ORIGINAL ARTICLE

Ladder relations for a class of matrix valued orthogonal polynomials

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Abstract

Using the theory introduced by Casper and Yakimov, we investigate the structure of algebras of differential and difference operators acting on matrix valued orthogonal polynomials (MVOPs) on $\mathbb{R}$, and we derive algebraic and differential relations for these MVOPs. A particular case of importance is that of MVOPs with respect to a matrix weight of the form $W(x) = e^{-v(x)}e^{xA}e^{xA^*}$ on the real line, where v is a scalar polynomial of even degree with positive leading coefficient and A is a constant matrix.

KEYWORDS

integrable systems, ladder relations, mathematical physics, non–Abelian Toda lattice, orthogonal polynomials

1 | INTRODUCTION

Matrix valued orthogonal polynomials (MVOPs) were introduced by Krein in the 1940s and they appear in different areas of mathematics and mathematical physics, including spectral theory,[1] scattering theory,[2] tiling problems,[3] integrable systems,[4–7] and stochastic processes.[8–10] There is also a fruitful interaction between harmonic analysis of matrix valued functions on compact symmetric pairs and MVOPs. The first example of such an interaction is a family of MVOPs related with the spherical functions of the compact symmetric pair (SU(3), S(U(2) × U(1))), which

appeared in Ref. 11. Inspired by Ref. 12, the case of (SU(2) × SU(2), diag) gave a direct approach[13,14] leading to a general set-up in the context of multiplicity free pairs.[15,16] In this context, some properties of the orthogonal polynomials such as orthogonality, recurrence relations, and differential equations are understood in terms of the representation theory of the corresponding symmetric spaces, see also Ref. 17 for the quantum group case and Ref. 18 for multivariable matrix orthogonal polynomials.

The interpretation of MVOPs in terms of the representation theory of a certain symmetric pair is typically only for a limited (discrete) number of the parameters involved. It is then necessary to develop analytic tools to extend to a general set of parameters. In this context, shift operators for MVOPs turned out to be very useful.[7,19–22]

In the last two decades, there has been significant progress in understanding how the differential and algebraic properties of the classical scalar orthogonal polynomials can be extended to the matrix valued setting. A. Durán and M. Ismail[23] introduced first-order lowering and raising operators for MVOPs, and these results were rederived later on using the Riemann–Hilbert formulation by Grünbaum and coauthors.[24] This Riemann–Hilbert formulation is a powerful methodology to obtain algebraic and differential identities for MVOPs, as well as for the functions of the second kind, and it has been extensively used in the last few years, we refer the reader to Refs. 25–27 and to Refs. 28, 29 for matrix orthogonal polynomials of Hermite and Laguerre type on the real line or the positive half-line. From the perspective of integrable systems, a very relevant result is the connection with matrix analogues of Painlevé equations for the recurrence coefficients, a theme that is well explored in the scalar case, see, for instance, the monograph.[30]

There is also extensive work on orthogonal polynomial solutions of matrix valued differential equations of second order from an analytic point of view, we refer the reader, for instance, to Refs. 31–36.

Very recently, Casper and Yakimov[37] developed a general framework to solve the matrix Bochner problem, that is, the classification of $N \times N$ weight matrices $W(x)$ whose associated MVOPs are eigenfunctions of a second-order differential operator. The main purpose of this paper is to apply the theory proposed in Ref. 37 to MVOPs defined on the real line. This approach is an alternative to the Riemann–Hilbert methodology and has the advantage of being more transparent in the derivation of differential and difference identities for MVOPs.

Given $N \in \mathbb{N}$, we denote by $M_N(\mathbb{C})$ the space of all $N \times N$ matrices with complex entries. Let $W : \mathbb{R} \to M_N(\mathbb{C})$ be a positive definite matrix weight supported in the (possibly infinite) interval $[a, b]$. For $M_N(\mathbb{C})$-valued functions H, G, we define the matrix valued inner product

$$\langle H, G \rangle = \int_a^b H(y)W(y)G(y)^* \, dy \in M_N(\mathbb{C}). \tag{1}$$

Using standard arguments it can be shown that there exists a unique sequence $(P(x, n))_n$ of monic MVOPs with respect to W, in the following sense:

$$\langle P(x, n), P(x, m) \rangle = \mathcal{H}(n)\delta_{n,m}, \tag{2}$$

where the squared norm $\mathcal{H}(n)$ is a positive definite matrix, see, for instance, Refs.[38,39] As a direct consequence of orthogonality, the polynomials $P(x, n)$ satisfy the following three-term recurrence relation:

$$xP(x, n) = P(x, n + 1) + B(n)P(x, n) + C(n)P(x, n - 1), \tag{3}$$

where $B(n), C(n) \in M_N(\mathbb{C})$. Note that these matrix coefficients multiply the MVOPs from the left. From the orthogonality relations, we also obtain that

$$B(n) = X(n) - X(n+1), \qquad C(n) = \mathcal{H}(n)\mathcal{H}(n-1)^{-1},$$

where $X(n)$ is the one-but-leading coefficient of $P(x, n)$, i.e., $P(x, n) = x^n + x^{n-1}X(n) + \cdots$.

We note that the previous MVOPs can be related to the matrix biorthogonal polynomials presented recently in Ref. 25, namely, the Hermitian case (Section 2.4) because the weight satisfies $W(x) = W(x)^*$. As a consequence, the MVOPs that we study coincide with P_n^L in their notation.

The structure of this paper is the following: in Section 2, following the approach of Casper and Yakimov in Ref. 37, we discuss differential and difference operators for these MVOPs. In this noncommutative setting, operators can act both from the right and from the left. We consider two isomorphic algebras of operators acting on MVOPs, one algebra of matrix valued differential operators acting from the right, $\mathcal{F}_R(P)$, and a second algebra of matrix valued discrete operators acting from the left, $\mathcal{F}_L(P)$. In this construction, a differential operator $\mathcal{D} \in \mathcal{F}_R(P)$ acts naturally on the variable of the MVOPs, whereas a difference operator $M \in \mathcal{F}_L(P)$ acts on its degree.

In Section 3, we fix the form of two differential operators $\mathcal{D} = \partial_x + A$, with A an arbitrary matrix, and $\mathcal{D}^\dagger = -\mathcal{D} + v'(x)$, with $v(x)$ a scalar polynomial, and we work out the corresponding difference operators M and $M^\dagger$ using the techniques in Section 2; the form of these operators is motivated by the exponential-type weights on $\mathbb{R}$ that appear later on the paper, but the results that we obtain in Section 3 hold in a more general setting, just by prescribing the form of the operators. We also investigate the structure of the Lie algebra generated by $\mathcal{D}$ and $\mathcal{D}^\dagger$, as well as discrete string equations for the recurrence coefficients of the corresponding family of MVOPs.

The approach proposed in Ref. 37 is particularly explicit in the case of exponential weights defined on the real line; these weights are studied in Section 4 and written in the form $W(x) = e^{-v(x)}e^{xA}e^{xA^*}$, with $x \in \mathbb{R}$, where the potential $v(x)$ is an even polynomial with positive leading coefficient and A is a constant matrix. In this case, the differential operator $\mathcal{D} = \partial_x + A$ has an uncomplicated adjoint $\mathcal{D}^\dagger = -\mathcal{D} + v'(x)$, with respect to the matrix valued inner product given by W. By adjoint we mean that

$$\langle P \cdot \mathcal{D}, Q \rangle = \langle P, Q \cdot \mathcal{D}^\dagger \rangle, \tag{4}$$

for all matrix-valued polynomials P and Q. The actions of $\mathcal{D}$ and $\mathcal{D}^\dagger$ on the MVOPs are

$$(P \cdot \mathcal{D})(x, n) = P'(x, n) + P(x, n)A,$$

$$(P \cdot \mathcal{D}^\dagger)(x, n) = -P'(x, n) - P(x, n)A - v'(x)P(x, n),$$

which will imply that $\mathcal{D}, \mathcal{D}^\dagger \in \mathcal{F}_R(P)$. Our first result states that $\mathcal{D}, \mathcal{D}^\dagger$ induce ladder relations:

$$P \cdot \mathcal{D}(x, n) = \sum_{j=-k+1}^{0} A_j(n)P(x, n+j), \quad P \cdot \mathcal{D}^\dagger(x, n) = \sum_{j=0}^{k-1} \widetilde{A}_j(n)P(x, n+j),$$

where $k = \deg v$, with some matrix coefficients $A_j(n)$ and $\widetilde{A}_j(n)$. These operators are closely related to the creation and annihilation operators given in Ref. 23, with the advantage that $\mathcal{D}$ and $\mathcal{D}^\dagger$ are each other's adjoint. This property is crucial to show that the Lie algebra generated

by the operators $\mathcal{D}$ and $\mathcal{D}^{\dagger}$ is finite dimensional and it is isomorphic to the algebra generated by the ladder operators for the scalar weight $w(x) = e^{-\upsilon(x)}$, see, for instance,[40] [Ref. 41, Chapter 3]. From the ladder relations, we obtain nonlinear algebraic equations for the coefficients of the recurrence relation (3). In the literature, these identities are often called discrete (or Freud) string equations. We include two examples: Hermite-type weights with $\upsilon(x) = x^2 + tx$ and $t \in \mathbb{R}$, and Freud-type weights with $\upsilon(x) = x^4 + tx^2$, and in this last case the discrete string equations can be seen as a matrix analogue of the discrete Painlevé I equation.[30] We remark that this kind of identity, which is very relevant in integrable systems, is obtained here as a result of the relation between the two Fourier algebras of operators and in particular from the fact that $\mathcal{F}_L(P)$ and $\mathcal{F}_R(P)$ are isomorphic.

Section 5 is devoted to the detailed study of Hermite-type matrix valued weights. In this setting, we show first that the ladder relations, written in terms of the squared norms of the monic MVOPs, in fact characterize this matrix valued weight. Next, for a Hermite-type weight of the form $W(x) = e^{-x^2}L(x)L(x)^*$, with L a lower triangular matrix constructed from scalar Hermite polynomials, we find a second-order differential operator D that, together with $\mathcal{D}$, $\mathcal{D}^{\dagger}$ and I, generate a Lie algebra of dimension 4 known as the Harmonic oscillator algebra. Using the Casimir $\mathcal{C}$ of this Lie algebra, we can diagonalize the norms $\mathcal{H}(n)$ of the MVOPs and combining this with a ladder relation characterization, we propose a computational method for these Hermite MVOPs that is more efficient that the standard Gram–Schmidt procedure.

In Section 6, we further specify a Hermite type weight in such a way that there exists a matrix valued Pearson equation for the weight W, for specific choices of the matrix A. This setting gives extra ladder relations for the corresponding MVOPs.

Complementing the previous results, in Section 7 we investigate similar identities of differential and algebraic type for a deformation of the matrix weight with respect to extra parameters. Examples include the non–Abelian Toda and Langmuir lattice equations, which appear, for instance, in Refs. 42–45. For the particular case of a multitime Toda deformation, we give a Lax pair formulation, analogous to [Ref. 41, (2.8.5)] for the scalar case.

In the Appendix, we establish the link between the ladder relations obtained with this methodology and the ladder operators previously considered by A. Durán and M. Ismail in Ref. 23.

2 | PRELIMINARIES

In this section, we introduce the left and right Fourier algebras related to the sequence of monic MVOPs, following a recent work of Casper and Yakimov.[37] Some of the results in this section have already appeared in a more general form in Ref. 37, but we include them to keep our description self-contained.

We view the sequence $P(x,n)$ as a function $P : \mathbb{C} \times \mathbb{N}_0 \to M_N(\mathbb{C})$. It is, therefore, natural to consider the space of functions

$$\mathcal{P} = \{Q : \mathbb{C} \times \mathbb{N}_0 \to M_N(\mathbb{C}) : \quad Q(x,n) \text{ is rational in } x \text{ for fixed n}\}.$$

A differential operator of the form

$$\mathcal{D} = \sum_{j=0}^{n} \partial_x^j F_j(x), \qquad \partial_x^j := \frac{d^j}{dx^j}, \tag{5}$$

where $F_j : \mathbb{C} \to M_N(\mathbb{C})$ is a rational function of x, acts on an element $Q \in \mathcal{P}$ from the right by

$$(Q \cdot D)(x, n) = \sum_{j=0}^{n} (\partial_x^j Q)(x, n) F_j(x).$$

We denote the algebra of all differential operators of the form (5) by $\mathcal{M}_N$. Now we consider a left action on $\mathcal{P}$ by discrete operators. For $j \in \mathbb{Z}$, let δ^j be the discrete operator, which acts on a sequence $A : \mathbb{N}_0 \to M_N(\mathbb{C})$ by

$$(\delta^j \cdot A)(n) = A(n + j),$$

where we take the value of a sequence at a negative integer to be equal to the zero matrix. A discrete operator

$$M = \sum_{j=-\ell}^{k} A_j(n)\delta^j, \tag{6}$$

where $A_{-\ell}, \dots, A_k$ are sequences, acts on elements of $\mathcal{P}$ from the left by

$$(M \cdot Q)(x, n) = \sum_{j=-\ell}^{k} A_j(n)(\delta^j \cdot Q)(x, n) = \sum_{j=-\ell}^{k} A_j(n) Q(x, n + j).$$

We denote the algebra of all discrete operators of the form (6) by $\mathcal{N}_N$, and we adapt the construction given in [Ref. 37, Definition 2.20] to our setting:

Definition 1. For the sequence $(P(x, n))_n$ of MVOPs we define:

$$\begin{aligned} \mathcal{F}_L(P) &= \{M \in \mathcal{N}_N : \exists D \in \mathcal{M}_N, \, M \cdot P = P \cdot D\} \subset \mathcal{N}_N, \\ \mathcal{F}_R(P) &= \{D \in \mathcal{M}_N : \exists M \in \mathcal{N}_N, \, M \cdot P = P \cdot D\} \subset \mathcal{M}_N. \end{aligned} \tag{7}$$

Using these Fourier algebras, we prove the following uniqueness result:

Lemma 1. *Given $D \in \mathcal{F}_R(P)$, there exists a unique $M \in \mathcal{F}_L(P)$ such that $M \cdot P = P \cdot D$. Conversely, given $M \in \mathcal{F}_L(P)$, there exists a unique $D \in \mathcal{F}_R(P)$ such that $M \cdot P = P \cdot D$.*

Proof. Let us assume that there exist $M_1, M_2 \in \mathcal{F}_L(P)$ such that

$$(M_1 \cdot P)(x, n) = (P \cdot D)(x, n), \qquad (M_2 \cdot P)(x, n) = (P \cdot D)(x, n),$$

then $((M_1 - M_2) \cdot P)(x, n) = 0$. Suppose that $M_1 - M_2$ has the following expression:

$$((M_1 - M_2) \cdot P)(x, n) = \sum_{j=-\ell}^{k} A_j(n) P(x, n + j). \tag{8}$$

By taking the leading coefficient of (8) we obtain that $A_k(n) = 0$. Proceeding recursively we conclude that $A_j(n) = 0$ for all $j = -\ell, \dots, k$. The converse is proven in a similar way. ∎

It follows directly from the definition that the elements of $\mathcal{F}_L(P)$ are related to the elements of $\mathcal{F}_R(P)$. Lemma 1 shows that the map

$$\varphi : \mathcal{F}_L(P) \to \mathcal{F}_R(P), \qquad \text{defined by} \quad M \cdot P = P \cdot \varphi(M),$$

is in fact a bijection. In Ref. 37 this map is called the *generalized Fourier map*. As in Ref. 37, we introduce the bispectral algebras $\mathcal{B}_L(P)$ and $\mathcal{B}_R(P)$:

$$\begin{aligned}
\mathcal{B}_L(P) &= \{M \in \mathcal{F}_L(P) : \text{order}(\varphi(M)) = 0\}, \\
\mathcal{B}_R(P) &= \{D \in \mathcal{F}_R(P) : \text{order}(\varphi^{-1}(D)) = 0\},
\end{aligned} \tag{9}$$

where a differential operator of order zero is a rational function $F : \mathbb{C} \to M_N(\mathbb{C})$ and a discrete operator of order zero is a sequence $A : \mathbb{N}_0 \to M_N(\mathbb{C})$.

Remark 1. For $M_1, M_2 \in \mathcal{F}_L(P)$ we have that

$$M_1 M_2 \cdot P = M_1 \cdot P \cdot \varphi(M_2) = P \cdot \varphi(M_1)\varphi(M_2), \tag{10}$$

which implies that $M_1 M_2 \in \mathcal{F}_L(P)$. Therefore the linear space $\mathcal{F}_L(P)$ is a subalgebra of $\mathcal{N}_N$. A similar computation shows that $\mathcal{F}_R(P)$ is an algebra. We shall refer to $\mathcal{F}_L(P)$ and $\mathcal{F}_R(P)$ as the left and right Fourier algebras, respectively.

Now it follows from (10) that $M_1 M_2 \cdot P = P \cdot \varphi(M_1)\varphi(M_2)$ for all $M_1, M_2 \in \mathcal{F}_L(P)$. On the other hand, by the definition of φ, we have that $M_1 M_2 \cdot P = P \cdot \varphi(M_1 M_2)$ and, because φ is bijective, we conclude that φ is an isomorphism of algebras.

In Ref. 37, it is shown that this map φ is an isomorphism of algebras in a more general setting as well. The crucial requirement there is that P only has trivial left and right annihilators.

Remark 2. We can write the three-term recurrence relation (3) as

$$xP = P \cdot x = L \cdot P, \qquad \text{where} \quad L = \delta + B(n) + C(n)\delta^{-1}.$$

Therefore, $x \in \mathcal{F}_R$, $L \in \mathcal{F}_L$ and $\varphi(L) = x$. Moreover, for every polynomial $v \in \mathbb{C}[x]$, we have

$$P \cdot v(x) = P \cdot v(\varphi(L)) = v(L) \cdot P.$$

The main result from Ref. 37 that we use in this paper is the existence of an adjoint operation †
in the Fourier algebras $\mathcal{F}_L(P)$ and $\mathcal{F}_R(P)$, see [Ref. 37, §3.1]. To introduce the adjoint on $\mathcal{F}_L(P)$, we first note that the algebra of discrete operators $\mathcal{N}_N$ has a $*$-operation given by

$$\left(\sum_{j=-\ell}^{k} A_j(n) \delta^j \right)^* = \sum_{j=-\ell}^{k} A_j(n-j)^* \delta^{-j}, \tag{11}$$

where $A_j(n-j)^*$ denotes the conjugate transpose of $A_j(n-j)$. The adjoint of $M \in \mathcal{N}_N$ is

$$M^\dagger = \mathcal{H}(n) M^* \mathcal{H}(n)^{-1}, \tag{12}$$

where the squared norm $\mathcal{H}(n)$, given by (2), is viewed as a sequence. The following relation holds:

$$\langle (M \cdot P)(x,n), P(x,m) \rangle = \langle P(x,n), (M^\dagger \cdot P)(x,m) \rangle.$$

In Ref. 39, A. Grünbaum and J. Tirao introduce an adjoint in the algebra of all differential operators having the orthogonal polynomials as eigenfunctions. This was recently extended in [Ref. 37, Corollary 3.8] where the authors show that for every differential operator $D \in \mathcal{F}_R(P)$, there exists a unique operator $\mathcal{D}^\dagger \in \mathcal{F}_R(P)$ such that

$$\langle P \cdot D, Q \rangle = \langle P, Q \cdot D^\dagger \rangle,$$

for all $P, Q \in M_N(\mathbb{C})[x]$. We say that $\mathcal{D}^\dagger$ is the adjoint of $\mathcal{D}$. Moreover, $\mathcal{F}_L(P)$ is closed under the adjoint operation $\dagger$ and $\varphi(M^\dagger) = \varphi(M)^\dagger$ for all $M \in \mathcal{F}_L(P)$.

Definition 2. Given a pair $(M, \mathcal{D})$ with $M \in \mathcal{F}_L(P)$ and $\mathcal{D} \in \mathcal{F}_R(P)$, a relation of the form

$$M \cdot P = P \cdot \mathcal{D}, \qquad \text{where} \quad M = \sum_{j=-\ell}^{k} A_j(n)\, \delta^j.$$

is called a ladder relation. If the operator M only contains nonpositive (nonnegative) powers of δ, we say that it is a lowering (raising) relation.

Observe that if a pair $(M, \mathcal{D})$ gives a raising relation, then it follows from (12) and (11) that $(M^\dagger, \mathcal{D}^\dagger)$ gives a lowering relation and viceversa.

3 | LADDER RELATIONS FOR MVOPs WITH PRESCRIBED DIFFERENTIAL OPERATORS

In this section, we study ladder relations for MVOPs in a general setting, where we only prescribe the form of the differential operators $\mathcal{D}$ and $\mathcal{D}^\dagger$. Our choice is motivated by the exponential weights on the real line that we study next in Section 4, but we emphasize that the results that we obtain hold in a more general setting. First we need the following notation:

Remark 3. Given the difference operator L corresponding to the three-term recurrence relation, see (2), and any polynomial q, we denote by $(q(L))_j(n)$ the coefficient of the difference operator $q(L)$ of order j in δ. In other words, we have

$$q(L) = \sum_{j=-\deg q}^{\deg q} (q(L))_j(n)\, \delta^j. \tag{13}$$

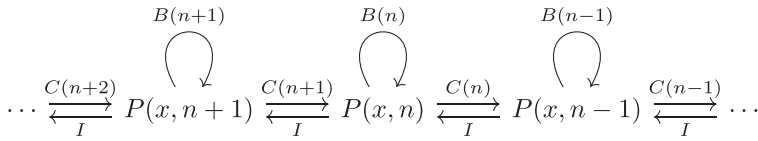The calculation of $(q(L))_j(n)$ can be carried out following the scheme shown in Figure 1:

FIGURE 1 Scheme for the calculation of $(q(L))_j(n)$

$(q(L))_j(n)$ is equal to the sum over all possible paths from $P(x,n)$ to $P(x, n+j)$ in $\deg q$ steps, where in each path we multiply the coefficients corresponding to each arrow.

For example, if $q(x) = x^3$ we have $q(L) = L^3$, and to compute $(L^3)_{-1}(n)$ we have a total of six paths from $P(x,n)$ to $P(x, n-1)$ in three steps:

$$\left(L^3\right)_{-1}(n) = C(n)B(n-1)^2 + B(n)C(n)B(n-1) + B(n)^2 C(n)$$
$$+ C(n+1)C(n) + C(n)C(n-1) + C(n)^2.$$

Theorem 1. *Let W be a matrix weight with monic MVOPs $P(x,n)$ such that*

$$\mathcal{D} = \partial_x + A, \qquad and \qquad \mathcal{D}^\dagger = -\mathcal{D} + v'(x), \tag{14}$$

for some polynomial $v(x)$ of degree k and some constant matrix A. Then the monic polynomials $P(x,n)$ satisfy the lowering relation

$$P \cdot \mathcal{D} = M \cdot P, \qquad M = \sum_{j=-k+1}^{0} A_j(n)\delta^j, \tag{15}$$

where, using the notation (13), we have

$$A_0(n) = A, \qquad A_j(n) = \left(v'(L)\right)_j(n), \quad j \neq 0. \tag{16}$$

Proof. It follows from [Ref. 37, Theorem 3.7] that $\mathcal{D} \in \mathcal{F}_R(P)$. Notice that $(P \cdot \partial_x)(x,n)$ is a polynomial of degree $n-1$. Furthermore $(P \cdot \mathcal{D})(x,n)$ is a polynomial of degree n with A as its leading coefficient. Therefore

$$(P \cdot \mathcal{D})(x,n) = \sum_{j=-n}^{0} A_j(n)\left(\delta^j \cdot P\right)(x,n).$$

Using the definition of $\mathcal{D}$ given in (29), it is clear that $A_0(n) = A$. Moreover for $j < 0$, we have

$$A_j(n) = \langle P \cdot \mathcal{D}, \delta^j \cdot P \rangle \mathcal{H}(n-j)^{-1} = \langle P, \delta^j \cdot P \cdot \mathcal{D}^\dagger \rangle \mathcal{H}(n-j)^{-1}$$

$$= \langle P, \delta^j \cdot P \cdot v'(x) \rangle \mathcal{H}(n-j)^{-1} = \langle P \cdot v'(x), \delta^j \cdot P \rangle \mathcal{H}(n-j)^{-1}$$

$$= \langle v'(L) \cdot P, \delta^j \cdot P \rangle \mathcal{H}(n-j)^{-1},$$

where we have used that $\langle P, \delta^j \cdot P \cdot D \rangle = 0$ for $j < 0$ in the third equality, and the fact that $v'(x)$ is a scalar function in the fourth equality. Using (13) we get

$$A_j(n) = \left(v'(L)\right)_j(n).$$

To complete the proof we note that $(v'(L))_j(n) = 0$ for all $j \leq -k$. ∎

Corollary 1. *There exists a unique $M^\dagger \in \mathcal{F}_L(P)$ such that*

$$M \cdot P = P \cdot D \qquad and \qquad M^\dagger \cdot P = P \cdot D^\dagger. \tag{17}$$

Moreover, we can write

$$M^\dagger = \mathcal{H}(n) \sum_{i=0}^{k-1} A_{-i}(n+i)^* \mathcal{H}(n+i)^{-1} \delta^i, \tag{18}$$

in terms of the coefficients of M in (15) and the squared norms of the MVOPs given by (2).

Proof. The result follows directly from formulas (11) and (12) applied to the operator M. ∎

If a Riemann–Hilbert formulation for matrix orthogonal polynomials can be used, then the previous theorem can be compared with the results in Ref. 24. We note that our approach gives an elementary proof for the lowering relation and we also obtain the exact degree of the lowering operator M and the raising operator $M^\dagger$.

Next we investigate the properties of the Lie algebra generated by the operators $\mathcal{D}$ and $\mathcal{D}^\dagger$ constructed before. Using the explicit expressions of $\mathcal{D}$ and $\mathcal{D}^\dagger$, we find that

$$\left[\mathcal{D}^\dagger, \mathcal{D}\right] = v''(x), \qquad \mathcal{D} + \mathcal{D}^\dagger = v'(x). \tag{19}$$

In the scalar case $A = 0$, $N = 1$, the differential operators $\mathcal{D}$ and $\mathcal{D}^\dagger$ generate a finite dimensional Lie algebra, see, for instance, [Ref. 40, Theorem 3.1]. Moreover it is conjectured in [Ref. 41, §24.5] that this is a characterizing property of the weight. In the following proposition, we prove that in the matrix valued setting, the operators $\mathcal{D}$ and $\mathcal{D}^\dagger$ generate a finite dimensional Lie algebra $\mathfrak{g}$ which is independent of the matrix A and is isomorphic to the Lie algebra corresponding to the scalar case.

Proposition 1. *The differential operators $\mathcal{D}$ and $\mathcal{D}^\dagger$ generate a Lie algebra $\mathfrak{g}$ of dimension $k + 1$.*

Proof. Let $v^{(j)}$ be the j-th derivative of v. Using that $v^{(j)}$ is scalar, and so it commutes with the matrix A, we first observe that $[\mathcal{D}, v^{(j)}] = -v^{(j+1)}$. Then for any $M_N(\mathbb{C})$-valued smooth function F we have

$$F \cdot [\mathcal{D}, v^{(j)}] = F \cdot \mathcal{D} v^{(j)} - (F v^{(j)}) \cdot \mathcal{D} = -v^{(j+1)} F.$$

Because $\mathcal{D}^\dagger = -\mathcal{D} + v'(x)$, we obtain that the Lie algebra generated by $\mathcal{D}$ and $\mathcal{D}^\dagger$ is generated by $\{\mathcal{D}, v'(x), \dots, v^{(k)}\}$, and is, therefore, $(k + 1)$-dimensional. ∎

Remark 4. If we take $v(x) = x^2$, the Lie algebra $\mathfrak{g}$ generated by $\mathcal{D}$ and $\mathcal{D}^\dagger$ is three-dimensional and we have the following relations:

$$\mathcal{D} + \mathcal{D}^\dagger = 2x + t, \qquad [\mathcal{D}^\dagger, \mathcal{D}] = 2. \tag{20}$$

In this case, $\mathfrak{g}$ is isomorphic to the Lie algebra of the three-dimensional Heisenberg group, which can be identified with the 3×3 strictly upper triangular matrices. We can map the operators $\mathcal{D}, \mathcal{D}^\dagger$ and the identity to a basis of the Lie algebra as follows:

$$\mathcal{D} \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{D}^\dagger \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I \longmapsto \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In Section 5.2, we extend $\mathfrak{g}$ to a four-dimensional Lie algebra $\mathfrak{h}$ with a nonrivial Casimir element in the center of $\mathcal{U}(\mathfrak{h})$. This operator induces a new difference operator having the MVOPs as eigenfunctions. Whether $\mathfrak{g}$ can be exploited in a similar way in the case of $v(x)$ of degree greater than two will require further investigation.

From the previous results, we obtain nonlinear relations for the coefficients of the three-term recurrence relation. These identities can be seen as a non-Abelian analogue of the discrete string or Freud equations, see, for instance, [Ref. 46, §4.1.1.5]. To obtain them, we observe that the second equation of (19) and Remark 2 imply that

$$\varphi^{-1}(\mathcal{D} + \mathcal{D}^\dagger) = M + M^\dagger = v'(L).$$

explicitly in terms of the difference operator coming from the three-term recurrence relation. Using the explicit formula for $\mathcal{D}$ in Proposition 1 and the definition of $M^\dagger$ in (12), we verify

$$\begin{aligned} (v'(L))_0(n) = \left(\varphi^{-1}(\mathcal{D} + \mathcal{D}^\dagger)\right)_0(n) &= \left(M + M^\dagger\right)_0(n) \\ &= A + \mathcal{H}(n)A^*\mathcal{H}(n)^{-1}, \end{aligned} \tag{21}$$

using the notation (13) again.

Theorem 2. *Let W be a matrix weight with monic MVOPs $P(x, n)$ such that*

$$\mathcal{D} = \partial_x + A \in \mathcal{F}_R(P), \qquad and \qquad \mathcal{D}^\dagger = -\mathcal{D} + v'(x),$$

for some polynomial $v(x)$ of degree k. Then the coefficients of the three term recurrence relation for $P(x, n)$ satisfy the following discrete string relations:

$$\begin{aligned} [B(n), A] &= I + \left(v'(L)\right)_{-1}(n) - \left(v'(L)\right)_{-1}(n+1), & n \geq 0, \\ [C(n), A] &= C(n)\left(v'(L)\right)_0(n-1) - \left(v'(L)\right)_0(n)C(n), & n \geq 1. \end{aligned} \tag{22}$$

Proof. The coefficients of x^{n-1} in

$$(P \cdot D)(x, n) = (M \cdot P)(x, n).$$

give

$$nI + X(n)A = AX(n) + A_{-1}(n). \tag{23}$$

Taking the difference of (23) for $n + 1$ and n we obtain

$$[B(n), A] - I = A_{-1}(n) - A_{-1}(n + 1),$$

which together with Proposition 1 gives the first desired result.

For the second commutation relation we take (21) with n replaced by $n - 1$ and multiplied by $C(n)$ from the left, and we subtract from it (21) with parameter n and multiplied by $C(n)$ from the right. The result follows after cancelation of $\mathcal{H}(n)A^*\mathcal{H}(n-1)^{-1}$ terms. ∎

Example 1. Let $v(x) = x^2 + tx$ and A a generic matrix, then (14) gives the operators

$$\mathcal{D} = \partial_x + A, \qquad \mathcal{D}^\dagger = -\partial_x - A + 2x + t. \tag{24}$$

Using Theorem 1 with $v'(x) = 2x + t$, we obtain

$$A_{-1}(n) = 2C(n) = 2\mathcal{H}(n)\mathcal{H}(n-1)^{-1}.$$

Therefore,

$$M = A + 2C(n)\delta^{-1}, \qquad M^\dagger = 2\delta + \mathcal{H}(n)A^*\mathcal{H}(n)^{-1} = 2\delta + 2B(n) - A + tI, \tag{25}$$

using (21) and the fact that $(v'(L))_0(n) = 2B(n) + tI$ in the last equality.

The discrete string equations from Theorem 2 are

$$[B(n), A] = 2(C(n) - C(n+1)) + I, \qquad C(0) := 0$$
$$[C(n), A] = 2(C(n)B(n-1) - B(n)C(n)). \tag{26}$$

Additionally, if we sum the first identity from 0 to $n - 1$, we obtain

$$\sum_{k=0}^{n-1} [B(k), A] = n - 2C(n).$$

because $C(0) = 0$.

Example 2. Let $v(x) = x^4 + tx^2$ and A a generic matrix, then (14) gives the operators

$$\mathcal{D} = \partial_x + A, \qquad \mathcal{D}^\dagger = -\partial_x - A + 4x^3 + 2tx.$$

The following relations hold true:

$$[D^\dagger, D] = 12x^2 + 2t, \quad [[D^\dagger, D], D] = 24x, \quad [[[D^\dagger, D], D], D] = 24.$$

The relation $P \cdot D = M \cdot P$ in Theorem 1 is written explicitly as follows:

$$P'(x, n) + P(x, n)A = (A + A_{-1}(n)\delta^{-1} + A_{-2}(n)\delta^{-2} + A_{-3}(n)\delta^{-3})P(x, n).$$

where the coefficients are computed using that $A_j(n) = (v'(L))_j(n)$ and the scheme in Figure 1:

$$A_{-1}(n) = 4(C(n)C(n-1) + C(n)^2 + C(n+1)C(n) + B(n)^2 C(n)$$

$$+ B(n)C(n)C(n-1) + C(n)B(n-1)^2) + 2tC(n)$$

$$A_{-2}(n) = 4(B(n)C(n)C(n-1) + C(n)B(n-1)C(n-1))$$

$$(+ C(n)C(n-1)B(n-2)),$$

$$A_{-3}(n) = 4C(n)C(n-1)C(n-2).$$

Furthermore, we use Theorem 2 to compute the discrete string equations

$$[C(n), A] = C(n)(v'(L))_0(n-1) - (v'(L))_0(n)C(n), [B(n), A]$$

$$= I + (v'(L))_{-1}(n) - (v'(L))_{-1}(n+1).$$

If we replace $(v'(L))_0(n) = A + \mathcal{H}(n)A^*\mathcal{H}(n)^{-1}$ in the first equation, we obtain a trivial identity, however in terms of the coefficients of the recurrence relation we have

$$(v'(L))_0(n) = B(n)(C(n) + C(n+1)) + (C(n) + C(n+1) + B(n)^2 + 2t)B(n),$$

which implies the identity

$$[C(n), A] = C(n)B(n-1)(C(n-1) + C(n))$$

$$+ C(n)(C(n-1) + C(n) + B(n-1)^2 + 2t)B(n-1)$$

$$- B(n)(C(n) + C(n+1))C(n)$$

$$- (C(n) + C(n+1) + B(n)^2 + 2t)B(n)C(n).$$

On the other hand, we can sum the second identity from 0 to $n-1$, to obtain

$$\sum_{k=0}^{n-1}[B(k), A] = n + (v'(L))_{-1}(0) - (v'(L))_{-1}(n) = n - (v'(L))_{-1}(n) = n - A_{-1}(n),$$

using (16) as well as the fact that $C(0) = 0$. Therefore, we obtain

$$\sum_{k=0}^{n-1} [B(k), A] = n - 4\Big(C(n)C(n-1) + C(n)^2 + C(n+1)C(n) + B(n)^2 C(n)$$

$$+ B(n)C(n)C(n-1) + C(n)B(n-1)^2 \Big) - 2tC(n).$$

If $A = 0$ then the weight is scalar and even, therefore $B(n) = 0$ and the recurrence coefficients commute. In this case, the previous equality reduces to

$$n = 4C(n)(C(n-1) + C(n) + C(n+1) + 2t),$$

which is the discrete Painlevé I equation, see, e.g., [Ref. 30, §1.2.2].

4 | LADDER RELATIONS FOR EXPONENTIAL WEIGHTS

In this section, we investigate the existence of lowering and raising relations for a class of matrix valued weights of exponential type on the real line. This is an important example that fits into the general theory presented in Section 2.

Definition 3. An $N \times N$ weight matrix supported on $\mathbb{R}$ is called an *exponential type* weight if it is of the form

$$W(x) = e^{-v(x)} e^{xA} e^{xA^*}, \qquad v(x) = x^k + v_{k-1} x^{k-1} + \cdots + v_0, \tag{27}$$

where v is a scalar polynomial, k is even and A is a constant matrix.

Remark 5. If we consider more general weights

$$W_{(A,T,L)}(x) = e^{-v(x)} L e^{xA} T e^{xA^*} L^*, \tag{28}$$

for some constant positive definite matrix T and constant invertible matrix L, then using the Cholesky decomposition $T = KK^*$, we get the following similarity:

$$W_{(A,T,L)}(x) = e^{-v(x)} LK e^{xK^{-1}AK} e^{x(K^{-1}AK)^*} (LK)^* = LK W_{(K^{-1}AK,I,I)}(x)(LK)^*.$$

If we denote by $P(x, n; A, T, L)$ the monic MVOPs with respect to $W_{(A,T,L)}(x)$, then

$$P(x, n; A, T, L) = LK P(x, n; K^{-1}AK, I, I)(LK)^{-1}.$$

Because of this, the weight $W_{(A,T,L)}(x)$ will be called exponential weight as well, and we conclude that it suffices to consider only weights of the form (27).

If we start with the matrix valued differential operator used before,

$$\mathcal{D} = \partial_x + A, \tag{29}$$

we can prove the form of the adjoint in accordance with Section 3:

Proposition 2. *The adjoint of the differential operator (29) with respect to W is*

$$\mathcal{D}^\dagger = -\partial_x - A + v'(x) = -\mathcal{D} + v'(x).$$

Proof. For polynomials $P, Q \in M_N(\mathbb{C})[x]$, we have from (1) and (29) that

$$\langle P \cdot \mathcal{D}, Q \rangle = \langle P', Q \rangle + \langle PA, Q \rangle.$$

Integrating $\langle P', Q \rangle$ by parts and using that W is invertible and self-adjoint, we obtain

$$\langle P', Q \rangle = -\langle P, Q' \rangle - \langle P, QW'W^{-1} \rangle = \langle P, -Q' - QW'W^{-1} \rangle. \tag{30}$$

Observe that the boundary terms on the right-hand side of (30) vanish because of the exponential decay of the matrix weight at $\pm\infty$. Taking into account that $\langle PA, Q \rangle = \langle P, QWA^*W^{-1} \rangle$, we have

$$\langle P \cdot \mathcal{D}, Q \rangle = \langle P, -Q' - QW'W^{-1} + QWA^*W^{-1} \rangle.$$

Therefore $\mathcal{D}^\dagger = -\partial_x - W'(x)W(x)^{-1} + W(x)A^*W(x)^{-1}$, which is the essentially the formal W-adjoint used in Ref. 37 for our weight. Putting back the explicit expression for W completes the proof of the proposition. ∎

Corollary 2. *Let W be a matrix weight as in (27), then $\mathcal{D} = \partial_x + A \in \mathcal{F}_R(P)$.*

Proof. This follows directly from Proposition 2 and [Ref. 37, Theorem 3.7]. ∎

Remark 6. If a weight $\widetilde{W}(x)$ as in Theorem 2 is differentiable, then the equation $\mathcal{D}^\dagger = -\mathcal{D} + v'(x)$ implies that $\widetilde{W}(x)$ solves the following first-order linear matrix differential equation:

$$\widetilde{W}'(x) = (A - v'(x))\widetilde{W}(x) + \widetilde{W}(x)A^*,$$

and it follows that $\widetilde{W}(x)$ is an exponential type weight.

Remark 7. Following the lines of Ref. 25, it is clear that a weight of the form (27) can be factorized in a very natural way as $W(x) = W^L(x)W^R(x)$, with

$$W^L(x) = e^{-\frac{v(x)}{2}}e^{xA}, \qquad W^R(x) = e^{-\frac{v(x)}{2}}e^{xA^*}. \tag{31}$$

We note that this factorization is not unique. This leads to the following logarithmic derivatives:

$$h^{\text{L}}(x) = \left(W^{\text{L}}(x)\right)' W^{\text{L}}(x)^{-1} = -\frac{v'(x)}{2} + A,$$

$$h^{\text{R}}(x) = \left(W^{\text{R}}(x)\right)' W^{\text{R}}(x)^{-1} = -\frac{v'(x)}{2} + A^*. \tag{32}$$

This would allow us to rederive some of the results obtained later in the paper, using the Riemann–Hilbert formulation. However, in this paper we adopt a different approach, based on the use of a specific differential operator $\mathcal{D}$ and its adjoint, which is very well suited to this type of exponential weights and bypasses the lengthy calculations needed in the Riemann–Hilbert formulation of more general cases of MVOPs.

In particular, if we let $v(x) = x^2 + tx$ as in Example 1, and we set $t = 0$, then (32) gives $h^{\text{L}}(x) = A - x$, $h^{\text{R}}(x) = A^* - x$. We identify $A^{\text{L}} = A^{\text{R}} = -I$ and $B^{\text{L}} = A$, $B^{\text{R}} = A^*$ in the notation used in [Ref. 25, Section 6.1]. Then, the general equations at the end of that section simplify considerably and are fully consistent with (26).

5 | HERMITE-TYPE WEIGHTS

If we replace $v(x) = x^2 + tx$ in (27), then we recover the results in Example 1. We say that this is a Hermite-type weight. In this section, we investigate further properties of these Hermite-type matrix weights. First, we will show that an arbitrary matrix weight having operators $\mathcal{D}$ and $\mathcal{D}^\dagger$ as in Theorem 2 and with a specific moment of order zero is equivalent to a Hermite-type weight, in a sense that we specify later on. We also investigate a particular case of the matrix A that leads to the Harmonic oscillator algebra and has a link with a quantum mechanical composite system.

5.1 | Characterization of Hermite-type weights with a ladder relation

The proof of this characterization follows the lines of the main result in Ref. 47, where the authors discuss a scalar Freud weight. First, we present a recursive equation for the norms $\widetilde{\mathcal{H}}(n)$:

Lemma 2. *Let $\widetilde{W}$ be a matrix weight, supported on $\mathbb{R}$, and let $(\widetilde{P}(x,n))_n$ be the sequence of monic orthogonal polynomials. Let A be a matrix such that*

$$\mathcal{D} = \partial_x + A \in \mathcal{F}_R(P), \qquad \mathcal{D}^\dagger = -\mathcal{D} + 2x + t. \tag{33}$$

Then the square norms,

$$\widetilde{\mathcal{H}}(n) = \int_{-\infty}^{\infty} \widetilde{P}(n,x)\widetilde{W}(x)\widetilde{P}(n,x)^* dx, \qquad n \geq 0$$

satisfy the following recursion:

$$\widetilde{\mathcal{H}}(n+1) = \frac{1}{2}\widetilde{\mathcal{H}}(n) + \widetilde{\mathcal{H}}(n)\widetilde{\mathcal{H}}(n-1)^{-1}\widetilde{\mathcal{H}}(n)$$

$$- \frac{1}{4}\widetilde{\mathcal{H}}(n)A^*\widetilde{\mathcal{H}}(n)^{-1}A\widetilde{\mathcal{H}}(n) + \frac{1}{4}A\widetilde{\mathcal{H}}(n)A^*, \tag{34}$$

for $n > 0$. Moreover, $\widetilde{C}(0) = 0$ gives the initial condition

$$\widetilde{\mathcal{H}}(1) = \frac{1}{2}\widetilde{\mathcal{H}}(0) - \frac{1}{4}\widetilde{\mathcal{H}}(0)A^*\widetilde{\mathcal{H}}(0)^{-1}A\widetilde{\mathcal{H}}(0) + \frac{1}{4}A\widetilde{\mathcal{H}}(0)A^*.$$

Proof. We write the string equations (22) in terms of the squared norms. Using the isomorphism φ^{-1} and (20) we verify that $2L = M + M^\dagger - t$ and using (25) we obtain

$$2\widetilde{B}(n) = A + \widetilde{\mathcal{H}}(n)A^*\widetilde{\mathcal{H}}(n)^{-1} - t. \tag{35}$$

If we replace (35) in the first identity of (26), we obtain

$$\widetilde{C}(n) - \widetilde{C}(n+1) = \frac{1}{4}\left[\widetilde{\mathcal{H}}(n)A^*\widetilde{\mathcal{H}}(n)^{-1}, A\right] - \frac{1}{2}. \tag{36}$$

Because $\widetilde{C}(n) = \widetilde{\mathcal{H}}(n)\widetilde{\mathcal{H}}(n-1)^{-1}$ for all $n > 0$, we get (34). ∎

Using this result, we can prove the following characterization of Hermite-type weights in terms of the $\mathcal{H}(n)$:

Theorem 3. *Let $\widetilde{W}$ be a matrix weight as in Lemma 2 and assume that*

$$\widetilde{\mathcal{H}}(0) = \int_{-\infty}^{\infty} e^{-x^2-xt}e^{xA}e^{xA^*}dx.$$

Then there is an invertible constant matrix K such that $W(x) = K\widetilde{W}(x)K^*$, where

$$W(x) = e^{-x^2-xt}e^{xA}e^{xA^*}, \tag{37}$$

almost everywhere with respect to Lebesgue measure on $\mathbb{R}$.

Proof. In this proof, we first show that the string equations determine uniquely, up to the zeroth moment, the coefficients of the recurrence relation for the monic MVOPs with respect to $\widetilde{W}$. Then the theorem follows by showing that the matrix weight (37) corresponds to a determinate moment problem, in the sense of Ref. 48.

From the previous lemma, the squared norms $\widetilde{\mathcal{H}}(n)$ for $n > 0$ are completely determined by the choice of $\widetilde{\mathcal{H}}(0)$. Furthermore, using the identities $\widetilde{C}(n) = \widetilde{\mathcal{H}}(n)\widetilde{\mathcal{H}}(n-1)^{-1}$ and $2\widetilde{B}(n) = A + \widetilde{\mathcal{H}}(n)A^*\widetilde{\mathcal{H}}(n)^{-1}$, we find that the coefficients of the recurrence relation and, thus, the monic orthogonal polynomials are completely determined as well.

Finally we need to prove that the moment problem for the Hermite weight W has a unique solution. By [Ref. 48, Theorem 3.6], it suffices to show that the diagonal entries of the matrix valued measure $W(x)dx$ are determinate. We follow the approach given by Freud in [Ref. 49, Theorem 5.1 and 5.2]: We observe that

$$\left|\int_{-\infty}^{\infty} e^{\beta|x|}W(x)_{i,i}dx\right| \leq \left\|\int_{-\infty}^{\infty} e^{\beta|x|}e^{-x^2-xt}e^{xA}e^{xA^*}dx\right\|_1$$

$$\leq \int_{-\infty}^{\infty} e^{\beta|x|}e^{-x^2-xt}e^{x\|A\|_1}e^{x\|A^*\|_1}dx \leq M < \infty,$$

for any $\beta > 0$. Therefore by [Ref. 49, Theorem 5.2], the diagonal measures $W(x)_{i,i}dx$ are determinate and so is W. ∎

5.2 | The harmonic oscillator algebra

We consider a weight related to the one studied in Ref. 7, of the form

$$W(x) = e^{-x^2}L(x)L(x)^*, \qquad L(x)_{j,k} = \begin{cases} \dfrac{H_{j-k}(x)}{(j-k)!}\dfrac{\alpha_j}{\alpha_k} & j \geq k, \\ 0 & j < k \end{cases} \tag{38}$$

where α is vector of positive parameters $(\alpha_j)_{j=1}^N$ and $H_n(x)$ are the standard scalar Hermite polynomials.

This weight matrix $W(x)$ in (38) is of exponential type, as defined in Definition 3; this result follows from the fact that $L(x)$ satisfies the matrix ODE

$$\frac{d}{dx}L(x) = L(x)A = AL(x), \qquad A_{j,k} = \begin{cases} \dfrac{2\alpha_{k+1}}{\alpha_k} & k = j-1 \\ 0 & \text{else} \end{cases}.$$

Then, we can write $L(x) = L(0)e^{xA}$, and we can recast (38) in the form of an exponential type weight as given in (28).

Remark 8. The exponential weight considered in (38) is taken in such a way that the 0-th square norm (of the monic MVOP) is diagonal; moreover, using the recurrence relation for the norms in Theorem 2, the special structure of the matrix A implies that all square norms are diagonal as well.

Lemma 3. *The squared norms $\mathcal{H}(n)$ of the monic MVOPs with respect to (38) and the three-term recursion coefficient $C(n)$ are diagonal for all $n \in \mathbb{N}_0$.*

Proof. We first show that $\mathcal{H}(0)_{j,j}$ is a diagonal matrix:

$$(\mathcal{H}(0))_{j,k} = \sum_{\ell=1}^N \int_{-\infty}^\infty e^{-x^2}L(x)_{j,\ell}(L(x)^*)_{\ell,k}dx$$

$$= \sum_{\ell=1}^{\min(j,k)} \int_{-\infty}^\infty e^{-x^2}\frac{H_{j-\ell}(x)}{(j-\ell)!}\frac{\alpha_j}{\alpha_\ell}\frac{H_{k-\ell}(x)}{(k-\ell)!}\frac{\alpha_k}{\alpha_\ell}dx$$

$$= 2^j\alpha_j^2\delta_{j,k}\sum_{\ell=1}^j \frac{2^{-\ell}}{(j-\ell)!}\frac{1}{\alpha_\ell^2}.$$

Next, if we denote the square norms in (34) with $t = 0$ as $\widetilde{\mathcal{H}}(n)$, and by $\mathcal{H}(n)$ the square norms of the monic polynomials with respect to the weight (38), then

$$L(0)\widetilde{\mathcal{H}}(n)L(0)^* = \mathcal{H}(n). \tag{39}$$

Therefore, the $\mathcal{H}(n)$ satisfies the same second-order recursion (34). Finally, because our matrix A in this section only has nonzero entries on the first subdiagonal, it follows that all the terms in the recursion (34) are diagonal. The result for $C(n)$ follows from the formula $C(n) = \mathcal{H}(n)\mathcal{H}(n-1)^{-1}$. ∎

The next proposition is a direct consequence of [Ref. 7, Proposition 3.5]:

Proposition 3. *The monic MVOPs $P(x, n)$ that correspond to (38) satisfy*

$$(P \cdot D)(x, n) = \Gamma(n)P(x, n), \tag{40}$$

with the self-adjoint second-order differential operator

$$D = -\frac{1}{2}\partial_x^2 + \partial_x(xI - A) + J, \qquad J := \mathrm{diag}(1, 2, \ldots, N), \tag{41}$$

that acts from the right and with eigenvalues $\Gamma(n) = nI + J$.

Proposition 4. *The identity operator and the differential operators $\mathcal{D}, \mathcal{D}^\dagger, D$ given in (33) and (41) generate a four-dimensional Lie algebra called the* harmonic oscillator algebra*, that we denote by $\mathfrak{h}$.*

Proof. The differential operators introduced in this section satisfy the following commutation relations:

$$[D, \mathcal{D}] = \mathcal{D}, \qquad [D, \mathcal{D}^\dagger] = -\mathcal{D}^\dagger, \qquad [\mathcal{D}, \mathcal{D}^\dagger] = -2, \tag{42}$$

that follow from direct calculation. Note that they act from the right. With the identification

$$\mathcal{D} \longleftrightarrow \sqrt{2}\mathcal{J}^+, \qquad \mathcal{D}^\dagger \longleftrightarrow \sqrt{2}\mathcal{J}^-, \qquad D \longleftrightarrow \mathcal{J}^3, \qquad I \longleftrightarrow \mathcal{E},$$

we find that $\mathfrak{h}$ is isomorphic to the four-dimensional Lie algebra $\mathcal{G}(a, b)$ given in [Ref. 50, §2.5] with parameters $a = 0$ and $b = 1$, see also [Ref. 51, Chapter 10, (1.1)]. ∎

Remark 9. Because $\mathcal{D}, D, \mathcal{D}^\dagger$ are elements of $\mathcal{F}_R(P)$, we have that $\mathfrak{h} \subset \mathcal{F}_R(P)$. Moreover the Lie algebra $\mathfrak{g}$ in Theorem 1 is a three-dimensional ideal of the Lie algebra $\mathfrak{h}$. The isomorphism φ immediately gives an isomorphic subalgebra $\varphi^{-1}(\mathfrak{h}) \subset \mathcal{F}_L(P)$.

The Casimir operator of this Lie algebra $\mathfrak{h}$ is given by

$$C = D - \frac{1}{2}\mathcal{D}^\dagger\mathcal{D} = J - xA + \frac{1}{2}A^2, \tag{43}$$

using the explicit expressions for $D, \mathcal{D}$, and $\mathcal{D}^\dagger$. It can be easily seen from the commutation relations (42) that C commutes with $\widetilde{\mathcal{D}}, \widetilde{\mathcal{D}}^\dagger$, and D. The Casimir operator is useful to derive another differential identity for the MVOPs, that we present below.

Lemma 4. *The Casimir operator is self-adjoint and it acts on the monic MVOPs as*

$$(P \cdot C)(x, n) = (\varphi^{-1}(C) \cdot P)(x, n),$$

with

$$\varphi^{-1}(C) = -A\delta + \left(nI + J - 2C(n) - AB(n) + \frac{1}{2}A^2\right)$$

$$+ (C(n)A - 2C(n)B(n-1))\delta^{-1}.$$

Proof. The fact that the Casimir operator is self-adjoint follows directly from the expression $C = D - \frac{1}{2}D^\dagger D$. Now applying the Lie algebra isomorphism,

$$\varphi^{-1}(C) = \varphi^{-1}(D) - \frac{1}{2}\varphi^{-1}(D^\dagger)\varphi^{-1}(D) = \Gamma(n) - \frac{1}{2}M^\dagger M.$$

Now the explicit expression of $\varphi^{-1}(C)$ follows by replacing the explicit expressions of $\Gamma(n)$ and $M, M^\dagger$ from Example 1, with $t = 0$ and noting that (39) respect the form of the the ladder relations. ∎

Observe that the Casimir operator $\varphi^{-1}(C)$ of the Lie algebra $\varphi^{-1}(\mathfrak{h})$ is a second-order difference operator having the sequence of monic MVOPs as eigenfunctions with a nondiagonal eigenvalue acting on $P(x, n)$ from the right. Therefore, $\varphi^{-1}(C)$ is an element of the left bispectral algebra $\mathcal{B}_L(P)$ given in (9).

The fact that the operator C commutes with $\widetilde{D}, \widetilde{D}^\dagger$, and D is translated via the isomorphism φ into the following relations

$$[\varphi^{-1}(C), \widetilde{M}] = 0, \qquad [\varphi^{-1}(C), \widetilde{M}^\dagger] = 0, \qquad [\varphi^{-1}(C), \Gamma(n)] = 0.$$

Remark 10. The scalar Hermite polynomials have a well-known application as part of the solution to the quantum harmonic oscillator, so it is natural to seek an analogous link in our matrix valued case. In the next section we also link the differential equation satisfied by our polynomials to a Schrödinger equation, but it is in fact several copies of the same Schrödinger equation.

5.3 | Computation of Hermite MVOPs

The nonlinear recurrence for the norms $\mathcal{H}(n)$ that we obtained before is important from a computational point of view as well. If we want to compute Hermite MVOPs, (34) is a very convenient alternative to Gram–Schmidt orthogonalization applied to the canonical basis; together with (35) and (36) to calculate the recurrence coefficients, this gives a double recursion (first (34) and then the recurrence relation) that can then be used to compute $P(x, n)$.

Two drawbacks of this approach are that we need to calculate the inverse $\mathcal{H}(n)^{-1}$ at each step to use (34), and also that the whole procedure can be slow because of the combination of those two matrix recursions, which generally involve full matrices. For this reason, we show next an alternative for the class of weights that we consider in this section: we can replace the matrix recurrence relation by a scalar recursion for some coefficients $\xi(n, N, k)$, and additionally the square norms $\mathcal{H}(n)$ can be easily made diagonal, significantly reducing the computation of the inverses that occur in (34).

This result is related to the results in Ref. 7, where the MVOPs are calculated explicitly but one needs to impose certain additional constraints on parameters in the weight matrix.

We denote by $P(x, n)$ the monic MVOPs with respect to (38), and we also define the following auxiliary functions, which will be useful later:

$$Q(x, n) = P(x, n)\Phi(x), \qquad \Phi(x) = e^{-x^2/2}L(x). \tag{44}$$

Lemma 5. *The differential operators $\mathcal{D}_Q$, $\mathcal{C}_Q$, and D_Q defined by*

$$\mathcal{D}_Q := \Phi(x)^{-1}\mathcal{D}\Phi(x), \qquad \mathcal{C}_Q := \Phi(x)^{-1}\mathcal{C}\Phi(x), \qquad D_Q := \Phi(x)^{-1}D\Phi(x), \tag{45}$$

are given explicitly as follows:

$$\mathcal{D}_Q = \partial_x + x, \qquad \mathcal{C}_Q = J, \qquad D_Q = \frac{1}{2}\left(-\partial_x^2 + x^2 - 1\right)I + J. \tag{46}$$

Proof. Because $L'(x) = AL(x)$, we immediately obtain $(L(x)^{-1})' = -AL(x)^{-1}$ and therefore $(\Phi(x)^{-1})' = (x - A)\Phi(x)^{-1}$. Using this expression we get

$$\mathcal{D}_Q = \partial_x + x.$$

We recall from [Ref. 7, Lemma 3.4] that $L(x)^{-1}JL(x) = J - \frac{1}{2}A^2 + xA$. Now by direct calculation we get the following equations:

$$[J, \Phi(x)] = xA\Phi(x) - \frac{1}{2}A^2\Phi(x), \qquad \Phi(x)^{-1}J\Phi(x) = J + xA - \frac{1}{2}A^2.$$

These imply that $\mathcal{C}_Q = J$. For the second-order operator we conjugate the terms separately.

$$-\frac{1}{2}\Phi(x)^{-1}\partial_x^2\Phi(x) = -\frac{1}{2}\partial_x^2 - \partial_x(x - A) - \left(\frac{1}{2}x^2 - xA + \frac{1}{2}A^2 + \frac{1}{2}\right),$$

$$\Phi(x)^{-1}\partial_x\Phi(x)(xI - A) = \partial_x(x - A) + (x^2 - 2xA + A^2).$$

When combined, they lead to the desired expression. ∎

Proposition 5. *The matrix elements of $Q(x, n)$ are multiples of scalar Hermite functions*

$$Q(x, n)_{j,k} = \xi(n, j, k)H_{n+j-k}(x)e^{-x^2/2}, \tag{47}$$

for $n + j - k \geq 0$ and equal to 0 otherwise.

Proof. Equation (45) implies that $Q(x, n)_{j,k}$ satisfies the Schrödinger equation for the quantum harmonic oscillator

$$-\frac{1}{2}Q(x, n)''_{j,k} + \frac{1}{2}x^2Q(x, n)_{j,k} = \left(n + j - k + \frac{1}{2}\right)Q(x, n)_{j,k}.$$

So the functions $Q(x, n)_{j,k}$ should each be linear combinations of the bounded and unbounded solutions to the above ODE. However, because we know $Q(x, n)$ is a matrix polynomial $P(x, n)L(x)$

multiplied by $e^{-x^2/2}$, the entries can only be equal to the bounded solution, which is the Hermite function $H_{n+j-k}(x)e^{-x^2/2}$ when $n + j - k \geq 0$ and the zero function otherwise. ∎

Proposition 6. *The constants in (47) have the special values:*

$$\xi(n, N, k) = \frac{2^{-n}}{(N-k)!} \frac{\alpha_N}{\alpha_k}, \tag{48}$$

$$\xi(0, j, k) = \frac{1}{(j-k)!} \frac{\alpha_j}{\alpha_k}, \quad \text{for } j \geq k, \quad \xi(0, j, k) = 0 \quad \text{for } j < k. \tag{49}$$

Proof. For $n = 0$ we simply have $Q(x, 0) = L(x)e^{-x^2/2}$, so we can directly read off the desired expression from (38).

We can determine the constants for $j = N$ by using the monicity of $P(x, n) = x^n I + \sum_{s=1}^{n} p_s(n)x^{n-s}$, comparing powers in x and considering specific entries. Writing out the matrix exponential $L(x) = L(0)e^{xA}$, which is a polynomial of degree $N - 1$ we have (for $k \leq N$)

$$e^{x^2/2}Q(x, n) = x^{n+N-1}\left(\frac{1}{(N-1)!}L(0)A^{N-1}\right)$$

$$+ x^{n+N-2}\left(\frac{1}{(N-1)!}p_1(n)L(0)A^{N-1} + \frac{1}{(N-2)!}L(0)A^{N-2}\right)$$

$$\vdots$$

$$+ x^{n+N-k}\left(\frac{1}{(N-r)!}p_q(n)L(0)A^{N-r} + \cdots + \frac{1}{(N-k)!}L(0)A^{N-k}\right)$$

$$\vdots$$

where $q = \min(k - 1, n)$ and $r = k - q$. For each power we only know one term explicitly: the right most one because it is without $p_s(n)$. However, the other terms only have nonzero entries in the first $k - 1$ columns[1] whereas the last term has exactly one nonzero entry in column k

$$\left(L(0)A^{N-k}\right)_{i,k} = \delta_{i,N}L(0)_{N,N}(A^{N-k})_{N,k} = \prod_{j=k+1}^{N}\frac{2\alpha_j}{\alpha_{j-1}} = 2^{N-k}\frac{\alpha_N}{\alpha_k}.$$

Comparing this to the leading coefficient in (47)

$$e^{x^2/2}Q(x, n)_{N,k} = \xi(n, N, k)2^{n+N-k}x^{n+N-k} + \dots$$

we conclude that

---

[1] This relies on the fact that A^r only has a nonzero r-th subdiagonal and $L(0)$ is lower triangular.

$$\xi(n, N, k) = \frac{2^{-n}}{(N-k)!} \frac{\alpha_N}{\alpha_k}.$$

∎

Proposition 7. *The auxiliary function satisfies the following first-order matrix ODE with coefficients that depend on the square norms:*

$$Q(x, n)J$$

$$= \left(nI + J - \frac{1}{2}xA - \frac{1}{2}\mathcal{H}(n)A^*\mathcal{H}(n)^{-1}(x - A) - 2\mathcal{H}(n)\mathcal{H}(n-1)^{-1} \right) Q(x, n)$$

$$+ \frac{1}{2}(A - \mathcal{H}(n)A^*\mathcal{H}(n)^{-1})Q'(x, n), \qquad n \geq 1,$$

and for $n = 0$

$$Q(x, 0)J = \left(J - \frac{1}{2}xA - \frac{1}{2}\mathcal{H}(0)A^*\mathcal{H}(0)^{-1}(x - A) \right) Q(x, 0)$$

$$+ \frac{1}{2}(A - \mathcal{H}(0)A^*\mathcal{H}(0)^{-1})Q'(x, 0).$$

Proof. We start off with the equation that we got from the Casimir and eliminate any $Q(x, m)$ with $m \neq n$ by using the three term recurrence in the first step and the lowering relation $M \cdot Q = Q \cdot D_Q$ in the second.

$$(Q \cdot (C_Q))(x, n) = -AQ(x, n+1) + \left(nI + J - 2C(n) - AB(n) + \frac{1}{2}A^2 \right) Q(x, n)$$

$$+ \left(C(n)A - 2C(n)B(n-1) \right) Q(x, n-1)$$

$$= \left(nI + J - xA + \frac{1}{2}A^2 - 2C(n) \right) Q(x, n)$$

$$+ \left(C(n)A + AC(n) - 2C(n)B(n-1) \right) Q(x, n-1)$$

$$= \left(nI + J - xA + \frac{1}{2}A^2 - 2C(n) \right) Q(x, n)$$

$$+ \frac{1}{2} \left(C(n)A + AC(n) - 2C(n)B(n-1) \right) C(n)^{-1}(x - A)Q(x, n)$$

$$+ \frac{1}{2} \left(C(n)A + AC(n) - 2C(n)B(n-1) \right) C(n)^{-1}Q'(x, n)$$

$$= \left(nI + J - B(n)x - 2C(n) + \frac{1}{2}\mathcal{H}(n)A^*\mathcal{H}(n)^{-1}A \right) Q(x, n)$$

$$+ (A - B(n))Q'(x, n)$$

In the final steps we clean up the expression using the expressions for $B(n)$ and $C(n)$ in terms of the square norms. If we leave everything in terms of the square norms we get the desired expression. ∎

Theorem 4. *We have the following (scalar) recursion for $2 \le j \le N - 1$ and $n \ge 1$*

$$\xi(n, j - 1, k)$$

$$= \left(\frac{\alpha_{j-1}}{\alpha_j}(n + j - k) + 2\frac{\alpha_{j-1}\alpha_{j+1}^2}{\alpha_j^3}\frac{\mathcal{H}(n)_{jj}}{\mathcal{H}(n)_{j+1,j+1}} - 2\frac{\alpha_{j-1}}{\alpha_j}\frac{\mathcal{H}(n)_{jj}}{\mathcal{H}(n-1)_{jj}} \right)\xi(n, j, k)$$

$$- 2(n + j - k + 1)\frac{\alpha_{j-1}\alpha_{j+1}}{\alpha_j^2}\frac{\mathcal{H}(n)_{jj}}{\mathcal{H}(n)_{j+1,j+1}}\xi(n, j + 1, k),$$

and for $j = N$ we get

$$\xi(n, N - 1, k) = \left(\frac{\alpha_{N-1}}{\alpha_N}(n + N - k) - 2\frac{\alpha_{N-1}}{\alpha_N}\frac{\mathcal{H}(n)_{NN}}{\mathcal{H}(n-1)_{NN}} \right)\xi(n, N, k).$$

Proof. The proof follows from looking at the entries of result of Proposition 7 when we will in $x = 0$. ∎

In conclusion, to compute $P(x, n)$ for a given n one must

1. Compute $\mathcal{H}(m)$ for $1 \le m \le n$ using (34) and the explicit form of $\mathcal{H}(0)$ from the proof of Lemma 3.
2. Compute the $\xi(n, j, k)$ using the recursion of Theorem 4 with the boundary condition from (48).
3. Then we have $Q(x, n)$ and so $P(x, n) = Q(x, n)L(x)^{-1}e^{x^2/2}$, where we take an explicit expression for $L(x)^{-1}$ from [Ref. 7, Proposition 3.1].

6 | MATRIX VALUED PEARSON EQUATIONS AND LADDER RELATIONS

In this section we specialize the matrix A in (38) in such a way that it satisfies a Pearson equation. This gives rise to new ladder relations and allows to obtain more explicit results for the case of Hermite and Freud-type weights.

Remark 11. A word should be said about the name Pearson equation in the context of *matrix* weights. In papers such as Refs. 7, 19, 20, 35, the name Pearson equation is used for an equation for the matrix weight of the form

$$(W\Phi)'(x) = W(x)\Psi(x), \tag{50}$$

with Φ and Ψ polynomials of degree two and one, respectively. In other sources, such as Ref. 25, 26, the same name is used for a more general equation of the form

$$W'(x) = A(x)W(x) + W(x)B(x),$$

with matrix polynomials A and B.

In this section we assume that we have a weight matrix W, which satisfies a Pearson equation of the form

$$W'(x) = -W(x)V(x), \tag{51}$$

where $V(x)$ is a matrix valued polynomial of degree k. The case where V is a polynomial of degree one is studied in detail in Ref. 35. Using integration by parts, we prove that there exist matrix valued sequences $M_{-2}(n), \ldots, M_{-k}(n)$ such that the monic orthogonal polynomials $P(x, n)$ with respect to W satisfy:

$$(P \cdot \partial_x)(x, n) = P'(x, n) = nP(x, n - 1) + M_{-2}(n)P(x, n - 2) + \cdots + M_{-k}(n)P(x, n - k). \tag{52}$$

Therefore, $\partial_x \in \mathcal{F}_R(P)$ and

$$\varphi^{-1}(\partial_x) = n\delta^{-1} + M_{-2}(n)\delta^{-2} + \cdots + M_{-k}(n)\delta^{-k}.$$

The operator ∂_x has an adjoint $\partial_x^\dagger$ given by

$$\partial_x^\dagger = -\partial_x + V(x)^*. \tag{53}$$

Moreover, $[D, \partial_x] = 0$ implies that $[\varphi^{-1}(D), \varphi^{-1}(\partial_x)] = 0$.

6.1 | Hermite-type example

We consider the weight of the previous section (38)

$$W(x) = e^{-x^2} L(x)L(x)^*, \tag{54}$$

but with a particular choice for the parameters

$$\frac{2\alpha_j}{\alpha_{j-1}} = \sqrt{(j-1)(N-j+1)}.$$

In contrast to the previous section, this *is* now in fact a particular case of the family of weights considered in Ref. 7. We do not seek the entries of the MVOPs as this has been done in Ref. 7, but instead we find a simple lowering relation, which is not obvious from the explicit MVOPs.

Proposition 8. *The weight W satisfies the Pearson equation (51) where V(x) is the following polynomial of degree two:*

$$-V(x) = (L(0)^*)^{-1}AL(0)^* + A^* + 2x\left(J + \frac{1}{2}(A^*)^2 - \frac{N+3}{2}\right) - x^2A^*,$$

where J is the diagonal matrix introduced in Subsection 5.2.

Proof. Using the structure of the matrix weight (54) we obtain:

$$W(x)^{-1}W'(x) = -2x + (L(x)^*)^{-1}AL(x)^* + A^*.$$

Now observe that

$$\frac{d}{dx}\left((L(x)^*)^{-1}AL(x)^*\right) = (L(x)^*)^{-1}[A, A^*]L(x)^*.$$

On the other hand, because $\frac{1}{4}[A, A^*]_{j,k} = (\frac{\alpha_j^2}{\alpha_{j-1}^2} - \frac{\alpha_{j+1}^2}{\alpha_j^2})\delta_{j,k}$, it follows that

$$4\left(\frac{\alpha_j^2}{\alpha_{j-1}^2} - \frac{\alpha_{j+1}^2}{\alpha_j^2}\right) = (2j - (N+1)) \implies [A, A^*] = 2J - (N+1),$$

and using the relation $(L(x)^*)^{-1}JL(x)^* = -xA^* + J + \frac{1}{2}(A^*)^2$ we get

$$\frac{d}{dx}\left((L(x)^*)^{-1}AL(x)^*\right) = 2(-xA^* + J + \frac{1}{2}(A^*)^2) - (N+1).$$

Integrating with respect to x we complete the proof of the proposition. ∎

It follows from Example 1 and (52) that

$$\varphi^{-1}(D) = A + 2C(n)\delta^{-1}, \qquad \varphi^{-1}(\partial_x) = n\delta^{-1} + M_{-2}(n)\delta^{-2}, \qquad (55)$$

for a certain sequence $M_{-2}(n)$.

Proposition 9. *The sequence in (55) is given by*

$$M_{-2}(n) = \mathcal{H}(n)A^*\mathcal{H}(n-2)^{-1}, \qquad n \geq 2.$$

Proof. We start by collecting the adjoints of the two lowering differential operators from (53) and (14)

$$\partial_x^\dagger = -\partial_x + V(x)^*, \qquad \mathcal{D}^\dagger = -\partial_x - A + 2x.$$

Because $V(x)^*$ is of degree 2 with A as a leading coefficient and $P(x, n)$ is monic, we have

$$(P \cdot (\mathcal{D}^\dagger - \partial_x^\dagger))(x, n) = -AP(x, n+2) + (\text{lower order terms})$$

On the other hand, we can write its corresponding difference operator using Corollary 1

$$\varphi^{-1}(\mathcal{D}^\dagger - \partial_x^\dagger) = -\mathcal{H}(n)M_{-2}(n+2)^*\mathcal{H}(n+2)^{-1}\delta^2 + (\text{lower order terms}).$$

So we must have

$$\mathcal{H}(n)M_{-2}(n+2)^*\mathcal{H}(n+2)^{-1} = A,$$

which leads to the desired result. ∎

Corollary 3. *The squared norms satisfy another second order recursion in n*

$$2(n+1)\mathcal{H}(n+1)^{-1} - 2(n+2)\mathcal{H}(n+2)^{-1}\mathcal{H}(n+1)\mathcal{H}(n)^{-1}$$
$$+ \mathcal{H}(n+2)^{-1}A\mathcal{H}(n+2)A^*\mathcal{H}(n)^{-1} - A^*\mathcal{H}(n)^{-1}A = 0.$$

Proof. Using $0 = [\varphi^{-1}(\mathcal{D}), \varphi^{-1}(\partial_x)]$, a direct computation gives

$$0 = (2(n-1)C(n) - 2nC(n-1) - M_{-2}(n)A + AM_{-2}(n))\delta^{-2}$$
$$+ 2(C(n)M_{-2}(n-1) - M_{-2}(n)C(n-2))\delta^{-3}.$$

From the δ^{-2} term we obtain

$$[M_{-2}(n), A] = 2((n-1)C(n) - nC(n-1)),$$

which gives the desired result when written in terms of the squared norms using Proposition 9. ∎

Remark 12. We note that we could reduce the order of the recursion by combining the result in Corollary 3 with (34). However, from the point of view of the scalar (diagonal) entries $\mathcal{H}(n)_{jj}$, this severely raises the order of the recursion in j.

6.2 | Freud-type example

We consider the weight matrix of Example 2 again:

$$W(x) = e^{-x^4 + tx^2} e^{xA} e^{xA^*}, \tag{56}$$

where A is the lower triangular nilpotent matrix defined by

$$A_{i,j} = \sqrt{\mu_i} \delta_{i-1,j}, \qquad 6\mu_i = (i-1)(N-i+1)(2N\alpha + 2\alpha i + 3\beta + \alpha),$$

and α, β are real numbers. As in Proposition 9, we prove that the weight W satisfies a Pearson equation of the form

$$W'(x) = -W(x)V(x),$$

where $V(x)$ is a polynomial of degree three. Therefore, there exist sequences $M_{-2}(n)$ and $M_{-3}(n)$ such that

$$P_n \cdot \partial_x = nP_{n-1} + M_{-2}(n)P(x, n-2) + M_{-3}(n)P(x, n-3).$$

7 | PARAMETER DEFORMATION OF THE WEIGHT AND MULTITIME TODA LATTICE

In this section, we consider an arbitrary matrix weight of the form

$$W(x, t) = e^{-\upsilon(x,t)} \widetilde{W}(x), \tag{57}$$

where $\upsilon(x; t)$ is a polynomial of even degree with positive leading coefficient depending smoothly on a parameter $t \geq 0$. In the following theorem, we study the effect of differentiating the recurrence coefficients with respect to t, an idea that is natural when one considers orthogonal polynomials in the context of integrable systems.

Theorem 5. *If we denote by ˙, the derivative with respect to t, then the recurrence coefficients in (3) satisfy the following deformation equations:*

$$\dot{B}(n) = (\dot{\upsilon}(L))_{-1}(n) - (\dot{\upsilon}(L))_{-1}(n+1)$$

$$\dot{C}(n) = (\dot{\upsilon}(L))_{-2}(n) - (\dot{\upsilon}(L))_{-2}(n+1) \tag{58}$$

$$+ (\dot{\upsilon}(L))_{-1}(n)B(n-1) - B(n)(\dot{\upsilon}(L))_{-1}(n),$$

where we use the same notation as in Remark 13.

Proof. Let $P(x, n)$ be the monic orthogonal polynomials with $W(x)$. Taking into account that $\langle P(x, n), P(x, m) \rangle = 0$ for $n > m$, we have

$$0 = \frac{\partial}{\partial t} \langle P(x, n), P(x, m) \rangle = \langle \dot{P}(x, n), P(x, m) \rangle - \langle P(x, n) \cdot \dot{v}(x), P(x, m) \rangle,$$

and then we can expand

$$\dot{P}(x, n) = \sum_{m=0}^{n-1} \langle P(x, n) \cdot \dot{v}(x), P(x, m) \rangle \mathcal{H}(m)^{-1} P(x, m)$$

$$= \sum_{m=0}^{n-1} (\dot{v}(L))_{m-n}(n) P(x, m),$$

where we use the notation in Remark 13 for $(\dot{v}(L))_k(n)$. On the other hand, if we differentiate the three-term recurrence relation (3) with respect to t, we obtain

$$x\dot{P}(x, n) = \dot{P}(x, n + 1) + B(n)\dot{P}(x, n) + C(n)\dot{P}(x, n - 1)$$

$$+ \dot{B}(n)P(x, n) + \dot{C}(n)P(x, n - 1)$$

$$= \sum_{m=0}^{n} (\dot{v}(L))_{m-n-1}(n + 1)P(x, m) + B(n) \sum_{m=0}^{n-1} (\dot{v}(L))_{m-n}(n)P(x, m) \tag{59}$$

$$+ C(n) \sum_{m=0}^{n-2} (\dot{v}(L))_{m-n+1}(n - 1)P(x, m)$$

$$+ \dot{B}(n)P(x, n) + \dot{C}(n)P(x, n - 1),$$

while on the left-hand side we get

$$x\dot{P}(x, n) = \sum_{m=0}^{n-1} (\dot{v}(L))_{m-n}(n)P(x, m + 1)$$

$$+ \sum_{m=0}^{n-1} (\dot{v}(L))_{m-n}(n)(B(m)P(x, m) + C(m)P(x, m - 1)). \tag{60}$$

Combining (59) and (60), isolating the derivatives of the recurrence coefficients, we get

$$\dot{B}(n)P(x,n) + \dot{C}(n)P(x,n-1)$$

$$= \sum_{m=0}^{n-1} (\dot{v}(L))_{m-n}(n)P(x,m+1) + \sum_{m=0}^{n-1} (\dot{v}(L))_{m-n}(n)B(m)P(x,m)$$

$$+ \sum_{m=1}^{n-1} (\dot{v}(L))_{m-n}(n)C(m)P(x,m-1) - \sum_{m=0}^{n} (\dot{v}(L))_{m-n-1}(n+1)P(x,m)$$

$$- B(n)\sum_{m=0}^{n-1} (\dot{v}(L))_{m-n}(n)P(x,m) - C(n)\sum_{m=0}^{n-2} (\dot{v}(L))_{m-n+1}(n-1)P(x,m).$$

Comparing coefficients of $P(x,n)$ and $P(x,n-1)$ we obtain (58) for $\dot{B}(n)$ and $\dot{C}(n)$. ∎

Example 3. If $\dot{v}(x) = x$, we obtain the non–Abelian Toda lattice

$$\dot{B}(n) = C(n) - C(n+1), \qquad \dot{C}(n) = C(n)B(n-1) - B(n)C(n).$$

Note that for $v(x) = x^2 + xt$ the relations (26) give

$$2\dot{B}(n) = [B(n), A] - I, \qquad 2\dot{C}(n) = [C(n), A].$$

Example 4. If $\dot{v}(x) = x^2$, we obtain the non–Abelian Langmuir lattice

$$\dot{B}(n) = B(n)C(n) - B(n+1)C(n+1) + C(n)B(n-1) - C(n+1)B(n),$$

$$\dot{C}(n) = C(n)C(n-1) - C(n+1)C(n) + C(n)B(n-1)^2 - B(n)^2C(n).$$

Next, we consider a weight $W(x,\vec{t})$ as in (57) with a multitime Toda deformation, namely, with a polynomial v of the form

$$v(x,\vec{t}) = v(x,t_1,\ldots,t_k) = \sum_{j=1}^{k} t_j x^j.$$

If we denote by $\dot{}$, the derivative with respect to t_j, then we have $\dot{v}(L) = L^j$. Theorem 57 gives the expressions for the derivatives of the recurrence coefficients, but if j is large, then the coefficients $(\dot{v}(L))_{-1,-2}$ can be difficult to compute, and a much more convenient formulation is given as a Lax pair.

In the spirit of [Ref. 41, §2.8], we identify the operator L with the block tridiagonal matrix with block entries (L_{nm}), $L_{n,n+1} = I$, $L_{n,n} = B(n)$, $L_{n,n-1} = C(n)$, and $L_{n,m} = 0I$ if $|n-m| \geq 2$. For a $N \times N$-block semi-infinite matrix $S = (S_{nm})$, we define S_+ as the matrix obtained by replacing all the $N \times N$ blocks of S below the main diagonal by zero. Analogously, we let S_- to be the matrix obtained by replacing all the $N \times N$ blocks above the subdiagonal by zero.

Then, we have the following result:

Theorem 6. *For $j = 1, \ldots, k - 1$, we have*

$$\dot{L} = \left[L, (L^j)_+\right] = -\left[L, (L^j)_-\right].$$

Proof. We first observe that

$$\left[L, (L^j)_+\right] + \left[L, (L^j)_-\right] = \left[L, (L^j)_+ + (L^j)_-\right] = \left[L, L^j\right] = 0,$$

which proves the second equality. Using that $\dot{B}(n) = (\dot{L})_{n,n}$ and $\dot{C}(n) = (\dot{L})_{n,n-1}$, we will complete the proof by showing that $[L, (L^j)_+]_{n,n}$ equals the right-hand side of the first equation in (58), that $[L, (L^j)_+]_{n,n-1}$ equals the right-hand side of the second equation of (58) and that $[L, (L^j)_+]_{n,m} = 0$ otherwise.

Note that $(\dot{v}(L))_m(n) = (L^j)_{n,n+m}$ for any indices m, n so the first equation in (58) reads

$$\dot{B}(n) = (L^j)_{n,n-1} - (L^j)_{n+1,n}, \qquad n \geq 1.$$

On the other hand, bearing in mind that L is block tridiagonal and L^j_- is lower triangular with zeros on the diagonal, we have

$$\left[L, (L^j)_-\right]_{n,n} = L_{n,n+1}(L^j)_{n+1,n} - (L^j)_{n,n-1}L_{n-1,n} = (L^j)_{n+1,n} - (L^j)_{n,n-1},$$

because $L_{n,n+1} = I$ for any $n \geq 0$, which proves the result for the main diagonal. The second equation in (58) is

$$\dot{C}(n) = (L^j)_{n,n-2} - (L^j)_{n+1,n-1} + (L^j)_{n,n-1}B(n-1) - B(n)(L^j)_{n,n-1}, \qquad n \geq 2.$$

We also have

$$\begin{aligned}
\left[L, (L^j)_-\right]_{n,n-1} &= L_{n,n}(L^j)_{n,n-1} + L_{n,n+1}(L^j)_{n+1,n-1} \\
&\quad - (L^j)_{n,n-1}L_{n-1,n-1} - (L^j)_{n,n-2}L_{n-2,n-1} \\
&= B(n)(L^j)_{n,n-1} + (L^j)_{n+1,n-1} \\
&\quad - (L^j)_{n,n-1}B(n-1) - (L^j)_{n,n-2},
\end{aligned}$$

which proves the result for the first subdiagonal.

Finally, if $k \geq n + 1$ we repeat the calculation using $[L, (L^j)_-]_{n,k}$, which gives 0, consistently with $(\dot{L})_{n,k}$, and if $k \leq n - 2$, we compute $[L, (L^j)_+]_{n,k}$, which gives 0 on both sides again. ∎

We remark that the multitime Toda lattice (6) coincides with the one given in [Ref. 42, Proposition 4.4].

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APPENDIX A: COMPARISON OF LADDER OPERATORS

In this article, we have taken a different approach to ladder operators for MVOPs than, for example, the one in Ref. 23. Their approach is inspired by Ref. 40 and Ref. 52 for the scalar orthogonal polynomials. This Appendix is meant to compare our approach with theirs for our class of weight functions, i.e., the exponential weights in (27).

The ladder relations for exponential weights are stated in Section 4 and can be formulated as

$$P'(x,n) = \sum_{j=1}^{k-1} A_{-j}(n)P(x,n-j) + [A,P(x,n)],$$

for monic polynomials P. For exponential weights on the real line, we have the following identity:

$$W'(x) = -W(x)V(x), \qquad V(x) = v'(x)I - A^* - \rho(x), \tag{A1}$$

where $\rho(x) = W^{-1}(x)AW(x)$. The ladder relation given in Ref. 23 for exponential weights reads

$$P'(x,n) = F(x,n)P(x,n) - E(x,n)P(x,n-1),$$

where the coefficients are

$$E(x,n)\mathcal{H}(n-1) = -\int_{\mathbb{R}} P(y,n)W(y)\frac{V(x)-V(y)}{x-y}P(y,n)^* \, dy, \tag{A2}$$

$$F(x,n)\mathcal{H}(n-1) = -\int_{\mathbb{R}} P(y,n)W(y)\frac{V(x)-V(y)}{x-y}P(y,n-1)^* \, dy.$$

These identities are obtained in the following way: we expand the derivative of $P(x,n)$ in the basis of MVOPs, with coefficients multiplying on the left:

$$P'(x,n) = \sum_{k=0}^{n-1} \langle P'(x,n), P(x,k)\rangle \mathcal{H}(k)^{-1}P(x,k)$$

$$= \sum_{k=0}^{n-1} \left(\int_{\mathbb{R}} P'(y,n)W(y)P(y,k)^* \, dy \right) \mathcal{H}(k)^{-1}P(x,k)$$

$$= \int_{\mathbb{R}} P'(n,y)W(y) \left(\sum_{k=0}^{n-1} P(y,k)^* \mathcal{H}(k)^{-1}P(x,k) \right) dy.$$

We integrate by parts, and the boundary terms vanish because of the decay of $W(x)$ at $\pm\infty$. This, together with (A1), gives

$$P'_n(x) = - \int_{\mathbb{R}} P(n,y)W(y)(-V(y)) \sum_{k=0}^{n-1} P(y,k)^* \mathcal{H}(k)^{-1}P(x,k)dy$$

$$= - \int_{\mathbb{R}} P(n,y)W(y)(V(x) - V(y)) \sum_{k=0}^{n-1} P(y,k)^* \mathcal{H}(k)^{-1}P(x,k)dy,$$

where we have used the fact that the integral with $-V(x)$ vanishes by orthogonality. If we now apply the Christoffel–Darboux formula

$$(x-y) \sum_{k=0}^{n-1} P(y,k)^* \mathcal{H}(k)^{-1}P(x,k) = P(y,n-1)^* \mathcal{H}(n-1)^{-1}P(x,n)$$

$$\qquad\qquad\qquad\qquad\qquad - P(y,n)^* \mathcal{H}(n-1)^{-1}P(x,n-1),$$

(A3)

we obtain the formulas (A2) for the coefficients $E(x,n)$ and $F(x,n)$. Furthermore, using the formula for $V(x)$ in (A1), we can write

$$F(x,n)\mathcal{H}(n-1) = - \int_{\mathbb{R}} P(y,n)W(y)S(x,y)P(y,n-1)^* \, dy, \qquad (A4)$$

$$E(x,n)\mathcal{H}(n-1) = - \int_{\mathbb{R}} P(y,n)W(y)S(x,y)P(y,n)^* \, dy,$$

where

$$S(x,y) = \frac{v'(x) - v'(y)}{x-y} - \frac{\rho(x) - \rho(y)}{x-y}$$

On the other hand, by direct computation using the fact that $P(x,n)$ is monic and (A1), we have

$$P(x,n)A - AP(x,n)$$

$$= \sum_{k=0}^{n-1} \langle P_n A, P_k \rangle \mathcal{H}(k)^{-1}P_k(x)$$

$$= \sum_{k=0}^{n-1} \left(\int_{\mathbb{R}} P(y,n)W(y)(\rho(y) - \rho(x))P(y,k)^* dy \right) \mathcal{H}(k)^{-1}P(x,k).$$

Therefore, applying (A3) again, we obtain

$$P(x, n)A - AP(x, n)$$

$$= \left(\int_{\mathbb{R}} P(y, n)W(y)\frac{\rho(y) - \rho(x)}{x - y}P(y, n - 1)^* dy \right) \mathcal{H}(n - 1)^{-1}P(x, n)$$

$$+ \left(\int_{\mathbb{R}} P(y, n)W(y)\frac{\rho(y) - \rho(x)}{x - y}P(y, n)^* dy \right) \mathcal{H}(n - 1)^{-1}P(x, n - 1).$$

Comparing this last equation with (A4), we find a relation between the two ladder operators, because

$$(F(x, n) + E(x, n))\mathcal{H}(n - 1) = P(x, n)A - AP(x, n)$$

$$- \int_{\mathbb{R}} P(y, n)W(y)\frac{v'(x) - v'(y)}{x - y}P(y, n - 1)^* dy$$

$$- \int_{\mathbb{R}} P(y, n)W(y)\frac{v'(x) - v'(y)}{x - y}P(y, n)^* dy.$$

Lastly, we note that $E(x, n)$ and $F(x, n)$ are explicit for the case treated in Section 6.1. In principle one could derive the expressions using Proposition 8, but it is simpler to deduce them from Proposition 9:

$$F(x, n) = -\mathcal{H}(n)A^*\mathcal{H}(n - 1)^{-1},$$

$$E(x, n) = -x\mathcal{H}(n)A^*\mathcal{H}(n - 1)^{-1} - nI$$

$$+ \frac{1}{2}\mathcal{H}(n)A^*\mathcal{H}(n - 1)^{-1}A + \frac{1}{2}\mathcal{H}(n)(A^*)^2\mathcal{H}(n - 1)^{-1}.$$