REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 54, No. 1, 2013, Pages 83–107
Published online: June 24, 2013

MAXIMAL OPERATORS ASSOCIATED WITH GENERALIZED HERMITE POLYNOMIAL AND FUNCTION EXPANSIONS

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Abstract. We study the weak and strong type boundedness of maximal heat–diffusion operators associated with the system of generalized Hermite polynomials and with two different systems of generalized Hermite functions. We also give a necessary background to define Sobolev spaces in this context.

1. Introduction

The generalized Hermite polynomials $H_n^\mu(x)$ of degree n were defined for $x \in \mathbb{R}$ by G. Szëgo in [21, problem 25, p. 380] as being an orthogonal family of polynomials with respect to the measure $\gamma_\mu(dx) = x^{2\mu}e^{-x^2}\,dx$ on $\mathbb{R}$, with $\mu > -1/2$. In this paper we are going to consider a normalization of these polynomials that were defined by Rosenblum [16]. When $\mu = 0$ these polynomials coincide with the classical Hermite polynomials and the behavior of these maximal operators have been studied for this particular case in [1, 6, 11, 12, 17].

To extend them to $x \in \mathbb{R}^d$ we can do it as a tensor product of the one-dimensional generalized Hermite polynomials. Indeed, set $\mu = (\mu_1, \ldots, \mu_d)$ with $\mu_k > -1/2$ for all $k = 1, \ldots, d$, $x \in \mathbb{R}^d$ and for the multi-index $n = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$ we define the d-dimensional generalized Hermite polynomial of degree $|n| = n_1 + \ldots + n_d$ as

$$H_n^\mu(x) = \prod_{k=1}^d H_{n_k}^{\mu_k}(x_k).$$

In this way these polynomials are orthogonal in L^2 with respect to the measure

$$\gamma_\mu(dx) = \prod_{k=1}^d x_k^{2\mu_k} e^{-|x|^2}\,dx;$$

and are eigenfunctions with eigenvalues equal to $-|n|$ for every $n \in \mathbb{N}_0^d$ of the differential-difference operator

$$\mathcal{L}_\mu = \sum_{k=1}^d L_{\mu_k}, \tag{1.1}$$

2010 *Mathematics Subject Classification.* 42C05, 42C15.
Key words and phrases. Generalized Hermite polynomials and functions; heat–diffusion semigroups; maximal operators.

with

$$L_{\mu_k}\phi(x) = \frac{1}{2}\mathfrak{D}^2_{\mu_k}\phi(x) - x\mathfrak{D}_{\mu_k}\phi(x) - \mu_k(\phi(x) - \phi\circ\sigma_k(x));$$

being

$$\mathfrak{D}_{\mu_k}\phi(x) = \frac{\partial\phi}{\partial x_k}(x) + \frac{\mu_k}{x_k}(\phi(x) - \phi\circ\sigma_k(x)),$$

and σ_k the reflection with respect to the hyperplane $\{x_k = 0\}$, i.e., $\sigma_k(x_1, \dots, x_k, \dots, x_d) = (x_1, \dots, -x_k, \dots, x_d)$.

Associated to this differential-difference operator we have the diffusion semigroup $T_t^\mu = e^{\mathcal{L}_\mu t}$ which applied to functions f in $L^p(\mathbb{R}^d, \gamma_\mu)$, $p \geq 1$, solves the heat–diffusion equation with initial data f, that is, if $u(x,t) = T_t^\mu f(x)$ then

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{L}_\mu u, & t > 0, \\ u(x,0) = f(x). \end{cases} \tag{1.2}$$

Our goal in this paper is to give sense to the second equality $u(x,0) = f(x)$, which means to prove the convergence $u(\cdot,t) \to f$ as $t \to 0^+$ in the almost everywhere sense. This is a consequence of the L^p-boundedness of the maximal operator

$$T_*^\mu f(x) = \sup_{t>0} |T_t^\mu f(x)|,$$

stated in Theorem 1.3, whose proof is given in section 2.

Theorem 1.3. *For $\mu \in (-1/2, \infty)^d$, the operator T_*^μ is of weak-type $(1,1)$ and strong-type (p,p) for $p > 1$ with respect to γ_μ.*

Corollary 1.4. *For $\mu \in (-1/2, \infty)^d$ and every $f \in L^1(\mathbb{R}^d, \gamma_\mu)$,*

$$\lim_{t\to 0^+} T_t^\mu f = f \quad a.e.$$

This corollary is an immediate consequence of the first part of Theorem 1.3 and the fact that generalized Hermite polynomials are dense in $L^1(\mathbb{R}^d, \gamma_\mu)$, see [19].

Let us point out that Theorem 1.3 was proved in [2] for $d = 1$.

It is known that to study the weak formulation of a second order differential-difference operator of type (1.1) it is required to obtain the appropriate Sobolev spaces associated with $\mathcal{L}_\mu$. To define these Sobolev spaces we need to study the boundedness of the Riesz and Bessel potentials defined respectively as:

$$I_\beta = (-\mathcal{L}_\mu)^{-\beta},$$

and

$$B_\beta = (I - \mathcal{L}_\mu)^{-\beta},$$

for $\beta > 0$. Boundedness properties of these operators for $\mu = 0$ can be found in [4, 5]. On the other hand, boundedness results for any μ are given in Corollary 1.6, which is a consequence of the hypercontractivity of T_t^μ given in the following theorem.

Theorem 1.5. *For $\mu \in (-1/2, \infty)^d$, the diffusion semigroup T_t^μ is hypercontractive.*

Corollary 1.6. *For $\mu \in (-1/2, \infty)^d$, the Bessel and Riesz potentials associated to $\mathcal{L}_\mu$ are of strong-type (p, p), $1 < p < \infty$, with respect to the measure γ_μ.*

As in the case of other orthogonal systems, one can study the heat–diffusion semigroups associated to differential-difference operators whose eigenfunctions are the functions associated with the generalized Hermite polynomials.

In the context of generalized Hermite polynomials there are at least two different families of orthogonal functions. Namely, the one-dimensional generalized Hermite functions introduced by M. Rosenblum in [16] and the one-dimensional generalized Hermite functions introduced by G. Szegö in [21, problem 25, p. 380]. In the case of $\mu = 0$ we have just one family of orthogonal functions, the so called Hermite functions, and the behavior of the maximal operators associated with these function expansions can be found in [20].

Here we are going to define directly d-dimensional Hermite functions as a tensor product of one-dimensional generalized Hermite functions. For $\mu \in (-1/2, \infty)^d$, $n \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$ we define

$$\psi_n^\mu(x) = \left(\prod_{k=1}^{d} \frac{\eta_{\mu_k}(n_k)}{\Gamma(\mu_k + 1/2)} \right)^{1/2} \frac{1}{2^{|n|/2} n!} H_n^\mu(x) e^{-|x|^2/2},$$

where $n! = n_1! \cdots n_d!$, and for $l \in \mathbb{N}_0$ and $\nu > -1/2$, $\eta_\nu(l)$ is the generalized factorial defined as

$$\eta_\nu(2m) = \frac{2^{2m} m! \Gamma(m + \nu + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} = (2m)! \frac{\Gamma(m + \nu + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})},$$
$$\eta_\nu(2m+1) = \frac{2^{2m+1} m! \Gamma(m + \nu + \frac{3}{2})}{\Gamma(\nu + \frac{1}{2})} = (2m+1)! \frac{\Gamma(m + \nu + \frac{3}{2})}{\Gamma(\nu + \frac{1}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(m + \frac{3}{2})}.$$

They are orthogonal with respect to the measure

$$\rho_\mu^\psi(dx) = \prod_{k=1}^{d} x_k^{2\mu_k} \, dx.$$

The other set of functions we can define is

$$\varphi_n^\mu(x) = \psi_n^\mu(x) \prod_{k=1}^{d} |x_k|^{\mu_k},$$

and they are orthogonal with respect to the Lebesgue measure

$$\rho_\mu^\varphi(dx) = dx.$$

These systems of generalized functions are respectively the eigenfunctions of the following differential-difference operators:

$$\mathcal{H}_\mu^\psi = \frac{1}{2}\bigg(\sum_{k=1}^d \mathfrak{D}_{\mu_k}^2 - |x|^2\bigg), \tag{1.7}$$

$$\mathcal{H}_\mu^\varphi = \frac{1}{2}\bigg(\sum_{k=1}^d \bigg[\mathfrak{D}_{\mu_k}^2 - \frac{2\mu_k}{x_k}\bigg(\frac{\partial}{\partial x_k} + \mu_k - 1\bigg)\bigg] - |x|^2\bigg). \tag{1.8}$$

Here and in the sequel, ϑ will denote either ψ or φ. It can be proved that the system $\{\vartheta_n^\mu\}$ is an orthonormal basis on $L^2(\mathbb{R}^d, \rho_\mu^\vartheta)$, for the one-dimensional case see [16, p. 13]. Also, by using the one-dimensional result in [16, (3.7.4)], we get

$$\mathcal{H}_\mu^\vartheta \vartheta_n^\mu(x) = -(|n| + |\mu| + d/2)\vartheta_n^\mu(x),$$

with $|\mu| = \mu_1 + \cdots + \mu_d$.

For these systems we study problems associated with the initial value problem similar to (1.2) for the operators defined in (1.7) and (1.8). We answer this in theorems 1.9, 1.11 and corollaries 1.10, 1.12.

Let

$$T_*^{\mu,\vartheta} f(x) = \sup_{t>0} |T_t^{\mu,\vartheta} f(x)|,$$

with $T_t^{\mu,\vartheta} f(x) = e^{t\mathcal{H}_\mu^\vartheta} f(x)$; then we have:

Theorem 1.9. *(a) For $\mu \in (-1/2, \infty)^d$, $T_*^{\mu,\psi}$ is of weak-type $(1,1)$ and strong-type (p,p) for $p > 1$ with respect to the measure ρ_μ^ψ.*

(b) For $\mu \in [0, \infty)^d$, $T_*^{\mu,\varphi}$ is of weak-type $(1,1)$ and strong-type (p,p) for $p > 1$ with respect to the Lebesgue measure dx.

And as an immediate consequence of this theorem we have:

Corollary 1.10. *With $\mu \in (-1/2, \infty)^d$ for the system $\{\psi_n^\mu\}$ and with $\mu \in [0, \infty)^d$ for the system $\{\varphi_n^\mu\}$, we obtain for $1 \le p < \infty$ and every $f \in L^p(\mathbb{R}^d, \rho_\mu^\vartheta)$,*

$$\lim_{t\to 0^+} T_t^{\mu,\vartheta} f = f \quad a.e.$$

We will see that for $\mu \notin [0, \infty)^d$ the maximal operator associated with $T_t^{\mu,\varphi}$ need not be weak-type $(1,1)$ (see comments after Theorem 1.11). This behavior is also present in one of the Laguerre function systems. For instance, Macías, Segovia and Torrea in [10] showed that the one-dimensional maximal semigroup associated with the system $\{L_n^\alpha(x) x^{\alpha/2} e^{-x/2}\}$ on $(0, \infty)$ for $-1 < \alpha < 0$ fails to be weak-type $(1,1)$, and investigated in detail its boundedness properties on L^p for a restrictive range of p's. Nowak and Sjögren in [14] observe that in higher dimensions, by using a simple argument, there is no weak-type inequality for this heat–diffusion semigroup maximal operator either. In a recent work by Nowak and Sjögren in [15, Theorem 1.3] they completely described the behavior of the maximal operator associated to the system of Laguerre functions $\{\prod_{k=1}^d L_{n_k}^{\alpha_l}(x_k) x_k^{\alpha_k/2} e^{-x_k/2}\}$ in higher dimensions for $\alpha \notin [0, \infty)^d$.

In section 2 we will see the connection between the generalized Hermite polynomials and Laguerre polynomials, and develop a transference method from generalized Hermite to Laguerre polynomials as the one introduced in [8] from Laguerre to Hermite. From that it comes as no surprise that the boundedness properties of the heat–diffusion maximal operator for the generalized Hermite functions $\{\varphi_n^\mu\}$ for $\mu \notin [0,\infty)^d$ are the same as the boundedness properties of the heat–diffusion maximal operator associated with those particular Laguerre functions.

As we said before, for $\mu \notin [0,\infty)^d$ the maximal operator associated with the second system of generalized Hermite functions is not bounded on the whole range of p's. It will depend on the parameter μ. Let us set some notations before writing the theorem. For $\mu \in (-1/2,\infty)^d$ we denote by

$$D = \{1,2,\dots,d\}, \qquad \tilde{\mu} = \min\{\mu_k : k \in D\}, \qquad \tilde{d}(\mu) = \#\{k \in D : \mu_k = \tilde{\mu}\}.$$

For $-1/2 < \tilde{\mu} < 0$ we set $p_1 = p_1(\tilde{\mu}) = -\frac{1}{\tilde{\mu}}$ and $p_0 = p_0(\tilde{\mu}) = (p_1)' = \frac{1}{1+\tilde{\mu}}$. Then we get

Theorem 1.11. *For $d \geq 1$ and $\mu \in (-1/2,\infty)^d$ such that $-1/2 < \tilde{\mu} < 0$ we have*

a) If $\tilde{d}(\mu) = 1$, then

i) $T_*^{\mu,\varphi}$ is bounded on $L^p(dx)$ for $p_0 < p < p_1$.

ii) $T_*^{\mu,\varphi}$ is of weak-type (p_1,p_1).

iii) $T_*^{\mu,\varphi}$ is of restricted weak-type (p_0,p_0).

b) If $\tilde{d}(\mu) \geq 2$, then

i) $T_*^{\mu,\varphi}$ is bounded on $L^p(dx)$ for $p_0 < p < p_1$.

ii) For $2 \leq d \leq 3$, $T_*^{\mu,\varphi}$ satisfies the logarithmic weak-type (p_1,p_1) inequality

$$|\{T_*^{\mu,\varphi} f > \lambda\}| \leq C \frac{\|f\|_{p_1}}{\lambda^{p_1}} \left[\log\left(2 + \frac{\lambda}{\|f\|_{p_1}}\right)\right]^{\tilde{d}(\mu)-1}, \quad \lambda > 0,$$

for $f \in L^{p_1}(dx)$.

For $d \geq 4$, there exists an $f \in L^{p_1}(dx)$ such that

$$|\{T_*^{\mu,\varphi} f > \lambda\}| = \infty$$

for all $\lambda > 0$. This function f can be taken in the smaller space $L^{p_1,1}(dx)$.

iii) For $2 \leq d \leq 3$, $T_*^{\mu,\varphi}$ satisfies the logarithmic restricted weak-type (p_0,p_0) inequality

$$|\{T_*^{\mu,\varphi} \chi_E > \lambda\}| \leq C \frac{|E|}{\lambda^{p_0}} \left[\log\left(2 + \frac{1}{|E|}\right)\right]^{\frac{p_0}{p_1}(\tilde{d}(\mu)-1)}, \quad \lambda > 0,$$

for all measurable sets $E \subset \mathbb{R}^d$ of finite measure.

For $d \geq 4$, this inequality does not hold, even if the exponent of the logarithmic factor is arbitrarily increased.

No boundedness holds for $p \notin [p_0,p_1]$. Indeed, as it was observed in [15, p. 214], in order to get the L^p-boundedness of the Laguerre heat–diffusion maximal operator associated with these particular Laguerre functions it is sufficient to look at it in dimension $d = 1$ and with $-1 < \alpha < 0$. In this case the boundedness occurs precisely when $p_0(\alpha) < p < p_1(\alpha)$, with $p_1(\alpha) = -\frac{2}{\alpha}$ and $p_0(\alpha) = p_1'(\alpha)$. As we will see in the proof of Theorem 1.11, this will occur in our case for $-1/2 < \mu < 0$ precisely when $p_0 := p_0(2\mu) < p < p_1(2\mu) =: p_1$. $T_*^{\mu,\varphi}$ is not strong-type (p_1, p_1) nor weak-type (p_0, p_0). Besides, inequalities (b) (ii) and (iii) for $d = 2, 3$ are sharp.

As an immediate consequence of this theorem we have:

Corollary 1.12. *Let $d \geq 1$ and let $\mu \in (-1/2, \infty)^d$ be such that $-1/2 < \tilde{\mu} < 0$. Then for every $f \in L^p(dx)$ with $p_0 < p < p_1$, or $f \in L^{p_1}(dx)$ and $d \leq 3$ or $\tilde{d}(\mu) = 1$, or $f \in L^{p_0,1} \log^{(\tilde{d}(\mu)-1)/p_1} L$ and $d \leq 3$ or $\tilde{d}(\mu) = 1$, we have*

$$\lim_{t \to 0^+} T_t^{\mu,\varphi} f(x) = f(x), \quad a.e.\ x \in \mathbb{R}^d.$$

Let us point out that the proof of Theorem 1.11 follows basically the proof of Theorem 1.3 from [15].

2. Proof of Theorems 1.3, 1.5 and Corollary 1.6

For $f \in L^2(\mathbb{R}^d, \gamma_\mu)$, taking into account that the family $\{H_n^\mu / \|H_n^\mu\|_{2,\mu}\}$ is an orthonormal basis on $L^2(\mathbb{R}^d, \gamma_\mu)$, we have

$$T_t^\mu f(x) = \sum_{n \in \mathbb{N}_0^d} \frac{\langle f, H_n^\mu \rangle}{\|H_n^\mu\|_{2,\mu}^2} H_n^\mu(x) e^{-|n|t},$$

with $\langle f, H_n^\mu \rangle = \int_{\mathbb{R}^d} f(y) H_n^\mu(y) \gamma_\mu(dy)$, and

$$\|H_n^\mu\|_{2,\mu}^2 = \langle H_n^\mu, H_n^\mu \rangle = \frac{2^{|n|}(n!)^2 \prod_{k=1}^d \Gamma(\mu_k + 1/2)}{\eta_\mu(n)},$$

with $\eta_\mu(n) = \prod_{k=1}^d \eta_{\mu_k}(n_k)$.

This series representing $T_t^\mu f$ converges on $L^2(\mathbb{R}^d, \gamma_\mu)$. However, the definition of $T_t^\mu f$ for $f \in L^p(\mathbb{R}^d, \gamma_\mu)$, $p \geq 1$, through generalized Hermite polynomial expansions would be unsatisfactory since the series may diverge for $1 \leq p < 2$ (see [12]).

Therefore, to avoid this kind of problems it can be proved that for $f \in L^2(\mathbb{R}^d, \gamma_\mu)$, $T_t^\mu f(x)$ can be written as an integral operator

$$T_t^\mu f(x) = \int_{\mathbb{R}^d} M_{\mathfrak{GH}}^{\mu,d}(x, y, t)\ f(y)\ \gamma_\mu(dy),$$

with

$$\begin{aligned} M_{\mathfrak{GH}}^{\mu,d}(x, y, t) &= \sum_{n \in \mathbb{N}_0^d} \frac{\eta_\mu(n)}{2^{|n|}(n!)^2 \prod_{k=1}^d \Gamma(\mu_k + 1/2)} H_n^\mu(x) H_n^\mu(y) e^{-|n|t} \\ &= \prod_{k=1}^d \sum_{n_k=0}^\infty \frac{\eta_\mu(n_k)}{2^{n_k}(n_k!)^2 \Gamma(\mu_k + 1/2)} H_{n_k}^{\mu_k}(x) H_{n_k}^{\mu_k}(y) e^{-n_k t}. \end{aligned} \tag{2.1}$$

But now this integral representation of T_t^μ makes sense for every function $f \in L^p(\mathbb{R}^d, \gamma_\mu)$ with $1 \leq p \leq \infty$.

Now, the one-dimensional generalized Hermite polynomials can be written in terms of the Laguerre polynomials. Namely, for n even ($n = 2m$)

$$H^{\mu}_{2m}(x) = (-1)^m (2m)! \frac{\Gamma(\mu + 1/2)}{\Gamma(m + \mu + 1/2)} L^{\mu-1/2}_m(x^2),$$

and for n odd ($n = 2m + 1$)

$$H^{\mu}_{2m+1}(x) = (-1)^m (2m + 1)! \frac{\Gamma(\mu + 1/2)}{\Gamma(m + \mu + 3/2)} x L^{\mu+1/2}_m(x^2),$$

(see [16]), where L^{α}_n stands for the one-dimensional Laguerre polynomial of degree n and type α.

For $\alpha > -1$, the one-dimensional Laguerre polynomial of type α and degree $n \in \mathbb{N}_0$ is defined as

$$L^{\alpha}_n(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}),$$

for $x > 0$. Now, let $\alpha = (\alpha_1, \ldots, \alpha_d) \in (-1, \infty)^d$ be given; we define the d-dimensional Laguerre polynomial L^{α}_n of type α and degree $|n|$ as

$$L^{\alpha}_n(x) = \prod_{k=1}^{d} L^{\alpha_k}_{n_k}(x_k)$$

for $x \in \mathbb{R}^d_+$ and $n \in \mathbb{N}^d_0$. Properly normalized, these polynomials are an orthonormal basis of $L^2(\mathbb{R}^d_+, \lambda_\alpha)$, being

$$\lambda_\alpha(dx) = \prod_{k=1}^{d} x_k^{\alpha_k} e^{-x_k} dx.$$

Also, they are eigenfunctions of the Laguerre differential operator

$$\mathfrak{L} = \sum_{k=1}^{d} \left[x_k \frac{\partial^2}{\partial x_k^2} + (\alpha_k + 1 - x_k) \frac{\partial}{\partial x_k} \right],$$

that is,

$$\mathfrak{L} L^{\alpha}_n(x) = -|n| L^{\alpha}_n(x).$$

As with the generalized Hermite polynomials, in this context we have the diffusion semigroup $\mathfrak{T}^{\alpha}_t = e^{t\mathfrak{L}}$ whose kernel in dimension 1 is

$$M^{\alpha,1}_{\mathfrak{L}}(v, u, t) = \sum_{k=0}^{\infty} \frac{\Gamma(k + 1)}{\Gamma(k + \alpha + 1)} L^{\alpha}_k(v) L^{\alpha}_k(u) e^{-kt},$$

for $u, v \in (0, \infty)$ and $\alpha > -1$, cf. [12].

According to the Hille-Hardy formula [9, (4.17.6)], for $u, v \in (0, \infty)$ and $\alpha > -1$

$$M^{\alpha,1}_{\mathfrak{L}}(v, u, t) = \frac{(e^{-t} vu)^{-\frac{\alpha}{2}}}{1 - e^{-t}} e^{-\frac{e^{-t}(v+u)}{1-e^{-t}}} I_\alpha \left(\frac{\sqrt{4e^{-t} vu}}{1 - e^{-t}} \right),$$

I_α being the modified Bessel function of first kind and order α, cf. [9, Sec. 5.7].

The symbol $f \lesssim g$ for a non-negative f stands for $f \leq Cg$ for some positive and finite constant that usually will depend only on d and α. We write $f \simeq g$ if $f \lesssim g$ and $g \lesssim f$. From [12] we get the following estimate:

$$M_{\mathfrak{L}}^{\alpha,1}(v,u,t) \simeq \begin{cases} (1-e^{-t})^{-\alpha-1} e^{-\frac{e^{-t}(v+u)}{1-e^{-t}}} & \text{if} \quad \sqrt{4e^{-t}vu} < 1-e^{-t} \\ \frac{(\sqrt{4e^{-t}vu})^{-\alpha-\frac{1}{2}}}{(1-e^{-t})^{1/2}} e^{\frac{-e^{-t}(v+u)+\sqrt{4e^{-t}vu}}{1-e^{-t}}} & \text{if} \quad \sqrt{4e^{-t}vu} \geq 1-e^{-t} \end{cases} \tag{2.2}$$

for $\alpha > -1$, $u, v \in (0,\infty)$.

The corresponding d-dimensional kernel associated with the Laguerre polynomial expansions is

$$M_{\mathfrak{L}}^{\alpha,d}(x,y,t) = \prod_{k=1}^{d} M_{\mathfrak{L}}^{\alpha_k,1}(x_k,y_k,t),$$

for $x, y \in \mathbb{R}_+^d$ and $t > 0$.

Associated to this semigroup we have its maximal operator

$$\mathfrak{T}_*^{\alpha} f(x) = \sup_{t>0} |\mathfrak{T}_t^{\alpha} f(x)|.$$

From E. Stein's maximal theorem (see [18]), we know that $\mathfrak{T}_*^{\alpha}$ is of strong-type (p,p) with respect to λ_α for $p > 1$. The weak-type $(1,1)$ of this operator was proved by B. Muckenhoupt in [12] for dimension 1 and by U. Dinger in [3] for higher dimensions.

Let us go back to the kernel defined in (2.1). We want to relate this d-dimensional Mehler-type formula with the one associated to the Laguerre polynomials. Let us start with the one-dimensional Mehler-type formula associated to the generalized Hermite polynomials, i.e.,

$$M_{\mathfrak{GH}}^{\mu,1}(x,y,t) = \sum_{n=0}^{\infty} \frac{\eta_\mu(n)}{2^n (n!)^2 \Gamma(\mu+1/2)} H_n^\mu(x) H_n^\mu(y) e^{-nt},$$

for $\mu > -1/2$, $x, y \in \mathbb{R}$ and $t > 0$. Then,

$$\begin{aligned} M_{\mathfrak{GH}}^{\mu,1}(x,y,t) &= \sum_{m=0}^{\infty} \frac{\eta_\mu(2m)}{2^{2m} (2m)!^2 \Gamma(\mu+1/2)} H_{2m}^\mu(x) H_{2m}^\mu(y) e^{-2mt} \\ &\quad + \sum_{m=0}^{\infty} \frac{\eta_\mu(2m+1)}{2^{2m+1} (2m+1)!^2 \Gamma(\mu+1/2)} H_{2m+1}^\mu(x) H_{2m+1}^\mu(y) e^{-(2m+1)t} \\ &= (I) + (II). \end{aligned}$$

Thus, by taking $\alpha = \mu - \frac{1}{2}$,

$$\begin{aligned}(I) &= \sum_{m=0}^{\infty} \frac{2^{2m} m!\Gamma(m+\mu+\frac{1}{2})}{2^{2m}((2m)!)^2(\Gamma(\mu+\frac{1}{2}))^2} \frac{(-1)^{2m}((2m)!)^2(\Gamma(\mu+1/2))^2}{(\Gamma(m+\mu+1/2))^2} \\ &\quad \times L_m^{\mu-1/2}(x^2)L_m^{\mu-1/2}(y^2)e^{-2mt} \\ &= \sum_{m=0}^{\infty} \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} L_m^{\alpha}(x^2)L_m^{\alpha}(y^2)e^{-m(2t)} \\ &= M_{\mathfrak{L}}^{\alpha,1}(x^2,y^2,2t),\end{aligned}$$

and

$$\begin{aligned}(II) &= \sum_{m=0}^{\infty} \frac{2^{2m+1} m!\Gamma(m+\mu+\frac{3}{2})}{2^{2m+1}((2m+1)!)^2(\Gamma(\mu+\frac{1}{2}))^2} \frac{(-1)^{2m}((2m+1)!)^2(\Gamma(\mu+1/2))^2}{(\Gamma(m+\mu+3/2))^2} \\ &\quad \times xyL_m^{\mu+1/2}(x^2)L_m^{\mu+1/2}(y^2)e^{-(2m+1)t} \\ &= e^{-t}xy \sum_{m=0}^{\infty} \frac{\Gamma(m+1)}{\Gamma(m+(\alpha+1)+1)} L_m^{\alpha+1}(x^2)L_m^{\alpha+1}(y^2)e^{-m(2t)} \\ &= e^{-t}xyM_{\mathfrak{L}}^{\alpha+1,1}(x^2,y^2,2t).\end{aligned}$$

Therefore,

$$M_{\mathfrak{GH}}^{\mu,1}(x,y,t) = M_{\mathfrak{L}}^{\alpha,1}(x^2,y^2,2t) + e^{-t}xyM_{\mathfrak{L}}^{\alpha+1,1}(x^2,y^2,2t), \tag{2.3}$$

for $x, y \in \mathbb{R}_* = \mathbb{R} \setminus \{0\}$.

According to (2.3),

$$M_{\mathfrak{GH}}^{\mu,d}(x,y,t) = \prod_{k=1}^{d} (M_{\mathfrak{L}}^{\alpha_k,1}(x_k^2,y_k^2,2t) + e^{-t}x_ky_kM_{\mathfrak{L}}^{\alpha_k+1,1}(x_k^2,y_k^2,2t)), \tag{2.4}$$

for $x, y \in \mathbb{R}_*^d$, and with $\alpha_k = \mu_k - 1/2 \in (-1,\infty)$ for $k = 1,\dots,d$.

Using (2.2) we obtain that for $u, v \in (0,\infty)$ and $\alpha > -1$,

$$\sqrt{4e^{-t}vu}\, M_{\mathfrak{L}}^{\alpha+1,1}(v,u,t) \lesssim M_{\mathfrak{L}}^{\alpha,1}(v,u,t).$$

Therefore,

$$M_{\mathfrak{L}}^{\alpha,1}(v,u,t) + \sqrt{4e^{-t}vu}M_{\mathfrak{L}}^{\alpha+1,1}(v,u,t) \lesssim M_{\mathfrak{L}}^{\alpha,1}(v,u,t). \tag{2.5}$$

Proof of Theorem 1.3. Without loss of generality we may assume $f \geq 0$. According to (2.4) we have

$$T_t^{\mu} f(x) = \int_{\mathbb{R}_*^d} \prod_{k=1}^{d} (M_{\mathfrak{L}}^{\alpha_k,1}(x_k^2,y_k^2,2t) + e^{-t}x_ky_kM_{\mathfrak{L}}^{\alpha_k+1,1}(x_k^2,y_k^2,2t))\ f(y)\ \gamma_\mu(dy),$$

with $\alpha_k = \mu_k - 1/2$ and $\mu_k > -1/2$ for $k = 1,\dots,d$. From (2.5) plus d applications of Tonelli's theorem together with an easy change of variable on each one-dimensional integral, we get

$$
\begin{aligned}
|T_t^\mu f(x)| &\lesssim \int_{\mathbb{R}^d_*} \prod_{k=1}^{d} M_{\mathfrak{L}}^{\alpha_k,1}(x_k^2, y_k^2, 2t)\ f(y)\ \gamma_\mu(dy) \\
&= \int_{\mathbb{R}^d_*} M_{\mathfrak{L}}^{\alpha,d}(x^2, y^2, 2t)\ f(y)\ \gamma_\mu(dy) \\
&= \int_{\mathbb{R}^d_+} M_{\mathfrak{L}}^{\alpha,d}(x^2, y^2, 2t)\ f_d(y)\ \gamma_\mu(dy),
\end{aligned}
\tag{2.6}
$$

with $\alpha = \mu - \frac{1}{2}(1,\dots,1)$, $x^2 = (x_1^2,\dots,x_d^2)$, and similarly for y^2, being

$$f_1(y) = f(y) + f \circ \sigma_1(y)$$

and

$$f_{k+1}(y) = f_k(y) + f_k \circ \sigma_{k+1}(y),$$

for $k = 1,\dots,d-1$. Since we will need an explicit form of f_d let us write some notations. Let us recall that for $k \in D$, σ_k stands for the reflection with respect to the k-th hyperplane $\{x_k = 0\}$. Let $\emptyset \neq A \subset D$; we denote by $\sigma_A := \prod_{k\in A} \sigma_k$, and the symbol $\prod$ represents the composition of the reflections indexed by A. Observe that σ_A is well-defined since $\sigma_k\sigma_j = \sigma_j\sigma_k$ for all $j,k \in D$; $\sigma_\emptyset =$ identity map on $\mathbb{R}^d$. Also since $\sigma_k^{-1} = \sigma_k$, for all $k \in D$, $\sigma_A^{-1} = \sigma_A$. We define the A-th hyper-octant as $O_A := \{\sigma_A(y) : y \in \mathbb{R}^d_+\}$. Thus for any measurable function f,

$$f_d(y)\chi_{\mathbb{R}^d_+}(y) = \sum_{A\subset D} f(\sigma_A(y))\chi_{\mathbb{R}^d_+}(y) = \sum_{A\subset D} f(x_A),$$

with $x_A = \sigma_A(y)$ and $y \in \mathbb{R}^d_+$. If $f \geq 0$, taking into account that $\gamma_\mu(x_A) = \gamma_\mu(y)$ and $dx_A = dy$, we have

$$\|f_d\chi_{\mathbb{R}^d_+}\|_{1,\gamma_\mu} = \sum_{A\subset D} \|(f_d \circ \sigma_A)\chi_{\mathbb{R}^d_+}\|_{1,\gamma_\mu} = \sum_{A\subset D} \|f\chi_{O_A}\|_{1,\gamma_\mu} = \|f\|_{1,\gamma_\mu}.$$

Similarly, for $1 < p < \infty$,

$$2^{-d/p}\|f\|_{p,\gamma_\mu} \leq \|f_d\chi_{\mathbb{R}^p_+}\|_{p,\gamma_\mu} \leq \sum_{A\subset D} \|f\chi_{O_A}\|_{p,\gamma_\mu} \leq 2^d\|f\|_{p,\gamma_\mu}.$$

Let us make a change of variables in (2.6). Set $u = \phi(y) = y^2$ for $y \in \mathbb{R}^d_+$, then

$$
\begin{aligned}
\int_{\mathbb{R}^d_+} M_{\mathfrak{L}}^{\alpha,d}(x^2, y^2, 2t)\ f_d(y)\ \gamma_\mu(dy) &= 2^{-d} \int_{\mathbb{R}^d_+} M_{\mathfrak{L}}^{\alpha,d}(x^2, u, 2t)\ f_d \circ \phi^{-1}(u)\ \lambda_\alpha(du) \\
&= 2^{-d}\ \mathfrak{T}_{2t}^{\alpha}(f_d \circ \phi^{-1})(x^2).
\end{aligned}
$$

Therefore, for every $x \in \mathbb{R}^d_*$

$$T_t^\mu f(x) \lesssim \mathfrak{T}_{2t}^{\alpha}(f_d \circ \phi^{-1})(\phi(x)), \quad t > 0 \tag{2.7}$$

and so

$$T_*^\mu f(x) \lesssim \mathfrak{T}_*^{\alpha}(f_d \circ \phi^{-1})(\phi(x)),$$

with $\mu \in (-1/2,\infty)^d$ and $\alpha = \mu - \frac{1}{2}(1,\dots,1)$.

To get the weak-type $(1,1)$ of T_*^μ it will be sufficient to prove the weak-type inequality for $\mathfrak{T}_*^\alpha(f_d\circ\phi^{-1})(x^2)$ with respect to γ_μ. Let $\theta>0$ be given and let $E_\theta=\{x\in\mathbb{R}_*^d:\mathfrak{T}_*^\alpha(f_d\circ\phi^{-1})(\phi(x))>\theta\}$; since $\mathfrak{T}_*^\alpha(f_d\circ\phi^{-1})((\sigma_A(x))^2)=\mathfrak{T}_*^\alpha(f_d\circ\phi^{-1})(x^2)$ for all $A\subset D$, then $E_\theta\cap O_A=\sigma_A(E_\theta\cap\mathbb{R}_+^d)$. Thus

$$\gamma_\mu(E_\theta)=\sum_{A\subset D}\gamma_\mu(E_\theta\cap O_A)=\sum_{A\subset D}\gamma_\mu(\sigma_A(E_\theta\cap\mathbb{R}_+^d)). \tag{2.8}$$

But

$$\begin{aligned}\gamma_\mu(\sigma_A(E_\theta\cap\mathbb{R}_+^d))&=\int_{\mathbb{R}_*^d}\chi_{\sigma_A(E_\theta\cap\mathbb{R}_+^d)}(y)\gamma_\mu(y)\,dy\\&=\int_{\mathbb{R}_*^d}\chi_{\sigma_A(E_\theta\cap\mathbb{R}_+^d)}(\sigma_A(x))\gamma_\mu(\sigma_A(x))\,dx\\&=\int_{\mathbb{R}_*^d}\chi_{E_\theta\cap\mathbb{R}_+^d}(x)\gamma_\mu(x)\,dx\\&=\gamma_\mu(E_\theta\cap\mathbb{R}_+^d).\end{aligned}$$

Therefore, taking into account (2.8), we have

$$\gamma_\mu(E_\theta)=2^d\gamma_\mu(E_\theta\cap\mathbb{R}_+^d)=\lambda_\alpha(\phi(E_\theta\cap\mathbb{R}_+^d)). \tag{2.9}$$

On the other hand, $\phi(E_\theta\cap\mathbb{R}_+^d)=\{v\in\mathbb{R}_+^d:\mathfrak{T}_*^\alpha(f_d\circ\phi^{-1})(v)>\theta\}$ and from [12] and [3] we obtain

$$\begin{aligned}\lambda_\alpha(\phi(E_\theta\cap\mathbb{R}_+^d))&\le\frac{C}{\theta}\|f_d\circ\phi^{-1}\|_{1,\lambda_\alpha}\\&=\frac{C2^d}{\theta}\|f_d\|_{1,\gamma_\mu}\le\frac{C_d}{\theta}\|f\|_{1,\gamma_\mu}.\end{aligned} \tag{2.10}$$

And from (2.9) together with (2.10) we get

$$\gamma_\mu(E_\theta)\le\frac{C_d}{\theta}\|f\|_{1,\gamma_\mu}.$$

The case $p=\infty$ follows immediately. And the strong-type (p,p), for $p>1$, follows from interpolation. □

In the previous proof we have just individuated the tools to prove Theorem 1.5 and its Corollary 1.6.

Proof of Theorem 1.5. Since the Laguerre semigroup is hypercontractive (see, for example, [7]), from (2.7) we can easily conclude that for all $t>0$ and $p\in(1,\infty)$, there exists a $q=q(p,t)$ such that

$$\begin{aligned}\|T_t^\mu f\|_{L^q(\gamma_\mu)}&\lesssim\|\mathfrak{T}_{2t}^\alpha(f_d\circ\phi^{-1})\circ\phi\|_{L^q(\gamma_\mu)}\\&=2^{d/q}\|(\mathfrak{T}_{2t}^\alpha(f_d\circ\phi^{-1})\circ\phi)\chi_{\mathbb{R}_+^d}\|_{L^q(\gamma_\mu)}\\&=\|\mathfrak{T}_{2t}^\alpha(f_d\circ\phi^{-1})\|_{L^q(\lambda_\alpha)}\lesssim\|f_d\circ\phi^{-1}\|_{L^p(\lambda_\alpha)}\\&=2^{d/p}\|f_d\chi_{\mathbb{R}_+^d}\|_{L^p(\gamma_\mu)}\lesssim\|f\|_{L^p(\gamma_\mu)},\end{aligned}$$

and this concludes the proof. □

Proof of Corollary 1.6. The strong-type (p,p), for $p \in (1,\infty)$, follows from the hypercontractivity of T_t^{μ} and from P. A. Meyer's multiplier theorem extended to any hypercontractive and $L^2(d\nu)$ symmetric diffusion semigroup $\{T_t\}$ related to an orthonornal basis $\{P_\beta\}_{\beta\in\mathbb{N}_o^d}$ of $L^2(d\nu)$ by the relation $T_tP_\beta = e^{-|\beta|t}P_\beta$. □

3. Proof of Theorems 1.9 and 1.11

Proof of Theorem 1.9. a) For $t>0$, the associated heat–diffusion semigroup $T_t^{\mu,\psi}$ is defined on $L^2(\mathbb{R}^d,\rho_\mu^\psi)$ by

$$T_t^{\mu,\psi} f(x) = \sum_{n\in\mathbb{N}_0^d} e^{-(|n|+|\mu|+d/2)t}\langle f,\psi_n^\mu\rangle\psi_n^\mu(x), \quad x\in\mathbb{R}^d, \quad f\in L^2(\mathbb{R}^d,\rho_\mu^\psi).$$

Though this series is convergent on $L^1(\mathbb{R}^d,\rho_\mu^\psi)$, we are going to consider the integral representation of this semigroup on L^1. Namely,

$$T_t^{\mu,\psi} f(x) = e^{-(|\mu|+d/2)t}\int_{\mathbb{R}^d} e^{-\frac{|x|^2+|y|^2}{2}} M_{\mathfrak{GH}}^{\mu,d}(x,y,t)\ f(y)\ \rho_\mu^\psi(dy).$$

From (2.4) together with (2.5) we obtain that

$$|M_{\mathfrak{GH}}^{\mu,d}(x,y,t)| \lesssim M_{\mathfrak{L}}^{\alpha,d}(x^2,y^2,2t), \tag{3.1}$$

with $\alpha = \mu - \frac{1}{2}(1,\dots,1)$, and therefore

$$\begin{aligned} |T_t^{\mu,\psi} f(x)| &\lesssim e^{-(|\mu|+d/2)t}\int_{\mathbb{R}_*^d} e^{-\frac{|x|^2+|y|^2}{2}} M_{\mathfrak{L}}^{\alpha,d}(x^2,y^2,2t)\ |f(y)|\ \rho_\mu^\psi(dy) \\ &= e^{-(|\mu|+d/2)t}\int_{\mathbb{R}_+^d} e^{-\frac{|x|^2+|y|^2}{2}} M_{\mathfrak{L}}^{\alpha,d}(x^2,y^2,2t)\ |f|_d(y)\ \rho_\mu^\psi(dy) \\ &= e^{-(|\mu|+d/2)t}\int_{\mathbb{R}_+^d} G^\alpha(x^2,y^2,2t)\ |f|_d(y)\ \rho_\mu^\psi(dy), \end{aligned} \tag{3.2}$$

where $G^\alpha(v,u,t) = \prod_{k=1}^d G^{\alpha_k}(v_k,u_k,t)$, and

$$G^a(\xi,\eta,t) = \frac{1}{1-e^{-t}} e^{-\frac{1}{2}\frac{1+e^{-t}}{1-e^{-t}}(\xi+\eta)} (e^{-t/2}\sqrt{\xi\eta})^{-a} I_a\left(\frac{2e^{-t/2}\sqrt{\xi\eta}}{1-e^{-t}}\right),$$

for $\xi,\eta\in(0,\infty)$, $a>-1$, and I_a stands for the modified Bessel function of the first kind and order a, see [14].

Using again the change of variables $u=\phi(y)=y^2$ in (3.2) we get that

$$|T_t^{\mu,\psi} f(x)| \lesssim e^{-(|\mu|+d/2)t}\int_{\mathbb{R}_+^d} G^\alpha(x^2,u,2t)\ (|f|_d\circ\phi^{-1})(u)\ \prod_{k=1}^d u_k^{\alpha_k}\ du.$$

Thus,

$$T_*^{\mu,\psi} f(x) \lesssim G_*^\alpha(|f|_d\circ\phi^{-1})(\phi(x)),$$

with

$$G_*^\alpha g(v) = \sup_{t>0}\left|\int_{\mathbb{R}_+^d} G^\alpha(v,u,t)\ g(u)\ \prod_{k=1}^d u_k^{\alpha_k} du\right|, \quad v\in\mathbb{R}_+^d,$$

which, according to [14], happens to be weak-type $(1,1)$ with respect to the measure $\prod_{k=1}^d v_k^{\alpha_k}\,dv$. Therefore, with a reasoning similar to the one done for getting the weak-type $(1,1)$ of the operator T_*^{μ}, we obtain also the weak-type $(1,1)$ for $T_*^{\mu,\psi}$ with respect to the measure $\prod_{k=1}^d x_k^{2\mu_k}\,dx$. On the other hand, $G_*^{\alpha}g$ can be estimated by a constant times $\mathcal{M}_S g$, where $\mathcal{M}_S$ is the strong maximal operator with respect to the measure $\prod_{k=1}^d v_k^{\alpha_k}\,dv$, see [13]; this implies the L^p-boundedness of G_*^{α} for $p>1$. In particular, we get boundedness on L^∞ of $T_*^{\mu,\psi}$. For the other p's the result follows either from interpolation or by calculating out the L^p-norm of $T_*^{\mu,\psi}$ and relating it with the L^p-norm of G_*^{α} with respect to the measure $\prod_{k=1}^d v_k^{\alpha_k}\,dv$.

b) For $\mu\in[0,\infty)^d$ and $t>0$, the associated semigroup $T_t^{\mu,\varphi}$ is defined on $L^2(\mathbb{R}^d,dx)$ by

$$T_t^{\mu,\varphi}f(x)=\sum_{n\in\mathbb{N}_0^d} e^{-(|n|+|\mu|+d/2)t}\langle f,\varphi_n^{\mu}\rangle\varphi_n^{\mu}(x),\quad x\in\mathbb{R}^d,\quad f\in L^2(\mathbb{R}^d,dx),$$

whose integral representation, valid also for $f\in L^1(\mathbb{R}^d,dx)$, is given by

$$\begin{aligned}T_t^{\mu,\varphi}f(x)&=e^{-(|\mu|+d/2)t}\int_{\mathbb{R}^d} e^{-\frac{|x|^2+|y|^2}{2}} M_{\mathfrak{GH}}^{\mu,d}(x,y,t)\prod_{k=1}^d |x_k|^{\mu_k}|y_k|^{\mu_k} f(y)\,dy\\&=\int_{\mathbb{R}^d}\mathcal{K}_{\varphi}^{\mu}(x,y,t)f(y)\,dy.\end{aligned}\tag{3.3}$$

From the boundedness results (3.1) and (3.2) we get for $\alpha=\mu-\frac12(1,\dots,1)$

$$\begin{aligned}|T_t^{\mu,\varphi}f(x)|&\lesssim e^{-(|\mu|+d/2)t}\int_{\mathbb{R}_+^d} G^{\alpha}(x^2,y^2,2t)\prod_{k=1}^d |x_k|^{\alpha_k+1/2}|y_k|^{\alpha_k+1/2}\ |f|_d(y)\,dy\\&=\int_{\mathbb{R}_+^d}\mathfrak{H}^{\alpha}(x^2,y^2,t)\ |f|_d(y)\,dy,\end{aligned}\tag{3.4}$$

with $\mathfrak{H}^{\alpha}(v,u,t)=\prod_{k=1}^d\mathfrak{H}^{\alpha_k}(v_k,u_k,t)=\prod_{k=1}^d\mathfrak{H}^{\mu_k-1/2}(v_k,u_k,t)$, and

$$\begin{aligned}\mathfrak{H}^{\nu-1/2}(\xi,\eta,t)&=\frac{e^{-(\nu+1/2)t}}{1-e^{-2t}}e^{-\frac12\frac{1+e^{-2t}}{1-e^{-2t}}(\xi+\eta)}(e^{-t}\sqrt{\xi\eta})^{-(\nu-1/2)}\\&\quad\times I_{\nu-1/2}\left(\frac{2e^{-t}\sqrt{\xi\eta}}{1-e^{-2t}}\right)(\sqrt{\xi\eta})^{\nu}\\&=\frac{e^{-t}}{1-e^{-2t}}e^{-\frac12\frac{1+e^{-2t}}{1-e^{-2t}}(\xi+\eta)}I_{\nu-1/2}\left(\frac{2e^{-t}\sqrt{\xi\eta}}{1-e^{-2t}}\right)(\sqrt{\xi\eta})^{1/2}\end{aligned}\tag{3.5}$$

$$=\frac{e^{-\frac12\coth t(\xi+\eta)}}{2\sinh t}I_{\nu-1/2}\left(\frac{\sqrt{\xi\eta}}{\sinh t}\right)(\sqrt{\xi\eta})^{1/2},\tag{3.6}$$

for $\xi,\eta\in(0,\infty)$ and $\nu>-1/2$.

By using the estimates of $I_{\nu-1/2}$ (cf. [12]), i.e.,

$$I_{\nu-1/2}(x)\simeq\begin{cases}x^{\nu-1/2} & 0<x<1\\ \frac{e^x}{x^{1/2}} & x\ge 1\end{cases},$$

we get that

$$I_{\nu-1/2}(x) \lesssim \frac{x^{\nu-1/2}}{(1+x)^{\nu}} e^{x}$$

for $x > 0$. Thus, taking into account (3.5), we obtain

$$\mathfrak{H}^{\nu-1/2}(\xi,\eta,t) \lesssim e^{-t/2} \left(\frac{e^{-t}}{1-e^{-2t}} \right)^{\nu} \frac{e^{-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(\xi+\eta)+\frac{2e^{-t}}{1-e^{-2t}}\sqrt{\xi\eta}}}{(1-e^{-2t})^{1/2}} \left(\frac{\sqrt{\xi\eta}}{1+\frac{2e^{-t}\sqrt{\xi\eta}}{1-e^{-2t}}} \right)^{\nu}. \tag{3.7}$$

Then, it is easily seen that for $\nu \geq 0$,

$$\mathfrak{H}^{\nu-1/2}(\xi,\eta,t) \lesssim \mathfrak{g}(\xi,\eta,t) \tag{3.8}$$

with

$$\mathfrak{g}(\xi,\eta,t) = \frac{e^{-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(\xi+\eta)+\frac{2e^{-t}}{1-e^{-2t}}\sqrt{\xi\eta}}}{(1-e^{-2t})^{1/2}}.$$

Now let us observe that for $\mu \in [0,\infty)^d$, taking into account estimate (3.8), we get that

$$\begin{aligned} T_*^{\mu,\varphi} f(x) &\lesssim \sup_{t>0} \int_{\mathbb{R}_+^d} G_t(\tilde{x},y)\ |f|_d(y)\,dy \\ &\lesssim \sup_{s>0} \int_{\mathbb{R}_+^d} \frac{e^{-\frac{|\tilde{x}-y|^2}{4s}}}{(4\pi s)^{d/2}}\ |f|_d(y)\,dy \\ &\lesssim \mathcal{M}(\chi_{\mathbb{R}_+^d}|f|_d)(\tilde{x}), \end{aligned} \tag{3.9}$$

with $\tilde{x} = (|x_1|,\ldots,|x_d|)$, and

$$G_t(x,y) = \prod_{k=1}^{d} \mathfrak{g}(x_k^2,y_k^2,t) = \frac{e^{-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(|x|^2+|y|^2)+\frac{2e^{-t}}{1-e^{-2t}}x\cdot y}}{(1-e^{-2t})^{d/2}}, \quad \text{for } x,y \in \mathbb{R}_+^d,$$

is the kernel defined in [20] which, as pointed out in that paper, is bounded by the usual Gauss-Weierstrass kernel on $\mathbb{R}^d$. $\mathcal{M}$ represents the Hardy-Littlewood maximal function which happens to be weak-type $(1,1)$ and strong type (p,p), $p > 1$, with respect to the Lebesgue measure. Let us see that $\mathcal{M}(\chi_{\mathbb{R}_+^d}|f|_d)(\tilde{x})$ satisfies also the weak-type $(1,1)$ and the strong-type (p,p), $p > 1$, inequalities. Indeed,

$$\begin{aligned} |\{x \in \mathbb{R}^d : \mathcal{M}(\chi_{\mathbb{R}_+^d}|f|_d)(\tilde{x}) > \lambda\}| &= 2^d |\{x \in \mathbb{R}_+^d : \mathcal{M}(\chi_{\mathbb{R}_+^d}|f|_d)(x) > \lambda\}| \\ &\leq 2^d |\{x \in \mathbb{R}^d : \mathcal{M}(\chi_{\mathbb{R}_+^d}|f|_d)(x) > \lambda\}| \\ &\leq \frac{C}{\lambda}\|\chi_{\mathbb{R}_+^d}|f|_d\|_1 = \frac{C}{\lambda}\|f\|_1. \end{aligned}$$

The boundedness on $L^\infty(dx)$ is immediate, and for the other p's follows from interpolation. $\square$

Proof of Theorem 1.11. The proof of this theorem follows the ideas given in [15] to prove the boundedness of the heat–diffusion semigroup associated with a special class of Laguerre functions: $\mathcal{L}_k^\alpha(x) = c_{k,\alpha} L_k^\alpha(x) x^{\alpha/2} e^{-x/2}$, $x > 0, \alpha > -1$, $k \in \mathbb{N}_0$, $L_k^\alpha(x)$ being the Laguerre polynomial of degree k and type α, and extended to $\mathbb{R}_+^d$ by tensor product.

We write here all the modified lemmas, propositions and theorems used in that paper in order to obtain the results stated in Theorem 1.11.

To agree with the notation of that paper for $\nu > -1/2$ we set

$$\mathcal{H}_t^{2\nu}(\xi,\eta) := \mathfrak{H}^{\nu-1/2}(\xi,\eta,t)$$

and $\mathcal{H}_t^\alpha(v,u) = \prod_{k=1}^d \mathcal{H}_t^{\alpha_k}(v_k,u_k)$, where now we redefine $\alpha_k = 2\mu_k$, for all $k = 1,\dots,d$. Then, for $x \in \mathbb{R}_*^d$ and $f \geq 0$,

$$T_*^{\mu,\varphi} f(x) \lesssim \sup_{t>0} \int_{\mathbb{R}_+^d} \mathcal{H}_t^\alpha(x^2,y^2)\ f_d(y)\,dy =: \mathcal{H}_*^\alpha f_d(x^2). \tag{3.10}$$

By using (3.6) and the estimates of $I_{\nu-1/2}$ we get for $a = 2\nu$

$$\mathcal{H}_t^a(\xi,\eta) \simeq \begin{cases} D_t^a(\xi,\eta) & \text{if} \quad \sqrt{\xi\eta} < \sinh t \\ E_t^a(\xi,\eta) & \text{if} \quad \sqrt{\xi\eta} \geq \sinh t \end{cases} \tag{3.11}$$

with

$$D_t^a(\xi,\eta) = \frac{(\sqrt{\xi\eta})^{a/2}}{2(\sinh t)^{(a+1)/2}} e^{-\frac{1}{2}\coth t(\xi+\eta)},$$

$$E_t^a(\xi,\eta) = \frac{e^{-\frac{1}{2}\frac{(\sqrt{\xi}-\sqrt{\eta})^2}{\sinh t}}}{(\sinh t)^{1/2}} e^{-\frac{\cosh t-1}{2\sinh t}(\xi+\eta)}.$$

For the positive results we need the following lemma which is the substitute of Lemma 2.1 from [15], and its proof follows similarly.

Lemma 3.12. *For $a > -1$,*

$$\mathcal{H}_t^a(\xi,\eta) \lesssim \begin{cases} \frac{e^{-c\frac{(\sqrt{\xi}-\sqrt{\eta})^2}{t}}}{t^{1/2}} e^{-c(\xi+\eta)t} + \frac{(\sqrt{\xi\eta})^{a/2}}{(\sqrt{t})^{a+1}} e^{-c\frac{\xi+\eta}{t}} & \textit{if} \quad 0 < t \leq 1 \\ (\sqrt{\xi\eta})^{a/2} e^{-c(\xi+\eta)} & \textit{if} \quad t > 1. \end{cases} \tag{3.13}$$

For the negative results we need the following lemma which is the substitute of Lemma 2.2 from [15, p. 221].

Lemma 3.14. *The following lower estimates hold:*

a) For $0 < t < 1/16$, $(4t)^{-1} < \xi^2 < 4t^{-1}$ and $|\xi - \eta| < \frac{1}{\xi}$,

$$\mathcal{H}_t^a(\xi^2,\eta^2) \gtrsim \xi.$$

b) For $0 < t \leq 1$, $0 < \xi^2 < 2t$ and $0 < \eta^2 < 2t$,

$$\mathcal{H}_t^a(\xi^2,\eta^2) \gtrsim \frac{(\xi\eta)^{a/2}}{(\sqrt{t})^{a+1}}.$$

Proof. From the assumptions of (a), we also have $\xi > 2$, $\xi/2 < \eta < 2\xi$ and $\sqrt{\xi\eta} > \xi/\sqrt{2} > \sqrt{2} > t \simeq \sinh t$, therefore

$$\mathcal{H}_t^a(\xi^2,\eta^2) \simeq E_t^a(\xi^2,\eta^2) \simeq \frac{e^{-\frac{1}{2}\frac{(\xi-\eta)^2}{\sinh t}}}{t^{1/2}} e^{-\frac{\cosh t-1}{2\sinh t}(\xi^2+\eta^2)} \gtrsim t^{-1/2} \gtrsim \xi.$$

As for (b), we have $\xi^2+\eta^2 \lesssim t \simeq \tanh t = \frac{1}{\coth t}$, and hence

$$\mathcal{H}_t^a(\xi^2,\eta^2) \simeq D_t^a(\xi^2,\eta^2) \simeq \frac{(\xi\eta)^{a/2}}{(\sqrt{t})^{a+1}} e^{-\frac{1}{2}\coth t(\xi^2+\eta^2)} \gtrsim \frac{(\xi\eta)^{a/2}}{(\sqrt{t})^{a+1}}. \qquad \square$$

3.1. **Boundedness of $T_*^{\mu,\varphi}$ on $L^p(dx)$ for $p_0 < p < p_1$.** We denote by M_k the standard centered one-dimensional maximal operator in $\mathbb{R}_+^d$ taken with respect to the k-th variable.

Let us see the one-dimensional case first. Let $\alpha = a = 2\nu = 2\mu \in (-1,0)$; we set $p_1 = -2/a = -1/\mu$ and $p_0 = p_1' = 2/(a+2) = 1/(\mu+1)$.

The following proposition is the substitute of Proposition 3.1 from [15, p. 223] and its proof follows similarly using Lemma 3.12 instead of their Lemma 2.1.

Proposition 3.15. *Let $d = 1$ and $-1 < a < 0$. Then there is a constant c such that for any suitable non-negative function g defined on $(0,\infty)$ and $x \in \mathbb{R}_+$ we have for $0 < t \leq 1$,*

$$\int_{\mathbb{R}_+} \mathcal{H}_t^a(x^2,y^2)\ g(y)\,dy \lesssim \begin{cases} e^{-ctx^2} M_1 g(x) + e^{-c\frac{x^2}{t}} x^{-1/p_1}\|g\|_{p_1} \\ e^{-ctx^2} M_1 g(x) + e^{-c\frac{x^2}{t}} x^{-1/p_0}\|g\|_{p_0,1} \end{cases} \tag{3.16}$$

For $t > 1$, the same inequalities hold with t replaced by 1 in the right-hand sides.

Remark 3.17. *Let us observe that throwing away the exponentials we obtain immediately the weak-type (p_1,p_1) and restricted weak-type (p_0,p_0) for the one-dimensional operator $\mathcal{H}_*^\alpha$. By interpolation $\mathcal{H}_*^\alpha$ turns out to be bounded on $L^p(\mathbb{R}_+)$ for $p_0 < p < p_1$. Now taking into account inequality (3.10) we obtain the same result for $T_*^{\mu,\varphi}$.*

For $-1 < a < \infty$ let us define the functions

$$p_1(a) = \begin{cases} -\frac{a}{2} & \text{if} \quad -1 < a < 0 \\ \infty & \text{if} \qquad\quad a \geq 0 \end{cases}$$

and

$$p_0(a) = p_1'(a) = \begin{cases} \frac{2}{a+2} & \text{if} \quad -1 < a < 0 \\ 1 & \text{if} \qquad\quad a \geq 0 \end{cases}$$

For $\mu \in (-1/2,\infty)^d$ we set $\alpha = 2\mu \in (-1,\infty)^d$, and if $-1/2 < \tilde{\mu} < 0$ then $\tilde{\alpha} = 2\tilde{\mu} \in (-1,0)$. We also set $p_1 = p_1(\tilde{\alpha})$ and $p_0 = p_1'$.

Theorem 3.18. *Let $d \geq 1$, $\mu \in (-1/2,\infty)^d$ and assume that $-1/2 < \tilde{\mu} < 0$. Then $T_*^{\mu,\varphi}$ is bounded on $L^p(dx)$ for $p_0 < p < p_1$.*

Proof. Let us recall that $T_*^{\mu,\varphi} f(x) \lesssim \mathcal{H}_*^\alpha f_d(x^2)$ with $\alpha = 2\mu$. So the result will follow once we prove the L^p-boundedness of $\mathcal{H}_*^\alpha$.

For $\alpha_k \geq 0$ it is sufficient to justify the boundedness on L^∞ and from L^1 into $L^{1,\infty}$, since then the strong-type (p,p) is obtained by interpolation. The proof of those boundedness results follows from the proof of Theorem 1.9 (b) for $d = 1$.

For the other case, the proof is similar to the proof of Theorem 3.2 from [15] using

$$\begin{aligned} T_*^{\mu,\varphi} f(x) &\lesssim \mathcal{H}_*^\alpha f_d(x^2) \\ &\lesssim \mathcal{H}_*^{\alpha_1} \circ \cdots \circ \mathcal{H}_*^{\alpha_d} f(x^2), \end{aligned}$$

and the L^p-boundedness of the one-dimensional operator. □

3.2. **The endpoint** p_1**.** Let f be a non-negative measurable function defined on $\mathbb{R}_*^d$, then as it was done in the proof of Theorem 1.3 we get for $1 < p < \infty$,

$$2^{-d/p} \|f\|_{p,dx} \leq \|f_d \chi_{\mathbb{R}_+^d}\|_{p,dx} \leq 2^d \|f\|_{p,dx}. \tag{3.19}$$

Let us suppose first that $\tilde{d}(\mu) = 1$, then $\tilde{d}(\alpha) = 1$ and $-1 < \tilde{\alpha} < 0$. Without loss of generality we may assume that $\tilde{\alpha} = \alpha_1$ is the only minimal α_k. Due to the product structure of $\mathcal{H}_t^\alpha(x^2, y^2)$ it is enough to use the strong-type (p_1, p_1) estimate in the variables $x_2, \ldots, x_d$ and then the weak-type (p_1, p_1) boundedness in the x_1 variable. This takes care of item (a ii) of Theorem 1.11.

For proving the remaining cases we proceed as it was done in [15]. We are going to write all the theorems involved and give a sketch of the proofs following what was done in [15].

3.2.1. *All* α_k *are minimal.* To prove this case we need the following theorem whose proof is analogous to the proof of Theorem 4.1 from [15].

Theorem 3.20. *Assume* $\alpha_k = a \in (-1, 0)$ *for all* k*. Then for* $d = 2, 3$ *the operator* $\mathcal{H}_*^\alpha$ *maps* $L^{p_1}(\mathbb{R}_+^d, dx)$ *boundedly into the space weak* $L^{p_1} \log^{(d-1)/p_1} L$*, i.e., there exists* $C > 0$*, such that*

$$\left|\{x \in \mathbb{R}_+^d : \mathcal{H}_*^\alpha g(x^2) > \lambda\}\right| \leq C \frac{\|g\|_{p_1}^{p_1}}{\lambda^{p_1}} \left[\log\left(2 + \frac{\lambda}{\|g\|_{p_1}}\right)\right]^{d-1},$$

for every $\lambda > 0$ *and* $g \in L^{p_1}(\mathbb{R}_+^d)$*.*

If we assume that all μ_k are minimal, using this theorem with $\alpha = 2\mu$, we get

$$\begin{aligned} |\{x \in \mathbb{R}_*^d : T_*^{\mu,\varphi} f(x) > \lambda\}| &\lesssim |\{x \in \mathbb{R}_*^d : \mathcal{H}_*^\alpha f_d(x^2) \gtrsim \lambda\}| \\ &= 2^d |\{x \in \mathbb{R}_+^d : \mathcal{H}_*^\alpha f_d(x^2) \gtrsim \lambda\}| \\ &\lesssim \frac{\|f_d \chi_{\mathbb{R}_+^d}\|_{p_1}^{p_1}}{\lambda^{p_1}} \left[\log\left(2 + \frac{\lambda}{\|f_d \chi_{\mathbb{R}_+^d}\|_{p_1}}\right)\right]^{d-1} \\ &\simeq \frac{\|f\|_{p_1}^{p_1}}{\lambda^{p_1}} \left[\log\left(2 + \frac{\lambda}{\|f\|_{p_1}}\right)\right]^{d-1}, \end{aligned}$$

where we use (3.19) in the last equality. This proves item (b ii) of Theorem 1.11 for $2 \leq d \leq 3$ and all μ_k minimal.

3.2.2. *Two minimal α_k in dimension* 3. Theorem 3.20 takes care of all cases with respect to the end point p_1 but one, when $d = 3$ and α has two minimal components. Without loss of generality we may assume $\alpha = (a, a, b)$ with $-1 < a < 0$ and $a < b$. In this case we have analogously to Theorem 4.4 of [15] the following theorem.

Theorem 3.21. *Let $d = 3$ and $\alpha = (a, a, b)$ with $-1 < a < 0$ and $a < b$. Then for every $g \in L^{p_1}$ the distribution function of $\mathcal{H}_*^\alpha g$ satisfies*

$$\left|\{x \in \mathbb{R}_+^d : \mathcal{H}_*^\alpha g(x^2) > \lambda\}\right| \le C \frac{\|g\|_{p_1}^{p_1}}{\lambda^{p_1}} \log\left(2 + \frac{\lambda}{\|g\|_{p_1}}\right), \quad \lambda > 0.$$

Taking into account this theorem and that μ has two minimal components which we may consider to be $\mu = (\nu, \nu, \beta)$ with $-1/2 < \nu < 0$ and $\nu < \beta$, then by setting $a = 2\nu$ and $b = 2\beta$ we obtain

$$\begin{aligned}
|\{x \in \mathbb{R}_*^3 : T_*^{\mu,\varphi} f(x) > \lambda\}| &\lesssim |\{x \in \mathbb{R}_*^3 : \mathcal{H}_*^\alpha f_3(x^2) \gtrsim \lambda\}| \\
&= 2^3 |\{x \in \mathbb{R}_+^3 : \mathcal{H}_*^\alpha f_3(x^2) \gtrsim \lambda\}| \\
&\lesssim \frac{\|f_3 \chi_{\mathbb{R}_+^3}\|_{p_1}}{\lambda^{p_1}} \log\left(2 + \frac{\lambda}{\|f_3 \chi_{\mathbb{R}_+^3}\|_{p_1}}\right) \\
&\simeq \frac{\|f\|_{p_1}^{p_1}}{\lambda^{p_1}} \log\left(2 + \frac{\lambda}{\|f\|_{p_1}}\right).
\end{aligned}$$

And with this we finish the proof of item (b ii) of Theorem 1.11 for $2 \le d \le 3$.

3.2.3. *Counterexamples.* To analyze the negative result at the end point p_1 for $d \ge 4$ we cannot apply directly the results given in [15], though we take some ideas from there. Assume now that $d \ge 4$ and $\mu \in (-1/2, \infty)^d$ is such that $\tilde{\mu} < 0$ and $\tilde{d}(\mu) \ge 2$.

For $x, y \in \mathbb{R}_+^d$ we have

$$M_{\mathfrak{G}\mathfrak{H}}^{\mu,d}(x, y, t) \gtrsim M_{\mathfrak{L}}^{\beta,d}(x^2, y^2, 2t)$$

with $\beta = \mu - \frac{1}{2}(1, 1, \dots, 1)$. Thus for $x \in \mathbb{R}_+^d$, f a suitable non-negative function with $\operatorname{supp}(f) \subset \mathbb{R}_+^d$ we obtain the following inequality

$$T_t^{\mu,\varphi} f(x) \gtrsim \int_{\mathbb{R}_+^d} \mathcal{H}_t^\alpha(x^2, y^2)\; f(y)\, dy, \tag{3.22}$$

with $\mathcal{H}_t^\alpha(x^2, y^2) = \prod_{k=1}^d \mathcal{H}_t^{\alpha_k}(x_k^2, y_k^2)$, $\alpha = 2\mu$, and we recall that for $\xi, \eta \in (0, \infty)$, $\mathcal{H}_t^a(\xi, \eta) = \mathfrak{H}^{\nu-1/2}(\xi, \eta, t)$ with $a = 2\nu$.

Now we state the negative result for p_1 with $d \ge 4$.

Theorem 3.23. *For $d \ge 4$, $\mu \in (-1/2, \infty)^d$ such that $\tilde{\mu} < 0$ and $d(\tilde{\mu}) \ge 2$, there exists a function $f \in L^{p_1,1}$ such that*

$$|\{T_*^\mu f > \lambda\}| = \infty,$$

for all $\lambda > 0$.

We will prove the theorem in the case when all μ_k are minimal. The same reasoning works in the general case by including the variables corresponding to non-minimal μ_k among the double-primed variables below.

Proof. We are assuming that all μ_k are minimal. The proof of this theorem follows the same steps of Theorem 4.6's proof from [15, p. 232]. There are some modifications to be made. For $d \geq 5$, we can choose d' so that $d = d' + d''$ and $2 \leq d' < d''$. Now for small t the set E_t in our context is $E_t := \{y \in \mathbb{R}^d_+ : t < y_k^2 < 2t \text{ for } k \leq d'; t^{-1} < y_k^2 < 2t^{-1} \text{ for } d' < k \leq d\}$. Let $f_t = (\sqrt{t})^{(d''-d')/p_1}\chi_{E_t}$, which has $L^{p_1,1}$ norm essentially one. For $x \in \mathbb{R}^d_+$ such that $x_k^2 < t$ for $1 \leq k \leq d'$ and $(4t)^{-1} < x_k^2 < 4t^{-1}$ for $d' < k \leq d$, and taking into account inequality (3.22), we have

$$T_*^{\mu,\varphi} f_t(x) \gtrsim \mathcal{H}_*^\alpha f_t(x^2) \geq (\sqrt{t})^{(d''-d')/p_1} \int_{E_t} \mathcal{H}_t^\alpha(x^2, y^2)\, dy,$$

with $\alpha = 2\mu$ and all α_k's are minimal. Now we call with a all α_k's and recall that $p_1 = -2/a$.

For the x taken above we restrict the integration to the set $F_t^x := \{y \in E_t : |y_k - x_k| < 1/x_k \text{ for } d' < k \leq d\}$ and apply Lemma 3.14, item (b) for the first d' variables and item (a) for the remaining ones so that by setting $\prod' = \prod_{k=1}^{d'}$ and $\prod'' = \prod_{k=d'+1}^{d}$ we have

$$\begin{aligned}
\mathcal{H}_t^\alpha(x^2, y^2) &\gtrsim \frac{\prod'(x_k y_k)^{a/2}}{(\sqrt{t})^{d'(a+1)}} \prod\nolimits'' x_k = \frac{\prod'(x_k y_k)^{-1/p_1}}{(\sqrt{t})^{d'(-2/p_1+1)}} \prod\nolimits'' x_k \\
&\simeq \frac{(\sqrt{t})^{-d'/p_1} \prod' x_k^{-1/p_1}}{(\sqrt{t})^{d'(-2/p_1+1)}} \prod\nolimits'' x_k = (\sqrt{t})^{d'/p_1 - d'} \prod\nolimits' x_k^{-1/p_1} \prod\nolimits'' x_k.
\end{aligned}$$

Thus

$$\begin{aligned}
T_*^{\mu,\varphi} f_t(x) \gtrsim \mathcal{H}_*^\alpha f_t(x^2) &\gtrsim (\sqrt{t})^{(d''-d')/p_1} (\sqrt{t})^{d'/p_1 - d'} \prod\nolimits' x_k^{-1/p_1} \prod\nolimits'' x_k \, |F_t^x| \\
&\simeq (\sqrt{t})^{d''/p_1} \prod\nolimits' x_k^{-1/p_1}.
\end{aligned}$$

From now on we can follow what was done in the proof of Theorem 4.6 in [15, p. 233] with t replaced by $\sqrt{t}$.

To cover also the case $d = 4$, we now consider d' with $2 \leq d' = d'' = d/2$.

For $R > 6$ we take the same set $E_R = \{y \in \mathbb{R}^d_+ : 1 < y_d < R, y_d^{-1} < y_k < 2y_d^{-1} \text{ for } k \leq d' \text{ and } y_d/8 < y_k < 8y_d \text{ for } d' < k < d\}$. Then $|E_R| \simeq \int_1^R y_d^{-d'+d''-1}\, dy_d = \log R$, and we define $f_R = |E_R|^{-1/p_1}\chi_{E_R}$, whose $L^{p_1,1}$-norm is essentially 1. For $x \in \mathbb{R}^d_+$, such that $4 < x_d < R - 1$, $0 < x_k < x_d^{-1}$ for $1 \leq k \leq d'$, and $x_d/2 < x_k < 2x_d$ for $d' < k < d$, we have

$$T_*^{\mu,\varphi} f_R(x) \gtrsim \mathcal{H}_*^\alpha f_R(x^2) \gtrsim (\log R)^{-1/p_1} \int_{E_R} \mathcal{H}_t^\alpha(x^2, y^2)\, dy,$$

with $\alpha = 2\mu$ and for all $0 < t \leq 1$.

We choose $t = x_d^{-2}$ and take $F_x := \{y \in \mathbb{R}^d_+ : x_d^{-1}/2 < y_k < x_d^{-1}$ for $1 \le k \le d'$ and $|x_k - y_k| < 1/x_k$ for $d' < k \le d\}$; it is easily proved that for x prescribed in the set above $F_x \subset E_R$. Thus,

$$\mathcal{H}^\alpha_* f_R(x^2) \gtrsim (\log R)^{-1/p_1} \int_{F_x} \mathcal{H}^\alpha_{x_d^{-2}}(x^2, y^2)\, dy.$$

Now for $y \in F_x$, $y_d \simeq x_d$ and by applying items (a) and (b) from Lemma 3.14 we get

$$\begin{aligned}\mathcal{H}^\alpha_{x_d^{-2}}(x^2, y^2) &\gtrsim x_d^{d'(a+1)} \prod{}' (x_k y_k)^{a/2} \prod{}'' x_k \\ &\simeq x_d^{d'(a+1)} x_d^{-d'a/2} \prod{}' x_k^{a/2} \prod{}'' x_k = x_d^{-d'/p_1 + d'} \prod{}' x_k^{-1/p_1} \prod{}'' x_k.\end{aligned}$$

From this,

$$\int_{F_x} \mathcal{H}^\alpha_{x_d^{-2}}(x^2, y^2)\, dy \gtrsim x_d^{-d'/p_1 + d'} \prod{}' x_k^{-1/p_1} \prod{}'' x_k\ |F_x| = x_d^{-d'/p_1} \prod{}' x_k^{-1/p_1}.$$

Thus

$$T^{\mu,\varphi}_* f_R(x) \gtrsim \mathcal{H}^\alpha_* f_R(x^2) \gtrsim (\log R)^{-1/p_1} x_d^{-d'/p_1} \prod{}' x_k^{-1/p_1},$$

which is inequality (13) from [15, p. 234] and from that point the proof follows likewise. $\square$

3.2.4. *Sharpness of the results.* In Theorem 1.11 (a ii) and (b ii) the weak-type inequality $L^{p_1,\infty} \log^{-(\tilde{d}(\mu)-1)} L$ is sharp in the following sense. There is a function $f \in L^{p_1}$ (as a matter of fact we can take f bounded and with compact support) such that for large λ,

$$|\{x \in \mathbb{R}^d_+ : T^{\mu,\varphi}_* f(x) > \lambda\}| \simeq \lambda^{-p_1} [\log(2+\lambda)]^{\tilde{d}(\mu)-1}.$$

Let us take $f = \chi_{(1/2,1)^d}$. And since $T^{\mu,\varphi}_* f(x) \gtrsim \mathcal{H}^\alpha_* f(x^2)$, the conclusion follows from the subsection 4.4 Comment on sharpness in [15, p. 234]; the only thing to change is Lemma 2.2 (b) by Lemma 3.14 (b).

The analysis done in that paper also shows that $T^{\mu,\varphi}_*$ is not bounded on L^{p_1} even if there is only one minimal μ_k.

3.3. **The endpoint** p_0**.** Let E be a measurable subset of $\mathbb{R}^d_*$ of finite measure. According to the notation within the proof of Theorem 1.3 we have

$$(\chi_E)_d(y)\chi_{\mathbb{R}^d_+}(y) = \sum_{A \subset D} \chi_E(\sigma_A(y))\chi_{\mathbb{R}^d_+}(y) = \sum_{A \subset D} \chi_{\sigma_A(E) \cap \mathbb{R}^d_+}(y).$$

Let us observe that for every $y \in \mathbb{R}^d_+$, $(\chi_E)_d(y) = \#\{A \subset D : \sigma_A(y) \in E\} \le 2^d$. Hence, $(\chi_E)_d \chi_{\mathbb{R}^d_+} \le 2^d \chi_F$ with $F = \bigcup_{A \subset D} (\sigma_A(E) \cap \mathbb{R}^d_+)$. On the other hand, there exists $A \subset D$ such that $|E| \le 2^d |\sigma_A(E) \cap \mathbb{R}^d_+|$. Thus, $2^{-d}|E| \le |\sigma_A(E) \cap \mathbb{R}^d_+| \le |F| \le \sum_{A \subset D} |\sigma_A(E) \cap \mathbb{R}^d_+| = \sum_{A \subset D} |E \cap O_A| = |E|$.

Let us recall that for $\mu \in (-1/2, \infty)^d$, $\alpha = 2\mu$, and $x \in \mathbb{R}^d_*$,

$$T^{\mu,\varphi}_* f(x) \lesssim \mathcal{H}^\alpha_* f(x^2).$$

So if for every measurable subset E of $\mathbb{R}^d_+$ with finite measure we prove that

$$|\{x \in \mathbb{R}^d_+ : \mathcal{H}_\alpha f(x^2) > \lambda\}| \leq C\frac{|E|}{\lambda^{p_0}}\left(\log\left(2+\frac{1}{|E|}\right)\right)^{\frac{p_0}{p_1}(\tilde{d}(\alpha)-1)}, \quad \lambda > 0,$$

then the same estimates hold for $T_*^{\mu,\varphi}$.

Indeed, let $E \subset \mathbb{R}^d_*$ with $|E| < \infty$; taking into account the above remarks and the fact that $\tilde{d}(\alpha) = \tilde{d}(\mu)$, we have

$$\begin{aligned}
|\{x \in \mathbb{R}^d_* : T_*^{\mu,\varphi}\chi_E(x) > \lambda\}| &\lesssim |\{x \in \mathbb{R}^d_* : \mathcal{H}_*^\alpha((\chi_E)_d\chi_{\mathbb{R}^d_+})(x^2) \gtrsim \lambda\}| \\
&= 2^d|\{x \in \mathbb{R}^d_+ : \mathcal{H}_*^\alpha((\chi_E)_d\chi_{\mathbb{R}^d_+})(x^2) \gtrsim \lambda\}| \\
&\lesssim |\{x \in \mathbb{R}^d_+ : \mathcal{H}_*^\alpha(\chi_F)(x^2) \gtrsim \lambda\}| \\
&\lesssim \frac{|F|}{\lambda^{p_0}}\log\left(2+\frac{1}{|F|}\right)^{\frac{p_0}{p_1}(\tilde{d}(\alpha)-1)} \\
&\simeq \frac{|E|}{\lambda^{p_0}}\log\left(2+\frac{1}{|E|}\right)^{\frac{p_0}{p_1}(\tilde{d}(\mu)-1)}.
\end{aligned}$$

We start with the case when there is only one minimal value α_k, which without loss of generality we may assume to be α_1. Then the maximal operator

$$\begin{aligned}
\mathcal{K}_*^{(\alpha_2,\ldots,\alpha_d)}f(x) = \sup_{t>0}\int_{\mathbb{R}^d_+} &\mathcal{H}_t^{(\alpha_2,\ldots,\alpha_d)}((x_2^2,\ldots,x_d^2),(y_2^2,\ldots,y_d^2)) \\
&\times f(x_1,y_2,\ldots,y_d)\,dy_2\cdots dy_d
\end{aligned}$$

is bounded on $L^p(\mathbb{R}^d_+)$ for p in an interval strictly containing the point $p_0 = p_0(\tilde{\alpha}) = p_0(\alpha_1)$. By interpolation it is also bounded on the Lorentz space $L^{p_0,1}(\mathbb{R}^p_+)$. Moreover, the one-dimensional maximal operator $\mathcal{H}_*^{\alpha_1}$ is of restricted weak-type (p_0,p_0) (see Remark 3.17), and the same is true for the d-dimensional operator

$$\mathcal{K}_*^{\alpha_1}f(x) = \int_{\mathbb{R}^d_+}\mathcal{H}_t^{\alpha_1}(x_1^2,y_1^2)\; f(y_1,x_2,\ldots,x_d)\,dy_1.$$

Since restricted weak-type (p_0,p_0) means boundedness from $L^{p_0,1}$ into weak L^{p_0} and

$$\mathcal{H}_*^\alpha f(x^2) \leq \mathcal{K}_*^{\alpha_1}\circ\mathcal{K}_*^{(\alpha_2,\ldots,\alpha_d)}f(x), \quad x \in \mathbb{R}^d_+,$$

item (a iii) in Theorem 1.11 follows.

Now the results for the end-point p_0 follow using

(1) *For all α_k minimal:* In this case $d \geq 2$, $\alpha_k = a$ for all k and $-1 < a < 0$. The critical exponents are $p_1 = -\frac{2}{a}$ and $p_0 = \frac{2}{a+2}$. Similarly to Theorem 5.1 from [15] we have

Theorem 3.24. *For $2 \leq d \leq 3$ and α as described above, the operator $\mathcal{H}_*^\alpha$ maps $L^{p-1}\log^{\frac{d-1}{p_1}}L$ into $L^{p_1,\infty}$, i.e., for every measurable set $E \subset \mathbb{R}^d_+$ of* *finite measure*

$$|\{x \in \mathbb{R}^d_+ : \mathcal{H}^\alpha_* \chi_E(x^2) > \lambda\}| \leq C \frac{|E|}{\lambda^{p_0}} \left(\log \left(2 + \frac{1}{|E|} \right) \right)^{\frac{p_0}{p_1}(d-1)},$$

for every $\lambda > 0$.

(2) *For two minimal* α_k *in dimension* 3*:* For this case we may assume without loss of generality that $\alpha = (a, a, b)$ with $-1 < a < 0$ and $a < b$. Then we have similarly to Theorem 5.7 from [15]

Theorem 3.25. *For* $d = 3$ *and* α *as described above, the operator* $\mathcal{H}^\alpha_*$ *maps* $L^{p_0,1} \log^{1/p_1} L$ *into* $L^{p_0,\infty}$*, i.e., there exists a constant* $C > 0$ *such that*

$$|\left\{x \in \mathbb{R}^3_+ : \mathcal{H}^\alpha_* \chi_E(x^2) > \lambda\right\}| \leq C \frac{|E|}{\lambda^{p_0}} \left[\log \left(2 + \frac{1}{|E|} \right) \right]^{p_0/p_1},$$

for every measurable $E \subset \mathbb{R}^3_+$ *of finite measure and every* $\lambda > 0$.

3.3.1. *Counterexamples.* As in the case of the endpoint p_1 we will find counterexamples in this case too by taking into account what was done in [15].

We know that for $E \subset \mathbb{R}^d_+$ with $|E| < \infty$ and $\mu \in (-1/2, \infty)^d$ we have

$$T^{\mu,\varphi}_* \chi_E \gtrsim \mathcal{H}^\alpha_* \chi_E(x^2),$$

with $\alpha = 2\mu$.

We will prove the following

Theorem 3.26. *Let* $d \geq 4$ *and* α *with at least two minimal* α_k*. Then there are neither* $C > 0$ *nor* $\gamma \in \mathbb{R}$ *such that the inequality*

$$|\left\{x \in \mathbb{R}^3_+ : \mathcal{H}^\alpha_* \chi_E(x^2) > \lambda\right\}| \leq C \frac{|E|}{\lambda^{p_0}} \left[\log \left(2 + \frac{1}{|E|} \right) \right]^{\gamma}, \quad \lambda > 0,$$

holds for all $E \subset \mathbb{R}^d_+$ *of finite measure.*

Once this theorem is proved then item (b iii) for $d \geq 4$ follows due to the estimate given at the beginning of this part.

Proof of Theorem 3.26. As before it is enough to prove this theorem when all α_k are minimal. Let $\alpha_k = a \in (-1, 0)$ for all $k = 1, \ldots, d$. For $d \geq 5$ we take $2 \leq d' < d''$ and define

$$\tilde{E}_t = \{y \in \mathbb{R}^d_+ : {\prod}' y_k^{-1/p_1} > \beta, y_k < \sqrt{t} \text{ for } k \leq d', \text{ and } \\ (\sqrt{t})^{-1} < y_k < \sqrt{2}(\sqrt{t})^{-1} \text{ for } d' < k \leq d\}.$$

Let us remark that $\tilde{E}_t$ is essentially the set $E_{\sqrt{t}}$ defined in [15, p. 245] where 2 is replaced by $\sqrt{2}$ in the double-primed coordinates. According to (18) from [15] we have

$$|\tilde{E}_t| \simeq |E_{\sqrt{t}}| \simeq \beta^{-p_1} \left[\log \left(2 + (\sqrt{t})^{d'} \beta^{p_1} \right) \right]^{d'-1} (\sqrt{t})^{-d''},$$

with β chosen in such a way that $(\sqrt{t})^{d'}\beta^{p_1} > 1$ (see Lemma 2.3 (a) and (b) from [15]).

We consider $x \in \mathbb{R}^d_+$ such that $x_k^2 < t$ for $k \le d'$ and $(4t)^{-1} < x_k^2 < 4t^{-1}$ for $d' < k \le d$, and estimate $\mathcal{H}^\alpha_* \chi_{\tilde{E}_t}$. In the integral defining this operator we further restrict the integration to the set

$$F_t^x = \{y \in \tilde{E}_t : |x_k - y_k| < 1/x_k \text{ for } d' < k \le d\}.$$

By Lemma 2.3 (a) and (b) from [15] we have

$$\begin{aligned}|F_t^x| &\simeq \beta^{-p_1}\left[\log\left(2 + (\sqrt{t})^{d'}\beta^{p_1}\right)\right]^{d'-1} \prod\nolimits'' \frac{1}{x_k}\\ &\simeq |\tilde{E}_t| \prod\nolimits'' \frac{\sqrt{t}}{x_k} \simeq |\tilde{E}_t|.\end{aligned}$$

According to items (a) and (b) from Lemma 3.14 for $t \le \frac{1}{16}$ we obtain that $\prod' \mathcal{H}^a_t(x_k^2, y_k^2) \gtrsim \prod' (x_k y_k)^{-1/p_1} (\sqrt{t})^{(2/p_1-1)d'}$ and $\prod'' \mathcal{H}^a_t(x_k^2, y_k^2) \gtrsim \prod'' x_k$.

Thus

$$\begin{aligned}\mathcal{H}^\alpha_* \chi_{\tilde{E}_t}(x^2) &\gtrsim \prod\nolimits' x_k^{-1/p_1} (\sqrt{t})^{(2/p_1-1)d'} \prod\nolimits'' x_k \int_{F_t^x} \prod\nolimits' y_k^{-1/p_1}\, dy\\ &\gtrsim \prod\nolimits' x_k^{-1/p_1} (\sqrt{t})^{(2/p_1-1)d'} \prod\nolimits'' x_k\ \beta\, |\tilde{E}_t|\\ &> (\sqrt{t})^{-d'/p_1} (\sqrt{t})^{(2/p_1-1)d'} (\sqrt{t})^{d''}\ \beta\, |\tilde{E}_t|\\ &\simeq (\sqrt{t})^{d''-d'/p_0}\ \beta\, |E_{\sqrt{t}}|,\end{aligned}$$

which is inequality (19) with t replaced by $\sqrt{t}$ in [15, p. 245]. And from now on we follow the steps of that proof.

Now for $d' = d'' \ge 2$ we take $d = 4$ and define $F_N = \bigcup_{j=2}^N \tilde{E}_{2^{-j}}$ and $\beta = N^{1/p_1}(\sqrt{t})^{-d'/p_1}$ with $t = 2^{-j}$. As before we follow the steps of Theorem 5.9's proof in [15, p. 246] to get the conclusion of this theorem for the case $d = 4$. □

Taking into account that for $x \in \mathbb{R}^d_+$

$$T_*^{\mu,\varphi} f(x) \gtrsim \mathcal{H}^\alpha_* f(x^2)$$

for any non-negative function f with $\operatorname{supp}(f) \subset \mathbb{R}^d_+$, and the comments on sharpness on page 247 from [15], we can conclude also in this case that in Theorem 1.11 (a iii) and (b iii) the space

$$L^{p_0,1} \log^{\tilde{d}(\mu)-1/p_1} L = \left\{ f : \int_0^\infty f^*(s) s^{1/p_0} \left[\log(2 + 1/s)^{(\tilde{d}(\mu)-1)/p_1}\right] ds/s < \infty \right\},$$

being f^* the decreasing rearrangement of f on $\mathbb{R}_+$, is the best possible in the sense of convergence at 0 of the above integral.

We obtain also from those comments that $T_*^{\mu,\varphi}$ is not weak-type (p_0, p_0) even if there is only one minimal μ_k. □

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Received: January 13, 2012
Accepted: August 21, 2012