NILPOTENCY DEGREE OF THE NILRADICAL OF SOLVABLE LIE ALGEBRAS ON TWO GENERATORS

LEANDRO CAGLIERO, FERNANDO LEVSTEIN, AND FERNANDO SZECHTMAN

Abstract. Given a field F of characteristic 0, we consider solvable Lie algebras $\mathfrak{g}$ of block upper triangular matrices on two generators. Imposing mild conditions on these generators, we prove that the nilpotency degree of the nilradicals $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ is as large as possible, namely the number of diagonal blocks minus one.

As an application when F is algebraically closed, let $\mathcal{N}_\ell(V)$ denote the free ℓ-step nilpotent Lie algebra associated to a given F-vector space V. As a consequence of the above degree, we obtain a complete classification of all uniserial representations of the solvable Lie algebra $\mathfrak{g} = \langle x \rangle \ltimes \mathcal{N}_\ell(V)$, where x acts on V via an arbitrary invertible Jordan block.

1. Introduction

We fix throughout a field F of characteristic 0. All Lie algebras and representations considered in this paper are assumed to be finite dimensional over F, unless explicitly stated otherwise.

Given a 5-tuple $(\ell, d, \alpha, \lambda, X)$, where ℓ is a positive integer, $d = (d_1, \dots, d_{\ell+1})$ is a sequence of $\ell + 1$ positive integers, $\alpha, \lambda \in F$, and $X = (X(1), \dots, X(\ell))$ is a sequence of ℓ matrices $X(i) \in M_{d_i \times d_{i+1}}$ such that $X(i)_{d_i,1} \neq 0$ for all i, consider the matrices $D, E \in \mathfrak{gl}(d)$, $d = d_1 + \cdots + d_{\ell+1}$, given in block form by

$$D = J^{d_1}(\alpha) \oplus J^{d_2}(\alpha - \lambda) \oplus \cdots \oplus J^{d_{\ell+1}}(\alpha - \ell\lambda),$$

where $J^p(\beta)$ denotes the upper triangular Jordan block of size p and eigenvalue β,

$$E = \begin{pmatrix} 0 & X(1) & 0 & \dots & 0 \\ 0 & 0 & X(2) & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \ddots & X(\ell) \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

The Lie subalgebra of $\mathfrak{gl}(d)$ generated by D and E is easily seen to be equal to $\langle D \rangle \ltimes \mathfrak{n}$, where $\mathfrak{n}$ is nilpotent. Theorem **??** proves that, except for a few extraordinary cases, the nilpotency degree of $\mathfrak{n}$ is exactly ℓ.

Suppose F is algebraically closed. Theorem **??** uses the above bound to give a complete classification of all uniserial representations of the solvable Lie algebra $\mathfrak{g} = \mathfrak{g}_{*****,\ell,n} = \langle x \rangle \ltimes \mathcal{N}_\ell(V)$, where V is a vector space of dimension $n \geq 1$, $\mathcal{N}_\ell(V)$

2010 *Mathematics Subject Classification.* 17B10, 17B30.

Key words and phrases. uniserial representation; free ℓ-step nilpotent Lie algebra.

This research was partially supported by grants from CONICET, FONCYT and SeCyT-UNCórdoba).

This research was partially supported by an NSERC grant.

is the free ℓ-step nilpotent Lie algebra associated to V, and x acts on V via a single Jordan block $J_n(\lambda)$, $\lambda \neq 0$.

A representation $R : \mathfrak{g} \to \mathfrak{gl}(U)$ is *relatively faithful* if $\ker(R) \cap V = 0$ and $\ker(R) \cap \mathfrak{n}^{\ell-1}$ is properly contained in $\mathfrak{n}^{\ell-1}$. It suffices to classify all uniserial representations of $\mathfrak{g}$ that are relatively faithful. Indeed, let $R : \mathfrak{g} \to \mathfrak{gl}(U)$ be a uniserial representation. If $V \subseteq \ker(R)$ then R is determined by a uniserial representation $\langle x \rangle \to \mathfrak{gl}(U)$. The Jordan normal form suffices to classify such representations. We may thus assume without loss of generality that V is not contained in $\ker(V)$. If $(0) \neq \ker(R) \cap V \neq V$, then R is determined by a uniserial representation $\overline{R} : \mathfrak{g}_{****,\ell,m} \to \mathfrak{gl}(U)$, where $\mathfrak{g}_{****,\ell,m} = \langle x \rangle \ltimes \mathcal{N}_\ell(\overline{V})$, $\overline{V}$ is a factor of V by an x-invariant subspace, x acts on $\overline{V}$ via an invertible Jordan block $J_m(\lambda)$, $1 \leq m < n$, and $\ker(\overline{R}) \cap \overline{V} = 0$. Hence, we may assume without loss of generality that $\ker(R) \cap V = 0$. Let $1 < s \leq \ell$ be the smallest positive integer such that $\mathfrak{n}^s$ is contained in $\ker(R)$. Then R is determined by a uniserial representation $\overline{R} : \mathfrak{g}_{****,s,n} :\to \mathfrak{gl}(U)$, where $\mathfrak{g}_{****,s,n} = \langle x \rangle \ltimes \overline{\mathfrak{n}}$, $\overline{\mathfrak{n}} = \mathfrak{n}/\mathfrak{n}^s$, and $\overline{\mathfrak{n}}^{s-1}$ is not contained in the kernel of $\overline{R}$. Therefore, we may assume without loss of generality that $\ker(R) \cap V = 0$ and that $\mathfrak{n}^{\ell-1} \not\subset \ker(R)$, that is, that R is relatively faithful.

The degenerate case $n = 1$ appears as a special case in [**?**]. The cases $\ell = 1$ and $\ell = 2$ have recently been solved in [**?**] and [**?**], respectively. Without resorting to any of these cases, we will obtain the following classification, valid for all ℓ and n.

Let $v_0, \dots, v_{n-1}$ be a basis of V such that

$$[x, v_0] = \lambda v_0 + v_1, [x, v_1] = \lambda v_1 + v_2, \dots, [x, v_{n-1}] = \lambda v_{n-1}.$$

Given a sequence $\vec{d} = (d_1, \dots, d_{\ell+1})$ of $\ell + 1$ positive integers satisfying

$$\max_{1 \leq i \leq l} \{d_i + d_{i+1}\} = n + 1,$$

and a scalar $\alpha \in F$, we define a representation $R = R_{\vec{d},X,\alpha} : \mathfrak{g} \to \mathfrak{gl}(d)$, where $d = d_1 + \cdots + d_{\ell+1}$, in block form, in the following manner:

$$R(x) = A = J^{d_1}(\alpha) \oplus J^{d_2}(\alpha - \lambda) \oplus \cdots \oplus J^{d_{\ell+1}}(\alpha - \ell\lambda),$$

$$R(v_j) = (\mathrm{ad}_{\mathfrak{gl}(d)} A - \lambda 1_{\mathfrak{gl}(d)})^j \begin{pmatrix} 0 & X(1) & 0 & \dots & 0 \\ 0 & 0 & X(2) & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \ddots & X(\ell) \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}, \quad 0 \leq j \leq n-1.$$

This extends uniquely to a representation $\mathfrak{g} \to \mathfrak{gl}(d)$ by the universal property that defines of $\mathcal{N}_\ell(V)$.

Conjugating all $R(y)$, $y \in \mathfrak{g}$, by a suitable block diagonal matrix commuting with A, we may normalize R, in the sense that the last row of every $X(i)$ is the first canonical vector of $F^{d_{i+1}}$ and the first column of $X(1)$ is the last canonical vector of F^{d_1}. The representation R is always uniserial. It is also relatively faithful, except for a few extraordinary cases that occur when $n > 1$. Theorem **??** proves that, when $n > 1$, every relatively faithful uniserial representation of $\mathfrak{g}$ is isomorphic to one and only one normalized representation $R_{\vec{d},X,\alpha}$ of non-extraordinary type (the degenerate case can be found in Theorem **??**).

2. Preliminaries and notation

2.1. The Lie algebras $\mathfrak{g}_{n,\lambda}$ and $\mathfrak{g}_{n,\lambda,\ell}$. If $\mathfrak{g}$ is a Lie algebra, let $\{\mathfrak{g}^i : i \geq 0\}$ be the lower central series, that is $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$.

Let V be a vector space of dimension $n \geq 2$ and let $\mathcal{L}(V)$ be the free Lie algebra associated to V (or the free Lie algebra on n generators). For $\ell \geq 1$, let

$$\mathcal{N}_\ell(V) = \mathcal{L}(V)/\mathcal{L}(V)^\ell$$

be the free ℓ-step nilpotent Lie algebra associated to V.

Given an integer $p \geq 1$ and $\alpha \in F$, we write $J_p(\alpha)$ (resp. $J^p(\alpha)$) for the lower (resp. upper) triangular Jordan block of size p and eigenvalue α. Let $x \in \mathrm{End}(V)$ the linear map acting on V via a single Jordan block $J_n(\lambda)$. In particular V has a basis $\{v_0, \ldots, v_{n-1}\}$ such that

$$(\mathrm{ad}\, x - \lambda)^k v_0 = \begin{cases} v_k, & \text{if } 0 \leq k < n; \\ 0, & \text{if } k = n. \end{cases} \tag{2.1}$$

We extend the action of x on V to $\mathcal{L}(V)$ so that x becomes a Lie algebra derivation. This action preserves $\mathcal{L}(V)^\ell$ and thus x also acts by derivations on $\mathcal{N}_\ell(V)$. Let

$$\mathfrak{g}_{n,\lambda} = \langle x \rangle \ltimes \mathcal{L}(V) \quad \text{and} \quad \mathfrak{g}_{n,\lambda,\ell} = \langle x \rangle \ltimes \mathcal{N}_\ell(V)$$

be the corresponding semidirect products.

2.2. Gradings in $\mathfrak{gl}(d)$ and the outer automorphism. If $\vec{d} = (d_1, \ldots, d_{\ell+1})$ is a sequence of $\ell + 1$ positive integers, we define $|\vec{d}| = |\vec{d}|_1 = d_1 + \cdots + d_{\ell+1}$. A sequence $\vec{d}$ provides $\mathfrak{gl}(d)$, $d = |\vec{d}|$, with a block structure and we define

$$p_{i,j} : \mathfrak{gl}(d) \to M_{d_i,d_j}$$

the projection onto the (i, j)-block.

We consider, in $\mathfrak{gl}(d)$ two 'diagonal' gradings: one associated to the actual diagonals of $\mathfrak{gl}(d)$, that is

$$\mathcal{D}_t = \{A \in \mathfrak{gl}(d) : A_{ij} = 0 \text{ if } j - i \neq t\}; \tag{2.2}$$

and the other one associated to the block-diagonals of $\mathfrak{gl}(d)$, that is

$$\bar{\mathcal{D}}_t = \{A \in \mathfrak{gl}(d) : p_{ij}(A) = 0 \text{ if } j - i \neq t\}. \tag{2.3}$$

We call the degrees (2.2) and (2.3) *diagonal-degree* and *block-degree* respectively. The proof of the following proposition is straightforward. Ojo con las a,b y t sue se usan después $A^t, a_{i,j}, etc$

Proposition 2.1. *If $A \in \mathcal{D}_t$ with $\left(p_{i,j}(A)\right)_{a,b} \neq 0$, (with $1 \leq a \leq d_i$ and $1 \leq b \leq d_j$) then*

$$t = d_{j-1} + \ldots d_i + (b - a).$$

In particular, if either

$$d_{j+1} - 1 < d_i - d_j + b - a \quad \text{or} \quad d_i - d_j + b - a < 1 - d_{i+1}$$

then $p_{i+1,j+1}(A) = 0$. Similarly, if either

$$d_{i-1} - 1 < b - a \quad \text{or} \quad b - a < 1 - d_{j-1}$$

then $p_{i-1,j-1}(A) = 0$.

Recall that the map $\phi : \mathfrak{gl}(d) \to \mathfrak{gl}(d)$ given by

$$\phi(A)_{i,j} = (-1)^{i-j+1} A_{d+1-j,d+1-i}$$

gives a representative of the unique nontrivial class of outer automorphisms of $\mathfrak{sl}(d)$. In fact, ϕ is in the class of $A \mapsto -A^t$, indeed, if $K = (a_{i,j}) \in \mathfrak{gl}(d)$ is the antidiagonal matrix with $a_{i,d+1-i} = (-1)^{i+1}$ ($a_{i,j} = 0$ if $i+j \neq d+1$), then $\phi(A) = -KA^tK^{-1}$. It is clear that

$$\phi|_{\mathcal{D}_t} = (-1)^{t+1}. \tag{2.4}$$

2.3. **The Lie algebra $\mathfrak{h}(\alpha, \lambda, S)$.** Given a 5-tuple $(\ell, \vec{d}, \alpha, \lambda, S)$, where

- $\vec{d} = (d_1, \dots, d_{\ell+1})$ is a sequence of $\ell + 1$ positive integers, $\ell \geq 1$,
- $\alpha, \lambda \in F$ are scalars,
- $S = (S(1), \dots, S(\ell))$ is a sequence of ℓ matrices satisfying

$$S(i) \in M_{d_i \times d_{i+1}} \text{ and } S(i)_{d_i,1} \neq 0 \text{ for all } i; \tag{2.5}$$

we consider the matrices $D(\alpha, \lambda), E(S) \in \mathfrak{gl}(d)$, $d = |\vec{d}|$, given in block form by

$$D(\alpha, \lambda) = J^{d_1}(\alpha) \oplus J^{d_2}(\alpha - \lambda) \oplus \cdots \oplus J^{d_{\ell+1}}(\alpha - \ell\lambda),$$

and

$$E(S) = \begin{pmatrix} 0 & S(1) & 0 & \dots & 0 \\ 0 & 0 & S(2) & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \ddots & S(\ell) \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

Let $\mathfrak{h}(\alpha, \lambda, S)$ be the Lie subalgebra of $\mathfrak{gl}(d)$ generated by $D(\alpha, \lambda)$ and $E(S)$.

Definition 2.2. Given $\vec{d} = (d_1, \dots, d_{\ell+1})$, let

$$C(i) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \in M_{d_i \times d_{i+1}}.$$

and set $C = (C(1), \dots, C(\ell))$; we say that C is the *canonical* sequence. Also, given a sequence $S = (S(1), \dots, S(\ell))$ as in (2.5), we say that S is *normalized* if all the following conditions are satisfied:

(1) $S(i)_{d_i,1} = 1$ for all $1 \leq i \leq \ell$;
(2) $S(i)_{d_i,j} = S(i+1)_{d_{i+1}+1-j,1}$ for $1 \leq j \leq d_{i+1}$ and $1 \leq i \leq \ell$;
(3) $S(1)_{j,1} = 0$ for $1 \leq j < d_1$, and $S(\ell)_{d_\ell,j} = 0$ for $1 < j \leq d_{\ell+1}$.

We say that S is *weakly normalized* if conditions (1) and (2) are satisfied (this last concept will be used only in §**??**).

Example 2.3. It is clear that the canonical sequence C is normalized. Also, if $\vec{d} = (3, 5, 3, 4)$ and $S = (S(1), S(2), S(3))$ is a normalizad sequence, then $E(S)$

looks like as follows (the $*$ might be any scalar)

$$E(S)=\left(\begin{array}{ccc|ccccc|ccc|cccc}
0&0&0&0&*&*&*&*&0&0&0&0&0&0&0\\
0&0&0&0&*&*&*&*&0&0&0&0&0&0&0\\
0&0&0&1&a_2&a_3&a_4&a_5&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&a_5&*&*&0&0&0&0\\
0&0&0&0&0&0&0&0&a_4&*&*&0&0&0&0\\
0&0&0&0&0&0&0&0&a_3&*&*&0&0&0&0\\
0&0&0&0&0&0&0&0&a_2&*&*&0&0&0&0\\
0&0&0&0&0&0&0&0&1&b_2&b_3&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&b_3&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&b_2&*&*&*\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right).$$

The following proposition is not difficult to prove.

Proposition 2.4. *Let $\vec{d}=(d_1,\dots,d_{\ell+1})$ and let $G(\vec{d})$ be the subgroup of $GL(d)$, $d=|\vec{d}|$, consisting of invertible matrices $P=P_1\oplus\cdots\oplus P_{\ell+1}\in GL(d)$, with P_i a polynomial (with non-zero constant term) in $J^{d_i}(0)$. Given a sequence $S=(S(1),\dots,S(\ell))$ as in (2.5), there is an unique invertible matrix $P\in G(\vec{d})$ such that $PE(S)P^{-1}$ is equal to $E(S')$ for a normalized sequence S'.*

Let us denote

$$E^{(l)}=\operatorname{ad}(D(0,0))^l(E(S)),\qquad \text{for } l\geq 0.$$

Since $\operatorname{char}\mathbb{F}=0$, a straightforward computation (or the representation theory of $\mathfrak{sl}(2)$) shows that the set $\{E^{(l)}\}_{l=0}^{\rho}$, with $\rho=\max\{d_i+d_{i+1}-2 : i=1,\dots,\ell\}$, is linearly independent. Let $\mathfrak{n}(S)$ be the Lie algebra generated by $\{E^{(l)}\}_{l=0}^{\rho}$, that is

$$\mathfrak{n}(S)=\operatorname{span}_{\mathbb{F}}[[[E^{(l_1)},E^{(l_2)}],E^{(l_3)}],\dots,E^{(l_q)}].$$

The following proposition shows that this nilpotent Lie algebra, which is independent of α and λ, is the nilradical of $\mathfrak{h}(\alpha,\lambda,S)$.

Proposition 2.5. *The Lie algebra $\mathfrak{h}(\alpha,\lambda,S)$ is a solvable Lie subalgebra of $\mathfrak{gl}(d)$. Additionally*

1. *$\mathfrak{h}(\alpha,\lambda,S)$ is the semidirect product $\mathfrak{h}(\alpha,\lambda,S)=\mathbb{F}D(\alpha,\lambda)\ltimes\mathfrak{n}(S)$.*
2. *$\mathfrak{n}(S)$ is graded by the block-degree and filtered by the diagonal-degree.*
3. *$\mathfrak{n}(C)$ is graded by both the block-degree and the diagonal-degree. Moreover, $\mathfrak{n}(C)$ is isomorphic to the associated graded Lie algebra $gr(\mathfrak{n}(S))$ corresponding to the filtration given by the diagonal-degree.*

Proof. (1) It is not difficult to see that, for $l\geq 1$,

$$\left(\operatorname{ad}_{\mathfrak{gl}(d)}D(\alpha,\lambda)-\lambda\right)^l(E(S))=E^{(l)}$$

and thus, the Lie subalgebra of $\mathfrak{h}(\alpha,\lambda,S)$ generated by

$$\{\operatorname{ad}_{\mathfrak{gl}(d)}(D(\alpha,\lambda))^l(E(S)) : l\geq 0\},$$

which is invariant under the action of $\operatorname{ad}(D(\alpha,\lambda))$, coincides with $\mathfrak{n}(S)$. Finally, since $\mathbb{F}D(\alpha,\lambda)\oplus\mathfrak{n}(S)$ is a Lie subalgebra of $\mathfrak{h}(\alpha,\lambda,S)$ containing $D(\alpha,\lambda)$ and $E(S)$, it follows that $\mathfrak{h}(\alpha,\lambda,S)=\mathbb{F}D(\alpha,\lambda)\ltimes\mathfrak{n}(S)$.

(2) and (3) These are straightforward. □

2.4. **The uniserial representations $R_{\vec{d},\alpha,S}$.** Recall that given a vector space V of dimension n, $\mathfrak{g}_{n,\lambda} = \langle x\rangle \ltimes \mathcal{L}(V)$ and $\mathfrak{g}_{n,\lambda,\ell} = \langle x\rangle \ltimes \mathcal{N}_\ell(V)$ (see §2.1).

Given a scalar $\alpha \in F$, a sequence of positive integers $\vec{d} = (d_1, \dots, d_{\ell+1})$ satisfying

$$d_i + d_{i+1} \le n + 1 \text{ for all } i \text{ and} \tag{2.6}$$

$$d_i + d_{i+1} = n + 1 \text{ for at least one } i, \tag{2.7}$$

and a sequence $S = (S(1), \dots, S(\ell))$ as in (2.5), we use (2.1), (2.6) and the universal property of $\mathcal{L}(V)$ to define a representation

$$R_{\vec{d},\alpha,S} : \mathfrak{g}_{n,\lambda} \to \mathfrak{gl}(d), \quad d = |\vec{d}|,$$

by setting

$$\begin{aligned} R_{\vec{d},\alpha,S}(x) &= D(\alpha,\lambda), \\ R_{\vec{d},\alpha,S}(v_j) &= \left(\operatorname{ad}_{\mathfrak{gl}(d)} D(\alpha,\lambda) - \lambda\right)^j (E(S)), \quad 0 \le j \le n-1. \end{aligned}$$

It follows from (2.7) that $V \cap \ker R_{\vec{d},\alpha,S} = 0$ and we also have

$$\begin{aligned} R_{\vec{d},\alpha,S}(\mathfrak{g}_{n,\lambda}) &= \mathfrak{h}(\alpha,\lambda,S), \\ \mathcal{L}(V)^\ell &\subset \ker R_{\vec{d},\alpha,S}. \end{aligned}$$

In particular, we also obtain a representation of the truncated Lie algebra

$$\bar{R}_{\vec{d},\alpha,S} : \mathfrak{g}_{n,\lambda,\ell} \to \mathfrak{gl}(d).$$

Since, for all $i = 1, \dots, d-1$, either $R(x)_{i,i+1} \neq 0$ or $R(v_0)_{i,i+1} \neq 0$, it follows that $R_{\vec{d},\alpha,S}$ and $\bar{R}_{\vec{d},\alpha,S}$ are uniserial representations of $\mathcal{L}(V)$ and $\mathcal{N}_\ell(V)$ respectively.

Definition 2.6. If the sequence S is normalized, we say that $R_{\vec{d},\alpha,S}$ and $\bar{R}_{\vec{d},\alpha,S}$ are *normalized.*

Proposition 2.7. *Assume $\lambda \neq 0$. The normalized representations $R_{\vec{d},\alpha,S}$ (resp. $\bar{R}_{\vec{d},\alpha,S}$) of $\mathfrak{g}_{n,\lambda}$ (resp. $\mathfrak{g}_{n,\lambda,\ell}$) are non-isomorphic to each other.*

Proof. It is enough to consider the case for the representations of $\mathfrak{g}_{n,\lambda}$. Considering the eigenvalues of the image of x as well as their multiplicities, the only possible isomorphisms are easily seen to be between $R_{\vec{d},\alpha,S}$ and $R_{\vec{d},\alpha,S'}$. Assume that $R_{\vec{d},\alpha,S}$ is isomorphic to $R_{\vec{d},\alpha,S'}$. Then there is $P \in \mathrm{GL}(|\vec{d}|)$ satisfying

$$P R_{\vec{d},\alpha,S}(y) P^{-1} = R_{\vec{d},\alpha,S'}(y), \quad \text{for all } y \in \mathfrak{g}_{n,\lambda}. \tag{2.8}$$

Considering $y = x$ in (2.8) we obtain that P must commute with $D(\alpha,\lambda)$, and hence $P \in G(\vec{d})$ (see Proposition 2.4). Finally, considering $y = v_0$ in (2.8), it follows from Proposition 2.4 that $S = S'$. □

3. Classification of all uniserial $\mathfrak{g}_{n,\lambda}$-modules

In this section we classify all uniserial (finite dimensional) representations of $\mathfrak{g}_{n,\lambda} = \langle x\rangle \ltimes \mathcal{L}(V)$, where V is a vector space of dimension n over an algebraically closed filed $\mathbb{F}$ of characteristic 0 on which x acts via a single Jordan block $J_n(\lambda)$. First we prove a proposition that provides information about the structure of a uniserial representation of certain class of Lie algebras.

Proposition 3.1. *Let $\mathfrak{n}$ be a solvable Lie algebra and let x be a derivation of $\mathfrak{n}$ such that $[\mathfrak{n},\mathfrak{n}]$ has an x-invariant complement, say $\mathfrak{p}$, in $\mathfrak{n}$, and x acts on $\mathfrak{p}$ via a single Jordan block $J_n(\lambda)$, $\lambda \neq 0$. Let $v_0,\dots,v_{n-1}$ be a basis $\mathfrak{p}$ such that*

$$x(v_0) = \lambda v_0 + v_1, x(v_1) = \lambda v_1 + v_2, \dots, x(v_{n-1}) = \lambda v_{n-1}. \tag{3.1}$$

Set $\mathfrak{g} = \langle x\rangle \ltimes \mathfrak{n}$ and let $T : \mathfrak{g} \to \mathfrak{gl}(U)$ be a uniserial representation of dimension d such that

$$\ker(T) \cap \mathfrak{p} = 0.$$

Then there is a basis $\mathcal{B}$ of U, a unique scalar $\alpha \in \mathbb{F}$, a unique sequence of positive integers $\vec{d} = (d_1,\dots,d_{\ell+1})$, $\ell \geq 1$, satisfying $|\vec{d}| = d$ and

$$d_i + d_{i+1} \leq n+1 \text{ for all } i,$$
$$d_i + d_{i+1} = n+1 \text{ for at least one } i;$$

and a unique normalized sequence $S = (S(1),\dots,S(\ell))$ of matrices such that the matrix representation $R : \mathfrak{g} \to \mathfrak{gl}(d)$ associated to $\mathcal{B}$ satisfies:

$$R(x) = J^{d_1}(\alpha) \oplus J^{d_2}(\alpha-\lambda) \oplus \cdots \oplus J^{d_{\ell+1}}(\alpha - \ell\lambda), \tag{3.2}$$

$$R(v_0) = \begin{pmatrix} 0 & S(1) & 0 & \dots & 0 \\ 0 & 0 & S(2) & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \ddots & S(\ell) \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}. \tag{3.3}$$

and every $R(y)$, $y \in \mathfrak{n}$, is block strictly upper triangular relative to $\vec{d}$. Moreover, if $\mathfrak{n}^{k-1}$ is not contained in $\ker(T)$, then $\ell \geq k$.

Proof. This proof follows the lines of the proof of [**?**, Theorem 3.2]. It follows from Lie's theorem that there is a basis $\mathcal{B} = \{u_1,\dots,u_d\}$ of U such that the corresponding matrix representation $R : \mathfrak{g} \to \mathfrak{gl}(d)$ consists of upper triangular matrices.

Set

$$D = R(x) \text{ and } E_k = R(v_k),\ 0 \leq k \leq n-1.$$

Conjugating by an upper triangular matrix (see [**?**, Lemma 2.2] for the details) we may assume that D satisfies:

$$D_{i,j} = 0 \text{ whenever } D_{i,i} \neq D_{j,j}. \tag{3.4}$$

Since $\lambda \neq 0$ we have that the action of x on $\mathfrak{p}$ is invertible and hence $\mathfrak{p} \subset [\mathfrak{g},\mathfrak{g}]$. This implies that

$$\begin{gathered} E_k \text{ is strictly upper triangular for all } 0 \leq k \leq n-1, \\ \text{and hence } R(v)_{i,i+1} = 0 \text{ for all } 1 \leq i < d \text{ and } v \in [\mathfrak{n},\mathfrak{n}]. \end{gathered} \tag{3.5}$$

On the other hand we know, from [**?**, Lemma 2.1], that for every $1 \leq i \leq d$ there is some $y \in \mathfrak{g}$ such that

$$R(y)_{i,i+1} \neq 0. \tag{3.6}$$

This, combined with (3.5) and (3.4), imply that

$$\text{if } D_{i,i} \neq D_{i+1,i+1} \text{ then } R(v)_{i,i+1} \neq 0 \text{ for some } v \in \mathfrak{p}. \tag{3.7}$$

Step 1. If $D_{i,i} \neq D_{i+1,i+1}$ then $D_{i,i} - D_{i+1,i+1} = \lambda$ and $(E_0)_{i,i+1} \neq 0$.

Indeed, since T is a representation, it follows from (3.1) that, for $1 \leq i < d$,

$$(\mathrm{ad}_{\mathfrak{gl}(d)} D - \lambda)^k E_0 = \begin{cases} E_k, & \text{if } 0 \leq k < n; \\ 0, & \text{if } k = n. \end{cases} \tag{3.8}$$

Since D is upper triangular and E_0 is strictly upper triangular, this implies that, for $1 \leq i < d$,

$$(D_{i,i} - D_{i+1,i+1} - \lambda)^k (E_0)_{i,i+1} = \begin{cases} (E_k)_{i,i+1}, & \text{if } 0 \leq k < n; \\ 0, & \text{if } k = n. \end{cases} \tag{3.9}$$

Now, if $D_{i,i} \neq D_{i+1,i+1}$ then it follows from (3.7) and (3.9) that $(E_0)_{i,i+1} \neq 0$ and case $k = n$ in (3.9) implies $D_{i,i} - D_{i+1,i+1} = \lambda$.

Step 2. For some integer $\ell \geq 0$, there is a unique sequence $\vec{d} = (d_1, \dots, d_{\ell+1})$ of positive integers, with $d = |\vec{d}|$, such that

$$D = D_1 \oplus \cdots \oplus D_{\ell+1}, \quad D_i \in \mathfrak{gl}(d_i),$$

where each D_i has scalar diagonal of scalar $\alpha_i = \alpha - (i-1)\lambda$ for some $\alpha \in \mathbb{F}$.

This follows at once from (3.4) and Step 1, uniqueness is a consequence of the arrangement of the eigenvalues of D.

Step 3. According to the block structure of $\mathfrak{gl}(d)$ given by $\vec{d}$, $p_{r,r}(E_k) = 0$ for all $1 \leq r \leq \ell + 1$ and $0 \leq k \leq n-1$.

Indeed, setting $U^j = \mathrm{span}\{u_1, \dots, u_j\}$ (each U^j is a $\mathfrak{g}$-submodule of U), we have to show that the endomorphism induced by E_k, say $\bar{E}_k$, in

$$\bar{U}^r = U^{d_1+\cdots+d_r} / U^{d_1+\cdots+d_{r-1}}$$

is zero. On the one hand, the endomorphism induced by $\mathrm{ad}_{\mathfrak{gl}(d)} D$ in $\mathfrak{gl}(\bar{U}^r)$ is nilpotent. On the other hand, it follows from (3.8) that $\bar{E}_k$ is a generalized eigenvector of eigenvalue λ of the endomorphism induced by $\mathrm{ad}_{\mathfrak{gl}(d)} D$. Since $\lambda \neq 0$ this is a contradiction.

Step 4. According to the block structure of $\mathfrak{gl}(d)$ given by $\vec{d}$, if $1 \leq i < j \leq \ell + 1$ and $j \neq i+1$, then $p_{i,j}(E_k) = 0$ for all $0 \leq k \leq n-1$.

The proof of this uses the same argument used in the proof of Step 3. The point is that $p_{i,j}(E_k)$ corresponds to an eigenvector of eigenvalue $(j-i)\lambda$ of $\mathrm{ad}_{\mathfrak{gl}(d)} D$ and, if $j - i \neq 1$, (3.8) implies that $p_{i,j}(E_k)$ must be zero.

Step 5. Let α as in Step 2. We may assume that D is in Jordan form

$$D = J^{d_1}(\alpha) \oplus J^{d_2}(\alpha - \lambda) \oplus \cdots \oplus J^{d_{\ell+1}}(\alpha - \ell\lambda).$$

Moreover, $\ell \geq 1$ and if $\mathfrak{n}^{k-1}$ is not contained in $\ker(T)$, then $\ell \geq k$.

Indeed, by (3.6) and Step 3, the first superdiagonal of every D_i consists entirely of non-zero entries. Thus, for each $1 \leq i \leq \ell + 1$, there is $P_i \in GL(d_i)$ such that

$$P_i D_i P_i^{-1} = J^{d_i}(\alpha - (i-1)\lambda).$$

Set $P = P_1 \oplus \cdots \oplus P_{\ell+1} \in GL(d)$, then PDP^{-1} is as stated and and PE_kP^{-1} is still strictly block upper triangular with $p_{i,j}(PE_kP^{-1}) = 0$ if $1 \leq i \leq j \leq \ell+1$ and $j - i \neq 1$. Since $\mathfrak{n}^{k-1}$ is obtained by bracketing elements of $\mathfrak{p}$, it follows from Step 3 that, if $\ell < k$, then $\mathfrak{n}^{k-1} \subset \ker(T)$. In particular, since $\ker(T) \cap \mathfrak{p} = 0$, we have $\ell \geq 1$.

Step 6. For all $1 \leq i \leq \ell$, $d_i + d_{i+1} \leq n + 1$ and the equality holds for some i.

Indeed, from Step 1 we know that $(E_0)_{d_i,d_i+1} \neq 0$ for all i. If $d_i+d_{i+1} > n+1$, for some i, it follows from the Clebsh-Gordan decomposition of the tensor product of irreducible representations of $\mathfrak{sl}(2)$ that $(\mathrm{ad}_{\mathfrak{gl}(d)}D-\lambda)^n E_0 \neq 0$, contradicting (3.8) (for the details, see [**?**, Proposition 2.2]). On the other hand, if $d_i+d_{i+1} < n+1$ for all i then Clebsh-Gordan implies that $E_n = (\mathrm{ad}_{\mathfrak{gl}(d)}D-\lambda)^{n-1}E_0 = 0$, which is impossible since $\ker(T)\cap\mathfrak{p} = 0$.

Final Step. We may assume $E_0 = \begin{pmatrix} 0 & S(1) & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ & & & \\ 0 & & \ddots & S(\ell) \\ 0 & 0 & \dots & 0 \end{pmatrix}$, for a unique normalized sequence $S = (S(1),\dots,S(\ell))$.

Indeed, it follows from Step 3 and 4 that $E_0 = E(S)$ for some sequence as in (2.5). It follows from Proposition 2.4 that there is a unique normalized sequence $S = (S(1),\dots,S(\ell))$ and an invertible matrix $P = P_1\oplus\cdots\oplus P_{\ell+1} \in GL(d)$, with P_i a polynomial in $J^{d_i}(0)$ (and thus commuting with D), such that $PE_0P^{-1} = E(S)$. □

Theorem 3.2. *Let $\lambda \neq 0$. Every finite dimensional uniserial representation $T : \mathfrak{g}_{n,\lambda} \to \mathfrak{gl}(U)$ satisfying $\ker(T)\cap V = 0$ is isomorphic to one and only one normalized representation $R_{\vec{d},\alpha,S}$ with $\vec{d}$ satisfying $|\vec{d}| = \dim U$ and*

$$d_i + d_{i+1} \leq n+1 \text{ for all } i,$$
$$d_i + d_{i+1} = n+1 \text{ for at least one } i.$$

Proof. This is a consequence of Propositions 2.7 and 3.1 □

4. The nilpotency degree of the nilradical $\mathfrak{n}(S)$

The goal of this section is to compute the nilpotency degree of the nilradical $\mathfrak{n}(S)$ of $\mathfrak{h}(\alpha,\lambda,S)$. We will see that, for generic $\vec{d}$ and S, the nilpotency degree of $\mathfrak{n}(S)$ is ℓ. The only exceptions will occur when $\vec{d}$ are *odd-symmetric* (as defined below) with $d_1 = d_{\ell+1} = 1$ and $\phi(E(S)) = E(S)$ (see §2.2).

From now on, set $k = \ell+1$.

Definition 4.1. Given $\vec{d} = (d_1,\dots,d_k)$, we say that $\vec{d}$ is *symmetric* if $d_i = d_{k+1-i}$ for all $i = 1,\dots,k$. We say that $\vec{d}$ is *odd-symmetric* if, in addition, k is odd and $d_{(k+1)/2}$ is odd. Also, if $S = (S(1),\dots,S(k-1))$ is a sequence satisfying (2.5), we say that S is *ϕ-invariant* if $E(S)$ is invariant by the automorphism ϕ. We notice that it follows from (2.4) that the canonical sequence (see Definition 2.2) is invariant.

Proposition 4.2. *Let $\vec{d} = (d_1,\dots,d_k)$ be odd-symmetric, set $d = |\vec{d}|$, and let $S = (S(1),\dots,S(k-1))$ be a ϕ-invariant sequence satisfying* (2.5)*. Then*

$$A_{i,d+1-i} = 0, \qquad i = 1,\dots,\tfrac{d+1}{2},$$

for all $A \in \mathfrak{h}(\alpha,\lambda,S)$. In particular, if in addition $d_1 = 1$ then $p_{1,k}(\mathfrak{h}(\alpha,\lambda,S)) = 0$.

Proof. It follows from Proposition 2.5 that it is enough to prove the result for $\alpha = \lambda = 0$. The hypothesis on $\vec{d}$ implies that $\phi(D(0,0)) = D(0,0)$ and the hypothesis on S says that $\phi(E(S)) = E(S)$ and since it follows that $\phi(A) = A$ for all $A \in \mathfrak{h}(0,0,S)$. Therefore, since $\vec{d}$ is odd-symmetric (and hence d is odd), the definition of ϕ implies that all the entries of A in the antidiagonal must be zero for all $A \in \mathfrak{h}(0,0,S)$. □

4.1. **The nilradical corresponding to the canonical sequence $S = C$.** In this subsection we will consider the case $(\alpha, \lambda, S) = (0, 0, C)$. In order to simplify the notation, let $\mathfrak{h} = \mathfrak{h}(0, 0, C)$ and $E = E(C)$.

Associated to the Lie algebra $\mathfrak{h}$ we define, for $1 \le i < j \le k$, the numbers

$$r_{i,j} = \begin{cases} 0, & \text{if } p_{i,j}(X) = 0 \text{ for all } X \in \mathfrak{h}; \\ \min\{\mathrm{rk}(p_{i,j}(X)) : 0 \neq X \in \mathfrak{h}\}, & \text{otherwise.} \end{cases}$$

Proposition 4.3. $r_{i,j} \in \{0, 1, 2\}$.

Proof. It follows from the definition of E that $r_{i,i+1} = 1$, for $1 \le i \le k-1$. For $l \ge 1$, $r_{i,i+l+1} \le 2$ is a consequence of the following two facts. First, if X is any element of block-degree l, then $\mathrm{rk}(p_{i,i+l+1}([E, X])) \le 2$, since all the elements of $p_{i,i+l+1}([E, X])$ are zero, with the possible exception of those in the first column and the last row.

On the other hand, set $j = i + l + 1$, we will prove that if $p_{i,j}([E, X]) = 0$ for all $X \in \mathfrak{h}$, then $r_{i,j} = 0$. By induction we will show that

$$p_{i,j}([\mathrm{ad}(D)^r E, X]) = 0, \qquad r \ge 0;\ X \in \mathfrak{h}.$$

The case $r = 0$ is given. Moreover, given the case r,

$$\begin{aligned} p_{i,j}([\mathrm{ad}(D)^{r+1} E, X]) &= p_{i,j}([[D, \mathrm{ad}(D)^r E], X]) \\ &= -p_{i,j}([\mathrm{ad}(D)^r E, [D, X]]) + p_{i,j}([D, [\mathrm{ad}(D)^r E, X]]) \\ &= p_{i,i}(D) p_{i,j}([\mathrm{ad}(D)^r E, X]) - p_{i,j}([\mathrm{ad}(D)^r E, X]) p_{j,j}(D) \\ &= 0. \end{aligned}$$

Since we know, from Proposition 2.5, that the elements $\mathrm{ad}(D)^r E$, $r \ge 0$, generates $\mathfrak{n}$, it follows that $r_{i,j} = 0$. □

Proposition 4.4. *If $A \in \mathfrak{gl}(d)$ has the property*

$$\big(p_{i,j}(A)\big)_{a,b} = \begin{cases} 1, & \text{if } a, b = a_0, b_0; \\ 0, & \text{otherwise;} \end{cases}$$

then the entries of $p_{i,j}(\mathrm{ad}(D)^k(A))$ are zero except those contained in the diagonal $b - a = b_0 - a_0 + k$, in which case:

$$\big(p_{i,j}(\mathrm{ad}(D)^k(A))\big)_{a_0 - i, b_0 + k - i} = (-1)^{k-i} \binom{k}{i}.$$

In particular, $\big(p_{i,j}(A)\big)_{d_i,1} = 1$ then all the entries of $p_{i,j}(\mathrm{ad}(D)^{d_i+d_j-1}(A))$ are zero except

$$\big(p_{i,j}(\mathrm{ad}(D)^{d_i+d_j-2}(A))\big)_{1,d_j} = (-1)^{d_j-1} \binom{d_i + d_j - 2}{d_i - 1}.$$

Proof. This is an straightforward computation. □

Proposition 4.5. *If there is $X \in \mathfrak{h}$ such that $\big(p_{i,j}(X)\big)_{d_i,1} \neq 0$, then $r_{i,j} = 1$.*

Proof. This is consequence of Proposition 4.4. □

Proposition 4.6. *If $r_{i,j} = 1$ then there exists $X \in \mathfrak{h}$ such that*

$$p_{i,j}(X) = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix} \tag{4.1}$$

and if $r_{i,j} = 2$ then there exists $X \in \mathfrak{h}$ such that

$$p_{i,j}(X) = \begin{pmatrix} 0 & \dots & 1 & * \\ 0 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}. \tag{4.2}$$

Moreover, for any $X \in \mathfrak{h}$ satisfying $p_{i,j}(X) \neq 0$ then there exists k_0 such that $p_{i,j}\big(\mathrm{ad}(D)^{k_0}(X)\big)$ is either as (4.1) *or as* (4.2).

Proof. Let $X \in \mathfrak{h}$ be such that $\mathrm{rk}(p_{i,j}(X)) = r_{i,j}$, and let

$$t_0 = \min\{t = b - a : \big(p_{i,j}(X)\big)_{a,b} \neq 0\},$$
$$T_0 = \{(a,b) : b - a = t_0 \text{ and } \big(p_{i,j}(X)\big)_{a,b} \neq 0\}.$$

If $r_{i,j} = 1$ then there is only one pair $(a_0, b_0) \in T_0$. If $k_0 = d_j - 1 - t_0$ then it follows from Proposition 4.4 that $\mathrm{ad}(D)^{k_0}(X)$ is, up to a non-zero scalar, as stated.

If $r_{i,j} = 2$ then there are at most two possible pairs $(a,b) \in T_0$. It follows from Proposition 4.4 that, if $k_0 = d_j - 2 - t_0$, then the only possible non-zero entries of $p_{i,j}(\mathrm{ad}(D)^{k_0}(X))$ are

$$\begin{matrix} \big(p_{i,j}(X)\big)_{1,d_j-1} & \big(p_{i,j}(X)\big)_{1,d_j} \\ & \big(p_{i,j}(X)\big)_{2,d_j}. \end{matrix}$$

Moreover, the pair

$$\Big(\big(p_{i,j}(X)\big)_{1,d_j-1},\ \big(p_{i,j}(X)\big)_{2,d_j}\Big) \tag{4.3}$$

is a linear combination of two pairs of two consecutive binomial numbers $\binom{k_0}{l}$, $0 \leq l \leq k$, that is

$$\Big(\big(p_{i,j}(X)\big)_{1,d_j-1},\ \big(p_{i,j}(X)\big)_{2,d_j}\Big) = x_1\Big(\binom{k_0}{l_1},\ \binom{k_0}{l_1+1}\Big) + x_2\Big(\binom{k_0}{l_2},\ \binom{k_0}{l_2+1}\Big)$$

with $0 \leq l_1 \neq l_2 \leq k_0$ for some $(x_1, x_2) \neq (0,0)$. Since $\Big(\binom{k_0}{l_1},\ \binom{k_0}{l_1+1}\Big)$ and $\Big(\binom{k_0}{l_2},\ \binom{k_0}{l_2+1}\Big)$ are linearly independent, it follows that the pair (4.3) is non-zero. Finally, we conclude that $\mathrm{ad}(D)^{k_0}(X)$ is, up to a non-zero scalar, as stated because otherwise we would have $r_{i,j} = 1$. □

Proposition 4.7. *If there exists $X \in \mathfrak{h}$ such that either*

$$p_{i,j}(X) = 0 \quad \textit{and} \quad p_{i+1,j+1}(X) \neq 0$$

or

$$p_{i,j}(X) \neq 0 \quad \textit{and} \quad p_{i+1,j+1}(X) = 0$$

then $r_{i,j+1} = 1$. Moreover, any of the following:

(a1) $r_{i,j} = 0$ *and* $r_{i+1,j+1} \neq 0$,

(a2) $r_{i,j} \neq 0$ *and* $r_{i+1,j+1} = 0$*,*
(b) $r_{i,j} = r_{i+1,j+1} = 1$ *and* $d_i \neq d_{j+1}$*,*
(c1) $r_{i,j} = 1$*,* $r_{i+1,j+1} = 2$ *and* $d_i + 1 \neq d_{j+1}$*,*
(c2) $r_{i,j} = 2$*,* $r_{i+1,j+1} = 1$ *and* $d_i \neq d_{j+1} + 1$*.*

implies the existence of such an X *and thus* $r_{i,j+1} = 1$*.*

Proof. First, if there exists $X \in \mathfrak{h}$ such that $p_{i,j}(X) = 0$ and $p_{i+1,j+1}(X) \neq 0$, then, by Proposition 4.6, we may assume that X isis either as (4.1) or as (4.2). In either case, it is clear that $\operatorname{rk}(p_{i,j+1}([E, X])) = 1$.

Now we prove the particular statements. By symmetry, it is enough to prove (a1), (b) and (c1).

Proof of (a1): it is immediate that (a1) implies the existence of $X \in \mathfrak{h}$ such that $p_{i,j}(X) = 0$ and $p_{i+1,j+1}(X) \neq 0$.

Proof of (b): let $X \in \mathfrak{h} \cap \mathcal{D}_{t_X}$ be homogeneous such that all the entries of $p_{i,j}(X)$ are zero except that $\big(p_{i,j}(X)\big)_{1,d_j} = 1$, as granted by Proposition 4.6. This implies that $t_X = d_{j-1} + \cdots + d_i + (d_j - 1)$.

Similarly, let $Y \in \mathfrak{h} \cap \mathcal{D}_{t_Y}$ be homogeneous such that all the entries of $p_{i+1,j+1}(X)$ are zero except that $\big(p_{i+1,j+1}(X)\big)_{1,d_{j+1}} = 1$. Now $t_Y = d_j + \cdots + d_{i+1} + (d_{j+1} - 1)$.

It follows from the hypothesis that

$$t_Y - t_X = d_{j+1} - d_i \neq 0.$$

Therefore, either $t_Y > t_X$, in which case $p_{i,j}(Y) = 0$ or $t_X > t_Y$, in which case $p_{i+1,j+1}(X) = 0$, and we are done.

Proof of (c1): This is analogous to the proof of (b). □

Proposition 4.8. *If* $r_{i,j} = 1$ *and one of the following hold:*

(a) $d_i, d_j > 1$*,*
(b1) $d_j > 1$ *and* $r_{i+1,j+1} = 0$*,*
(b2) $d_i > 1$ *and* $r_{i-1,j-1} = 0$*,*
(c) $r_{i+1,j+1} = r_{i-1,j-1} = 0$*.*

then $r_{i-1,j+1} = 1$*.*

Proof. Any of these conditions implies that, for any $X \in \mathfrak{h}$,

$$\big(p_{i-1,j+1}([[X, E], E]]\big)_{d_{i-1},1} = -2\big(p_{i,j}(X)\big)_{1,d_j}.$$

Since $r_{i,j} = 1$, it follows from Proposition 4.6 that there exists $X \in \mathfrak{h}$ such that $\big(p_{i,j}(X)\big)_{1,d_j} \neq 0$, and thus $\big(p_{i-1,j+1}([[X, E], E]]\big)_{d_{i-1},1} \neq 0$. Now Proposition 4.5 implies $r_{i-1,j+1} = 1$. □

Proposition 4.9. *If* $r_{i,j} = 2$*, then* $r_{i-1,j} \neq 2$ *and* $r_{i,j-1} \neq 2$*.*

Proof. By symmetry, it is enough to show $r_{i,j-1} \neq 2$. We use induction on k. For $k = 3$ there is nothing to prove, since $r_{1,2} = 1$. Let $k > 3$, we can assume $i = 1$ and $j = k$. Arguing by contradiction, we assume $r_{1,k-1} = 2$. By inductive hypothesis $r_{1,k-2}, r_{2,k-1} \neq 2$ and, since $r_{1,k-1} = 2$, Proposition 4.7 (a1), (a2), implies that

$$r_{1,k-2}, r_{2,k-1} = 1.$$

Since $r_{1,k-1} = 2$ then $d_{k-1} > 1$ and hence, since $r_{2,k-1} = 1$ and $r_{1,k} = 2$, Proposition 4.8 (a) implies $d_2 = 1$ and thus $r_{2,k} \neq 2$. Since $r_{1,k} = 2$, Proposition 4.7 (a2) implies $r_{2,k} = 1$. Proposition 4.7 (c2) implies $d_1 = d_k + 1$.

Now we have $r_{3,k} \neq 0$ since, otherwise, Proposition 4.8 (b1), applied to $(i,j) = (2, k-1)$ would imply that $r_{1,k} = 1$. Moreover, we claim $r_{3,k} = 2$.

If $r_{3,k} = 1$ we can find a homogeneous $X \in \mathfrak{h} \cap \mathcal{D}_t$ such that $p_{3,k}(X)$ is as stated in Proposition 4.6, that is $\big(p_{3,k}(X)\big)_{1,d_k} = 1$. Since $1 = d_2 < 2 \leq d_k$, Proposition 2.1 implies $p_{2,k-1}(X) = 0$. Since $r_{1,k-1} = 2$, Proposition 4.7 implies $p_{1,k-2}(X) = 0$. Therefore

$$p_{1,k-1}([X,E]) = 0 \quad \text{ and } \quad p_{2,k}([X,E]) \neq 0$$

and, once again, Proposition 4.7 implies $r_{1,k} = 1$, a contradiction. We have proved that $r_{3,k} = 2$ and hence $d_3 \geq 2$, $r_{3,k-1} \neq 2$ by the inductive hypothesis, and $r_{3,k-1} \neq 0$ by Proposition 4.7. Therefore $r_{3,k-1} = 1$ and, it follows from Proposition 4.6 that there is a homogeneous $X \in \mathfrak{h}$ as in (4.1), that is with $\big(p_{3,k-1}(X)\big)_{1,d_k-1} = 1$. Taking into account that $d_3, d_{k-1} \geq 2$ it is not difficult to see that

$$\big(p_{1,k}([[[X,E],E],E]))\big)_{1,d_1} = 3$$

which implies that $r_{1,k} = 1$, a contradiction. □

Proposition 4.10. $r_{1,k} = 2$ *implies* $d_1 = d_k > 1$*.*

Proof. Since $r_{1,k} = 2$, we have $d_{1,k} > 1$. We must show that $d_1 = d_k$. We know by Proposition 4.9 that $r_{1,k-1}, r_{2,k} \neq 2$. Also, by fact Proposition 4.7, it follows that $r_{1,k-1}, r_{2,k} \neq 0$. Therefore $r_{1,k-1} = r_{2,k} = 1$. Now Proposition 4.7 (b) implies $d_1 = d_k$. □

Proposition 4.11. *If* $r_{1,k} = 0$ *then* $d_1 = 1$ *or* $d_k = 1$*.*

Proof. We will consider all possible values for $r_{1,k-1}, r_{2,k}$.

Case $r_{1,k-1} = 0$, $r_{2,k} \neq 0$; or $r_{1,k-1} \neq 0$, $r_{2,k} = 0$: Impossible by Proposition 4.7.

Case $r_{1,k-1} = r_{2,k} = 1$: It follows from Proposition 4.7 (b) that $d_1 = d_k$ and it is clear that if $d_1 \neq 1$ then $r_{1,k} \neq 0$, thus $d_1 = 1$.

Case $r_{1,k-1} = r_{2,k} = 2$: This implies that $d_1, d_2, d_{k-1}, d_k \geq 2$. Consider $r_{2,k-1}$. It is not 0 by Proposition 4.7 and it can not be 2 by Proposition 4.9. Hence $r_{2,k-1} = 1$ and now Proposition 4.8 implies $r_{1,k} = 1$ contradicting our hypothesis.

Case $r_{1,k-1} = 2, r_{2,k} = 1$: This implies $d_1 > 1$. It follows from Propositions 4.9 and 4.7 (a2) that $r_{2,k-1} = 1$. Then, since $d_1 > 1$ if we also had $d_k > 1$, we would have $r_{1,k} = 1$ by Proposition 4.8. Thus $d_k = 1$.

Case $r_{1,k-1} = r_{2,k} = 0$: By the induction hypothesis, either the claim is true or $d_1, d_k > 1$ and $d_2 = d_{k-1} = 1$. We assume, by contradiction that

$$d_1, d_k > 1 \text{ and } d_2 = d_{k-1} = 1.$$

Let j_0 be the largest j such that

$$r_{i,k-j+i} = 0 \text{ for all } i = 1, \ldots, j.$$

Clearly $2 \leq j_0 \leq k-2$ and, again, the induction hypothesis imply

$$d_j = d_{k+1-j} = 1 \text{ for all } 2 \leq j \leq j_0. \tag{4.4}$$

Since, by definition of j_0, we have $r_{i,k-(j_0+1)+i} \neq 0$ for some i, it follows from Proposition 4.7 (a1) or (a2) that in fact $r_{i,k-(j_0+1)+i} \neq 0$ for all $i = 1, \ldots, j_0 + 1$. Moreover, (4.4) implies that

$$r_{i,k-(j_0+1)+i} = 1 \text{ for all } i = 2, \ldots, j_0.$$

Let $X \in \bar{\mathcal{D}}_{k-(j_0+1)}$, $X \neq 0$ such that $[D, X] = 0$. By the definition of j_0, we must have $[E, X] = 0$ and thus $[E + D, X] = 0$. Since $D + E$ is principal nilpotent, it follows that, up to scalar, X is a power of $D + E$. This implies that $d_1 = d_k = 2$ and $r_{1,k-j_0} = r_{j_0+1,k} = 2$. At this point we know that

$$X = \left(\begin{array}{c|ccc|c|c|c|cc}
\cdots & \cdots & 1 & 0 & 0 & & & 0 & 0 \\
\cdots & \cdots & 0 & 1 & 0 & & & 0 & 0 \\
\hline
 & & & 0 & 1 & & & & \\
\hline
 & & & & & \ddots & & & \\
\hline
 & & & & & 0 & 1 & 0 & 0 \\
\hline
 & & & & & & 0 & 1 & 0 \\
 & & & & & & & 0 & 1 \\
 & & & & & & & \vdots & \vdots \\
\hline
 & & & & & & & \vdots & \vdots
\end{array}\right)$$

Moreover, there must exists $Y \in \bar{\mathcal{D}}_{k-(j_0+2)}$ such that $[D, Y] = X$. This implies that $d_{j_0+1} = d_{k-j_0} = 2$ and

$$Y = \left(\begin{array}{c|c|cc|c|c|c|cc}
\cdots & a_0 & 0 & 0 & 0 & & & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 & & & 0 & 0 \\
\hline
 & & 0 & a_1 & 0 & & & & \\
\hline
 & & & & & \ddots & & & \\
\hline
 & & & & & a_{j_0-1} & 0 & 0 & 0 \\
\hline
 & & & & & & a_{j_0} & 0 & 0 \\
 & & & & & & 0 & 0 & 0 \\
\hline
 & & & & & & & 0 & a_{j_0+1} \\
\hline
 & & & & & & & \vdots & \vdots
\end{array}\right)$$

But with this Y it is impossible to satisfy the condition $[D, Y] = X$. □

Now we can prove the crucial step.

Proposition 4.12. *Let $k \geq 2$ and $\vec{d} = (d_1, \dots, d_k)$. Then*

(1) If $r_{2,k-1} = 2$ and $d_1 = d_k = 1$ then $r_{1,k} = 0$.
(2) If $r_{1,k} = 0$ then $d_1 = 1$ and $d_k = 1$.
(3) If $r_{2,k-1} = 1$ then $r_{1,k} = 1$, unless $k = 4$ and $\vec{d} = (1, 1, 1, 1)$.
(4) If $r_{1,k} = 2$ then k is odd and $\vec{d}$ is odd-symmetric with $d_1 = d_k > 1$.
(5) If $r_{1,k} = 0$ then either k is even and $\vec{d} = (1, \dots, 1)$, or k is odd and $\vec{d}$ is odd-symmetric with $d_1 = d_k = 1$.

Proof. We use induction on k. For $k = 2$ there is nothing to prove. We assume $k \geq 3$ and that the whole proposition is true for lower values of k.

Proof of part (1), we have $r_{2,k-1} = 2$ and $d_1 = d_k = 1$: By induction hypothesis on part (4), $r_{2,k-1} = 2$ implies that $k - 2$ is odd and $\vec{d}$ is odd-symmetric. Proposition 4.2 and $d_1 = d_k = 1$ imply $r_{1,k} = 0$.

Proof of part (2), we have $r_{1,k} = 0$: As in Proposition 4.11, we will consider all possible values for $r_{1,k-1}, r_{2,k}$.

The cases $r_{1,k-1} = 0$, $r_{2,k} \neq 0$ and $r_{1,k-1} \neq 0$, $r_{2,k} = 0$ are impossible by Proposition 4.7.

The case $r_{1,k-1} = r_{2,k} = 0$ follows by induction hypothesis on part (2).

The cases $r_{1,k-1} = r_{2,k} = 1$ and $r_{1,k-1} = r_{2,k} = 2$ are as in Proposition 4.11. In particular, $r_{1,k-1} = r_{2,k} = 1$ implies $d_1 = d_k = 1$.

Finally, let us prove that the case $r_{1,k-1} = 2$, $r_{2,k} = 1$ is impossible.

This case implies that $d_{k-1}, d_1 \geq 2$ and thus, by Proposition 4.11, $d_k = 1$. Proposition 4.7 (c2) implies that $d_1 = 2$. Proposition 4.9 implies $r_{2,k-1} \neq 2$, Proposition 4.7 implies $r_{2,k-1} \neq 0$, and thus $r_{2,k-1} = 1$. Since $d_{k-1} \geq 2$, if $d_2 > 1$, Proposition 4.8 (a) would imply that $r_{1,k} = 1$; hence $d_2 = 1$. Let $l \geq 2$ be the first index such that $d_{l-1} = 1$ but $d_l > 1$. Thus we have

$$2 = d_1,\ 1 = d_2 = \cdots = d_{l-1},\ 2 \leq d_l,\ \ldots,\ 2 \leq d_{k-1},\ 1 = d_k$$

Now we will show that $r_{l,k-1} \neq 0, 1, 2$, which is a contradiction.

Since $d_l, d_{k-1} \geq 2$, Proposition 4.11 implies $r_{l,k-1} \neq 0$. Let us show that $r_{l,k-1} \neq 1$. Otherwise there would be a homogeneous $X \in \mathcal{D}_t \cap \bar{\mathcal{D}}_{k-1-l}$ such that $\mathrm{rk}(p_{l,k-1}(X)) = 1$ and by Proposition 4.6 we may assume as in (4.1). Since $d_{k-1} \geq 2$ and $X \in \mathcal{D}_t$, it follows that

$$p_{j,k-1-l+j}(X) = 0, \text{ for all } j = 2, \ldots, l-1$$

and this implies that $\mathrm{rk}(p_{1,k}(\mathrm{ad}(E)^l(X))) = 1$, a contradiction.

Let us show that $r_{l,k-1} \neq 2$. If $r_{l,k-1} = 2$ then, by induction hypothesis on (1) we have $r_{l-1,k} = 0$. This implies that $r_{l,k} = r_{l-1,k-1} = 1$ and thus we have a homogeneous $X \in \mathcal{D}_t \cap \bar{\mathcal{D}}_{k-l}$ such that $\mathrm{rk}(p_{l,k}(X)) = 1$, but since $r_{l-1,k} = 0$, Proposition 4.7 implies that $\mathrm{rk}(p_{l-1,k-1}(X)) = 1$ and $p_{l-1,k}([X, E])) = 0$. By Proposition 4.6 we may assume that $p_{l,k}(X)$ and $p_{l-1,k-1}(X)$ are as in (4.1). Now $\mathrm{ad}^{l-1}(E)(X) \neq 0$ which is absurd.

Proof of part (3), we have $r_{2,k-1} = 1$: If $d_1 \neq d_k$, it follows from Proposition 4.10 and part (2) that $r_{1,k} = 1$. Therefore, we assume from now on $d_1 = d_k$. Let us consider now $r_{1,k-1}$ and $r_{2,k}$.

If $r_{1,k-1} = r_{2,k} = 0$, then $r_{1,k} = 0$ and the induction hypothesis on part (5) implies (this does not depend on the parity of k) that $d_i = 1$, for all $1 \leq i \leq k$ which in turn implies $r_{2,k-1} = 0$, and this can not happen unless $k = 4$.

The cases $r_{1,k-1} = 0$, $r_{2,k} \neq 0$ and $r_{1,k-1} \neq 0$, $r_{2,k} = 0$ imply that $r_{1,k} = 1$ by Proposition 4.7.

The case $r_{1,k-1} = 2$ implies that $r_{1,k}$ can not by 2 by Proposition 4.9 and that $r_{1,k}$ can not by 0 by part (2), and thus $r_{1,k} = 1$. Similarly, $r_{2,k} = 2$ then $r_{1,k} = 1$.

Therefore, we can assume $r_{1,k-1} = r_{2,k} = 1$ and thus $(d_1, \ldots, d_k) \neq (1, \ldots, 1)$. If $d_{k-1} > 1$ and $d_2 > 1$ then, by Proposition 4.8 (a), $r_{1,k} = 1$. Let $i < j$ be such that $d_i, d_j > 1$ and

$$d_l = 1 \text{ for } j < l \leq k \text{ and } 1 \leq l < i. \tag{4.5}$$

We have $r_{i,j} \neq 0$ by Proposition 4.11.

Assume first $r_{i,j} = 1$. This implies that we have a homogeneous $X \in \mathcal{D}_t \cap \bar{\mathcal{D}}_{j-i}$ such that $\mathrm{rk}(p_{i,j}(X)) = 1$ and, by Proposition 4.6, we may assume $p_{i,j}(X)$ as in (4.1). Since $d_i, d_j > 1$, we have

$$p_{i-q,j+\beta-q}(\mathrm{ad}(E)^\beta(X)) = (-1)^{\beta-q}\binom{\beta}{q}.$$

and this implies that $r_{1,k}=1$.

Now assume $r_{i,j}=2$. By the induction hypothesis on part (4) we have $j+1-i$ odd and

$$(d_i, d_{i+1}\dots, d_j) \quad \text{is odd-symmetric.} \tag{4.6}$$

Also, the induction hypothesis on part (1), implies $r_{i-1,j+1}=0$ and thus $r_{i-1,j}=r_{i,j+1}=1$. This implies that we have a homogeneous $X\in\mathcal{D}_t\cap\bar{\mathcal{D}}_{j+1-i}$ such that $\operatorname{rk}(p_{i-1,j}(X))=1$, but since $r_{i-1,j+1}=0$, Proposition 4.7 implies that $\operatorname{rk}(p_{i,j+1}(X))=1$ and $p_{i-1,j+1}([X,E]))=0$. By Proposition 4.6 we may assume that

$$p_{i-1,j}(X)=\begin{pmatrix}0 & \cdots & 0 & 1\end{pmatrix} \qquad \text{and} \qquad p_{i,j+1}(X)=\begin{pmatrix}x\\0\\\vdots\\0\end{pmatrix}$$

but since $p_{i-1,j+1}([X,E]))=0$ we have $x=1$. This implies that

$$p_{i-q,j+1+\beta-q}(\operatorname{ad}(E)^{\beta}(X))=\pm\left(\binom{\beta}{q}-\binom{\beta}{q-1}\right)$$

when $i-q>1$ and $j+1+\beta-q<k$ (in all these cases the size of block $p_{i-q,j+1+\beta-q}(\operatorname{ad}(E)^{\beta}(X))$ is 1×1) and

$$p_{1,k}(\operatorname{ad}(E)^{k-j+i-2}(X))_{d_1,1}=\pm\left(\binom{k-j+i-2}{i-1}-\binom{k-j+i-2}{i-2}\right)$$

If this number is not zero, then $\operatorname{rk}\big(p_{1,k}(\operatorname{ad}(E)^{k-j+i-2}(X))\big)=1$ and thus $r_{1,k}=1$. Otherwise

$$\binom{k-j+i-2}{i-1}=\binom{k-j+i-2}{i-2}$$

and hence $j=k+1-i$. This, together with (4.5) and (4.6), imply k odd and

$$(d_2, d_3\dots, d_{k-1}) \quad \text{is odd-symmetric.}$$

Now, Proposition 4.2 implies $r_{2,k-1}=0$, a contradiction.

Proof of parts (4) and (5), we have $r_{1,k}\neq 1$: It follows from part (3) that $r_{2,k-1}\neq 1$. We now apply the induction hypothesis on parts (4) and (5) and we consider the cases k even and k odd.

If k is even, then $(d_2,\dots,d_{k-1})=(1,\dots,1)$ (in particular $r_{2,k-1}=0$). We may assume $d_1\geq d_k$, we will show that $d_1=d_k=1$.

Let $X=\operatorname{ad}(D)^{d_1-1}(E)\in\mathcal{D}_{d_1}\cap\bar{\mathcal{D}}_1$, we have

$$p_{1,2}(X)=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$$

If $d_1>d_k$ then $p_{\alpha,1+\alpha}(X)=0$ for all $2\leq\alpha\leq k-1$ and it is clear that $\operatorname{rk}(\operatorname{ad}(E)^{k-1}(X))=1$ and hence $r_{1,k}=1$, a contradiction. Therefore $d_1=d_k$.

If $d_1=d_k>1$ then $p_{\alpha,\alpha+1}(X)=0$ for all $2\leq\alpha\leq k-2$ and

$$p_{k-1,k}(X)=\begin{pmatrix}0 & \cdots & 0 & (-1)^{d_k-1}\end{pmatrix}.$$

Now

$$p_{1,k}\big(\mathrm{ad}(D+E)^{(k-1)+(d_k-1)}(X)\big) = \begin{pmatrix} 0 & \dots & (-1)^{k+d_k} & 0 \\ 0 & \dots & 0 & (-1)^{d_k-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

and since k is even, we obtain $\mathrm{rk}\Big(p_{1,k}\big(\mathrm{ad}(D+E)^{(k-1)+(d_k-1)+1}(X)\big)\Big) = 1$, a contradiction. Therefore $d_1 = d_k = 1$.

If k is odd, then the induction hypothesis on parts (4) and (5) implies that $(d_2, \dots, d_{k-1})$ is odd-symmetric. If $r_{1,k-1} \neq 1$, the induction hypothesis on parts (4) and (5) implies $(d_1, \dots, d_{k-1}) = (1, \dots, 1)$ and thus $d_k = 1$ (otherwise we would obtain $r_{1,k} = 1$). Hence $r_{1,k-1} = 1$ and similarly $r_{2,k} = 1$. Now $r_{1,k} \neq 1$, part (2) and Proposition 4.10 imply $d_1 = d_k$ and thus $(d_1, \dots, d_k)$ is odd-symmetric. Finally, Proposition 4.2 and item (2) imply that $r_{1,k} = 2$ if and only if $d_1 = d_k > 1$. □

Summarizing, we have proved the following theorem.

Theorem 4.13. *Let $k \geq 2$ and $\vec{d} = (d_1, \dots, d_k)$. Then the nilpotency degree of $\mathfrak{n}(C)$ is $k-1$ except when $r_{1,k} = 0$. This occurs if and only if*

(1) $\vec{d} = (1, \dots, 1)$, in which case $\mathfrak{n}$ is 1-dimensional abelian.
(2) k is odd, $\vec{d}$ is odd-symmetric with $d_1 = d_k = 1$, in which case the nilpotency degree is $k-2$.

In addition, $r_{1,k} = 2$ if and only if k is odd, $\vec{d}$ is odd-symmetric with $d_1 = d_k > 1$.

Corollary 4.14. *If $l < k$ and $r_{i,l+i} = 0$ for $i = 1, \dots, k-l$, then $\vec{d} = (1, \dots, 1)$.*

Proof. By hypothesis, all sequences $(d_1, \dots, d_{l+1})$, $(d_2, \dots, d_{l+1})$, up to $(d_{k-l}, \dots, d_k)$, fall in the cases of parts (5) and (4) of Proposition 4.12. □

CIEM-CONICET, FAMAF-Universidad Nacional de Córdoba, Argentina.
E-mail address: cagliero@famaf.unc.edu.ar

CIEM-CONICET, FAMAF-Universidad Nacional de Córdoba, Argentina.
E-mail address: levstein@famaf.unc.edu.ar

Department of Mathematics and Statistics, Univeristy of Regina, Canada
E-mail address: fernando.szechtman@gmail.com