens in the sense that
   $W_a \subset W_b \Rightarrow A/(b) \to A/(a)$.

1. Background, terminology and notation

In this section we recall some facts about MV-algebra and sheaf (on topological space) theory, and in this way we fix notation and terminology.

For the definition and basic facts on MV-algebras the reader is advised to have at hand the book "Algebraic foundations of many-valued reasoning", Reference [1].

1.1. MV-algebras

MV-algebras are models of an equational theory in universal algebra.

1. A MV-algebra A has a 2-ary, a 1-ary and a 0-ary primitive operations, denoted $\oplus$, $\neg$, 0, subject to universal axioms.
2. It is convenient to introduce one 0-ary and two 2-ary derived operations, defined by the following formulae.

   $$1 = \neg 0, \quad x \odot y = \neg(\neg x \oplus \neg y), \quad x \ominus y = x \odot \neg y.$$

   We have $\neg\neg x = x$, $0 = \neg 1$, and $x \oplus y = \neg(\neg x \odot \neg y)$.
3. Given an integer n, we let $nx = x \oplus x \oplus x \cdots \oplus x$, n times, and $0x = 0$.
4. There is a partial order relation defined by $x \leq y \iff \exists z, x \oplus z = y$.
   A useful characterization is the following: $x \leq y \iff x \ominus y = 0$.
5. The partial order is a lattice, with *supremum* denoted $\vee$ and *infimum* denoted $\wedge$. The lattice operations are definable by formulae:

   $$x \vee y = (x \odot \neg y) \oplus y, \quad \text{and} \quad x \wedge y = x \odot (\neg x \oplus y).$$

   We have $x \wedge y = \neg(\neg x \vee \neg y)$ and $x \vee y = \neg(\neg x \wedge \neg y)$.
6. We have $x \wedge y \leq x \oplus y$ and $x \odot y \leq x \wedge y$.
7. There is a *distance* operation defined by $d(x, y) = (x \ominus y) \oplus (y \ominus x)$, and we have $d(x, y) = 0 \iff x = y$.
8. The following equations hold in any MV-algebra:

   $$x \oplus \neg x = 1, \quad (x \ominus y) \wedge (y \ominus x) = 0.$$

9. For any MV-algebra A, elements $x, y \in A$ and integer n, we have
   $n(x \wedge y) = nx \wedge ny$. It follows that $x \wedge y = 0 \Rightarrow nx \wedge ny = 0$.
10. $A = \{0\}$ is the trivial MV-algebra, A is said *nontrivial* if $1 \neq 0$.
11. The closed real unit interval $[0, 1] \subset \mathbb{R}$ is the basic example of an MV-algebra. The structure is given by:

    $$x \oplus y = \min(1, x + y), \quad \neg x = 1 - x, \quad x \ominus y = \max(0, x - y),$$
    $$x \odot y = \max(0, x + y - 1), \quad d(x, y) = |x - y|.$$

1.2. Ideals and morphisms of MV-algebras

1. We shall denote by $\mathcal{A}$ the category of MV-algebras. A *morphism* is a function that preserves the three primitive operations. It follows that it will preserve all the derived operations since these operations are defined by formulae. In particular it preserves 0 and 1. Thus the zero function $A \xrightarrow{0} B$ is a morphism only when B is the trivial algebra.
2. Given a morphism of MV-algebras $A \xrightarrow{\varphi} B$, the *image*, $Im(\varphi) \subset B$, is a subalgebra of B,
   $Im(\varphi) = \{y \in B \mid \exists x \in A, \varphi(x) = y\}$.

3. Congruences are associated with ideals.

   An *ideal* of an MV-algebra A is a subset $I \subset A$ satisfying:

   (I1) $0 \in I$

   (I2) $x \in I, \ y \in A, \ y \leq x, \Rightarrow y \in I$

   (I3) $x \in I, \ y \in I, \Rightarrow x \oplus y \in I$.

   An ideal is said to be *proper* if $1 \notin I$.

4. Given a finite number of elements in an MV-algebra, we write $(x, y \ldots z)$ for the ideal generated by $x, y \ldots z$. We have:

$$(x \wedge y) = (x) \cap (y), \quad (x \vee y) = (x \oplus y) = (x, y).$$

   Consequently, all finitely generated ideals are *principal*.

5. For any morphism $A \xrightarrow{\varphi} B$, the *kernel*, $Ker(\varphi) \subset A$, is an ideal of A, $Ker(\varphi) = \{x \in A \mid \varphi(x) = 0\}$. Any ideal $I \subset A$ determines a congruence by stipulating $x \sim y \iff d(x, y) \in I$. The quotient is denoted $A \xrightarrow{\rho} A/I$. We have $I = Ker(\rho)$. For any $x \in A$, the element $\rho(x)$ will be denoted $[x]_I$. Thus, $[x]_I = [y]_I \iff d(x, y) \in I$.

1.3. MV-chains and prime ideals

1. A MV-algebra is a *MV-chain* if it is nontrivial and its order relation is total. Thus, MV-chains are characterized by the axioms:

   (C1) $1 \neq 0$.

   (C2) $x \ominus y = 0$ *or* $y \ominus x = 0$.

2. From 1.1(8) it follows that an MV-algebra A is an MV-chain if and only if it satisfies:

   (C'1) $A \neq \{0\}$.

   (C'2) $x \wedge y = 0 \Rightarrow x = 0$ *or* $y = 0$.

3. An ideal P is *prime* if it satisfies the following conditions:

   (P1) $1 \notin P$.

   (P2) For each x, y in A, either $(x \ominus y) \in P$ or $(y \ominus x) \in P$.

4. From 1.1(8) it follows that an ideal P is prime if and only if it satisfies:

   (P'1) $P \neq A$

   and either one of the following two equivalent conditions

   (P'2) $x \wedge y \in P \Rightarrow x \in P$ *or* $y \in P$

   (P'2) $x \wedge y = 0 \Rightarrow x \in P$ *or* $y \in P$.

5. For any morphism $A \xrightarrow{\varphi} B$, the following holds by definition:

   $Ker(\varphi)$ is a prime ideal iff $Im(\varphi)$ is an MV-chain.

6. For any morphism $A \xrightarrow{\varphi} B$ and prime ideal P of B, the ideal $\varphi^{-1}(P)$ is a prime ideal of A.

7. For any MV-algebra A, and elements $x, y \in A$, The following equivalence holds (see [1, 1.2.14]):

$$(\forall P \text{ prime}) \ (x \in P \Rightarrow y \in P) \iff (y) \subset (x).$$

1.4. Simple MV-algebras and maximal ideals

1. Maximal ideals are prime. Given a prime ideal P, there exists a maximal ideal $M \supset P$ (see [1, 1.2.12]).

2. We have an important characterization of maximal ideals (see [1, 1.2.2]). An ideal M is maximal if and only if it satisfies the following conditions:

   (M1) $1 \notin M$.

   (M2) for each $x \in A$, $x \notin M \iff \exists n \geq 1 \neg nx \in M$.

3. It follows that for any morphism $A \xrightarrow{\varphi} B$ and a maximal ideal M of B, the ideal $\varphi^{-1}(M)$ is a maximal ideal of A (see [1, 1.2.16]).

4. Recall that an MV-algebra is *simple* iff it is nontrivial and $\{0\}$ is its only proper ideal. By definition, an ideal M is maximal if and only if the quotient A/M is a simple MV-algebra.

5. Given any maximal ideal M of an MV-algebra A:

   (a) A/M is isomorphic to a subalgebra of the real unit interval $[0, 1]$ (see [1, 1.2.10, 3.5.1]).

   (b) There is a unique embedding $A/M \hookrightarrow [0, 1]$ which determines by composition a morphism

   $\chi_M : A \longrightarrow A/M \hookrightarrow [0, 1]$. Furthermore $ker(\chi_M) = M$, and $\chi_{ker(\chi)} = \chi$ (see [1, 7.2.6]).

1.5. Presheaves and sheaves

For the following the reader may consult the book "Sheaves in Geometry and Logic", Reference [12]. Contravariant functors $\mathcal{X}^{op} \to \mathcal{S}$ from any small category $\mathcal{X}$ into the category $\mathcal{S}$ of sets will be called *presheaves*.

Given a topological space X, we denote by $\mathcal{O}(X)$ the lattice of open sets.

1. A *presheaf* E on X is a contravariant functor $\mathcal{O}(X)^{op} \xrightarrow{E} \mathcal{S}$. Given $U \subset V$ in $\mathcal{O}(X)$, and $s \in EV$, we denote $s|_U \in EU$ the image of s under the map $EV \to EU$.

2. A *sheaf* is a presheaf E such that given any open cover $U_\alpha \subset U$ in $\mathcal{O}(X)$, the following *sheaf axiom* holds for E:

$$\forall s_\alpha \in EU_\alpha, \quad such \ that \ \ s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}, \quad \exists! \, s \in EU, \ s|_{U_\alpha} = s_\alpha.$$

3. A *space E* over X is a continuous function $E \xrightarrow{\pi} X$.
   (a) Given $p \in X$, the *stalk* (or *fiber*) over p is the set $E_p = \pi^{-1}(p)$.
   (b) Given $U \in \mathcal{O}(X)$, a *section* defined on U is a continuous function $U \xrightarrow{\sigma} E$, $\pi \sigma = id$. If $U = X$, σ is said to be a *global* section. The set of sections defined on U is denoted $\Gamma(U, E)$.

4. An *etale space* is a space $E \xrightarrow{\pi} X$ over X, where π is a local homeomorphism. We shall also say that E is a *sheaf*.

5. An etale space $E \to X$ is said to be *global* if each point $e \in E$ is in the image of some global section.

6. For any etale space $E \to X$, given any two sections σ, μ, the set $[[\sigma = \mu]] = \{x \mid \sigma(x) = \mu(x)\}$ is open. When E is Hausdorff, it is also closed (hence *clopen*), and this fact characterizes Hausdorff sheaves.

7. (*Sheaf of sections,* see [12, Ch. II Section 5]) For any space over X, $E \to X$, the assignment $U \mapsto \Gamma(U, E)$ together with the restriction maps determine a presheaf $\Gamma(-, E)$ satisfying the sheaf axiom.

8. (*Godement construction,* [4], see also [12, Ch. II Section 5]) Given any presheaf $\mathcal{O}(X)^{op} \xrightarrow{E} \delta$, we can associate to E an etale space $E^{\mathcal{G}} \xrightarrow{\pi} X$.
   For each $p \in X$, we take the colimit of the system $\{EU\}_{U \in \mathcal{F}_p}$ indexed by the filter $\mathcal{F}_p$ of open neighborhoods of p. By setting $E_p^{\mathcal{G}} = \lim_{\overrightarrow{U \in \mathcal{F}_p}} EU$ we have maps $EU \to E_p^{\mathcal{G}}$. For any $U \ni p$ and $s \in EU$, we denote by $[s]_p$ the corresponding element in $E_p^{\mathcal{G}}$. By a slight abuse of notation we can write $E_p^{\mathcal{G}} = \{[s]_p \mid s \in EU, \ U \ni p\}$.
   We define the set $E^{\mathcal{G}}$ as the disjoint union of the sets $E_p^{\mathcal{G}}$, $p \in X$. $E^{\mathcal{G}} = \{([s]_p, p), \ s \in EU, \ U \ni p, \ p \in X\}$. The map $E^{\mathcal{G}} \xrightarrow{\pi} X$ is defined to be the projection, $\pi([a]_p, p) = p$.
   Given $U \in \mathcal{O}(X)$, each element $s \in EU$ determines a section $\widehat{s} : U \to E^{\mathcal{G}}$, $\widehat{s}(p) = ([s]_p, p)$. We topologize the set $E^{\mathcal{G}}$ by taking as a base of open sets all the image sets $\widehat{s}(U)$. Under this topology the map π becomes a local homeomorphism.

9. The Godement construction can be applied *mutatis mutandis* to any presheaf $\mathcal{B}^{op} \xrightarrow{E} \delta$ defined *only on a base* $\mathcal{B} \subset \mathcal{O}(X)$ of the topology. When E is the restriction to $\mathcal{B}$ of a presheaf $\mathcal{O}(X)^{op} \xrightarrow{E} \mathcal{E}$, it yields the same etale space $E^{\mathcal{G}}$. The reader should consider, among others, that the filter base $\mathcal{B}_p \subset \mathcal{F}_p$ is a cofinal poset in $\mathcal{F}_p$.

10. Given a presheaf E, the assignment $s \mapsto \widehat{s}$ defines a natural transformation $EU \to \Gamma(U, E^{\mathcal{G}})$ from E to the sheaf of sections of the Godement construction. This transformation is an isomorphism iff E satisfies the sheaf axiom.

11. Given a space E over X, and a point $p \in X$, for all $U \ni p$, evaluation at p defines a map $\Gamma(U, E) \to E_p$ from the set of sections defined on U to the stalk of E at p. This determines a continuous function $\Gamma(-, E)^{\mathcal{G}} \to E$ from the Godement construction of the sheaf of sections to E. This function is a homeomorphism iff E is an etale space.

12. We shall denote by $\delta h(X)$ the category of sheaves on a topological space X. The objects are etale spaces $E \xrightarrow{p} X$, and a morphism from $E \xrightarrow{p} X$ to $G \xrightarrow{q} X$ is a continuous function $E \xrightarrow{f} G$ such that $q \circ f = p$. The sheaf of sections and the Godement construction establish an equivalence between this category and the category which has as objects the presheaves $\mathcal{O}(X)^{op} \to \delta et$ satisfying the sheaf axiom, and as morphisms the natural transformations of functors.

Convention: *When we say that E is a sheaf on X, we will mean indistinctly that E as a presheaf $\mathcal{O}(X)^{op} \xrightarrow{E} \delta$ satisfying the sheaf axiom, or that E is an etale space $E \to X$, and we will use either one of the two sets of data. When the base space is not fixed, we will write $\mathsf{E} = (X, E)$.*

2. Prime spectrum of MV-algebras

THE PRIME SPECTRUM Spec_A.

Given an MV-algebra A, we can associate with A a topological space Z_A, and a global sheaf over Z_A, $\text{Spec}_A = (E_A \xrightarrow{\pi} Z_A)$, as follows:

2.1. Construction of Spec_A

The set of points of Z_A is the set of all prime ideals $P \subset A$. For each $a \in A$, let $W_a \subset Z_A$ be the set $W_a = \{P \mid P \ni a\}$. One immediately checks that $W_0 = Z_A$, $W_1 = \emptyset$, and that $W_a \cap W_b = W_{a \oplus b}$. So we can take the collection $\mathcal{W}_A$ of these sets as a base of a topology, the *co-Zariski topology* on Z_A, $\mathcal{W}_A \subset \mathcal{O}(Z_A)$.

We define the set E_A as the disjoint union of the MV-chains A/P, $P \in Z_A$. That is, $E_A = \{([a]_P, P), \ a \in A, \ P \in Z_A\}$. The map $E_A \xrightarrow{\pi} Z_A$ is defined to be the projection, $\pi([a]_P, P) = P$.

Each element $a \in A$ defines a global section (as a function of sets) $Z_A \xrightarrow{\widehat{a}} E_A$, $\widehat{a}(P) = ([a]_P, P)$.

Observation 2.2. Recalling 1.2(5), 1.5(6) we have: $W_a = [[\widehat{a} = 0]]$ and $W_{d(a, b)} = [[\widehat{a} = \widehat{b}]]$. *Thus, the basic open sets W_a are the Zero sets $Z(\widehat{a})$ of the sections $\widehat{a}$, where $Z(\widehat{a})$ is short for $[[\widehat{a} = 0]]$. We have $W_a = Z(\widehat{a})$.* □

We will now define a topology in E_A in such a way that π becomes a local homeomorphism, or *etale space* over Z_A. The open base for this topology consists of all the image sets $\widehat{a}(W_b)$. These sets are closed under intersections. In fact, given $\widehat{a}(W_b)$ and $\widehat{c}(W_d)$, we have $\widehat{a}(W_b) \cap \widehat{c}(W_d) = \widehat{s}(W_{b \oplus d \oplus d(a, c)})$, where $s = a$, or $s = b$ indistinctly.

With this topology π becomes a local homeomorphism, and every global section $\widehat{a}$ a continuous and open function. Furthermore, this topology is the final topology with respect to the functions $\widehat{a}$. The sections $\widehat{a}$ show that Spec_A is a global sheaf.

2.3. Sections of Spec_A

Given any open set $U \subset Z_A$, the sections over U of Spec_A are the sections of π, $U \xrightarrow{\sigma} Z_A$ which locally are of the form $\widehat{a}$. Specifically, given any $P \in U$, $\sigma P = ([s]_P, P)$ for some $s \in A$. Then, the fact that σ is in $\Gamma(U, E_A)$ means that there is an $a \in A$, $W_a \ni P$, $W_a \subset U$ such that $\sigma(Q) = ([s]_Q, Q)$, the same s for all $Q \in W_a$. That is, $\sigma = \widehat{s}$ on W_a. This is equivalent to the existence of a compatible family of sections $\widehat{s_i}$ defined on a open covering W_{a_i} of U, such that $\sigma|_{W_{a_i}} = \widehat{s_i}|_{W_{a_i}}$ for all i.

The construction above is reminiscent of the Godement construction. We show now that it actually is the Godement construction on a certain presheaf $\mathcal{W}_A^{op} \to \mathscr{S}$. The following is easy to check:

Proposition 2.4. *Any prime ideal P with the order relation of A is a filtered poset (given $a, b \in P$, $a \leq a \oplus b$, $b \leq a \oplus b$, $a \oplus b \in P$), and the assignment $a \mapsto A/(a)$ defines a directed system indexed by P. For each element $a \in P$, we have an induced morphism $A/(a) \to A/P$ which determine a filtered colimit cone. That is, $A/P = \lim_{\overrightarrow{a \in P}} A/(a)$.* $\square$

2.5. Godement construction for Spec_A

The topological space Z_A is defined as before. Recall now 1.5(9). The assignment $W_a \mapsto A/(a)$ defines a presheaf $\mathcal{W}_A^{op} \longrightarrow \mathscr{S}$ on the base of the co-Zariski topology. From 1.3 (7) it follows that if $W_a \subset W_b$ then $A/(b) \to A/(a)$. From Proposition 2.4 it immediately follows that E_A as constructed in 2.1 corresponds to the Godement construction applied to this presheaf. $\square$

3. Duality between MV-algebras and sheaves of MV-chains

We introduce now the notion of *MV-space*. This notion is inspired by the notion of *ringed space* in algebraic geometry (see for example [6, page 72]). The present section should be compared with [2, Section 4].

Definition 3.1 (*The Category $\mathcal{E}$ of MV-Spaces*). (1) A MV-space is a pair (X, E), where X is a topological space and $\mathsf{E} = (E \to X)$ is a sheaf of MV-chains on X (that is, the stalks E_x, $x \in X$ are MV-chains). It follows that for any open set $U \subset X$, the set of sections $\Gamma(U, E)$ is a MV-algebra, $\Gamma(U, E) \subset \prod_{x \in U} E_x$. We say that E is a *sheaf of MV-chains*, and we call E the *structure sheaf*.[1]

(2) A morphism $(f, \varphi) : (X, \mathsf{E}) \longrightarrow (Y, \mathsf{F})$ of MV-spaces is a continuous function $X \xrightarrow{f} Y$ together with a family $\varphi = (\varphi_x)_{x \in X}$ of morphisms $F_{f(x)} \xrightarrow{\varphi_x} E_x$ such that for every open $V \subset Y$, and section $s \in \Gamma(V, F)$, the composite

$$
\begin{array}{ccc}
E_x & \xleftarrow{\;\varphi_x\;} & F_{f(x)} \\
\Big\uparrow{\scriptstyle k} & & \Big\uparrow{\scriptstyle s} \\
x \in X & \xrightarrow{\;f\;} & Y
\end{array}
\qquad k(x) = (\varphi_x \circ s \circ f)(x)
$$

is a section $k \in \Gamma(U, E)$, $U = f^{-1}V$. This amounts to saying that φ is a morphism of sheaves of MV-chains $f^*F \xrightarrow{\varphi} E$, in the topos $\mathscr{S}h(X)$, where f^* denotes the inverse image functor $\mathscr{S}h(Y) \longrightarrow \mathscr{S}h(X)$.

Given any MV-algebra A, the sheaf of MV-chains $\mathrm{Spec}_A = (Z_A, E_A)$, determines a MV-space, and this assignment is (contravariantly) functorial $\mathrm{Spec} : \mathcal{A}^{op} \longrightarrow \mathcal{E}$. On the other hand, given any sheaf of MV-chains $\mathsf{E} = (X, E)$, the MV-algebra of global sections $\Gamma(\mathsf{E}) = \Gamma(X, E)$, determines a (contravariant) functor $\Gamma : \mathcal{E}^{op} \longrightarrow \mathcal{A}$. These two functors are *adjoints on the right*, in the sense that there is a natural bijection between the homsets

$$[(X, E), (Z_A, E_A)] \xrightarrow{\;\cong\;} [A, \Gamma(X, E)].$$

A map $(X, E) \xrightarrow{(f, \varphi)} (Z_A, E_A)$ corresponds under this bijection to the morphism assigning to every $a \in A$ the section $k \in \Gamma(X, E)$ defined by $k(x) = (\varphi_x \circ \widehat{a} \circ f)(x) = \varphi_x([a]_{f(x)})$.

As is well known [11] this adjunction amounts to the following proposition, whose proof is straightforward:

---

[1] Technically, E is a MV-chain object in the topos $\mathscr{S}h(X)$ of sheaves over X.

Proposition 3.2. *For any MV-algebra A, the stipulation*

$$\eta : A \longrightarrow \Gamma\mathrm{Spec}_A = \Gamma(Z_A, E_A), \quad a \mapsto \widehat{a}, \quad \widehat{a}(P) = [a]_P \quad (see\ 2.1)$$

defines a morphism of MV-algebras, which is natural in A, and has the following universal property:
For any MV-space $(X,\ E)$ *and morphism* $A \to \Gamma(X, E)$,

$$
\begin{array}{ccc}
A \xrightarrow{\ \eta\ } \Gamma(Z_A, E_A) & \qquad & (Z_A, E_A) \\
\ \searrow \qquad \downarrow & & \uparrow \\
\qquad \Gamma(X, E) & & \mid\exists! \\
& & (X,\ E) \quad \square
\end{array}
$$

In the following proposition we describe explicitly the other unit corresponding to this adjunction.

Proposition 3.3. *For any MV-space* $(X,\ E) \in \mathcal{E}$, *the stipulation*

$$(h,\ \varepsilon) : (X,\ E) \longrightarrow \mathrm{Spec}\Gamma(X,\ E) = (Z_{\Gamma(X, E)},\ E_{\Gamma(X, E)})$$

$$
\begin{array}{ccc}
E_x \xleftarrow{\ \varepsilon_x\ } \Gamma(X,\ E)/h(x), & \qquad & \varepsilon_x([\sigma]_{h(x)}) = \widehat{x}(\sigma) = \sigma(x) \\
\Big\downarrow \qquad\qquad \Big\downarrow & & \\
x \in X \xrightarrow{\ h\ } Z_{\Gamma(X, E)} & & h(x) = \{\sigma \mid \sigma(x) = 0\}
\end{array}
$$

defines a morphism of MV-spaces, which is natural in $(X,\ E)$, *and has a corresponding universal property.*
[note that ε_x is well defined since clearly $\varepsilon_x(h(x)) = \{0\}$, and h is continuous since $h^{-1}(W_\sigma) = [[\sigma = 0]]$, which is open, 1.5 (6)] $\square$

Our general theory of representation of MV-algebras as algebras of global sections of sheaves of MV-chains is based on the analysis of the morphism η. *In* 3.12 *we establish that for any MV-algebra A,* η *is an isomorphism.* Equivalently, the functor Spec $: \mathcal{A}^{op} \hookrightarrow \mathcal{E}$ is full and faithful. It follows that *the dual of the category of MV-algebras can be considered to be a category of MV-spaces*, which are objects of a geometrical nature. This result is similar to Grothendieck's method to dualize rings by affine schemes [6].

The injectivity of η is an immediate consequence of [1, 1.2.14], stating that every proper ideal in an MV-algebra is an intersection of prime ideals. Equivalently, that the zero ideal $\{0\}$ is the intersection of all prime ideals. We record this fact in the following

Proposition 3.4. *For any MV-algebra A, the morphism* $A \xrightarrow{\ \eta\ } \Gamma(Z_A, E_A)$ *is injective.* $\square$

Notation 3.5. *For any MV-algebra A, the image of the morphism* η *will be denoted by* $\widehat{A}$. *Thus,* $A \xrightarrow{\cong} \widehat{A} \subset \Gamma(Z_A,\ E_A)$.

By [1, 1.2.10], for any $a \in A$, the prime ideals of the quotient algebra $A/(a)$ are in one to one correspondence with the prime ideals which belong to W_a. Thus $Z_{A/(a)} = W_a$. From Proposition 3.4 applied to the algebra $A/(a)$ we obtain:

Proposition 3.6. *Given any MV-algebra A and elements* $a,\ b,\ c$ *in A,*

$$\widehat{b}\,|_{W_a} = \widehat{c}\,|_{W_a} \iff [b]_{(a)} = [c]_{(a)}, \quad that\ is,\ A/(a) \cong \widehat{A}\,|_{W_a}. \quad \square$$

The fact that the ideal $\{0\}$ is an intersection of prime ideals implies:

Remark 3.7. Given any MV-algebra A and an element $a \in A$, we have $Z_A = W_a \iff a = 0$. Given any two elements $a,\ b \in A$, $W_a \cup W_b = W_{a \wedge b}$ (see 1.3(8)). Thus $Z_A = W_a \cup W_b \iff a \wedge b = 0$). $\square$

We state now the following lemma, that will be proved in Section 8:

Lemma 3.8 (*Compactness Lemma*). *Given any MV-algebra A, the spectral space* Z_A *is sober, compact, and has a base of compact open sets.* $\square$

In what follows we need only the compactness of the space Z_A. We analyze now the surjectivity of η. With reference to (2.3), a global section is determined by a compatible family of sections $\widehat{a_i}$ on an open cover W_{b_i}, which, can be assumed finite. As a consequence, the surjectivity of η is equivalent to the following property:

3.9. *For any finite number of elements* $a_1, \ldots, a_n \in A$ *with zero intersection,* $a_1, \wedge \cdots \wedge a_n = 0$, *we have: Given* $b_1, \ldots, b_n \in A$ *such that* $(\widehat{b_i} = \widehat{b_j})|_{Z(\widehat{a_i}) \cap Z(\widehat{a_j})}$, *there exist* $b \in A$ *such that* $(\widehat{b} = \widehat{b_i})|_{Z(\widehat{a_i})}$. *By* 3.4 *this b is unique.*

In turn, 3.9 follows by induction from the following property:

3.10. *For any two elements a_1, $a_2 \in A$, we have: Given b_1, $b_2 \in A$ such that $(\widehat{b_1} = \widehat{b_2})|_{Z(\widehat{a_1}) \cap Z(\widehat{a_2})}$, there exists $b \in A$, unique upon restriction to $Z(\widehat{a_1}) \cup Z(\widehat{a_2})$, such that such that $(\widehat{b} = \widehat{b_1})|_{Z(\widehat{a_1})}$ and $(\widehat{b} = \widehat{b_2})|_{Z(\widehat{a_2})}$. In other words, the following diagram is a pullback square:*

$$\begin{array}{ccc}
\widehat{A}|_{Z(\widehat{a_1}) \cup Z(\widehat{a_2})} & \longrightarrow & \widehat{A}|_{Z(\widehat{a_1})} \\
\downarrow & & \downarrow \\
\widehat{A}|_{Z(\widehat{a_2})} & \longrightarrow & \widehat{A}|_{Z(\widehat{a_1}) \cap Z(\widehat{a_2})}
\end{array}$$

Combining Proposition 3.6 with the observation that $Z(\widehat{a_1} \vee \widehat{a_2}) = Z(\widehat{a_1}) \cap Z(\widehat{a_2})$ and $Z(\widehat{a_1} \wedge \widehat{a_2}) = Z(\widehat{a_1}) \cup Z(\widehat{a_2})$, 3.10 has the following equivalent reformulation, that will be proved in Section 7:

Lemma 3.11 (*Pushout–Pullback Lemma*). *Given an MV-algebra A and elements a_1, $a_2 \in A$, the following pushout diagram is also a pullback diagram.*

$$\begin{array}{ccc}
A/(a_1 \wedge a_2) & \longrightarrow & A/(a_1) \\
\downarrow & & \downarrow \\
A/(a_2) & \longrightarrow & A/(a_1 \vee a_2)
\end{array}$$

Note that the diagram above is always a pushout square, because, by 1.2(4), $(a_1 \vee a_2)$ is the supremum of (a_1) and (a_2) in the lattice of congruences of A. □

We have then the following theorem:

Theorem 3.12 (*Representation Theorem*). *Given any MV-algebra A, the morphism*

$$\eta : A \longrightarrow \Gamma\mathsf{Spec}_A = \Gamma(Z_A, E_A), \quad a \mapsto \widehat{a}, \quad \widehat{a}(P) = [a]_P$$

(see 3.2) is an isomorphism.

Proof. Follows from Lemma 3.8 (compactness) and 3.11 (pushout–pullback) as indicated above. □

3.13. Related constructions in the theory of l-groups

Taking into account the equivalence between the categories of MV-algebras and of lattice ordered abelian groups with a distinguished strong unit, any result on MV-algebras has a counterpart as a result on these groups, and conversely.

We will say, for short, *l-group* for lattice ordered abelian group, and *ul-group* for lattice ordered abelian group with a distinguished strong unit. Morphisms of *ul*-groups are *l*-group morphisms preserving the distinguished unit, thus *ul*-groups are not a full subcategory.

A proof of an statement on *ul*-groups does not furnish a proof of the corresponding statement on MV-algebras, neither guaranties its validity. And conversely. Statements should be examined in each case under the explicit definition and special properties of Mundici's functors Γ and Ξ ([1, chapter 7]) which establish the equivalence of the categories (see for example 4.10).

There are many general representations of lattice ordered groups and rings as sections of sheaves. To make an explicit comparison we focus our attention in the traditional construction of [10], and the newer one of [17]. By direct inspection and using the ideal correspondence of Cignoli–Torrens, [1, 7.2.2, 7.2.3 and 7.2.4], we can examine their constructions dealing directly with their translation to MV-algebras.

In [10] it is considered the set of all prime ideals, but endowed this time with the *Zariski topology*. A base for this topology is given by the sets $D_a = \{P \mid a \notin P\}$. We will denote this space by K_A. Let $G_A \xrightarrow{\pi} K_A$ be any sheaf of MV-algebras defined over K_A, such that the stalks G_P are quotient MV-algebras $A \to G_P$, with kernel denoted $\circ_P$, $G_P = A/\circ_P$. If this sheaf defines a representation of A as an MV-algebra of global sections by means of the morphism $\eta : A \to \Gamma(K_A, G_A)$, $a \mapsto \widehat{a}$, where $\widehat{a}$ is the section defined by $\widehat{a}(P) = [a]_{\circ_P}$, then for any $a \in A$ the section $\widehat{a}$ must be continuous. It follows (see 1.5(6)) that the set $H_a = [[\widehat{a} = 0]] = \{P \mid a \in \circ_P\}$ must be a Zariski open set. This is not so when $\circ_P = P$, in which case $H_a = W_a$ is precisely a co-Zariski open set. Thus, *the Zariski topology forces to abandon the requirement that the stalks be MV-chains.*

The Zariski openness of the set H_a is achieved by defining an ideal $\circ_P = \{a \mid \exists b \notin P, \mid \forall Q : (b \notin Q \Rightarrow a \in Q)\}$. That is, $\circ_P \subset P$ is the ideal of all the $a \in A$ such that the section $Z_A \xrightarrow{\widehat{a}} E_A$ of the spectrum Spec_A (as defined in 2.1) not only is null at P, but it has null Zariski germ at P, $\circ_P = \{a \mid \exists b \notin P \mid \widehat{a}|_{D_b} = 0\}$ (note that tautologically if $\widehat{a}(P) = 0$, then $\widehat{a}$ has null co-Zariski germ since $\widehat{a}|_{W_a} = 0$). The reader can check that in this way the sets H_a are Zariski open, in fact, they are the Zariski interior of the co-Zariski open sets W_a.

In [10, 3.11] it is established that the *l*-group counterpart of the morphism $\eta : A \to \Gamma(K_A, G_A)$, $a \mapsto \widehat{a}$, is an isomorphism. The stalks are not totally ordered. These results for MV-algebras are explicitly worked out in [5].

When A is a hyperarchimedean MV-algebra (see Section 5), every prime ideal is maximal, and the sets W_a are Zariski open (4.2). Thus if $\widehat{a}$ is null at P, W_a is a Zariski open neighborhood of P, and since by definition $\widehat{a}|_{W_a} = 0$, $\widehat{a}$ has null germ at P. Thus, $\circ_P = P$. Our construction coincides in this case with the translation of its *l*-group counterpart in [10].

In [17] it is considered the set of prime ideals of an *l*-group G, denoted $Sp(G)$, endowed, like here, with the co-Zariski topology. It is constructed a sheaf of totally ordered *l*-groups, denoted $(Sp(G), \tilde{G})$, and in Proposition 5.1.2 it is established that the morphism $G \longrightarrow \Gamma(Sp(G), \tilde{G})$ is an isomorphism. The equivalence between this statement in the case of *ul*-groups, and our statement 3.12 on MV-algebras establishing that $A \longrightarrow \Gamma(Z_A, E_A)$ is an isomorphism is not immediate. It requires an inspection on the behavior of Mundici's functors Γ and Ξ with respect to the functors Spec and Γ of Section 3, that will be done elsewhere.

4. Maximal and [0, 1]-valued morphism spectra

THE MAXIMAL SPECTRUM Spec_A^M.
We consider now maximal ideals. Recall that maximal ideals are prime.

4.1. Construction of Spec_A^M

The base space M_A is defined to be the subspace $M_A \subset Z_A$ determined by the maximal ideals. Recall then that a base for the topology is given by the sets $W_a^M = W_a|_{M_A} = \{M \mid M \ni a\}$. The etale space $E_A^M \to M_A$, $E_A^M \subset E_A$, is the restriction of E_A to M_A. $\square$

A salient feature of the maximal spectral space is the following:

Proposition 4.2. *For any MV-algebra A, the sets W_a^M are closed (thus clopen) subsets of M_A.*

Proof. We shall show that $M_A \backslash W_a^M$ is open. Let $P \in M_A$, $P \notin W_a^M$. By 1.4(2), take an integer n such that $\neg\, na \in P$. Then $P \in W_{\neg\, na}^M$. To finish the proof we have to show that $W_a^M \cap W_{\neg\, na}^M = \emptyset$. We do as follows: Let $Q \in M_A$ be such that $\neg\, na \in Q$ and $a \in Q$. Then $na \in Q$, thus $\neg\, na \oplus na \in Q$, so $1 \in Q$ (see 1.3(8)), but Q is proper. $\square$

Corollary 4.3. *Given any MV-algebra A, the maximal spectrum $\mathrm{Spec}_A^M = E_A^M \to M_A$ is a Hausdorff sheaf of simple MV-algebras.*

Proof. Let P, $Q \in M_A$, $P \neq Q$. Take $a \in P$, $a \notin Q$. Then W_a^M and $M_A \backslash W_a^M$ separate P and Q. This shows that M_A is Hausdorff.

Now let $([a]_P, P) \neq ([b]_Q, Q)$ in E_A^M (see 2.1). If $P \neq Q$, take $U \ni P$ and $V \ni Q$ be disjoint open sets in M_A. Then $\widehat{a}(U) \ni ([a]_P, P)$ and $\widehat{b}(V) \ni ([b]_Q, Q)$ are disjoint open sets in E_A. If $P = Q$, then $[a]_P \neq [b]_P$. Consider the closed set in M_A, $W_{d(a,b)}^M = [[\widehat{a} = \widehat{b}]]$ (see 2.2). Notice that $P \notin W_{d(a,b)}^M$. Then $\widehat{a}(M_A \backslash W_{d(a,b)}^M) \ni ([a]_P, P)$ and $\widehat{b}(M_A \backslash W_{d(a,b)}^M) \ni ([b]_P, P)$ are disjoint open sets in E_A^M. This completes the proof that E_A^M is Hausdorff.

Finally, by 1.4(4), we know that the fibers are simple MV-algebras. $\square$

Given a prime ideal P, there exists a maximal ideal $M \supset P$, 1.4(1). Thus, for any basic open set W_a with $P \in W_a$ we have $M \in W_a$. As a consequence:

Observation 4.4. *The spectral space Z_A is never Hausdorff unless it is equal to M_A.* $\square$

Observation 4.5. *The inclusion $M_A \subset Z_A$ is dense. Namely, the prime spectral space Z_A is the closure of the maximal spectral space M_A.* $\square$

We are now concerned with the injectivity of the morphism $A \stackrel{\eta}{\longrightarrow} \Gamma(M_A, E_A^M)$, $a \mapsto \widehat{a}$, $\widehat{a}(M) = [a]_M$. While Proposition 3.4 holds for all MV-algebras, we must now restrict to *semisimple* MV-algebras, those MV-algebras A such that the intersection of all maximal ideals M is the zero ideal, in symbols, $(0) = \bigcap_{M \in M_A} M$. We then have:

Proposition 4.6. *A MV-algebra is semisimple if and only if the morphism $A \stackrel{\eta}{\longrightarrow} \Gamma(M_A, E_A^M)$ is injective.* $\square$

By 1.4(3), for any $a \in A$, the maximal ideals of the quotient algebra $A/(a)$ are in one to one correspondence with the maximal ideals belonging to the set W_a^M, in symbols, $M_{A/(a)} = W_a^M$. The counterpart of Proposition 3.6 does not hold in general for semisimple algebras, and the following stronger condition is necessary.

Definition 4.7. A MV-algebra A is *strongly semisimple* if for any $a \in A$, the intersection of all maximal ideals M containing a is the ideal generated by a, in symbols, $(a) = \bigcap_{M \ni a} M$. Equivalently, by 1.4(3), iff A is semisimple together with all its principal quotients $A/(a)$.

Clearly if A is strongly semisimple, so are all its quotients $A/(a)$. Hyperarchimedean algebras (Section 5) are strongly semisimple. As we shall see, finitely presented algebras (Section 6) provide examples of non-hyperarchimedean strongly semisimple algebras. In particular, free algebras are strongly semisimple but not hyperarchimedean. The example following Corollary 3.4.4 in [1] shows that the semisimple MV-algebra $Cont([0, 1], [0, 1])$ is not strongly semisimple. The following is immediate:

Proposition 4.8. *A MV-algebra A is strongly semisimple if and only if given elements a, b, c in A,*

$$\widehat{b}|_{W_a^M} = \widehat{c}|_{W_a^M} \iff [b]_{(a)} = [c]_{(a)}, \quad \text{that is } A/(a) \cong \widehat{A}|_{W_a^M}. \quad \square$$

The reader will have no difficulty in proving the following:

Corollary 4.9. *Let A be a strongly semisimple MV-algebra. Then the restriction morphism* $\Gamma(Z_A, E_A) \xrightarrow{\ \rho\ } \Gamma(M_A, E_A^M)$ *is injective.* $\square$

Notice that from the representation Theorem 3.12 it immediately follows that the statement in this corollary holds for any semisimple algebra. However, it is worth to pay the price of the stronger hypothesis to have a proof independent of the validity of the representation theorem. Interesting applications of this corollary are given by Proposition 4.21 and Theorems 9.9 and 9.10.

4.10. Strong semisimplicity and Yosida frames

It is easy to see that a MV-algebra is strongly semisimple if and only if the lattice of ideals of A is a Yosida frame in the sense of [13]. By means of the categorical equivalence between MV-algebras and *ul*-groups discussed in 3.13, and using the Cignoli–Torrence ideal correspondence [1, 7.2.2, 7.2.3 and 7.2.4], we can translate statements on the lattice of *l*-ideals of a *l*-group into statements on the lattice of ideals of a MV-algebra, and conversely. This works well for hyperarchimedean objects since a MV-algebra is hyperarchimedean if and only if its corresponding *ul*-group is hyperarchimedean [17, 4.3.13].
Our results show that the lattice of l-ideals of a hyperarchimedean l-group is a Yosida frame.

While the free *l*-group on n generators is a *ul*-group ([14, Lemma 14]), its corresponding MV-algebra is not a free MV-algebra, and for $n > 1$ its strong semisimplicity is not established. Thus, while our results show that the lattice of ideals of the free MV-algebra on n generators is a Yosida frame, it does not follow the same result for the lattice of *l*-ideas of the free *l*-group on n generators. This last fact is true and easy for $n = 1$, true and hard to prove for $n = 2$, and not known for $n > 2$, [13, 5.6].
THE SPECTRAL SPACE OF $[0, 1]$-VALUED MORPHISMS.
We consider now a different construction, that we denote X_A, of a topological space associated to a MV-algebra A. This construction is more akin to functional analysis than to algebraic geometry, and has been considered in particular in [2].

4.11. Construction of X_A

The points of X_A are all the $[0, 1]$-valued morphisms, $X_A = [A, [0, 1]] = \{\chi : A \longrightarrow [0, 1]\} \subset [0, 1]^A$. The topology is the subspace topology inherited from the product space, with the unit interval endowed with the usual topology. A subbase for this topology is given by the sets $W_{a, U}^X = \{\chi \mid \chi(a) \in U\}$, for $a \in A$ and U an open set, $U \subset [0, 1]$. $\square$

Remark 4.12. Given any $x \in [0, 1]$, $W_{a, x}^X = \{\chi \mid \chi(a) = x\}$ is a closed set of X_A. When $x = 0$ we write $W_a^X = \{\chi \mid \chi(a) = 0\}$. $\square$

Since X_A is closed in $[0, 1]^A$, X_A is a compact Hausdorff space.
From 1.4(5) it follows:

Proposition 4.13. *Maximal ideals M are in bijection with morphisms $A \xrightarrow{\chi} [0, 1]$. If M corresponds to χ,*
$A/M \xrightarrow{\cong} \chi(A) \subset [0, 1]$, *and* $a \in M \iff \chi(a) = 0$. $\square$

We denote $X_A \xrightarrow{\kappa} Z_A$ the injection defined by $\kappa(\chi) = ker(\chi)$. Its image is the subspace $M_A \subset Z_A$ of maximal ideals. By abuse of notation we will denote by χ the inverse map $\chi = \kappa^{-1} : M_A \longrightarrow X_A$, and write $\chi(M) = \chi_M$.

Each element $a \in A$ defines a continuous function $X_A \xrightarrow{\widehat{a}} [0, 1]$, $\widehat{a}(\chi) = \chi(a)$. Thus, $W_{a, U}^X = \widehat{a}^{-1}U$. By definition, *the topology in X_A is the initial topology with respect to the functions $\widehat{a}$, $a \in A$.*
For any topological space X we denote by $Cont(X, [0, 1])$, the MV-algebra of all $[0, 1]$-valued continuous functions on X. Recall that a subalgebra $A \subset Con(X, [0, 1])$ is said to be *separating* if given any two points $\chi \neq \xi$ in X, there is $f \in A$ such that $f(\chi) = 0$ and $f(\xi) > 0$.
If two $[0, 1]$-valued morphisms are different, by Proposition 4.13 their kernels must be different. It follows:

Proposition 4.14. *Given any MV-algebra A, the functions $X_A \to [0, 1]$ of the form $\widehat{a}$, $a \in A$, form a separating subalgebra of the MV-algebra $Cont(X_A, [0, 1])$. That is, given any two points $\chi \neq \xi$ in X_A, there is $a \in A$ such that $\chi(a) = 0$ and $\xi(a) > 0$.* $\square$

For any $a \in A$ we will write $W_a^X = \{\chi \mid \chi(a) = 0\} = X_A \backslash W_{a, (0, 1]}^X$. With this notation we have $\kappa(W_a^X) = W_a^M$ and $\chi(W_a^M) = W_a^X$.

Proposition 4.15. *Given any MV-algebra A, the sets $W_{a, (0, 1]}^X == \{\chi \mid \widehat{a}(\chi) > 0\} = \{\chi \mid \widehat{a}(\chi) \neq 0\}$ form a base for the topology of X_A.*

Proof. Since X_A is a compact Hausdorff space, the result follows from Proposition 4.14 and ([1, Remark to Theorem 3.4.3]). $\square$

Thus, the open base $W^X_{a,(0,1]}$ consists of the complements of the zerosets of the functions $\hat{a}$.

Under the bijection $M_A \cong X_A$ determined by (κ, χ), we have $\kappa(W^X_{a,(0,1]}) = \{M \in M_A \mid a \notin M\}$. So the topology of X_A corresponds to the *Zariski* topology in M_A, while the topology of M_A was defined to be the *co-Zariski* topology.

From Proposition 4.2 it follows:

Proposition 4.16. *For any MV-algebra A, the bijection $M_A \xrightarrow{\chi} X_A$ is continuous. That is, in the set M_A of maximal ideals, the co-Zariski topology is finer than the Zariski topology.* □

The maximal spectral space in not a compact space in general. We have

Remark 4.17. *M_A is compact if and only if it is homeomorphic to X_A via the bijection χ.*

Proof. One implication is clear. For the other, assume M_A to be compact. Then the continuous map $M_A \xrightarrow{\chi} X_A$ is also a closed map (thus a homeomorphism) because X_A is Hausdorff. □

In general the co-Zariski topology will be strictly finer than the Zariski topology. As will be seen in Section 5, they coincide if and only if the MV-algebra is *hyperarchimedean*.

Proposition 4.18. *The injection $X_A \xrightarrow{\kappa} Z_A$ is continuous if and only if the sets $W^X_a = \{\chi \mid \chi(a) = 0\}$ are open (thus clopen) sets in X_A. When this is the case, κ and χ establish a homeomorphism $X_A \xrightarrow{\cong} M_A$.*

Proof. $W^X_a = \kappa^{-1}W_A$, which are an open base of Z_A, thus κ is continuous if and only if the sets W^X_a are open in X_A. In this case then, the continuous bijection χ of Proposition 4.16 has a continuous inverse. The proof also can be completed without using this proposition. In fact, if κ is continuous, it is a closed map as a map $X_A \xrightarrow{\kappa} M_A$, because X_A is compact and M_A is Hausdorff. □

The MV-algebra of global sections of $\mathrm{Spec}^M_A = E^M_A \to M_A$ is related with the MV-algebra of continuous functions $Cont(M_A, [0,1])$. Define a map $E^M_A \xrightarrow{\lambda} [0,1]$ by writing $\lambda([a]_M, M) = \chi_M(a))$ for each $([a]_M, M) \in E_M$. Given any $a \in A$, observe that the following diagram is commutative:

$$
\begin{array}{ccc}
E^M_A & \xrightarrow{\lambda} & [0,1] \\
\hat{a} \uparrow & & \uparrow \hat{a} \\
M_A & \xrightarrow{\chi} & X_A
\end{array}
$$

Since the topology of E^M_A is the final topology with respect to the functions $\hat{a}(M) = ([a]_M, M)$, and the topology of X_A is the initial topology with respect to the functions $\hat{a}(\chi) = \chi(a)$, it follows that λ *is continuous if and only if so is χ*. By Proposition 4.16, λ is continuous. Summing up:

Proposition 4.19. *Given any MV-algebra A, the map $E^M_A \xrightarrow{\lambda} [0,1]$ defined by $\lambda([a]_M, M) = \chi_M(a)$ is continuous and establishes:*

(a) *A continuous injection $E^M_A \hookrightarrow [0,1] \times M_A$ over M_A, where $[0,1]$ is endowed with the usual topology.*

(b) *By composition, an injective morphism $\Gamma\mathrm{Spec}^M_A = \Gamma(M_A, E^M_A) \xrightarrow{\lambda_*} Cont(M_A, [0,1])$, $\lambda_*(\hat{a})(M) = \chi_M(a)$.*
   Furthermore, this morphism establishes an isomorphism of $\Gamma\mathrm{Spec}^M_A$ onto a separating subalgebra of $Cont(M_A, [0,1])$.

Proof. As indicated above, the function λ is continuous by Proposition 4.16. Then, Proposition 4.14 completes the proof. A more general result is given in Proposition 5.9. □

Consider now a global section $M_A \xrightarrow{\sigma} E^M_A$ which is the restriction of a global section of the prime spectrum $Z_A \supset M_A$. Since by Corollary 8.9 Z_A is a compact topological space, there is a finite open cover $W^M_{a_i} \subset M_A$ over which σ is of the form $\hat{b_i}$ for some $b_i \in A$. Thus the composite $X_A \xrightarrow{\kappa} M_A \xrightarrow{\sigma} E^M_A$ is of the form $\hat{b_i}$ over a finite cover of X_A by the closed sets $W^X_{a_i}$ (Remark 4.12). It follows that the composite $X_A \xrightarrow{\sigma\kappa} E^M_A$ is continuous, and so is the composite $X_A \xrightarrow{\lambda\sigma\kappa} [0,1]$. We then have a morphism

$$\Gamma(Z_A, E_A) \xrightarrow{\kappa^*\lambda_*} Cont(X_A, [0,1]) \qquad \kappa^*\lambda_*(\sigma) = \lambda\sigma\kappa, \quad \lambda\hat{a}\kappa(\chi) = \chi(a) \qquad (4.20)$$

rendering commutative the following diagram (recall $\kappa\chi = id$):

$$
\begin{array}{ccc}
\Gamma(Z_A, E_A) & \xrightarrow{\kappa^*\lambda_*} & Cont(X_A, [0,1]) \\
\downarrow{\rho} & & \downarrow{\chi^*} \\
\Gamma(M_A, E^M_A) & \xhookrightarrow{\lambda_*} & Cont(M_A, [0,1])
\end{array}
$$

Recalling Corollary 4.9 we then have:

Proposition 4.21. *For every strongly semisimple MV-algebra A, the morphism in* (4.20) *above is injective.*

$$\Gamma(Z_A, E_A) \xrightarrow{\quad \kappa^* \lambda_* \quad} Cont(X_A, [0, 1]) . \quad \square$$

5. Hyperarchimedean algebras

Before proving the representation theorem for a general MV-algebra, it is instructive to consider *hyperarchimedean* MV-algebras [2,1]. In this section we prove the compactness and the pushout–pullback lemmas in this case.

For a hyperarchimedean MV-algebra A, every prime ideal P is maximal, the Zariski and co-Zariski topologies coincide, and the spectrum space Z_A is Hausdorff. Further, A/P is uniquely isomorphic to a subalgebra of the real unit interval $[0, 1]$. We are then in a position to apply the classical techniques used in [2] for a representation theorem of locally finite MV-algebras, and we will generalize this theorem to general hyperarchimedean MV-algebras.

Hyperarchimedean MV-algebras are strongly semisimple MV-algebras where not only the principal ideals are intersection of maximal ideals, but every ideal I is the intersection of all maximal ideals M containing I, in symbols, $I = \bigcap_{M \supset I}$. Equivalently, by 1.4(3), an MV-algebra A is hyperarchimedean iff A is semisimple together with all its quotients A/I.

Definition 5.1 ([1] 6.2.2). An element $a \in A$ in an MV-algebra is *archimedean* if there is an integer $n \geq 1$ such that $na = (n + 1)a$. A MV-algebra A is *hyperarchimedean* if every $a \in A$ is archimedean.

Proposition 5.2 ([1] 6.3.2). *A MV-algebra is hyperarchimedean if and only if every prime ideal is maximal. A MV-chain is hyperarchimedean if and only if is a subalgebra of the real unit interval* $[0, 1]$. $\square$

Remark 5.3. The real unit interval 1.1(11) is a hyperarchimedean MV-algebra, and for $x \in [0, 1]$, $nx = (n + 1)x$ if and only if $x \geq 1/n$. $\square$

Proposition 5.4. *Given an archimedean element $a \in A$ in an MV-algebra A, the set $W_a^X = \{\chi \mid \chi(a) = 0\}$ is open in X_A, thus a clopen set.*

Proof. We take from [2, 4.5] the following argument: Suppose that $a \in A$ is hyperarchimedean. Take an integer $n \geq 1$ such that $na = (n + 1)a$, whence $X_A \setminus W_a^X = \{\chi \mid \chi(a) > 0\} = \{\chi \mid \chi(a) \geq 1/n\}$ (Remark 5.3), which is closed, thus W_a^X is open. $\square$

Observation 5.5. Given a semisimple MV-algebra A, and $a \in A$, if $\chi(a)$ is archimedean with *the same n* for all $\chi \in X_A$, then a is archimedean. Actually, it is enough that the assumption holds for all $\chi \notin W_a^X$. $\square$

For semisimple MV-algebras the converse of Proposition 5.4 is also valid.

Proposition 5.6. *Given a semisimple MV-algebra A, if the set W_a^X is open in X_A, then the element a is archimedean.*

Proof. Suppose that W_a^X is open. Then $X_A \setminus W_a^X$ is closed, thus compact. It follows that the set $\{\chi(a) \mid \chi \in X_A \setminus W_a^X\}$ is separated from 0. Let $n \geq 1$ be such that $X_A \setminus W_a^X = \{\chi \mid \chi(a) \geq 1/n\}$. Then for all χ in $X_A \setminus W_a^X$, $n\chi(a) = (n + 1)\chi(a)$. The result follows by 5.5. $\square$

Recall that a *Stone* space (also called *boolean* space) is a totally disconnected compact Hausdorff space, or, equivalently, a compact Hausdorff space with an open base of clopen sets.

We list now a series of conditions that characterize hyperarchimedean MV-algebras.

Theorem 5.7.

(a) *The following conditions in an arbitrary MV-algebra A are equivalent:*
1. *A is hyperarchimedean.*
2. *$Z_A = M_A$ (that is, every prime ideal is maximal).*
3. *The prime spectral space Z_A is Hausdorff.*
4. *The prime spectrum* $\mathsf{Spec}_A = (E_A \to Z_A)$ *is a Hausdorff sheaf.*
5. *For all $a \in A$, the sets $W_a \subset Z_A$ are closed (thus clopen) in Z_A.*
6. *The map $\kappa : X_A \xrightarrow{\cong} Z_A$ is an homeomorphism (in particular X_A is homeomorphic to M_A, $\kappa : X_A \xrightarrow{\cong} M_A$).*
7. *The prime spectral space Z_A is a Stone space.*

(b) *The following conditions in a semisimple MV-algebra A are equivalent:*
1. *A is hyperarchimedean.*
8. *For all $a \in A$, the sets $W_a^X \subset X_A$ are open (thus clopen) in X_A.*
9. *The map $\kappa : X_A \longrightarrow Z_A$ is continuous.*
10. *The maximal spectral space M_A is compact.*
11. *The map $\kappa : X_A \longrightarrow M_A$ is a homeomorphism.*

Proof. (a) (1) $\Longleftrightarrow$ (2): By 5.2.

(2) $\Longleftrightarrow$ (3) and (2) $\Longleftrightarrow$ ((4)): By 4.3, 4.4.

(2) $\Rightarrow$ (5): By 4.2.

(5) $\Rightarrow$ (3): Let P, $Q \in Z_A$, $P \neq Q$. Take $a \in P$, $a \notin Q$. Then, W_a and $Z_A \backslash W_a$ are open sets that separate P and Q.

(1) $\Rightarrow$ (6): By 4.18, 5.4.

(6) $\Rightarrow$ (7): Recall that X_A is a compact Hausdorff space.

(7) $\Rightarrow$ (3): A Stone space, in particular, is Hausdorff.

(b) (1) $\Rightarrow$ (8): By 5.4.

(8) $\Rightarrow$ (1): By 5.6.

(8) $\Longleftrightarrow$ (9): Notice that $W_a^X = \kappa^{-1} W_a$.

(9) $\Rightarrow$ (11): By 4.18.

(11) $\Rightarrow$ (9): Since the inclusion $M_A \subset Z_A$ is continuous.

(11) $\Longleftrightarrow$ (10): By 4.17. □

From Theorem 5.7 it clearly follows:

Theorem 5.8. *A MV-algebra A is hyperarchimedean if and only if the spectrum sheaf* Spec_A *is a Hausdorff sheaf of simple MV-algebras over a Stone space.* □

Let $\mathsf{E} = (E \rightarrow X)$ be any Hausdorff sheaf of simple MV-algebras. Consider the MV-algebra of global sections $\Gamma \mathsf{E} = \Gamma(X, E)$. The following proposition is essentially proved in [2, Section 6]:

Proposition 5.9. *For each $\sigma \in \Gamma(X, E)$ define $f_\sigma(x) = \lambda_x(\sigma(x))$, where λ_x is the unique embedding $E_x \hookrightarrow [0, 1]$ (1.4(5)). The assignment $\sigma \mapsto f_\sigma$ defines a embedding $\Gamma(X, E) \hookrightarrow Con(X, [0, 1])$ into a hyperarchimedean subalgebra S of $Con(X, [0, 1])$. S is separating if X has a base of clopen sets.*

Proof. In[2, 6.4] it is proved that the continuity of f_σ follows from the Hausdorff property of E. Since the zeroset $Z(f_\sigma) = [[\sigma = 0]] \subset X$ is open (1.5(6)), it follows, by [2, 4.5], that each f_σ is an archimedean element. Finally the last assertion follows because the characteristic function (as a section) of any clopen set is a continuous section. □

A global Hausdorff sheaf of simple MV-algebras over a Stone space is completely determined by its algebra of global sections.

Theorem 5.10. *Suppose we are given an MV-space (X, E) with E a global Hausdorff sheaf of simple MV-algebras, and X a Stone space. Then the unit of the adjunction*

$$(h, \varepsilon) : (X, E) \longrightarrow \mathsf{Spec}\,\Gamma(X, E) = (Z_{\Gamma(X, E)}, E_{\Gamma(X, E)})$$

given by Proposition 3.3 is an isomorphism of MV-spaces.

Proof. We refer to Proposition 3.3. Consider the map $X \xrightarrow{h} Z_{\Gamma(X, E)}$, $h(x) = \{\sigma \mid \sigma(x) = 0\}$. By Proposition 5.9 $Z_{\Gamma(X, E)} = M_{\Gamma(X, E)}$. The ideal $h(x)$ corresponds to the ideal $\{f_\sigma \mid f_\sigma(x) = 0\}$. Then, [1, Theorem 3.4.3 (i)] shows that h is a bijection. Since X is compact, $M_{\Gamma(X, E)}$ is Hausdorff and h is continuous, then h is a homeomorphism. Finally, the morphism ε_x is injective since the stalks $\Gamma(X, E)/h(x)$ are simple MV-chains, and surjective by definition of global sheaf. □

From Proposition 5.9 and Theorem 5.10 we immediately obtain the following companion of Theorem 5.8.

Theorem 5.11. *A sheaf $\mathsf{E} = (E \rightarrow X)$ is a global Hausdorff sheaf of simple MV-algebras over a Stone space if and only if the MV-algebra of global sections $\Gamma \mathsf{E} = \Gamma(X, E)$ is hyperarchimedean.* □

THE PUSHOUT–PULLBACK LEMMA FOR HYPERARCHIMEDEAN ALGEBRAS.

We prove now the pushout–pullback Lemma 3.11 for hyperarchimedean algebras. Recall that in this case $Z_A = M_A$, and that by Proposition 4.19 (b) we have an injective morphism $\widehat{A} \overset{\lambda_*}{\hookrightarrow} Cont(M_A, [0, 1])$. With reference to 3.10 we now prove the lemma in the following form:

Lemma 5.12. *Given any hyperarchimedean algebra A, and any two elements a_1, $a_2 \in A$, the following diagram of $[0, 1]$-valued functions is a pullback.*

$$
\begin{array}{ccc}
\lambda_*(\widehat{A})|_{W_{a_1} \cup W_{a_2}} & \longrightarrow & \lambda_*(\widehat{A})|_{W_{a_1}} \\
\downarrow & & \downarrow \\
\lambda_*(\widehat{A})|_{W_{a_2}} & \longrightarrow & \lambda_*(\widehat{A})|_{W_{a_1} \cap W_{a_2}}
\end{array}
$$

Proof. Let b_1, $b_2 \in A$ be such that $(\lambda_* \widehat{b_1})|_{W_{a_1}}$, $(\lambda_* \widehat{b_2})|_{W_{a_2}}$ are compatible in the intersection $W_{a_1} \cap W_{a_2}$. We have to show that there exists $b \in A$, unique upon restriction of $\lambda_* \widehat{b}$ to $W_{a_1} \cup W_{a_2}$, such that $(\lambda_* \widehat{b})|_{W_{a_1}} = (\lambda_* \widehat{b_1})|_{W_{a_1}}$ and $(\lambda_* \widehat{b})|_{W_{a_2}} = (\lambda_* \widehat{b_2})|_{W_{a_2}}$.

Take an integer $n \geq 1$ such that $na_1 = (n+1)a_1$, $na_2 = (n+1)a_2$. Then

$$b = (b_1 \wedge b_2) \vee (na_1 \wedge b_2) \vee (na_2 \wedge b_1) \tag{5.13}$$

is the required element.

To check this first we simplify notation: Let $f_1 = \lambda_* \widehat{a_1}, f_2 = \lambda_* \widehat{a_2}, g_1 = \lambda_* \widehat{b_1}, g_2 = \lambda_* \widehat{b_2}$, and $g = \lambda_* \widehat{b}$. Since η and λ_* are morphisms, we have $nf_1 = (n+1)f_1$, $nf_2 = (n+1)f_2$ and

$$g = (g_1 \wedge g_2) \vee (nf_1 \wedge g_2) \vee (nf_2 \wedge g_1).$$

Also note that $W_{a_1} = Z(f_1)$ and $W_{a_2} = Z(f_2)$.

Let x be any element of M_A. Then, $x \in Z(f_1)$ or $x \in Z(f_2)$. Suppose $x \in Z(f_1)$, thus $f_1(x) = 0$. Consider two cases:

$x \notin Z(f_2)$, $f_2(x) > 0$, thus $nf_2(x) = 1$. Then: $g(x) = (g_1(x) \wedge g_2(x)) \vee (g_1(x) = g_1(x))$.

$x \in Z(f_2)$, $f_2(x) = 0$ and $g_1(x) = g_2(x)$. Then: $g(x) = g_1(x) \wedge g_2(x) = g_1(x)$.

For $x \in Z(f_2)$ we proceed in the same way.

Direct inspection shows that this proof works equally well with the following formula for the element b:

$$b = (b_1 \odot \neg na_1) \vee (b_2 \odot \neg na_2). \quad \square \tag{5.14}$$

Combining Theorem 5.7(7) and Lemma 5.12 we have (see Theorem 3.12):

Theorem 5.15. *Given any hyperarchimedean MV-algebra A, the unit of the adjunction (see 3.2):*

$$\eta : A \xrightarrow{\eta} \Gamma \mathrm{Spec}_A = \Gamma(Z_A, E_A)$$

is an isomorphism of MV-algebras. $\square$

Algebras of global sections of global Hausdorff sheaves over Stone spaces are known as *boolean products*. Observe that this Theorem, together with Theorem 5.7(7) yields the characterization of hyperarchimedean MV-algebras as boolean products of simple MV-algebras, [1, 6.5.6].

Combining Theorems 5.10 and 5.15, with reference to Section 3, we have:

Theorem 5.16. *The functors $\mathcal{A}^{op} \xrightarrow{\mathrm{Spec}} \mathcal{E}$ and $\mathcal{E}^{op} \xrightarrow{\Gamma} \mathcal{A}$ establish a contravariant equivalence between the category of hyperarchimedean MV-algebras and the category of MV-spaces which are global Hausdorff sheaves over Stone spaces.* $\square$

In view of Theorem 5.7(6), this result extends to all hyperarchimedean MV-algebras the representation theorem originally proved in [2, Theorem 6.9.] for the subclass of locally finite MV-algebras.

6. Finitely presented MV-algebras

In this section we prove the pushout–pullback lemma for *finitely presented* MV-algebras, that is, quotients of finitely generated free MV-algebras by finitely generated ideals.

Free MV-algebras. As is well known in universal algebra, the free n-generator MV-algebra $F_n = F[x_1, \ldots, x_n]$, is the quotient by an equivalence relation of the *set of terms* in the variables $x_1, \ldots, x_n$, [1, Section 1.4]. Two terms f, g are considered equal in F_n if the equation $f = g$ follows from the defining axioms of MV-algebras.

The universal property of free MV-algebras states that for any MV-algebra A, a term $f \in F_n$ determines by substitution and evaluation a function $A^n \xrightarrow{f_A} A$. A n-tuple $(a_1, \ldots, a_n)$ determines uniquely a morphism $F_n \xrightarrow{\varphi} A$ by defining $\varphi(f) = f_A(a_1, \ldots, a_n)$. This defines an inverse function to the assignment $\varphi \mapsto (\varphi(x_1), \ldots, \varphi(x_n))$. There is a bijection

$$\ell : [F_n, A] \xrightarrow{\cong} A^n, \quad \ell^{-1}(a_1, \ldots, a_n)(f) = f_A(a_1, \ldots, a_n). \tag{6.1}$$

The functions of the form f_A are called *term functions on A*, and the assignment $f \mapsto f_A$ is a morphism $F_n \xrightarrow{\delta} A^{A^n}$, where the exponential notation stands for the MV-algebra of all functions $A^n \to A$ with the pointwise structure. The term functions corresponding to the variables are the projections $\delta x_i = x_{iA} = \pi_i : A^n \longrightarrow A$. The fact that an equation $f = g$ holds in a MV-algebra A means that the term functions f_A and g_A are equal. Clearly, for $A = F_n, f = f_{F_n}(x_1, x_2, \ldots, x_n)$, and F_n is isomorphic to the MV-algebra of term functions on F_n. When the morphism $F_n \xrightarrow{\delta} A^{A^n}$ is injective, it establishes an isomorphism between the free algebra and the algebra of term functions on A. Thus, the injectivity of the morphism δ amounts to saying that *A generates* the variety of MV-algebras, and its mathematical meaning is a completeness theorem with respect to the algebra A. In particular, this is the case for $A = [0, 1]$, and it is known as *Chang's completeness theorem*, [1, 2.5.3].

We shall abuse notation and write $f_{[0, 1]} = f$. The projections and the primitive operations of the MV-algebra $[0, 1]$ are continuous (for the usual topology), so all term functions on $[0, 1]$ are continuous functions. Thus we have an injective morphism

$$F_n \xrightarrow{\delta} Cont([0, 1]^n, [0, 1]) \subset [0, 1]^{[0, 1]^n}, \tag{6.2}$$

where continuity is with respect to the usual topology.

By definition the set of morphisms $[F_n, [0, 1]]$ is equal to the spectral space X_{F_n}. Thus, ℓ establishes a bijection $\ell : X_{F_n} \xrightarrow{\cong} [0, 1]^n$. Under this bijection the evaluations $\widehat{x}_i : X_{F_n} \longrightarrow [0, 1]$ correspond to the projections $\pi_i : [0, 1]^n \longrightarrow [0, 1]$ ($\pi_i \circ \ell = \widehat{x}_i$, $i = 1, 2, \ldots, n$) so that ℓ is continuous for the usual topology in $[0, 1]$. Since bothspaces are compact Hausdorff, we have:

Remark 6.3. The bijection $\ell : X_{F_n} \xrightarrow{\cong} [0, 1]^n$ establishes a homeomorphism of topological spaces. For any $f \in F_n$ and $p \in [0, 1]^n$, $\ell^{-1}(p)(f) = f(p)$. The evaluation map $X_{F_n} \xrightarrow{\widehat{f}} [0, 1]$ corresponds via ℓ to the term function $[0, 1]^n \xrightarrow{f} [0, 1]$. $\square$

Finitely presented MV-algebras. Since all finitely generated ideals are principal (1.2 (4)), a finitely presented MV-algebra is always of the form $R = F_n/(f)$, for some $f \in F_n$. A morphism φ as in (6.1) factors through the quotient if and only if $f_A(a_1, a_2, \ldots, a_n) = 0$. Thus, we have a commutative diagram:

$$\ell : [F_n, A] \xrightarrow{\cong} A^n \tag{6.4}$$
$$\cup \qquad\qquad \cup$$
$$\ell : [R, A] \xrightarrow{\cong} Z_{f_A}$$

where $Z(f_A) \subset A^n$ is the zeroset of the term function f_A.

When $A = [0, 1]$, we have $Z(f) \subset [0, 1]^n$.

Remark 6.5. The bijection ℓ in 6.3 restricts to a homeomorphism of topological spaces $\ell : X_{F_n/(f)} \xrightarrow{\cong} Z(f)$. $\square$

The injective arrow δ in ((6.2)) factorizes as follows:

$$\begin{array}{ccc} F_n & \xrightarrow{\delta} & Cont([0, 1]^n, [0, 1]) \\ \downarrow & & \downarrow \\ R & \xrightarrow{\overline{\delta}} & Cont(Z(f), [0, 1]) \end{array} \tag{6.6}$$

A key nontrivial result here is that the arrow $\overline{\delta}$ is also injective.

Proposition 6.7. Given any $f, g, h \in F_n$: $[g]_{(f)} = [h]_{(f)} \iff g|_{Z(f)} = h|_{Z(f)}$, that is $F_n/(f) \cong F_n|_{Z(f)}$.

Proof. The statement is equivalent to [1, lemma 3.4.8] which says: $g \in (f) \iff Z(f) \subset Z(g)$.

In fact, it reduces to the lemma when $h = 0$, and it follows from the lemma applied to the function $d(g, h)$, where d is the distance operation (1.1(7), see also 1.2(5)). $\square$

The reader should compare the following remark with [1, Theorem 3.6.9].

Remark 6.8. In view of Remark 6.5 the injectivity of $\overline{\delta}$ amounts to the semisimplicity of finitely presented MV-algebras. Thus finitely presented MV-algebras are strongly semisimple (Definition 4.7).

We can safely assume the following:

Convention 6.9. (1) *We will henceforth identify the free MV-algebra $F_n = F[x_1, x_2, \ldots, x_n]$ with the MV-algebra of term functions on $[0, 1]$. We then have $F_n \subset Cont([0, 1]^n, [0, 1])$.*
(2) *We similarly identify any finitely presented MV-algebra $R = F_n/(f)$ with the MV-algebra of term functions on $[0, 1]$ restricted to the subset $Z(f) \subset [0, 1]^n$. We then have $R = F_n|_{Z(f)} \subset Cont(Z(f), [0, 1])$.*

THE PUSHOUT–PULLBACK LEMMA FOR FINITELY PRESENTED MV-ALGEBRAS.

For the proof of this lemma we need the following well known result (first observed by McNaughton). It is related to the proof of [1, Lemma 3.4.8]. It can be proved as in [1, Proposition 3.3.1], see also [15, Lemmas 5.2 and 5.3].

Proposition 6.10. *Given any $f \in F_n$ and a finite set $H \subset F_n$, there exists a set of convex polyhedra $\{T_1, \ldots, T_m\}$ whose union coincides with $Z(f)$, and such that all the functions $h \in H$ are linear over each T_i.* $\square$

Given any finitely presented MV-algebra $R = F_n/(f)$, and $g \in F_n$, we have $R/([g]_f) = (F_n/(f))/([g]_f) = F_n/(f, g) = F_n/(f \vee g)$. It follows that the pushout–pullback Lemma 3.11 for finitely presented MV-algebras can be stated as follows: Given f_1, $f_2 \in F_n$, the following diagram is a pullback.

$$\begin{array}{ccc} F_n/(f_1 \wedge f_2) & \longrightarrow & F_n/(f_1) \\ \downarrow & & \downarrow \\ F_n/(f_2) & \longrightarrow & F_n/(f_1 \vee f_2) \end{array}$$

Since $Z(f_1 \wedge f_2) = Z(f_1) \cup Z(f_2)$ and $Z(f_1 \vee f_2) = Z(f_1) \cap Z(f_2)$, by convention 6.9 this is equivalent to the following:

Lemma 6.11. *Given f_1, $f_2 \in F_n$, the following diagram of $[0, 1]$-valued functions is a pullback:*

$$\begin{array}{ccc} F_n|_{Z(f_1) \cup Z(f_2)} & \longrightarrow & F_n|_{Z(f_1)} \\ \downarrow & & \downarrow \\ F_n|_{Z(f_2)} & \longrightarrow & F_n|_{Z(f_1) \cap Z(f_2)} \end{array}$$

That is, given g_1, $g_2 \in F_n$ such that $g_1|_{Z(f_1) \cap Z(f_2)} = g_2|_{Z(f_1) \cap Z(f_2)}$, there exists $g \in F_n$ (necessarily unique upon restriction to $Z(f_1) \cup Z(f_2)$), such that $g|_{Z(f_1)} = g_1|_{Z(f_1)}$ and $g|_{Z(f_2)} = g_2|_{Z(f_2)}$.

Proof. As in case of hyperarchimedean algebras, Lemma 5.12, we are dealing with algebras of $[0, 1]$-valued functions. We now use the formula (5.14). Set

$$h = (g_1 \odot \neg \, nf_1) \vee (g_2 \odot \neg \, nf_2).$$

Given x, take $n \geq 1$ such that $nf_1(x) = (n+1)f_1(x)$, $nf_2(x) = (n+1)f_2(x)$, and check (as in 5.12, (5.14)) that for $x \in Z(f_1)$, $h(x) = g_1(x)$, and for $x \in Z(f_2)$, $h(x) = g_2(x)$. The problem now is that we do not have a single n that works for all the x in $Z(f_1) \cup Z(f_2)$. To make n independent of x we proceed as follows:

Assume $x \in Z(f_1)$. Then $h(x) = g_1(x) \vee (g_2(x) \odot \neg \, nf_2(x))$. We shall see there is a $n \geq 1$ such that $g_2(x) \odot \neg \, nf_2(x) \leq g_1(x)$ for all $x \in Z(f_1)$. Since $g_2(x) \odot \neg \, nf_2(x) = max\{0, \ g_2(x) - nf_2(x)\}$ (see 1.1(11)), and $g_1(x) \geq 0$, we have to prove $g_2(x) - nf_2(x) \leq g_1(x)$.

Let $\{T_1, \ldots, T_m\}$ be a set of convex polyhedra as in Proposition 6.10, for $f = f_1$, and $H = \{f_2, g_1, g_2\}$. Let $x_{i0}, \ldots, x_{in_i}$ be the vertices of the polyhedron T_i. For each $1 \leq i \leq m$ and $0 \leq j \leq n_i$ there is an integer n_{ij} such that $g_2(x_{ij}) - n_{ij}f_2(x_{ij}) \leq g_1(x_{ij})$ for x_{ij}. In fact, if $x \in Z(f_2)$, then $g_1(x_{ij}) = g_2(x_{ij})$ and any number n_{ij} will do. If $f_2(x_{ij}) > 0$, then the inequality will hold if we take a sufficiently large n_{ij}. Let n be such that $n_{ij} \leq n$ for all i, j. Then $g_2(x_{ij}) - nf_2(x_{ij}) \leq g_1(x_{ij})$ for all i, j. Since each $x \in Z(f_1)$ is a convex combination of the vertices of T_i for some i, and since g_1 and the function $g_2 - nf_2$ are linear over T_i, we get $g_2(x) - nf_2(x) \leq g_1(x)$.

For $x \in Z(f_2)$ we proceed in the same way. □

7. Pushout–pullback lemma

The general pushout–pullback Lemma 3.11 follows from the particular case of finitely presented MV-algebras 6.11. This is so by categorical nonsense because finite limits commute with filtered colimits in the category of MV-algebras. However we find that in this paper a sketch of an explicit proof is convenient.

Any MV-algebra B is a filtered colimit of finitely presented MV-algebras. Explicitly, the diagram of all morphisms $R \xrightarrow{\alpha} B$, for all finitely presented MV-algebras R, is a filtered colimit diagram (with transition morphisms $(R, \alpha) \to (S, \beta)$ all $R \xrightarrow{\varphi} S$ such that $\beta \circ \varphi = \alpha$). Moreover, given a_1, $a_2 \in B$, a diagram of the form $B/(a_1) \longleftarrow B \longrightarrow B/(a_2)$ is in a similar way a filtered colimit of diagrams $R/(r_1) \longleftarrow R \longrightarrow R/(r_2)$, r_1, $r_2 \in R$, of finitely presented algebras. It follows that the corresponding pushout squares conform a filtered colimit of squares. With this in mind we proceed to prove the lemma.

Lemma 7.1 (*Pushout–Pullback Lemma*). *Given any MV-algebra A, and two elements a_1, $a_2 \in A$, the following pushout diagram is also a pullback diagram (recall that $(a_1, a_2) = (a_1 \vee a_2)$).*

$$\begin{array}{ccc} A/(a_1 \wedge a_2) & \longrightarrow & A/(a_1) \\ \downarrow & & \downarrow \\ A/(a_2) & \longrightarrow & A/(a_1, a_2) \end{array}$$

Proof. Let b_1, $b_2 \in A$, and suppose that they are identified by the quotient map onto $A/(a_1, a_2)$. We have to show there is an element $c \in A$, unique modulo $(a_1 \wedge a_2)$, such that $c \mapsto b_1$ in $A/(a_1)$, and $c \mapsto b_2$ in $A/(a_2)$.

Let $F = F[x_1, x_2, y_1, y_2]$ be the free MV-algebra on four generators, and consider the morphism $F \longrightarrow A/(a_1 \wedge a_2)$ determined by the assignments $x_1 \mapsto a_1$, $x_2 \mapsto a_2$, $y_1 \mapsto b_1$, $y_2 \mapsto b_2$. This morphism induces the four vertical arrows in the diagram below.
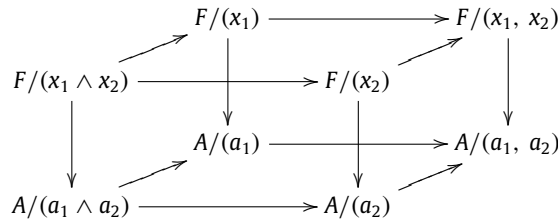
The upper square is a pullback, and while the elements y_1, y_2 are not identified in $F/(x_1, x_2)$, by the assumption made on b_1, b_2, they do so downstairs in $A/(a_1, a_2)$. Consider the following diagram:
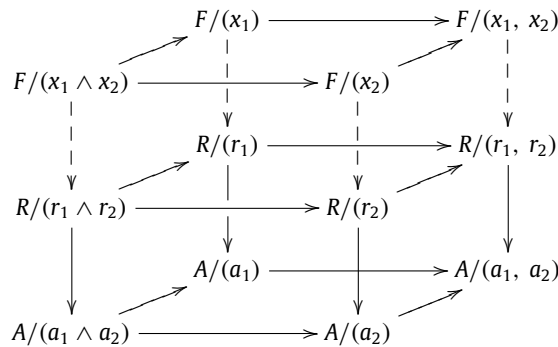
where the square in the bottom is a filtered colimit of the middle squares of finitely presented MV-algebras. By the construction of filtered colimits of MV-algebras, it follows there is one of them where y_1, y_2 are already identified in $R/(r_1, r_2)$. Let $y_1 \mapsto s_1 \in R/(r_1), y_2 \mapsto s_2 \in R/(r_2)$. Since by 6.11 the square is a pullback, there exists an element $s \in R$, unique modulo $(r_1 \wedge r_2)$, such that $s \mapsto s_1$, and $s \mapsto s_2$. Let $s \mapsto c \in A$, then c is the required element. □

8. Compactness lemma

To prove the compactness of the prime spectrum Z_A we will construct first its lattice of open sets along the lines developed in the Appendix. This construction yields a compact locale, whose set of points we identify with Z_A. This method guarantees the compactness of Z_A provided the locale has enough points. This latter property will be guaranteed by a standard application of Zorn's Lemma.

Sheaves of posets and a construction of the prime spectrum of MV-algebras

As is well known, the underlying poset of any MV-algebra A is a distributive lattice that we denote also by A. Further, the principal ideals of A under inclusion form another distributive lattice which is a quotient lattice of A. In fact, it is the quotient lattice determined by the following equivalence relation:

Definition 8.1. Given any MV-algebra A and two elements $a, b \in A$,

$$a \sim b \iff (a) = (b) \iff \exists n \mid a \leq nb \quad and \quad b \leq na.$$

From 1.2(4) it immediately follows

Proposition 8.2. *The relation defined in 8.1 is a lattice congruence:*

$$(a_1) = (a_2), \ (b_1) = (b_2) \implies (a_1 \wedge b_1) = (a_2 \wedge b_2), \ (a_1 \vee b_1) = (a_2 \vee b_2). \quad \square$$

Given any distributive lattice, the opposite order also defines a distributive lattice. We shall denote by A^{op} the distributive lattice determined by the opposite order in an MV-algebra A. We consider the opposite lattice of the lattice of principal ideals defined above:

Definition 8.3. Given any MV-algebra A, we denote by V_A the quotient of the lattice A^{op} by the congruence defined in 8.1, $A^{op} \longrightarrow V_A$.

The quotient map will be denoted by an over-lining, $a \mapsto \bar{a}$. We then have:

$$a \leq b \Rightarrow \bar{b} \leq \bar{a}, \quad \overline{a \wedge b} = \bar{a} \vee \bar{b}, \quad and \quad \overline{a \vee b} = \bar{a} \wedge \bar{b}.$$

We refer to A.6 for the definition of *point* of an inf-lattice.

Proposition 8.4. *Any point p of the lattice A^{op}, $A^{op} \xrightarrow{p} 2$ satisfies the equation*

$$p(a \oplus b) = p(a) \wedge p(b) \tag{8.5}$$

if and only if p factorizes as shown in the following diagram:

$$\tag{8.6}$$

Proof. Assume the factorization (8.6), then by abuse of notation we can write $p = \bar{p}$. From Proposition 8.2 it follows that $\overline{a \oplus b} = \overline{a \vee b} = \bar{a} \wedge \bar{b}$. Then, $p(a \oplus b) = p(\overline{a \oplus b}) = p(\bar{a}) \wedge p(\bar{b}) = p(a) \wedge p(b)$. Conversely, assume the Eq. (8.5). Then, $p(x) = p(nx)$. There remains to be proved that if $a \sim b$, then $p(a) = p(b)$. By hypothesis, $b \leq na$ and $a \leq mb$, whence $p(a) = p(na) \leq p(b)$, and $p(b) = p(mb) \leq p(a)$, as required to complete the proof. $\square$

Throughout, both lattices A^{op} and V_A are equipped with the Grothendieck topology J_f of finite suprema defined in A.7, A.10. By Proposition 8.4 we then have.

Proposition 8.7. *The points of the site (V_A, J_f) are exactly the prime ideals of the MV-algebra A.* $\square$

An application of Theorem A.12(2) now yields

Proposition 8.8. *The topological space Z_A (see 2.1) is the same as the space $P_J(V_A)$ of points of the site (V_A, J_f).* $\square$

Then, Theorem A.16 completes the proof of the compactness Lemma 3.8.

Corollary 8.9. *Given any MV-algebra A, the spectral space Z_A is sober, compact, and has a base of compact open sets.* $\square$

We conclude this section with a characterization of the open sets of Z_A. In view of A.10, these sets are in one to one correspondence with the elements of the locale $I_J(V_A)$, given by the lattice ideals of V_A. The latter are, in turn, in one to one correspondence with certain lattice ideals of A^{op}, or equivalently, lattice filters of A. Using that for any $x \in A$ and integer $n \geq 0$, the ideals (x) and (nx) coincide, we can prove the following:

Proposition 8.10. *There is a one to one correspondence between the open sets $W \subset Z_A$ and the lattice filters $U \subset A$ having the following property: "$na \in U \Rightarrow a \in U$". For any such lattice filter U, its corresponding open set is given by $W = \{P \mid \exists a \in U, \ a \in P\} = \bigcup_{a \in U} W_a$ (see 2.1).*

9. McNaughton theorem

In this final section we prove that McNaughton theorem is equivalent to (in particular it follows from) the representation Theorem 3.12 for free MV-algebras. A key fact is the realization that finite co-Zariski open covers of the prime spectral space of the free MV-algebra correspond with finite covers of the cube by convex polyhedra with rational vertices.

Recall our identification of the free MV-algebra $F_n = F[x_1, x_2, \ldots, x_n]$ with the algebra of term functions $F_n \subset [0, 1]^{[0, 1]^n}$ (6.9).

Proposition 9.1 *([1, 3.1.9]). A linear polynomial with integer coefficients $h = s_0 + s_1 x_1 + \cdots + s_n x_n, s_i \in \mathbb{Z}$, determines a term function, denoted $h^{\sharp}$, by means of the definition $h^{\sharp} = (h \vee 0) \wedge 1$.* $\square$

Following [1, Definition 3.1.6]) a continuous function $[0, 1]^n \xrightarrow{\tau} [0, 1]$ is said to be a *McNaughton function* if there are linear polynomials $h_1, \ldots, h_k$ with integer coefficients such that for each point $x \in [0, 1]^n$, $\tau(x) = h_i(x)$ for some i, $1 \leq i \leq k$. Each h_i is said to be a *linear constituent* of τ.

Clearly the projections x_i and the constant function 0 are McNaughton functions, and directly from the definition it can be easily seen that McNaughton functions form a MV-subalgebra of $[0, 1]^n$. It follows:

Proposition 9.2 *([1, 3.1.8]). Term functions are McNaughton functions. That is, F_n is a subalgebra of M_n, $F_n \subset M_n$ (where M_n denotes the MV-algebra of McNaughton functions).* $\square$

McNaughton's Theorem establishes the converse result, that is, that every McNaughton function is a term function, $F_n \supset M_n$.

Since any convex polyhedron $P \subset [0, 1]^n$ is the intersection of $[0, 1]^n$ and a finite set of closed half spaces defined by linear polynomials, we have:

Proposition 9.3. *Any convex polyhedron $P \subset [0, 1]^n$ with rational vertices is the zeroset of a term function $f = h_1^{\sharp} \vee \cdots \vee h_k^{\sharp}$, $P = Z(f)$, where $h_1, \ldots, h_k$ are linear polynomials with integer coefficients.* $\square$

As a particular case of Proposition 6.10 we have

Proposition 9.4. *Given any $f \in F_n$, there are convex polyhedra $T_1, \ldots, T_m$ whose union coincides with $Z(f)$.*

Given term functions $f_1, \ldots, f_m \in F_n$, the open sets $W_{f_i} = Z(\widehat{f_i}) \subset Z_{F_n}$ cover the prime spectral space Z_{F_n} exactly when $f_1 \wedge \cdots \wedge f_m = 0$, which in turn is equivalent to the fact that the zerosets $Z(f_i) \subset [0, 1]^n$ cover the cube $[0, 1]^n$.

By refining the covers if necessary, we have

Remark 9.5. *The (finite) open covers of the prime spectral space Z_{F_n} correspond to the (finite) covers of the cube $[0, 1]^n$ by convex polyhedra with rational vertices.*

The following is not difficult to prove (compare with Proposition 6.10).

Proposition 9.6 ([1, 3.3.1]). *Given a McNaughton function τ with linear constituents $h_1, \ldots, h_k$, there are convex polyhedra with rational vertices $T_1, \ldots, T_m$ whose union coincides with $[0, 1]^n$, and such that for each T_i there is a h_j with $(\tau = h_j)|_{T_i}$.*

In conclusion we have:

Proposition 9.7. *Any McNaughton function τ is determined by a cover of $[0, 1]^n$ by convex polyhedra $T_i = Z(f_i)$, $f_1, \ldots, f_m \in F_n, f_1 \wedge \cdots \wedge f_m = 0$, and a compatible family $g_1, \ldots, g_m \in F_n$ of term functions, $(g_i = g_j)|_{T_i \cap T_j}$. Conversely, any such set of data determines a McNaughton function by setting $(\tau = g_i)|_{T_i}$ (the second assertion is justified by 9.2).*

It is convenient now to expand 2.3 and write in detail the definition of a global section of the prime spectrum of the free MV-algebra on n generators.

Fact 9.8. *A global section of the prime spectrum $\sigma \in \Gamma(Z_{F_n}, E_{F_n})$ is determined by a cover of Z_{F_n} by open sets $W_{f_i} = Z(\widehat{f_i})$, $f_1, \ldots, f_m \in F_n, f_1 \wedge \cdots \wedge f_m = 0$, and a compatible family $g_1, \ldots, g_m \in F_n$ of term functions, $(\widehat{g_i} = \widehat{g_j})|_{W_{f_i} \cap W_{f_j}}$. Then, $(\sigma = \widehat{g_i})|_{W_{f_i}}$.* □

We see that a pair of families of term functions $f_1, \ldots, f_m, g_1, \ldots, g_m, f_1 \wedge \cdots \wedge f_m = 0$, determines either a McNaughton function or a global section, according as $(g_i = g_j)|_{Z(f_i) \cap Z(f_j)}$ or $(\widehat{g_i} = \widehat{g_j})|_{Z(\widehat{f_i}) \cap Z(\widehat{f_j})}$. By [1, Lemma 3.4.8] this two conditions are equivalent, also see 3.6 and 6.7. This immediately yields the identity

$$M_n = \Gamma(\mathsf{Spec}_{F_n}) \qquad (recall \ (Z_{F_n}, E_{F_n}) = \mathsf{Spec}_{F_n}).$$

We establish now a precise statement of this fact:

Theorem 9.9. *The composite morphism*

$$\Gamma(Z_{F_n}, E_{F_n}) \xrightarrow{\kappa^* \lambda_*} Cont(X_{F_n}, [0, 1]) \xrightarrow{(\ell^{-1})^*} Cont([0, 1]^n, [0, 1])$$

sends a global section σ into the function $\tau(p) = \lambda \sigma \kappa(\ell^{-1}(p))$, and establishes an isomorphism $\Gamma \mathsf{Spec}_{F_n} \xrightarrow{\cong} M_n$ between the MV-algebra of global sections of the prime spectrum of the free algebra and the MV-algebra of McNaughton functions. Given $g \in F_n$, this isomorphism sends the global section $\widehat{g}$ into the term function g, and the global section determined by a pair of families $f_1, \ldots, f_m, g_1, \ldots, g_m, f_1 \wedge \cdots \wedge f_m = 0$, into the McNaughton function determined by the same pair of families. □

Proof. Let $p \in [0, 1]^n$. Recalling (4.20) and 6.3 we have $(\ell^{-1})^* \kappa^* \lambda_*(\sigma)(p) = \kappa^* \lambda_*(\sigma)(\ell^{-1}(p)) = \lambda \sigma \kappa(\ell^{-1}(p))$. This shows that the composite morphism sends a global section σ into the function τ given by $\tau(p) = \lambda \sigma \kappa(\ell^{-1}(p))$. In particular, for $g \in F_n$, we have $\lambda \widehat{g} \kappa(\ell^{-1}(p)) = \ell^{-1}(p)(g) = g(p)$. Thus it sends $\widehat{g}$ into g. It follows it sends the global section determined by a pair of families $f_1, \ldots, f_m, g_1, \ldots, g_m, f_1 \wedge \cdots \wedge f_m = 0$, into the McNaughton function determined by the same pair of families. This shows that it is surjective into the MV-algebra M_n. From Proposition 4.21 and Remark 6.8 it follows that it is also injective. □

By means of an identification of $[0, 1]^n$ with a subset of Z_{F_n}, and of the fibers of E_{F_n} over a maximal ideal with the interval $[0, 1]$, this isomorphism can be interpreted as the morphism which sends a global section σ to its restriction to $[0, 1]^n$. With this proviso, a McNaughton function has a unique extension into the whole prime spectrum Z_{F_n}.

One may also identify the maximal spectrum M_{F_n} with the cube $[0, 1]^n$. In this case, however, the latter is equipped with the co-Zariski topology, which has as a base of open sets the Zero sets of the term functions. The global sections of the maximal spectrum, just as the McNaughton functions, are given by (now a possibly infinite family of) linear polynomials on convex polyhedra but, unlike the McNaughton functions, they are only continuous for the (much) finer co-Zariski topology. Moreover, they do not extend to the whole prime spectrum. Indeed, an example of a global section is the function which is equal to 1 in $\{0\} = [0, 0]$, and constantly zero on each interval $[1/(n + 1), 1/n], n \in \mathbb{N}$.

Theorem 9.9 has the following immediate corollary

Theorem 9.10. *McNaughton theorem is equivalent to the representation Theorem 3.12 for free MV-algebras.* □

This shows that the representation theorem can be viewed as a vast generalization of McNaughton theorem, from free MV-algebras to arbitrary MV-algebras. In particular, 3.12 yields a proof of McNaughton theorem.

Acknowledgements

Appendix. Sheaf theory of posets

In this appendix we fix notation, terminology, and prove a number of general results of the theory of posets that are needed in this paper. All these results are known folklore of the subject, and can be found in the literature in one form or another. However, to the best of our knowledge, no preexisting treatment of these topics follow the basic idea we stress here, namely, that of a sheaf theory of posets. Posets are viewed as {0, 1}-based categories, and we examine Grothendieck theory of sheaves on *Set*-based categories, but we deal directly with posets. In particular, since our categories are posets, inf-lattices play the role of categories with finite limits, and locales that of Grothendieck topoi.

We shall consider a partial order to be a reflexive and transitive relation, not necessarily antisymmetric.[2] A set furnished with such a relation will be called a *poset*. This is equivalent to a category taking its homsets in the poset $2 = \{0, 1\} = \{\emptyset, \{*\}\}$. As usual, $x \leq y \iff hom(x, y) \neq \emptyset$. Under this equivalence a functor is the same thing that an order preserving function. Given any poset H, under the usual bijection between subsets and characteristic functions, functors $H \xrightarrow{p} 2$ correspond with poset filters $P \subset H$. Functors $H^{op} \xrightarrow{u} 2$ are called *presheaves*, and correspond with poset ideals $U \subset H$. The set $I(H)$ of all ideals, ordered by inclusion, $I(H) = 2^{H^{op}}$, is a *locale* (see A.1). The locale structure is given by the *union* and *intersection* of subsets. There is a *Yoneda* functor $H \xrightarrow{h} I(H)$, sending an element $a \in H$ to the principal ideal $(a]$. This functor is *full*, meaning that for any $x, y \in H$, we have $x \leq y \iff h(x) \leq h(y)$.

Following Joyal–Tierney [9], we think of locales as dual objects for generalized (possibly pointless) topological spaces, the locale being its lattice of open sets. In the same vein, we think of *inf-lattices* as open bases for locales.

Recall:

Definition A.1. A locale is a complete lattice in which finite infima distribute over arbitrary suprema. A morphism of locales $L \xrightarrow{f^*} R$ is a function f^* preserving finite infima and arbitrary suprema.[3] (The upper star is meant to indicate that such an arrow is to be considered as the inverse image of a morphism between the formal duals $\bar{R} \xrightarrow{f} \bar{L}$). The formal dual of a locale is called a *space* in [9], but we shall call it a *localic space*.

Remark A.2. Given a locale L, each element $u \in L$ determines a locale $L_u = \{x \mid x \leq u\}$. Notice that the inclusion $L_u \subset L$ is not a morphism of locales since does not preserve 1. There is a quotient morphism of locales $L \to L_u$ given by $x \mapsto x \wedge u$. This determines the open subspace $\bar{L}_u \hookrightarrow \bar{L}$.

For any topological space X, the lattice $\mathcal{O}(X)$ of open sets yields a locale. A continuous function $Y \xrightarrow{f} X$ determines a morphism of locales in the other direction $\mathcal{O}(X) \xrightarrow{f^*} \mathcal{O}(Y)$, the usual inverse image of f.

The poset $2 = \{0, 1\} = \mathcal{O}(\{*\})$ is the *singleton* or *terminal* localic space. Given any locale L, there exists a unique locale morphism $\{0, 1\} \to L$, $1 = \bar{2}$.

Definition A.3. A *point* p of a locale L is a morphism $1 \xrightarrow{p} \bar{L}$, that is, a locale morphism $L \xrightarrow{p^*} 2$.

Proposition A.4. *Given any locale L, the set of points P_L has a canonical topology whose open sets are the subsets $W_u \subset P_L$, $W_u = \{p \mid p^*u = 1\}$, for $u \in L$.* $\square$

There is a surjective morphism of locales $L \xrightarrow{\rho} \mathcal{O}(P_L)$.

Definition A.5. We say that a locale L has *sufficiently many (or enough)* points when ρ is injective, $u \neq v \Rightarrow W_u \neq W_v$. That is:

$$u \neq v \Rightarrow \exists p \mid p^*u = 1, \ p^*v = 0.$$

In this case, the localic space $\bar{L}$ is topological, $L \xrightarrow{\cong} \mathcal{O}(P_L)$.

---

[2] While this concept is usually known as a *pre-order*, our reason for departing from the classical nomenclature is that we prefer to use the non-compound name for the more important notion.

[3] There are two terminologies in the literature, in one we have *frames* and they formal duals *locales*, and in the other we have *locales* and their formal duals *(localic) spaces*. The translation is the following: *frame = locale*, and for the formal duals *locale = space*.

The topological space P_L is a *sober* space (that is, every nonempty irreducible closed subset has a unique generic point, [8, Ch. II]). The category of sober topological spaces is dual to the category of locales with enough points.

An *inf-lattice* is a poset with finite infima (in particular, the empty infimum or top element 1). A morphism of inf-lattices is an inf-preserving (whence, an order preserving) function.

Definition A.6. A *point* of an inf-lattice V is a morphism $V \xrightarrow{p} 2$. A *presheaf* is an order reversing map $V^{op} \xrightarrow{u} 2$. Points correspond to inf-lattice filters, and presheaves to poset (not necessarily inf-lattice) ideals (see A.8 and A.9).

Definition A.7. Let H be an inf-lattice. A *Grothendieck topology*[4] J on H is defined by specifying, for each $a \in H$, a set $J(a)$ of families $a_i \leq a$, called *covers*, such that:

  (i) $x \cong a \in J(a)$.
 (ii) $a_{i,j} \leq a_i \in J(a_i), \ a_i \leq a \in J(a) \Rightarrow a_{i,j} \leq a \in J(a)$.
(iii) $a_i \leq a \in J(a), b \in H \Rightarrow a_i \wedge b \in J(a \wedge b)$.
      The topology is said to be *subcanonical* if the covers are suprema, that is:
(iv) for every $a_i \leq a \in J(a), \ a = \bigvee_i a_i$.
      An inf-lattice furnished with a Grothendieck topology is called a *site*.

We shall often say *topology* instead of Grothendieck topology. There is a minimal or trivial topology whose covers are the isomorphisms. In general, it is possible for some elements a to be covered by the empty family, in symbols $\emptyset \in J(a)$. This is the case when H has a bottom element $0 \in H$, the empty supremum. In this case, it is usually assumed that $\emptyset \in J(0)$.

Definition A.8. Let (H, J) be a site. A *point* is a inf-preserving functor $H \xrightarrow{p} 2$ which sends covers into epimorphic families. Writing $P = \{a \mid p(a) = 1\}$, points correspond to J-prime inf-lattice filters. These are subsets $P \subset H$ such that:

$$
\begin{array}{lll}
\text{(i)} & 1 \in P, & \\
\text{(ii)} & a \geq b \in P & \Rightarrow a \in P, \\
\text{(iii)} & a, \ b \in P & \Rightarrow a \wedge b \in P, \\
\text{(iv.a)} & a_i \leq a \in J(a) \text{ and } a \in P & \Rightarrow \exists i \mid a_i \in P. \\
\text{(iv.b)} & \emptyset \in J(a) & \Rightarrow a \notin P.
\end{array}
$$

Note that a filter P need not be *proper*, for example, in the case of the trivial topology. However, in most cases there is a bottom element $0 \in H$, and $\emptyset \in J(0)$, so that $0 \notin P$ for any J-prime filter P.

Definition A.9. Let (H, J) be a site. A *sheaf* is a presheaf $H^{op} \xrightarrow{u} 2$ satisfying the following:
   Sheaf axiom: $a_i \leq a \in J(a)$ and $\forall i \, u(a_i) = 1 \Rightarrow u(a) = 1$.
   Writing $U = \{a \mid u(a) = 1\}$, sheaves correspond to J-ideals. These are poset (not inf-lattice) ideals satisfying the sheaf axiom. That is, subsets $U \subset H$ such that:

$$
\begin{array}{ll}
\text{(i)} & a \leq b \in U \Rightarrow a \in U. \\
\text{(ii.a)} & a_i \leq a \in J(a) \text{ and } \forall i \, a_i \in U \Rightarrow a \in U. \\
\text{(ii.b)} & \emptyset \in J(a) \Rightarrow a \in U.
\end{array}
$$

Usually there is a bottom element $0 \in H$, and $\emptyset \in J(0)$, so that $0 \in U$ for any J-ideal U.

Example A.10. Given any distributive lattice V, the *finite suprema* form a subcanonical Grothendieck topology (distributivity amounts to axiom (iii)), that we will denote J_f, $a_i \leq a \in J_f(a) \iff a = a_1 \vee a_2 \vee \cdots \vee a_n$. The points p of the site (V, J_f) correspond to the prime filters $P \subset V$ of the lattice, and a J_f-ideal $U \subset V$ is just a lattice ideal (notice that $0 \in U$ since 0 is the empty supremum). The generated lattice ideal, A.11(1), is given by:

$$
\#S = \{x \mid \exists \, a_1, \ a_2, \ldots, a_n \ \in S, \ n \geq 0, \ x \leq a_1 \vee a_2 \vee \cdots \vee a_n\}. \quad \square
$$

We next consider the poset $I_J(H)$ of all J-ideals, ordered by inclusion, $I_J(H) \subset I(H)$. In the next two theorems we collect the basic properties and the universal property which characterizes this construction.

Theorem A.11. *For any site* (H, J) *we have:*

1. *For any subset* $S \subset H$, *the set*

$$
\#S = \{x \mid \exists \, a_i \leq a \in J(a), \ a_i \in S \ \forall i, \ x \leq a\}
$$

is a J-ideal, called the J-ideal generated *by S.*

---

[4] Some authors call *pre-topology* what we call here *topology*.

2. *The poset $I_J(H)$ is a locale, $I_J(H) \subset I(H)$. The generated J-ideal determines a morphism of locales $I(H) \xrightarrow{\#} I_J(H)$, such that for $S \in I(H)$, $U \in I_J(H)$:*

$$\#S \leq U \iff S \leq U \quad (\# \text{ is left adjoint to the inclusion}).$$

The locale structure of $I_J(H)$ is given by the following:

$$U \wedge V = U \cap V, \quad \bigvee_i U_i = \# \bigcup_i U_i$$

.
3. *The bottom element is the J-ideal $\#\emptyset = \{a \mid \emptyset \in J(a)\}$, and the top element is the whole set H. Given a J-ideal U, $U = H \iff 1 \in U$.*
4. *The composite $H \xrightarrow{h} I(H) \xrightarrow{\#} I_J(H)$ determines a inf-lattice morphism $\varepsilon = \#h$ sending covers into suprema. Given $a \in H$,*

$$\varepsilon(a) = \#[a] = \{x \mid \exists\, b_i \leq b \in J(b), \; b_i \leq a \,\forall i, \; x \leq b\}.$$

5. *Given any $U \in I_J(H)$, $U = \bigvee_{a \in U} \varepsilon(a)$. Thus the elements of the form $\varepsilon(a)$, $a \in H$, are a base of the locale $I_J(H)$ (notice that $\varepsilon(a \wedge b) = \varepsilon(a) \wedge \varepsilon(b)$).*
6. *The topology is subcanonical if and only if the segment $(a]$ is already a J-ideal. That is, $\varepsilon(a) = (a]$. This is the case if and only if ε is full, that is, for any $x, y \in H, x \leq y \iff \varepsilon(x) \leq \varepsilon(y)$.*

Proof. The proof is routine, we give the guidelines and let the reader check the details. Clearly $I(H) = 2^{H^{op}}$ is a locale (in fact, it has the pointwise structure determined by the locale 2). Next, check that the generated sheaf $\#$ preserves finite infima and that it is left adjoint to the inclusion. From this it easily follows that $I_J(H)$ is a locale. The rest is straightforward. □

Theorem A.12. *For any site (H, J) we have:*

1. *Given any locale L and a inf-lattice morphism $H \xrightarrow{f} L$ sending covers into suprema, there exists a unique morphism of locales $I_J(H) \xrightarrow{f^*} L$ such that $f^*\varepsilon = f$. f^* is determined by the formula $f^*(U) = \bigvee_{a \in U} f(a)$. Furthermore, for any two $f, g, f \leq g \iff f^* \leq g^*$.*
2. *In particular composition with ε establishes a bijection*

   $P_{I_J(H)} \xrightarrow{\cong} P_J(H) = \{P \subset H \mid P \text{ is a J-prime inf-lattice filter}\}$.
   The topology of $P_{I_J(H)}$ induces a topology in the set $P_J(H)$. A base for this topology is given by the sets $W_a = W_{\varepsilon(a)} = \{P \mid a \in P\}$, for each $a \in H$ (see Proposition A.4 and Theorem A.11(5)).

Proof. We check that f^* preserves finite infima. Any order preserving map satisfies $f^*(U \cap V) \leq f^*(U) \wedge f^*(V)$. For the converse direction we proceed as follows:

$$f^*(U) \wedge f^*(V) = \bigvee_{a \in U} f(a) \,\wedge\, \bigvee_{a \in V} f(a) = \bigvee_{a \in U, b \in V} f(a) \wedge f(b) = \bigvee_{a \in U, b \in V} f(a \wedge b) \leq f^*(U \cap V).$$

The rest is clear. □

Remark A.13. Let $H \xrightarrow{p} 2$ be a point with corresponding J-prime inf-lattice filter $P \subset H$, and $U \subset H$ be any J-ideal. Then (see A.8 and A.9):

$$p^*(U) = 1 \iff U \cap P \neq \emptyset. \quad □$$

Proposition A.14. *If all covers of a site (H, J) are finite families, then for every $a \in H$, $\varepsilon(a) \in I_J(H)$ is compact. That is:*

$$\varepsilon(a) \leq \bigvee_i U_i \Rightarrow \exists\, i_1, i_2, \ldots, i_n \mid \varepsilon(a) \leq U_{i_1} \vee U_{i_2} \vee \cdots \vee U_{i_n}.$$

Thus, the locales $I_J(H)_{\varepsilon(a)}$ are compact. In particular (for $a = 1$), $I_J(H)$ is a compact locale with a base of compact elements (see A.11(5)).

Proof. The following chain of equivalences, which is justified by A.11(1), (2) and (4), proves the proposition:

$$\frac{\varepsilon(a) \leq \bigvee_i U_i}{\#[a] \leq \bigvee_i U_i \;\equiv\; (a] \leq \bigvee_i U_i \;\equiv\; a \in \bigvee_i U_i \;\equiv\; a \in \# \bigcup_i U_i}$$

$$\frac{\exists\, a_{i_1}, \;\ldots\; a_{i_n} \leq b \in J(b), \; a_{i_1}, \;\ldots\; a_{i_n} \in \bigcup_i U_i, \; a \leq b}{\exists\, a_{i_1}, \;\ldots\; a_{i_n} \leq b \in J(b), \; a_{i_1}, \;\ldots\; a_{i_n} \in U_{i_1} \cup \ldots \cup U_{i_n}, \; a \leq b}$$

$$\frac{a \in \#(U_{i_1} \cup \ldots \cup U_{i_n}) \;\equiv\; a \in U_{i_1} \vee \ldots \vee U_{i_n}}{(a] \leq U_{i_1} \vee \ldots \vee U_{i_n} \;\equiv\; \#[a] \leq U_{i_1} \vee \ldots \vee U_{i_n}}$$

$$\varepsilon(a) \leq U_{i_1} \vee \ldots \vee U_{i_n} \qquad\qquad □$$

By definition, the formal dual of any compact locale L is compact localic space. The topological space P_L of A.4 need not be compact unless L has enough points, in which case $L \cong \mathcal{O}(P_L)$. As is usual with the statements asserting the existence of points, the following theorem follows by an application of the Axiom of Choice:

Theorem A.15. *If all covers of a site* $(H, \jmath)$ *are* finite *families, then the locale* $I_\jmath(H)$ *has enough points. By* Remark A.13, *this amounts to the following statement:*

Given any two j-ideals U, $V \subset H$, *if* $U \neq V$, *then there exists a j-prime inf-lattice filter* $P \subset H$ *such that* $U \cap P = \emptyset$, *and* $V \cap P \neq \emptyset$.

Proof. Take an element $a \in H$ such that $a \in V$, $a \notin U$. Consider the set $\mathcal{F}$ of inf-lattice filters $F \subset H$, $\mathcal{F} = \{F \mid a \in F,\ U \cap F = \emptyset\}$. Clearly, if $F \in \mathcal{F}$, $U \cap F = \emptyset$, and $V \cap F \neq \emptyset$. We shall see that there is a $\jmath$-prime filter in $\mathcal{F}$.

The inf-lattice filter $[a) \subset H$, $[a) = \{x \mid a \leq x\}$ is in $\mathcal{F}$, so $\mathcal{F} \neq \emptyset$. On the other hand, given any chain $F_i, F_i \in \mathcal{F}$, the union $F = \bigcup_i F_i$ is an inf-lattice filter such that $a \in F$. But $U \cap F = U \cap \bigcup_i F_i = \bigcup_i (U \cap F_i) = \bigcup_i \emptyset = \emptyset$. Thus $F \in \mathcal{F}$. The Axiom of Choice then yields a maximal element $P \in \mathcal{F}$. We show now that P is $\jmath$-prime.

Given an inf-lattice filter $F \subset H$, and an element $a \in H$, we denote by $(F, a) = \{x \mid \exists b \in F,\ b \wedge a \leq x\}$ the inf-lattice filter *generated* by $F \cup \{a\}$.

Let $a_i \leq a \in \jmath(a)$ be a cover. We can assume it is nonempty because if not this would contradict $a \notin U$. Suppose
(1) $a \in P$.
Assume that $\forall i\ a_i \notin P$. Then, $(P, a_i) \cap U \neq \emptyset$. Take $x_i \in U$, $x_i \in (P, a_i)$, $x_i \leq b_i \wedge a_i$, $b_i \in P$. It follows that
(2) $b_i \wedge a_i \in U$.
Let $c = \bigwedge_i b_i$. Then
(3) $c \in P$.
But $c \leq b_i$, thus $c \wedge a_i \leq b_i \wedge a_i$. From (2) it follows that
(4) $c \wedge a_i \in U$.
Since $a_i \leq a \in \jmath(a)$, we have $c \wedge a_i \leq c \wedge a \in \jmath(c \wedge a)$ (see A.7 iii). Thus, from (4) it follows that (see A.9 (ii))
(5) $c \wedge a \in U$.
But from (1) and (3), we have
(6) $c \wedge a \in P$.
Finally, (5) and (6) contradict $U \cap P = \emptyset$. $\quad\square$

As a corollary of the last two theorems we have

Theorem A.16. *Let* $(H, \jmath)$ *be a site whose covers are* finite *families. Let* $P_\jmath(H)$ *be the set of $\jmath$-prime inf-lattice filters* $P \subset H$. *Then, the sets* $W_a = \{P \mid a \in P\}$ *are compact and form an open base for a topology. The resulting topological space* $P_\jmath(H)$ *is sober, compact, and has a base of compact open sets. Its locale of open sets is (isomorphic to) the locale* $I_\jmath(H)$ *of $\jmath$-ideals of* H. $\quad\square$

Compact sober topological spaces with a basis of compact opens are called *spectral spaces* [7]. They arose as an abstraction of spaces of prime ideals in ring theory.

All the results in this appendix apply to Example A.10. In Section 8 we have considered the particular case of this example given by the lattice V_A of principal ideals of a MV-algebra A.

References

[1] R. Cignoli, I. D'Ottaviano, D. Mundici, Algebraic Foundations of Many-valued Reasoning, in: Trends in Logic, vol. 7, Kluwer Academic Publishers, 2000.
[2] R. Cignoli, E.J. Dubuc, D. Mundici, Extending Stone duality to multisets and locally finite MV-algebras, Journal of Pure and Applied Algebra 189 (2004) 37–59.
[3] M. Coste, Localization, Spectra and Sheaf Representation, in: Springer Lecture Notes in Mathematics, vol. 753, 1977.
[4] R. Godement, Topologie Algebrique et Theorie des Faiseaux, Hermann, Paris, 1958.
[5] A. Di Nola, I. Esposito, B. Gerla, Local algebras in the representation of MV-algebras, Algebra Universalis 56 (2007) 133–164.
[6] R. Hartshorne, Algebraic Geometry, Springer Verlag, New York, 1977.
[7] M. Hochster, Prime ideal structure in commutative rings, Transactions on American Mathematical Society 142 (1969) 43–60.
[8] P.T. Johnstone, Stone Spaces, Cambridge University Press, 1982.
[9] A. Joyal, M. Tierney, An Extension of the Galois Theory of Grothendieck, in: Memoirs of the American Mathematical Society, vol. 151, 1984.
[10] K. Keimel, The Representation of Lattice Ordered Groups and Rings by Sections in Sheaves, in: Lectures on the Applications of Sheaves to Ring Theory, Springer Lecture Notes in Mathematics, vol. 248, 1971.
[11] S. Mac Lane, Categories for the Working Mathematician, 2nd ed., Springer Verlag, 1998.
[12] S. Mac Lane, I. Moerdijk, Sheaves in Geometry and Logic, Springer Verlag, 1992.
[13] J. Martinez, E.R. Zenk, Yosida Frames, Journal of Pure and Applied Algebra 204 (2006) 473–492.
[14] S.H. McCleary, Free lattice-ordered groups represented as o-2 transitive l-permutation groups, Transactions on American Mathematical Society 290 (1985) 69–79.
[15] V. Novak, I. Perfilieva, J. Mockor, Mathematical Principles of Fuzzy Logic, Kluwer Academic Publishers, Boston, 1999.
[16] Y. Poveda, Una Teoria General de Representacion para MV-algebras, Universidad de Buenos Aires, Facultad de Ciencias Exactas y Naturales, 2007.
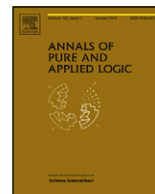[17] Y.C. Yang, l-Groups and Bezout Domains, Von der Fakultat Math. und Physik der Universitat Stuttgart, 2006, http://elib.uni-stuttgart.de/opus/volltexte/2006/2508.

Update

Annals of Pure and Applied Logic

Erratum

Erratum to "Representation theory of MV-algebras" [Ann. Pure Appl. Logic 161 (8) (2010)]

Eduardo J. Dubuc

Departamento de Matematicas, F.C.E. y N., UBA, Buenos Aires, Argentina

We detected an important omission: in the definition of the category of *MV-spaces* (Definition 3.1), the morphisms $F_{f(x)} \xrightarrow{\varphi_x} E_x$ should be required to be *injective*. Consequently $f^*F \xrightarrow{\varphi} E$ a *monomorphism* in the topos $Sh(X)$.

The following simple counterexample was send to us by Mike Hampton.

Let C be Chang's MV-algebra (*Algebraic analysis of many-valued logics, Trans. Amer. Math. Soc. 88 (1958) 467–490*), C has exactly two prime ideals, $0 = \{0\}$ and a maximal ideal $M \neq 0$, $C/0 = C$, $C/M = \{0, 1\}$. Let T be the *MV*-space $T = (\{*\}, \{0, 1\})$, $\Gamma(T) = \{0, 1\}$. Without requiring the morphisms of *MV*-spaces to be injective on the fibers, $[T, \text{Spec}_C]$ has two elements, while $[C, \Gamma(T)]$ has only one, thus contradicting the adjointness between the functors Spec and Γ.