Vibration analysis of rotating Timoshenko beams by means of the differential quadrature method

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Abstract. Vibration analysis of rotating beams is a topic of constant interest in mechanical engineering. The differential quadrature method (DQM) is used to obtain the natural frequencies of free transverse vibration of rotating beams. As it is known the DQM offers an accurate and useful method for solution of differential equations. And it is an effective technique for solving this kind of problems as it is shown comparing the obtained results with those available in the open literature and with those obtained by an independent solution using the finite element method. The beam model is based on the Timoshenko beam theory.

Keywords: rotating beam; Timoshenko beam theory; free vibration; differential quadrature method.

1. Introduction

Rotating beams are widely used in many engineering applications such as turbine blades, helicopters rotors, aircrafts propellers and robot manipulators.

It is necessary to accurately predict the beam natural frequencies since vibrational properties in engineering structures are often limiting factors in their performance. Many methods have been used to obtain the free vibration characteristics of rotating beams, such as Rayleigh, Galerkin, Ritz, finite element, modeling method, dynamic stiffness method, power series approach, differential transformation technique, the use of Genetic Programming to create an approximate model and also experimental approaches, Senatore (2006). Hodges and Rutkowski (1981), used the finite element method of variable order to analyze the problem. Du *et al.* (1994), presented a convergent power series expression to solve analytically for the exact natural frequencies and modal shapes of rotating Timoshenko beams. Naguleswaran (1994), studied the dimensionless natural frequencies of a rotating beam for different boundary conditions. He solved the mode shape equation by the

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Frobenius method and the frequency equations by trial and error. Al-Ansary (1998) used the perturbation technique and Galerkin's method to analyze flexural vibrations of rotating beams considering rotary inertia. Banerjee (2000), derived the dynamic stiffness matrix of a non-uniform, rotating, Euler-Bernoulli beam using the Frobenius method of solution in power series to model a tapered beam assembling the dynamic stiffness matrices of uniform beams by approximating the tapered beam as an assembly of many different uniform beams. Banerjee (2001), developed the dynamic stiffness method for a rotating cantilever Timoshenko beam. Lin and Hsiao (2001) used a method based on the power series solution to solve the natural frequency of the rotating Timoshenko beam, they considered the effect of Coriolis force on the natural frequency of the rotating beam. Chung and Yoo (2002), presented a finite element analysis for rotating cantilever beams based on a dynamic modelling method using the stretch deformation instead of the conventional axial deformation. Gunda *et al.* (2007), introduced a low degree of freedom model for dynamic analysis of rotating tapered beams based on a numerically efficient superelement, developed using a combination of polynomials and Fourier series as shape functions. Lee and Sheu (2007), used the Hamilton's principle and the consistent linearization of the fully nonlinear beam theory to derived two coupled governing differential equations for the natural frequencies of a rotating inclined beam and they utilized the method of Frobenius to establish the exact series solutions of the structural system. Ouyang and Wang (2007) developed a dynamic model for the vibration of a rotating beam subjected to axially moving forces. Singh, Mani and Ganguli (2007) used the Genetic Programming to create an approximate model of ratoting beams. Vinod, Gopalakrishnanand and Gangul (2007) presented a paper about spectral finite element formulation for a rotating beam subjected to small duration impact. A new rotating beam finite element was developed by Gunda and Ganguli (2008), in which the basis functions were obtained by the exact solution of the governing static homogenous differential equation of a stiff string, which resulted from an approximation in the rotating beam equation. Recently Mei presented a new approach called differential transformation to analyze free lateral vibrations of a stiffened rotating beam. As it was shown in his paper, Mei (2008), this technique gives accurate solutions to the vibrating problem. In 2009, an interesting paper was presented by Kumar and Ganguli (2009), they looked for rotating beams whose eigenpair (frequency and mode-shape) is the same as that of uniform nonrotating beams for a particular mode.

In this study the differential quadrature method (DQM) is presented to solve the mentioned structural problem. As it is known the differential quadrature method proposed by Bellman in the seventies, is a technique to solve differential equations. Some of its applications can be found in the papers by Bellman and Casti (1971), Bert and Malik (1996), Karami *et al.* (2003), Laura and Gutiérrez (1993), Choi *et al.* (2000), Liu and Wu (2001), Shu and Chen (1999) among many others and in treatises like as Bellman and Roth (1986), Shu (2000).

Numerical results of the examples have been obtained for the lateral free frequencies of vibration of a uniform beam with doubly symmetric cross-section that is attached to a rotating hub. The Euler-Bernoulli and Timoshenko theories (Seon Han *et al.* 1999) are considered comparatively to describe the lateral behavior of the beam. The accuracy with published results is very good. The solutions presented here have been obtained by means of Mathematica, Wolfram (1996).

On the other hand independent results have been obtained using the software (ALGOR 2007), by the finite element method, FEM.

The authors think that an advantage of the current paper is to present an efficient and simple model for vibration analysis of rotating beams, which requires relatively small computational effort than other methods. It would be helpful for the dynamic analysis of rotating beams for optimization and control applications where low order models are useful.

2. Governing equations

Fig. 1 shows a cantilever beam attached at one end to a rigid hub with radius $\overline{R}$. The hub rotates around its central axis, with a rotational speed $\overline{\eta}$. A Cartesian coordinate system is chosen with its origin at the clamped end of the beam (see Fig. 1). $\overline{x}$ is the radial coordinate which axis is considered to be coincidental with the centroidal axis of the rotating beam. The $\overline{Y}$-axis is parallel to the axis of rotation and the $\overline{Z}$-axis lies in the plane of rotation.

Only vibration in the $\overline{X}-\overline{Y}$ plane is considered and the Coriolis effects are ignored.

The flexural displacement in the $\overline{Y}$ direction is denoted as $\overline{w}$ and $\overline{\psi}$ denotes the section rotation.

The governing differential equations of motion for free lateral vibration of a Timoshenko rotating beam are the equations obtained by Banerjee (2001)

$$\frac{\partial \overline{Q}(\overline{x},t)}{\partial \overline{x}} - \rho A \frac{\partial^2 \overline{w}(\overline{x},t)}{\partial t^2} = 0 \tag{1}$$

$$\overline{Q}(\overline{x},t) - \overline{N}(\overline{x})\frac{\partial \overline{w}(\overline{x},t)}{\partial \overline{x}} + \frac{\partial \overline{M}(\overline{x},t)}{\partial \overline{x}} - \rho I \frac{\partial^2 \overline{\psi}(\overline{x},t)}{\partial t^2} + \overline{\eta}^2 \rho I \overline{\psi}(\overline{x},t) = 0 \tag{2}$$

$\overline{N}$ is the axial force due to the rotational speed

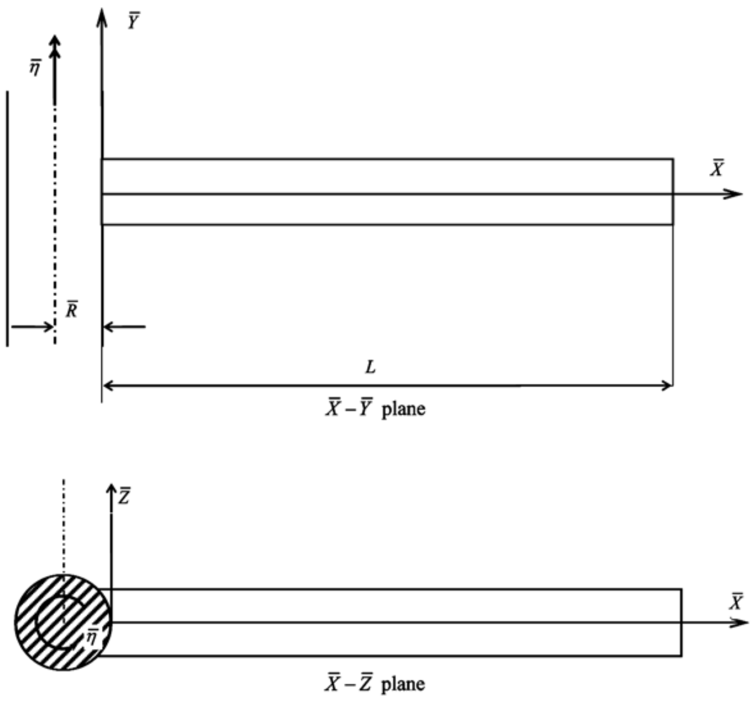

Fig. 1 Rotating cantilever beam model

$$\overline{N}(\overline{x}) = \overline{\eta}^2\rho A\left(\frac{L^2}{2}+\overline{R}L-\frac{\overline{x}^2}{2}-\overline{R}\overline{x}\right) \tag{3}$$

and $\overline{M}$ and $\overline{Q}$ are the bending moment and the shear force

$$\overline{M}(\overline{x},t) = EI\frac{\partial\overline{\psi}(\overline{x},t)}{\partial\overline{x}} \tag{4}$$

$$\overline{Q}(\overline{x},t) = \kappa GA\left(\frac{\partial\overline{w}(\overline{x},t)}{\partial\overline{x}}-\overline{\psi}(\overline{x},t)\right)+\overline{N}(\overline{x})\frac{\partial\overline{w}(\overline{x},t)}{\partial\overline{x}} \tag{5}$$

where A is the area of the cross section, I is the moment of inertia of the cross section about a central axis parallel to $\overline{Z}$, ρ is the density of the beam material, E is the Young's modulus, G is the shear modulus and κ is the shear factor.

Substituting Eqs. (3), (4) and (5) into Eqs. (1) and (2) yields

$$\begin{aligned}&\kappa GA\left(\frac{\partial^2\overline{w}(\overline{x},t)}{\partial\overline{x}^2}-\frac{\partial\overline{\psi}(\overline{x},t)}{\partial\overline{x}}\right)-\rho A\frac{\partial^2\overline{w}(\overline{x},t)}{\partial t^2}\\&+\overline{\eta}^2\rho A\left[\left(\frac{L^2}{2}+\overline{R}L-\frac{\overline{x}^2}{2}-\overline{R}\overline{x}\right)\frac{\partial\overline{w}(\overline{x},t)}{\partial\overline{x}}-(\overline{R}+\overline{x})\frac{\partial^2\overline{w}(\overline{x},t)}{\partial\overline{x}^2}\right] = 0\end{aligned} \tag{6}$$

$$EI\frac{\partial^2\overline{\psi}(\overline{x},t)}{\partial\overline{x}^2}+\kappa AG\left(\frac{\partial\overline{w}(\overline{x},t)}{\partial\overline{x}}-\overline{\psi}(\overline{x},t)\right)-\rho I\frac{\partial^2\overline{\psi}(\overline{x},t)}{\partial t^2}+\overline{\eta}^2\rho I\overline{\psi}(\overline{x},t) = 0 \tag{7}$$

For free vibrations at ω frequency and assuming a simply harmonic motion of the beam, it can be written

$$\overline{w}(\overline{x},t) = \overline{W}(\overline{x})e^{i\omega t};\quad \overline{\psi}(\overline{x},t) = \overline{\Psi}(\overline{x})e^{i\omega t} \tag{8a,b}$$

The length scales are adimensionalized by the length of the beam so that dimensionless quantities are given by

$$W = \frac{\overline{W}(\overline{x})}{L};\quad x = \frac{\overline{x}}{L};\quad R = \frac{\overline{R}}{L} \tag{9a-c}$$

Eqs. (6) and (7) may be expressed in dimensionless form with the choice

$$s = \frac{L}{\sqrt{I/A}};\quad \eta = \sqrt{\frac{\rho A}{EI}}L^2\overline{\eta};\quad \Omega = \sqrt{\frac{\rho A}{EI}}L^2\omega \tag{10a-c}$$

$$\frac{\kappa s^2}{2(1+\nu)}\left(\frac{\partial^2 W}{\partial x^2}-\frac{\partial\Psi}{\partial x}\right)+\eta^2\left[\left(\frac{1}{2}+R-\frac{x^2}{2}-Rx\right)\frac{\partial W^2}{\partial x^2}-(R+x)\frac{\partial W}{\partial x}\right]+\Omega^2 W = 0 \tag{11}$$

$$\frac{\kappa}{2(1+\nu)}s^4\frac{dW}{dx}+s^2\frac{d^2\Psi}{dx^2}-\frac{\kappa}{2(1+\nu)}s^4\Psi+\eta^2\Psi+\Omega^2\Psi = 0 \tag{12}$$

where s is the slenderness ratio, η is the non-dimensional rotational speed parameter and Ω is the dimensionless natural frequency coefficient.

The axial force, the shear force and the bending moment in their dimensionless form are expressed as follows

$$N = \frac{\overline{N}(\bar{x})}{EA}; \quad Q = \frac{\overline{Q}(\bar{x})}{EA}; \quad M = \frac{L}{EI}\overline{M}(\bar{x}) \tag{13a-c}$$

The boundary conditions of the beam are

$$W|_{x=0} = 0, \quad \Psi|_{x=0} = 0 \tag{14a,b}$$

$$Q|_{x=1} = 0, \quad M|_{x=1} = 0 \tag{15a,b}$$

The Eq. (15) can be written as

$$\left[\left(N + \frac{\kappa}{2(1+\nu)}\right)\frac{dW}{dx} - \frac{\kappa}{2(1+\nu)}\Psi\right]_{x=1} = 0 \tag{16a}$$

$$\left.\frac{d\Psi}{dx}\right|_{x=1} = 0 \tag{16b}$$

3. Differential quadrature method

In order to obtain the DQM analog equations to the governing equations of the rotating beam and its boundary conditions, the beam domain is discretized in a grid of points (Fig. 2) using the Chebyshev-Gauss-Lobato expression (Shu 1999)

$$x_i = \frac{1 - \cos[(i-1)\pi/(n-1)]}{2}; \quad i = 1, 2, \ldots, n \tag{17}$$

where n is the number of discrete points or nodes and x_i is the coordinate of node i.

The discretization of a derivative of order q at a node i of the grid, based on the quadrature laws (Bert and Malik 1996), can be expressed by

$$\left.\frac{d^{(q)}W}{dx^q}\right|_{x_i} = \sum_{j=1}^{n} C_{ij}^{(q)} W_j \tag{18}$$

$$\left.\frac{d^{(q)}\Psi}{dx^q}\right|_{x_i} = \sum_{j=1}^{n} C_{ij}^{(q)} \Psi_j \tag{19}$$

where the C_{ij} are the weighting coefficients of those linear algebraic equations (Bert and Malik 1996).

The C_{ij} coefficients may be arranged in a matrix [**C**] of order n. They are obtained using Lagrange interpolating functions (Karami *et al.* 2003)

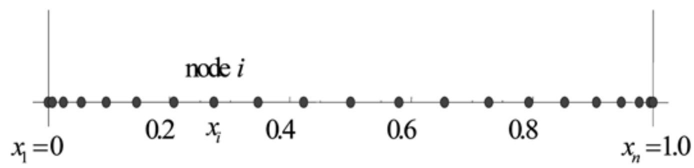

Fig. 2 Grid of n points

$$\Pi(x_i) = \prod_{j=1\ j\neq i}^{n} (x_i - x_j) \tag{20}$$

The DQM weighting coefficients of the matrix [**C**] can be calculated by

$$C_{ij}^{(1)} = \frac{\Pi(x_i)}{(x_i - x_j)\Pi(x_j)} \tag{21a}$$

$$C_{ij}^{(q)} = C_{ii}^{(q-1)} C_{ij}^{(1)} - \frac{C_{ij}^{(q-1)}}{x_i - x_j} \tag{21b}$$

with $i,j = 1, 2, \ldots, n$, for $i = j$ and

$$C_{ii}^{(1)} = -\sum_{j=1\ j\neq i}^{n} C_{ij}^{(1)} \tag{22a}$$

$$C_{ii}^{(q)} = -\sum_{\substack{j=1 \\ \text{with } j\neq i}}^{n} C_{ij}^{(q)} \tag{22b}$$

with $i,j = 1, 2, \ldots, n$, for $i = j$.

Using the quadrature laws, (18), (19), the DQ analogs of the governing Eqs. (11), (12) of node i become

$$\eta^2 (R + x_i) \sum_{j=1}^{n} (C_{ij}^{(1)}) W_j - s^2 N_i \sum_{j=1}^{n} (C_{ij}^{(2)}) W_j - \frac{\kappa}{2(1+\nu)} s^2 \sum_{j=1}^{n} C_{ij}^{(2)} W_j + \frac{\kappa}{2(1+\nu)} s^2 \sum_{j=1}^{n} C_{ij}^{(1)} \Psi_j = \Omega^2 W_i \tag{23}$$

$$-\frac{\kappa}{2(1+\nu)} s^4 \sum_{j=1}^{n} C_{ij}^{(1)} W_j - s^2 \sum_{j=1}^{n} C_{ij}^{(2)} \Psi_j + \left(\frac{\kappa}{2(1+\nu)} s^4 - \eta^2\right) \sum_{j=1}^{n} C_{ij}^{(0)} \Psi_j = \Omega^2 \Psi_i \tag{24}$$

with $i = 2, 3, \ldots, n-1$

In a similar way the DQ analogs of the boundary conditions are for $i = 1$

$$\sum_{j=1}^{n} C_{1j}^{(0)} W_j = 0 \tag{25a}$$

$$\sum_{j=1}^{n} C_{1j}^{(0)} \Psi_j = 0 \tag{25b}$$

and for $i = n$

$$\frac{\kappa}{2(1+\nu)} \sum_{j=1}^{n} C_{nj}^{(1)} W_j - \frac{\kappa}{2(1+\nu)} \sum_{j=1}^{n} C_{nj}^{(0)} \Psi_j = 0 \tag{26a}$$

$$\sum_{j=1}^{n} C_{nj}^{(1)} \Psi_j = 0 \tag{26b}$$

The linear equation system obtained by the DQM is used to determine the natural frequencies of the rotating beam.

The number of terms n taken in the summations had been studied for many situations and the system has acceptable convergence by $n = 21$ terms.

4. Finite element method

The independent results for the natural frequencies, were obtained by a finite element code ALGOR. The finite element model employed in the analysis has 1000 beam elements of two nodes in the longitudinal direction. The number of elements was proved to be enough to capture accurately the dynamic behaviour of the rotating beam by a previous convergence analysis. This beam model also takes into account the axial deformation induced by the centrifugal force.

5. Numerical results

The DQ summation with $n = 23$ terms or more, shows none significant improvement in accuracy over the summation with $n = 21$. Therefore, an approximation of twenty one nodes is used through out this work.

The results obtained by the present procedure are compared with published results by Hodges and Rutkowski (1981), Mei (2008), Banerjee (2001), Naguleswaran (1994) and Senatore (2006).

In Table 1 the first three natural frequency coefficients of an Euler-Bernoulli rotating beam are given. The results correspond to four combinations of the dimensionless radius R and rotational speeds η. It was taken $s \to \infty$ to represent an Euler-Bernoulli beam model with the present formulation. As it is seen, all the Ω coefficients are in good agreement with those of Hodges and Rutkowski (1981) and Mei (2008).

Table 1 Natural frequency coefficients of a rotating Bernoulli-Euler beam for different values of R and η (a comparison of results with references Hodges and Rutkowski (1981) and Mei (2008)

R	η		Present study	Hodges and Rutkowski (1981)	Mei (2008)
0	0	Ω_1	3.5160	*3.5160*	*3.5160*
		Ω_2	22.0343	*22.0345*	*22.0345*
		Ω_3	61.6957	*61.6972*	*61.6972*
0	12	Ω_1	13.1701	*13.1702*	--
		Ω_2	37.6030	*37.6031*	--
		Ω_3	79.6141	*79.6145*	--
3	4	Ω_1	10.2366	--	*10.2368*
		Ω_2	31.6047	--	*31.6049*
		Ω_3	72.5825	--	*72.5831*
1	15	Ω_1	24.4091	--	*24.4092*
		Ω_2	61.4370	--	*61.4371*
		Ω_3	113.4883	--	*113.4889*

Table 2 Natural frequencies of a rotating beam. with $R = 0$ (comparison between theoretical DQM, Naguleswaran (1994) and experimental results Senatore (2006)

η	f_i (Hz)	Theoretical results: Present study	Theoretical results: Naguleswaran (1994)	Experimental results Senatore (2006): Two weights system	Experimental results Senatore (2006): Three weights system
1	f_1	26.3	26.3	28.0	29.0
	f_2	158.8	158.5	158.0	158.1
	f_3	442.0	442.0	440.0	441.0
2	f_1	29.6	29.6	37.0	34.5
	f_2	162.0	161.6	163.3	161.3
	f_3	446.0	445.0	446.0	444.5

Table 3 Natural frequency coefficients of a rotating Timoshenko beam for various values of $1/s$ and η, with $E/\kappa G = 4$ and $R = 0$

$\frac{1}{s}$		$\eta = 0$ DQM	$\eta = 0$ Banerjee (2001)	$\eta = 4$ DQM	$\eta = 4$ Banerjee (2001)	$\eta = 8$ DQM	$\eta = 8$ Banerjee (2001)	$\eta = 12$ DQM	$\eta = 12$ Banerje (2001)
0	Ω_1	3.5160	*3.5160*	5.5850	*5.5850*	9.2568	*9.2568*	13.1701	*13.1700*
	Ω_2	22.0342	-	24.2730	-	29.9951	-	37.6028	-
	Ω_3	61.6953	-	63.9649	-	70.2911	-	79.6126	-
	Ω_4	120.8950	-	123.2546	-	130.0423	-	140.5277	-
	Ω_5	199.8413	-	202.2586	-	209.3207	-	220.5189	-
	Ω_6	298.5158	-	300.9731	-	308.2127	-	319.8658	-
0.03	Ω_1	3.4799	*3.4799*	5.5332	*5.5332*	9.1549	*9.1549*	12.9979	*12.9980*
	Ω_2	20.5903	-	22.8621	-	28.5854	-	36.0862	-
	Ω_3	53.3288	-	55.7723	-	62.4421	-	71.9944	-
	Ω_4	95.2015	-	97.9650	-	105.7240	-	117.2672	-
	Ω_5	142.8889	-	145.9937	-	154.8324	-	168.2568	-
	Ω_6	194.0228	-	197.4923	-	207.4388	-	222.7148	-
0.06	Ω_1	3.3787	*3.3787*	5.3954	*5.3954*	8.9209	*8.9208*	12.6724	*12.6720*
	Ω_2	17.5470	-	19.9662	-	25.8362	-	33.2672	-
	Ω_3	40.7447	-	43.7365	-	51.4154	-	61.6011	-
	Ω_4	66.3623	-	70.1298	-	79.9414	-	93.0672	-
	Ω_5	93.1125	-	97.6079	-	109.3074	-	124.5620	-
	Ω_6	119.1146	-	123.9631	-	135.3113	-	144.2550	-
0.09	Ω_1	3.2302	*3.2302*	5.2104	*5.2104*	8.6588	*8.6588*	12.3532	*12.3530*
	Ω_2	14.5410	-	17.1973	-	23.1957	-	30.1831	-
	Ω_3	31.6693	-	35.3058	-	43.6601	-	53.0860	-
	Ω_4	48.5298	-	52.8246	-	60.9317	-	65.7861	-
	Ω_5	64.2692	-	67.4909	-	73.6548	-	82.8283	-
	Ω_6	70.9776	-	74.2543	-	81.0611	-	87.0248	-

Table 2 shows a comparison between theoretical, DQM, Naguleswaran (1994), and experimental Senatore (2006) results performed on aluminum beams with sizes 310 mm × 30 mm × 3 mm (rectangular section with I = 6450 mm^4). There is a good agreement for the second and third natural frequencies with both simulated rotational speeds $\eta = 1$ and $\eta = 2$. The differences between theoretical and experimental first natural frequencies are probably caused by the way the rotational speed was simulated in the experimental procedure.

Table 4 Natural frequency coefficients of a Timoshenko beam with $s = 60$ and $\kappa = 0.907484$ ($\eta = 0$)

	DQM	FEM
Ω_1	3.507	3.507
Ω_2	21.661	21.661
Ω_3	59.307	59.306
Ω_4	112.709	112.707
Ω_5	179.636	179.632
Ω_6	257.629	257.623

Table 5 Natural frequency coefficients of a rotating Timoshenko beam with $s = 60$ and $\kappa = 0.90748406$ for various values of the rotating speed η and R = 0, 1 and 2

		$\eta = 1$		$\eta = 5$		$\eta = 10$		$\eta = 15$	
R		DQM	FEM	DQM	FEM	DQM	FEM	DQM	FEM
0	Ω_1	*3.6727*	3.6730	*6.4338*	6.4388	*11.1667*	11.1769	*16.0768*	16.0923
	Ω_2	*21.8082*	21.8083	*25.0738*	25.0846	*33.1240*	33.2691	*43.4488*	43.5129
	Ω_3	*59.4534*	59.4526	*62.8524*	62.8634	*72.3310*	72.3738	*85.5566*	85.6433
	Ω_4	*112.8600*	112.8608	*116.4790*	116.4938	*127.0068*	127.0677	*142.5433*	142.6687
	Ω_5	*179.7960*	179.7934	*183.5970*	183.6203	*194.9093*	195.0065	*212.1742*	212.3757
	Ω_6	*257.7970*	257.7923	*261.7760*	261.8134	*273.7660*	273.9218	*292.4359*	292.7616
1.00	Ω_1	*3.8798*	3.8804	*8.9195*	8.9260	*16.5502*	16.5639	*24.3044*	24.3270
	Ω_2	*22.0031*	22.0036	*28.9782*	28.9956	*43.8887*	43.9406	*60.7725*	60.8675
	Ω_3	*59.6581*	59.6579	*67.4510*	67.4724	*86.7856*	86.8592	*110.7750*	110.9196
	Ω_4	*113.0830*	113.0817	*121.6340*	121.6671	*144.4770*	144.5949	*174.7380*	174.9639
	Ω_5	*180.0290*	180.0278	*189.1660*	189.2245	*214.6540*	214.8588	*250.1030*	250.4859
	Ω_6	*258.0420*	258.0401	*267.7050*	267.8016	*295.3550*	295.7007	*335.1210*	335.7619
2.00	Ω_1	*4.0763*	4.0771	*10.8350*	10.8427	*20.5352*	20.5523	*30.3287*	30.3585
	Ω_2	*22.1963*	22.1971	*32.3772*	32.4004	*52.2384*	52.3058	*73.8578*	73.9824
	Ω_3	*59.8620*	59.8622	*71.6920*	71.7227	*98.7175*	98.8187	*130.3302*	130.5293
	Ω_4	*113.3020*	113.3021	*126.5210*	126.5714	*159.5250*	159.6901	*200.5160*	200.8254
	Ω_5	*180.2620*	180.2619	*194.5290*	194.6187	*232.2130*	232.5040	*281.4570*	281.9747
	Ω_6	*258.2870*	258.2876	*273.4630*	273.6151	*314.9790*	315.4739	*371.3390*	372.1979

The two theoretical results (DQM and reference Naguleswaran (1994)) exhibit an excellent agreement.

The first six dimensionless frequency coefficients of a rotating Timoshenko beam for four rotational speeds and various values of s are tabulated in Table 3. Note that the fundamental frequency Ω_1 is in complete agreement with those of reference (Banerjee 2001) for $R = 0$, $\kappa = 2/3$ and $E/G = 8/3$.

In Table 4 the first six natural frequencies obtained through the present analysis are compared to those obtained by employing a finite element code Algor for $\eta = 0$, $s = 60$ and $\kappa = 0.907484$. The

Table 6 First six frequency coefficients for rotating beams considering Timoshenko and Euler-Bernoulli beam theories ($R = 0$ and $\eta = 0$, 1, 2 and 3)

		Timoshenko						Euler-Bernoulli	
		$s = 30$			$s = 70$				
η		$\kappa = 0.84967$	$\kappa = 0.88636$	$\kappa = 0.90748$	$\kappa = 0.84967$	$\kappa = 0.88636$	$\kappa = 0.90748$	$s \to \infty$	Chung (2002)
0	Ω_1	3.4798	3.4809	3.4815	3.5093	3.5095	3.5096	3.5160	*3.5160*
	Ω_2	20.5888	20.6298	20.6519	21.7448	21.7536	21.7584	22.0331	*22.0345*
	Ω_3	53.3382	53.5483	53.6626	59.8241	59.8798	59.9100	61.6878	*61.6972*
	Ω_4	95.2734	95.8385	96.1471	114.4060	114.5930	114.6940	120.8670	-
	Ω_5	143.1290	144.2260	144.8280	183.6600	184.1080	184.3500	199.8650	-
	Ω_6	194.6150	196.3940	197.3720	265.3890	266.2600	266.7340	298.5350	-
1	Ω_1	3.6445	3.6456	3.6462	3.6747	3.6750	3.6750	3.6816	*3.6816*
	Ω_2	20.7373	20.7780	20.8000	21.8916	21.9004	21.9051	22.1809	*22.1810*
	Ω_3	53.4926	53.7020	53.8159	59.9701	60.0258	60.0558	61.8406	*61.8450*
	Ω_4	95.4454	96.0091	96.3170	114.5590	114.7450	114.8460	121.0470	-
	Ω_5	143.3200	144.4160	145.0160	183.8190	184.2660	184.5080	200.0960	-
	Ω_6	194.8270	196.6020	197.5790	265.5520	266.4230	266.8960	298.8600	-
2	Ω_1	4.0971	4.0982	4.0988	4.1298	4.1300	4.1301	4.1373	*4.1373*
	Ω_2	21.1762	21.2163	21.2379	22.3261	22.3348	22.3395	22.6148	*22.6149*
	Ω_3	53.9528	54.1601	54.2729	60.4060	60.4613	60.4911	62.2721	*62.2732*
	Ω_4	95.9591	96.5187	96.8245	115.0160	115.2010	115.3020	121.4930	-
	Ω_5	143.8920	144.9810	145.5780	184.2940	184.7390	184.9810	200.5510	-
	Ω_6	195.4590	197.2260	198.1980	266.0410	266.9100	267.3820	299.3160	-
3	Ω_1	4.7516	4.7528	4.7534	4.7887	4.7889	4.7891	4.7971	*4.7973*
	Ω_2	21.8880	21.9270	21.9481	23.0322	23.0407	23.0454	23.3201	*23.3203*
	Ω_3	54.7096	54.9137	55.0247	62.1246	61.1793	61.2089	62.9838	*62.9850*
	Ω_4	96.8077	97.3607	97.6629	115.7730	115.9570	116.0570	122.2320	-
	Ω_5	144.8380	145.9170	146.5090	185.0820	185.5260	185.7660	201.3060	-
	Ω_6	196.5060	198.2590	199.2230	266.8550	267.7200	268.1900	300.0740	-

coefficients of the two sets of results, show only minor discrepancies.

In Table 5 two sets of results for the first six natural frequency coefficients of rotating beams with $s = 60$ and $\kappa = 0.907484$ for a combination of R and η are compared.

The first column presents the numerical results obtained by using the DQM and the second column provides the finite element results (FEM). Again like in Table 4, the differences between them show no practical differences.

The next Tables 6, 7, 8 and 9 present a collection of frequency coefficients for various values of η with $R = 0$. The coefficients of the last two columns of the Tables correspond to the Euler-Bernoulli

Table 7 First six frequency coefficients for rotating beams considering Timoshenko and Euler-Bernoulli beam theories ($R = 0$ and $\eta = 4$, 5, 6, and 7)

		Timoshenko						Euler-Bernoulli	
		$s = 30$			$s = 70$				
η		$\kappa = 0.84967$	$\kappa = 0.88636$	$\kappa = 0.90748$	$\kappa = 0.84967$	$\kappa = 0.88636$	$\kappa = 0.90748$	$s \to \infty$	Chung (2002)
4	Ω_1	5.5314	5.5327	5.5335	5.5749	5.5752	5.5753	5.5851	*5.5850*
	Ω_2	22.8464	22.8841	22.9046	23.9856	23.9940	23.9985	24.2732	*24.2733*
	Ω_3	55.7489	55.9486	56.0572	62.1150	62.1690	62.1981	63.9657	*63.9668*
	Ω_4	97.9804	98.5244	98.8218	116.8230	117.0050	117.1040	123.2580	-
	Ω_5	146.1500	147.2140	147.7980	186.1790	186.6200	186.8590	202.3580	-
	Ω_6	197.9590	199.6920	200.6460	267.9880	268.8500	269.3170	301.1330	-
5	Ω_1	6.3858	6.3874	6.3883	6.4375	6.4378	6.4380	6.4496	*6.4495*
	Ω_2	24.0209	24.0573	24.0770	25.1579	25.1660	25.1704	25.4459	*25.4460*
	Ω_3	57.0526	57.2470	57.3528	63.3628	63.4159	63.4446	65.2040	*65.2050*
	Ω_4	99.4626	99.9958	100.2870	118.1560	118.3370	118.4340	124.5630	-
	Ω_5	147.8140	148.8610	149.4360	187.5790	188.0160	188.2520	203.7010	-
	Ω_6	199.8060	201.5160	202.4560	269.4370	270.2930	270.7580	302.4870	-
6	Ω_1	7.2846	7.2865	6.3883	7.3459	7.3463	7.3465	7.3605	*7.3604*
	Ω_2	25.3803	25.4153	24.0770	26.5192	26.5271	26.5314	26.8089	*26.8089*
	Ω_3	58.5996	58.7881	57.3528	64.8513	64.9033	64.9315	66.6829	*66.6829*
	Ω_4	101.2370	101.7580	100.2870	119.7630	119.9410	120.0370	126.1370	-
	Ω_5	149.8150	150.8420	149.4360	189.2720	189.7050	189.9390	205.3290	-
	Ω_6	202.0330	203.7150	202.4560	271.1960	272.0460	272.5070	304.1340	-
7	Ω_1	8.2102	8.2124	8.2136	8.2824	8.2829	8.2831	8.2995	*8.2996*
	Ω_2	26.8948	26.9286	26.9469	28.0409	28.0487	28.0529	28.3339	*28.3341*
	Ω_3	60.3677	60.5500	60.6493	66.5619	66.6129	66.6405	68.3849	*68.3860*
	Ω_4	103.2860	103.7930	104.0700	121.6290	121.8050	121.9000	127.9700	-
	Ω_5	152.1350	153.1400	153.6910	191.2510	191.6790	191.9110	207.2350	-
	Ω_6	204.6230	206.2720	207.1800	273.2580	274.1000	274.5570	306.0680	-

beam theory, the others columns contain coefficients obtained by using the Timoshenko beam theory. As it is known the Euler-Bernoulli model tends to overestimate the natural frequencies. This situation is increased for the natural frequencies of the higher modes. It can be stated that Euler-Bernoulli theory's prediction is better for slender beams.

On the other hand as Timoshenko models consider the effects of shear deformation and rotatory inertia, its estimate of the natural frequencies improves considerably for non-slender beams and for higher frequencies.

Table 8 First six frequency coefficients for rotating beams considering Timoshenko and Euler-Bernoulli beam theories. ($R = 0$ and $\eta = 8$, 9, 10 and 11)

		Timoshenko						Euler-Bernoulli	
		$s = 30$			$s = 70$				
η		$\kappa = 0.84967$	$\kappa = 0.88636$	$\kappa = 0.90748$	$\kappa = 0.84967$	$\kappa = 0.88636$	$\kappa = 0.90748$	$s \to \infty$	Chung (2002)
8	Ω_1	9.1523	9.1549	9.1564	9.2365	9.2370	9.2373	9.2569	*9.2568*
	Ω_2	28.5382	28.5709	28.5886	29.6973	29.7050	29.7091	29.9952	*29.9954*
	Ω_3	62.3346	62.5105	62.6062	68.4758	68.5257	68.5527	70.2919	*70.2930*
	Ω_4	105.5870	106.0790	106.3480	123.7420	123.9140	124.0080	130.0470	-
	Ω_5	154.7570	155.7360	156.2740	193.5040	193.9260	194.1550	209.4080	-
	Ω_6	207.5570	209.1710	210.0590	275.6150	276.4480	276.9000	308.2830	-
9	Ω_1	10.1049	10.1079	10.1096	10.2019	10.2025	10.2029	10.2256	*10.2257*
	Ω_2	30.2878	30.3195	30.3367	31.4658	31.4734	31.4774	31.7703	*31.7705*
	Ω_3	64.4784	64.6477	64.7400	70.5740	70.6228	70.6492	72.3857	*72.3867*
	Ω_4	108.1220	108.5980	108.8590	126.0850	126.2550	126.3470	132.3560	-
	Ω_5	157.6580	158.6120	159.1350	196.0200	196.4370	196.6630	211.8410	-
	Ω_6	210.8150	212.3910	213.2589	278.2570	279.0810	279.5280	310.7720	-
10	Ω_1	11.0643	11.0678	11.0697	11.1747	11.1755	11.1760	11.2023	*11.2023*
	Ω_2	32.1249	32.1557	32.1724	33.3272	33.3348	33.3389	33.6401	*33.6404*
	Ω_3	66.7785	66.9414	67.0301	72.8379	72.8858	72.9116	74.6479	*74.6493*
	Ω_4	110.8680	111.3280	111.5800	128.6450	128.8120	128.9020	134.6479	-
	Ω_5	160.8198	161.7470	162.2550	198.7870	199.1980	199.4200	214.5180	-
	Ω_6	214.3758	215.9140	216.7600	281.1760	281.9890	282.4310	313.5200	-
11	Ω_1	12.0282	12.0322	12.0344	12.1527	12.1536	12.1541	12.1843	*12.1842*
	Ω_2	34.0338	34.0639	34.0803	35.2658	35.2733	35.2774	35.5888	*35.5890*
	Ω_3	69.2160	69.3726	69.4580	75.2504	75.2972	75.3226	77.0628	*77.0638*
	Ω_4	113.8060	114.2510	114.4940	131.4060	131.5690	131.6580	137.6120	-
	Ω_5	164.2230	165.1210	165.6140	201.7930	202.1970	202.4160	217.4410	-
	Ω_6	218.2210	219.7180	220.5420	284.3610	285.1630	285.5990	316.5430	-

Table 9 First six frequency coefficients for rotating beams considering Timoshenko and Euler-Bernoulli beam theories. ($R = 0$ and $\eta = 12$, 13, 14 and 15)

		Timoshenko						Euler-Bernoulli	
		$s = 30$			$s = 70$				
η		$\kappa = 0.84967$	$\kappa = 0.88636$	$\kappa = 0.90748$	$\kappa = 0.84967$	$\kappa = 0.88636$	$\kappa = 0.90748$	$s \to \infty$	Chung (2002)
12	Ω_1	12.9954	12.9998	13.0022	13.1341	13.1352	13.1358	13.1701	*13.1702*
	Ω_2	36.0022	36.0317	36.0477	37.2683	37.2759	37.2801	37.6029	*37.6031*
	Ω_3	71.7739	71.9245	72.0066	77.7951	77.8411	77.8659	79.6134	*79.6145*
	Ω_4	116.9170	117.3460	117.5810	134.3520	134.5120	134.5990	140.5320	-
	Ω_5	167.8450	168.7150	169.1930	205.0240	205.4220	205.6370	220.5860	-
	Ω_6	222.3290	223.7840	224.5850	287.8010	288.5930	289.0230	319.8100	-
13	Ω_1	13.9648	13.9697	13.9723	14.1181	14.1193	14.1200	14.1586	*14.1587*
	Ω_2	38.0198	38.0487	38.0645	39.3240	39.3318	39.3360	39.6717	*39.6720*
	Ω_3	74.4370	74.5818	74.6608	80.4574	80.5026	80.5270	82.2855	*82.2866*
	Ω_4	120.1840	120.5980	120.8240	137.4690	137.6270	137.7120	143.6280	-
	Ω_5	171.6680	172.5110	172.9730	208.4680	208.8590	209.0710	223.9470	-
	Ω_6	226.6790	228.0920	228.8700	291.4860	292.2660	292.6900	323.3180	-
14	Ω_1	14.9359	14.9412	14.9441	15.1038	15.1052	15.1060	15.1493	*15.1494*
	Ω_2	40.0784	40.1070	40.1225	41.4243	41.4322	41.4364	41.7864	*41.7867*
	Ω_3	77.1919	77.3312	77.4072	83.2239	83.2684	83.2924	85.0659	*85.0670*
	Ω_4	123.5900	123.9880	124.2060	140.7440	140.8980	140.9810	146.8860	-
	Ω_5	175.6750	176.4890	176.9370	212.1120	212.4960	212.7040	227.5120	-
	Ω_6	231.2510	232.6230	233.3780	295.4040	296.1730	296.5900	327.0610	-
15	Ω_1	15.9084	15.9141	15.9172	16.0909	16.0925	16.0933	16.1416	*16.1416*
	Ω_2	42.1715	42.1997	42.2151	43.5618	43.5699	43.5743	43.9396	*43.9399*
	Ω_3	80.0270	80.1610	80.2342	86.0827	86.1266	86.1503	87.9425	*87.9436*
	Ω_4	127.1200	127.5040	127.7130	144.1620	144.3130	144.3950	150.2940	-
	Ω_5	179.8470	180.6340	181.0660	215.9430	216.3200	216.5250	231.2700	-
	Ω_6	236.0280	237.3580	238.0900	299.5450	300.3020	300.7130	331.0280	-

6. Conclusions

The convergence demonstrated by the presented results and its close agreement to published results validates the efficiency of the differential quadrature development and its implementation. The differential quadrature method has the same advantage as the finite element method and it needs less computer memory requirements than the FEM. In free vibration problems the DQM matrices are simply to construct and to use, they are also easy to implement in a PC and the computational effort can be considered trivial.

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