

ORBITS OF HOMOGENEOUS POLYNOMIALS ON BANACH SPACES

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ABSTRACT. We study the dynamics induced by homogeneous polynomials on Banach spaces. It is known that no homogeneous polynomial defined on a Banach space can have a dense orbit. We show, a simple and natural example of a homogeneous polynomial with an orbit that is at the same time d -dense (the orbit meets every ball of radius d), weakly dense and such that $\Gamma \cdot \text{Orb}_P(x)$ is dense for every $\Gamma \subset \mathbb{C}$ that is either unbounded or that has 0 as an accumulation point. Moreover we generalize the construction to arbitrary infinite dimensional separable Fréchet spaces. To prove this we study Julia sets of homogeneous polynomials on Banach spaces.

1. INTRODUCTION

Let X be a Banach space. A function $F : X \rightarrow X$ is said to be *hypercyclic* if there exists $x \in X$ such that its orbit $\text{Orb}_F(x) = \{F^n(x) : n \in \mathbb{N}_0\}$ is dense in X . In this case x is called a *hypercyclic* vector. The theory of linear hypercyclic operators had a great development in the last decades (see for example the books on the subject [3, 15]).

As a natural extension of the linear theory one may study orbits of (non linear) homogeneous operators, or polynomials. The first result in this direction was due to Bernardes [4], where he proved that no (non linear) homogeneous polynomial on a Banach space can be hypercyclic. From his result, it can be deduced that associated to each homogeneous polynomial there is a ball (afterwards *the limit ball*) which is invariant under the action of the polynomial. Moreover, any orbit which enters the limit ball converges to 0. Maybe this result was one of the reasons why the theory of the dynamics of non linear, and in particular, homogeneous polynomials has had a much smaller development than the linear counterpart. However, the behavior of the orbits induced by a homogeneous polynomial can be highly nontrivial and it is far from being understood. For example, in [4] Bernardes showed that the orbits may oscillate between infinity and the boundary of the limit ball. He also proved that every infinite dimensional and separable Banach space supports supercyclic homogeneous polynomials. More recently Peris, Kim and Song [16, 17] proved that every separable Banach space of dimension greater than one supports *numerically hypercyclic* homogeneous polynomials. This means that there are vectors $x \in S_X, x^* \in S_{X^*}$ for which its numerical orbit, $\text{Norb}_P(x, x^*) := \{x^*(P^n(x)) : n \in \mathbb{N}_0\}$ is dense in \mathbb{C} .

2010 *Mathematics Subject Classification.* 47H60, 37F10, 47A16, 32H50.

Partially supported by ANPCyT PICT 2015-2224, UBACyT 20020130300052BA, PIP 11220130100329CO and CONICET.

Since any orbit that meets the limit ball converges to 0, no orbit induced by a homogeneous polynomial is dense in X . But, it is natural to ask how big and complicated can the orbits that never meet the limit ball be. For example, we can ask whether there exist homogeneous polynomials with orbits which imply some weaker types of hypercyclicity. In particular we can formulate the following questions:

- (1) The Bourdon-Feldman Theorem [8] states that somewhere dense orbits of linear operators are actually dense. Does there exist homogeneous polynomials in Banach spaces with somewhere dense orbits?
- (2) Feldman [13] showed that if a linear operator is d -hypercyclic for some $d > 0$ (i.e. an operator with an orbit that meets every ball of radius d), then it is hypercyclic. Does there exist d -hypercyclic homogeneous polynomials in Banach spaces?
- (3) Are there weakly hypercyclic homogeneous polynomials (i.e. polynomials with an orbit which is dense with respect to the weak topology)?
- (4) Recently, Charpentier, Ernst and Menet [11] characterized the subsets $\Gamma \subset \mathbb{C}$ for which every Γ -supercyclic linear operator (i.e. operators T such that $\Gamma \cdot \text{Orb}_T(x)$ is dense for some $x \in X$) is a hypercyclic linear operator. For which $\Gamma \subset \mathbb{C}$ are there Γ -supercyclic homogeneous polynomials?

To study these questions we propose a notion of Julia set associated to homogeneous polynomials in Banach spaces. From its basic properties we deduce that every orbit of a homogeneous polynomial is nowhere dense. Thus, the Bourdon-Feldman Theorem still holds for homogeneous polynomials. Then, a careful study of the Julia set of a very simple homogeneous polynomial on ℓ_p , which is the product of the backward shift operator and a linear functional, allows us to prove that it is, at the same time, d -hypercyclic, weakly hypercyclic and Γ -supercyclic for every $\Gamma \subseteq \mathbb{C}$ that is either unbounded or not bounded away from zero. Moreover all this properties are achieved by the same orbit. Finally, we also show that such a homogeneous polynomial exists on every separable infinite dimensional Banach space.

It should be mentioned that if the space is a non-normable Fréchet space, then it can support hypercyclic homogeneous polynomials. The first to notice it was Peris [21], who exhibited a chaotic homogeneous polynomial on $\mathbb{C}^{\mathbb{N}}$. Later, other examples were presented, on some Köthe Echelon spaces (including $H(\mathbb{D})$) [20], on some spaces of differentiable functions on the real line [2] and more recently on $H(\mathbb{C})$, the space of entire functions on the complex plane [9]. There are also positive results for hypercyclicity of non-homogeneous polynomials. In [19] the authors proved that every complex separable infinite dimensional Banach space supports a non-linear hypercyclic polynomial. More recently Bernardes and Peris showed in [5] the existence of frequently hypercyclic, chaotic and distributionally chaotic (non-homogeneous) polynomials for a very wide class of infinite dimensional separable Banach spaces. There where also some attempts to extend the concept of hypercyclicity to multilinear operators, see [14, 6].

2. JULIA SETS FOR HOMOGENEOUS POLYNOMIALS ON BANACH SPACES

2.1. Preliminaries. Given a topological space X , a function $F : X \rightarrow X$ is said to be *transitive* if for every pair of non empty sets U, V , there exists $n \in \mathbb{N}$ such that $F^n(U) \cap V \neq \emptyset$. If X is a complete separable metric space without isolated points, then transitivity is equivalent to hypercyclicity. This means that there exists $x \in X$ whose orbit, $Orb_F(x) = \{F^n(x) : n \in \mathbb{N}\}$, is dense in the space. If F is hypercyclic and the periodic vectors are dense in the space, then F is said to be *chaotic*.

Given an F -space X , $P : X \rightarrow X$ is an m -homogeneous polynomial if there exist an m -linear operator $A : \overbrace{X \times \dots \times X}^m \rightarrow X$ such that $P(x) = A(x, \dots, x)$. We will deal with continuous polynomials. If X is a normed space then we can define a norm in the space of m -homogeneous polynomials $\mathcal{P}(^m X; X)$,

$$\|P\| := \sup_{x \in \overline{B}_X} \|P(x)\|.$$

Bernardes showed that if X is a normed space, then no (non linear) homogeneous polynomial is hypercyclic [4] (see also [23, Lemma 1.2] for a related finite dimensional statement). For the sake of completeness and because we need to define the notion of limit ball, we give a proof of (slightly modified version of) the mentioned result.

Proposition 2.1. *Let X be a normed space, let P be an m -homogeneous polynomial with $m \geq 2$ and let $r_P = \|P\|^{\frac{1}{1-m}}$. Then $P(r_P \overline{B}_X) \subseteq r_P \overline{B}_X$. Moreover for all $y \in r_P B_X$, $\lim_{n \rightarrow \infty} P^n(y) = 0$.*

Proof. Let $x \in r_P \overline{B}_X$, then

$$\begin{aligned} \|P(x)\| &\leq \|P\| \|x\|^m < \|P\| \|P\|^{\frac{m}{1-m}} \\ &= \|P\|^{1+\frac{m}{1-m}} = r_P. \end{aligned}$$

Let x with $\|x\| < r_P$, then

$$\|P^n(x)\| < \left(\frac{\|x\|}{r_P}\right)^{m^n} r_P \rightarrow 0.$$

□

Corollary 2.2. *No homogeneous polynomial of degree ≥ 2 defined on a normed space can be hypercyclic.*

Definition 2.3. *Let P be an m -homogeneous polynomial on a normed space X . We will say that $r_P = \|P\|^{\frac{1}{1-m}}$ is the limit radius of P and that $r_P B_X$ is the limit ball of P .*

2.2. Julia sets for homogeneous polynomials. One distinctive feature of the dynamics induced by a holomorphic function on the complex plane is that we may partition its domain into two completely invariant sets: the Julia set, which is closed, non-empty, perfect, with chaotic behavior (and thus, where the interesting dynamics occur) and the Fatou set, which is open and has regular behavior. Along the years many generalizations to several variables appeared and there is not a uniform consensus on what the Julia set means in this context.

Note that we can naturally partition the dynamical system (P, X) induced by a homogeneous polynomial in two different systems. On the one hand we have $A_P := \bigcup_{n \geq 0} P^{-n}(r_P B_X) = \{x \in X : P^n(x) \rightarrow 0\}$. This set is clearly an open set, P -invariant and is by definition the basin of attraction of zero. On the opposite side we have its complement, namely $A_P^c = \{x : \|P^n(x)\| \geq r_P \forall n \geq 0\}$. Clearly A_P^c is closed and it is also P -invariant. For the aims of this article, we propose the following.

Definition 2.4. *The Julia set associated to the homogeneous polynomial P on a normed space is the set $J_P := \partial A_P$.*

As the following easy result shows, any orbit with interesting dynamics must lie in the Julia set.

Proposition 2.5. *Let P be an m -homogeneous polynomial on a normed space X . If $x \notin J_P$ then either $\lim P^n(x) = \infty$ or $\lim P^n(x) = 0$.*

Proof. Let $x \notin J_P$ and suppose $P^n(x) \not\rightarrow 0$. Then $x \notin A_P$ and consequently there is some $0 < t < 1$ with $tx \notin A_P$. This implies by Proposition 2.1 that, for every n , $\|P^n(tx)\| \geq r_P$ and therefore $\|P^n(x)\| = \frac{1}{t^{m^n}} \|P^n(tx)\| \geq \frac{1}{t^{m^n}} r_P \rightarrow \infty$. \square

Notice that, in particular, periodic vectors and orbits having nonzero accumulation points must belong to J_P .

In the linear operator setting, an irregular vector is a vector that satisfies $\liminf_n \|T^n(x)\| = 0$ while $\limsup_n \|T^n(x)\| = \infty$. This conditions are incompatible for a non-linear homogeneous polynomial on a Banach space. In view of Proposition 2.1, it is natural to give the following definition of irregular vectors for homogeneous polynomials. Note that any irregular vector must belong to J_P .

Definition 2.6. *A vector is irregular for a homogeneous polynomial P if $\liminf_n \|P^n(x)\| = r_P$ and $\limsup_n \|P^n(x)\| = \infty$.*

We may also consider the basin of attraction of infinity. We define R_P as the maximal open subset such that every orbit tends in norm to ∞ , that is, $R_P = \{x \in X : \text{there exist } r > 0 \text{ such that } \|P^n(y)\| \rightarrow \infty \text{ for every } y \in B_r(x)\}$. By definition R_P is open and contained in $\overline{A_P^c}$.

Proposition 2.7. *Let P be an m -homogeneous polynomial defined over a normed space X . Then, X is the disjoint union of A_P , J_P and R_P .*

Proof. It suffices to prove that $(A_P^c)^\circ = R_P$. Since R_P is open $R_P \cap A_P = \emptyset$ we have that $R_P \subseteq (A_P^c)^\circ$. Reciprocally, if $x \in (A_P^c)^\circ$, there exists an $\epsilon > 0$ with $B_\epsilon(x) \cap \overline{A_P} = \emptyset$. Let $y \in B_\epsilon(x)$, by the last proposition either $P^n(y) \rightarrow 0$ or $\|P^n(y)\| \rightarrow \infty$. Since $y \notin A_P$, $P^n(y) \rightarrow \infty$. Therefore $x \in R_P$. \square

Recall that a function $F : Y \rightarrow Y$ is said to be *quasiconjugate* to a function $G : X \rightarrow X$ if there exist a continuous factor $\phi : X \rightarrow Y$, with dense range, such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{G} & X \\ \downarrow \phi & & \downarrow \phi \\ Y & \xrightarrow{F} & Y \end{array}$$

Most of the dynamical properties are preserved under quasiconjugacy.

The dynamical system induced by a linear operator T and λT may be completely different. Indeed, if $\|\lambda T\| < 1$ then every orbit tends to zero while T can support dense orbits. This is not the case if P is a homogeneous polynomial. The same phenomenon occurs for bilinear operators (see [14]).

Proposition 2.8. *Let P be an m -homogeneous polynomial, $m > 1$. Then for every $\lambda \neq 0$, λP is quasiconjugate to P under a linear isomorphism.*

Proof. Let $\lambda^{\frac{1}{m-1}}$ be any $\frac{1}{m-1}$ -root for λ . The following factor works, $\phi(x) = \frac{1}{\lambda^{\frac{1}{m-1}}}x$. Indeed,

$$\lambda P\left(\frac{x}{\lambda^{\frac{1}{m-1}}}\right) = \frac{\lambda}{\lambda^{\frac{m}{m-1}}}P(x) = \frac{1}{\lambda^{\frac{1}{m-1}}}P(x).$$

□

The Julia set is preserved under quasiconjugacy provided that the factor is a linear isomorphism.

Lemma 2.9. *Let Q and P be homogeneous polynomials such that Q is quasiconjugate to P under a linear isomorphism. Then $\phi(A_P) = A_Q$, $\phi(J_P) = J_Q$, $\phi(R_P) = R_Q$.*

Proof. Since ϕ is an isomorphism, it is enough to prove that $\phi(A_P) \subseteq A_Q$ and $\phi(R_P) \subseteq R_Q$. Let $x \in A_P$, then $\|Q^n(\phi(x))\| = \|\phi(P^n(x))\| \leq \|\phi\| \|P^n(x)\| \rightarrow 0$. So that $\phi(x) \in A_Q$. Consider now $x_0 \in R_P$, and let $\epsilon' > 0$ such that $\|P^n(x)\| \rightarrow \infty$ for all $x \in B_{\epsilon'}(x_0)$. Let $\epsilon > 0$ such that $\phi^{-1}(B_\epsilon(\phi(x_0))) \subseteq B_{\epsilon'}(x_0)$. Thus for $y \in B_\epsilon(\phi(x_0))$, $\|Q^n(y)\| = \|Q^n(\phi(\phi^{-1}(y)))\| = \|\phi(P^n(\phi^{-1}(y)))\|$, and thus $\|Q^n(y)\| \rightarrow \infty$ because ϕ is an isomorphism and $\phi^{-1}(y) \in B_{\epsilon'}(x_0)$. □

Corollary 2.10. *Let P be a non linear m -homogeneous polynomial and $\lambda \neq 0$. Then $J_{\lambda P} = \frac{1}{\lambda^{\frac{1}{m-1}}}J_P$.*

Lemma 2.11. *If $x \in A_P$ then $tx \in A_P$ for all $|t| \leq 1$, if $x \in R_P$ then $tx \in R_P$ for all $|t| \geq 1$.*

Proof. Let $x \in A_P$ and $|t| \leq 1$. Then $\|P^n(tx)\| \leq \|P^n(x)\| \rightarrow 0$. Consequently we have that $tx \in A_P$. Suppose now that $x \in R_P$ and $|t| \geq 1$. There exists $\epsilon > 0$ such that $\|P^n(y)\| \rightarrow 0$ for all $y \in B_\epsilon(x)$. Let $y \in B_{\epsilon t}(tx)$, therefore

$$\left\| \frac{y}{t} - x \right\| \leq \epsilon \text{ and } \|P^n(y)\| \geq \|P^n\left(\frac{y}{t}\right)\| \rightarrow \infty.$$

□

The following result shows that the Julia set of a homogeneous polynomial share some of the properties satisfied by the Julia set of a holomorphic function on the complex plane.

Proposition 2.12 (Properties of J_P). *Let P be a non linear homogeneous polynomial. The Julia set J_P satisfies the following properties:*

- i) *it is a closed set with empty interior;*
- ii) *it is P -invariant;*
- iii) *it is perfect;*
- iv) $J_{P^n} = J_P$.

Proof. i) This is clear since J_P is the boundary of an open set.

ii) First note that $y \in A_P$ if and only if $P(y) \in A_P$. Thus, if $x \in J_P$ then $P(x)$ is not in A_P . On the other hand, there exists $(a_n)_n \subseteq A_P$ such that $a_n \rightarrow x$. Since $P(a_n)$ belongs also to A_P and $P(a_n) \rightarrow P(x)$, we have that $P(x) \in J_P$.

iii) Just note that if $x \in J_P$ then $\lambda x \in J_P$ for every $|\lambda| = 1$.

iv) It suffices to show that $A_{P^n} = A_P$. Clearly if $P^k(x) \rightarrow 0$ then $(P^n)^k(x) \rightarrow 0$ and therefore $A_P \subseteq A_{P^n}$. The converse follows thanks to the existence of the limit ball (Proposition 2.1). If $P^{nk}(x) \rightarrow 0$ then $Orb_{P^n}(x)$ meets eventually the limit ball of P and therefore $P^n(x)$ must tend to zero. \square

In contrast to the one dimensional case, the Julia set J_P may be empty. Indeed, if $P \in \mathcal{P}(^2\ell_2; \ell_2)$ is defined as $P(x) = (x_2^2, x_3^2, x_4^2, \dots)$, then $P^n(x) \rightarrow 0$ for every $x \in \ell_2$. Thus $A_P = \ell_2$ and hence $J_P = R_P = \emptyset$.

Recall that a set A is *completely invariant* under P if $P(A) \subseteq A$ and $P^{-1}(A) \subseteq A$. The Julia set of a homogeneous polynomials need not to be completely invariant, as we will show in Example 3.19. On the other hand, under certain conditions on P , J_P results completely invariant.

Proposition 2.13. *Let $P \in \mathcal{P}(^dX; X)$. If P is open or if $R_P = \emptyset$ then J_P is completely invariant.*

Proof. Let $x \in X$ such that $P(x) \in J_P$. Suppose that $x \notin J_P$, so that $x \in A_P$ or $x \in R_P$. If $x \in A_P$ we have that $P^n(x) \rightarrow 0$ and hence $P(x) \in A_P$. If $R_P = \emptyset$ this implies that $x \in J_P$. If $R_P \neq \emptyset$ and $x \in R_P$, there is some $\epsilon > 0$ so that $\|P^n(y)\| \rightarrow \infty$ for every $y \in B_\epsilon(x)$. Since P is open, $P(B_\epsilon(x))$ is an open neighborhood of $P(x)$ and $P^n(z) \rightarrow \infty$ for every $z \in P(B_\epsilon(x))$ and hence $P(x) \in R_P$. \square

Since J_P has empty interior, we can extract an easy but important corollary, which may be seen as an extension to the nonlinear case of the Bourdon-Feldman Theorem [8].

Corollary 2.14. *Let X be a normed space, and $P \in \mathcal{P}(^mX; X)$, $m \geq 2$. Then the following subsets are always nowhere dense: any orbit induced by P , the set of irregular vectors, the set of periodic vectors.*

Proof. The periodic and irregular vectors are contained in the Julia set, which is nowhere dense. In order to be a somewhere dense orbit, the orbit can not tend to infinity or to zero. Therefore it must be contained in the Julia set, which has empty interior. \square

It is natural to ask now how big the Julia sets (or the closure of an orbit) can be. We already know that it cannot contain interior points. In the next subsection we will see some examples of Julia sets, but let us first show that no multiple of the limit sphere can be contained in the Julia set. We will need the following.

Proposition 2.15. *Let P be an m -homogeneous polynomial, and let $z \in J_P$ such that $tz \in J_P$ with $0 < |t| < 1$. Then $P^n(z) \rightarrow \infty$. Consequently if $x \in X$, $y \in \text{Orb}_P(x)$ and $0 < |t| < 1$, then ty is not an accumulation point of $\text{Orb}_P(x)$.*

Proof. If $tz \in J_P$ then $\|P^n(tz)\| \geq r_P$ for every $n \in \mathbb{N}$. Thus,

$$\|P^n(z)\| = \frac{1}{|t|^{m^n}} \|P^n(tz)\| \geq \frac{r_P}{|t|^{m^n}} \rightarrow \infty.$$

For the last assertion, suppose otherwise. It follows that $y \in \text{Orb}_P(x) \subset J_P$. Since J_P is closed, $ty \in J_P$. Thus, $P^n(y) \rightarrow \infty$. This is a contradiction since ty is an accumulation point of $\text{Orb}_P(y)$ \square

We will see in the next section that it is possible that the orbit of a point y accumulates at a multiple ty , for $|t| \geq 1$ (we will give an example that accumulates at every such a multiple).

Corollary 2.16. *Let P be an m -homogeneous polynomial and let $s \neq r_P = \|P\|^{-\frac{1}{m-1}}$. Then $sS_X \not\subset J_P$. In particular, sS_X is not contained in the closure of any orbit of P .*

Proof. For $s < r_P$ the result is a consequence of Proposition 2.1. Suppose that $sS_X \subset J_P$, for some $s > r_P$. By Proposition 2.15, if $\|z\| > s$ then $P^n(z) \rightarrow \infty$. Thus $\{\|z\| > s\} \subset R_P$, and since J_P is invariant, $P(s\overline{B}_X) \subset s\overline{B}_X$.

Since $s^{m-1}\|P\| = \frac{s^{m-1}}{r_P^{m-1}} > 1$, there is some $x_0 \in S_X$ with $\|P\| < s^{m-1}\|P\|\|P(x_0)\|$. Then,

$$\|P(sx_0)\| > \frac{s^m}{s^{m-1}\|P\|} \|P\| = s,$$

which is a contradiction. \square

In the last proposition it was important that $s \neq r_P$, and it is not true for $s = r_P$. Indeed, we have the following easy example: let $X = \ell_\infty^n = (\mathbb{C}^n, \|\cdot\|_\infty)$ and let $P : X \rightarrow X$ be the 2-homogeneous polynomial defined as $P(z_1, \dots, z_n) = (z_1^2, \dots, z_n^2)$. Since $\|P\| = 1$, the limit sphere is the unit sphere and it is easy to see that it coincides with J_P . If $n = 1$ the polynomial is $P(z) = z^2$ defined in \mathbb{C} . In this trivial example $J_P = \mathbb{T}$ and, furthermore $P|_{\mathbb{T}}$ is the doubling map on the circle, which is mixing and therefore has dense orbits. We do not know if there exists a non-trivial example of a Banach space X and a homogeneous polynomial P on X having a dense orbit in its limit sphere. Note that such a polynomial must satisfy

that $r_P S_X = J_P$ is completely invariant (thus for every $x \in X$, $\|P(x)\| = \|P\| \|x\|^m$) and that $P|_{r_P S_X}$ is transitive.

2.3. Some examples of Julia sets. In the following example, the Julia set contains an affine hyper-space.

Example 2.17. Let X be a separable infinite dimensional Banach space. Take $\varphi \in X^*$ and x_0 such that $\varphi(x_0) = 1$. Since $\text{Ker}(\varphi)$ is an infinite dimensional Banach space, there exists a hypercyclic operator $T : \text{Ker}(\varphi) \rightarrow \text{Ker}(\varphi)$. We define $P \in \mathcal{P}({}^m X; X)$ as

$$P(x) = \varphi(x)^m x_0 + \varphi(x)^{m-1} T(x - \varphi(x)x_0).$$

Then $J_P = \mathbb{T}x_0 \oplus \text{Ker}(\varphi)$ and $P|_{J_P}$ is transitive.

Proof. We start by studying A_P and R_P . Let $x = \lambda x_0 + y$ with $|\lambda| < 1$ and $y \in \text{Ker}(\varphi)$. Since

$$(1) \quad P^n(x) = \lambda^{m^n} x_0 + \lambda^{m^n-1} T^n(y)$$

we have that $\|P^n(x)\| \leq |\lambda|^{m^n} \|x_0\| + |\lambda|^{m^n-1} \|T^n(y)\| \rightarrow 0$. Thus $A_P \subseteq \{x : |\varphi(x)| < 1\}$.

Consider in X the following equivalent norm $\|x\|_\infty := \max\{|\varphi(x)| \|x_0\|, \|x - \varphi(x)x_0\|\}$. By Lemma 2.9, R_P , A_P and J_P are invariant under equivalent norms. The set $\{x : |\varphi(x)| > 1\}$ is open and, by (1), if $|\varphi(x)| > 1$, then $\|P^n(x)\|_\infty \rightarrow \infty$. Therefore $\{x : |\varphi(x)| > 1\} \subseteq R_P$, and $\{x : \varphi(x) \in \mathbb{T}\} = \partial A_P = J_P$.

Finally, let us see that $P|_{\mathbb{T}x_0 \oplus \text{Ker}(\varphi)}$ is transitive. Recall that for every open set $W \subseteq \mathbb{T}$, there exists n_0 such that $\mathbb{T} \subseteq W^{m^n}$ for every $n \geq n_0$. Let U and V be open sets in $\mathbb{T}x_0 + \text{Ker}(\varphi)$. By considering the norm $\|\cdot\|_\infty$ in X , we can suppose that $U = (U_1, U_2)$, $V = (V_1, V_2)$ with $U_1, V_1 \subseteq \mathbb{T}x_0$, $U_2, V_2 \subseteq \text{Ker}(\varphi)$, all of them being open sets. Let $n_0 \in \mathbb{N}$ be with $\mathbb{T} \subseteq U_1^{m^n-1}$ for every $n \geq n_0$. Since T is transitive, there is $n_1 > n_0 \in \mathbb{N}$ with $V_2 \cap T^{n_1}(U_2) \neq \emptyset$.

Note that, by (1),

$$P^{n_1}(U_1 \times U_2) = \{\lambda^{m^{n_1}} x_0 + \lambda^{m^{n_1}-1} T^{n_1}(y) : \lambda x_0 \in U_1, y \in U_2\} = \mathbb{T}x_0 \times \mathbb{T}T^{n_1}(U_2).$$

Therefore, $P^{n_1}(U_1 \times U_2) \cap V_1 \times V_2 \neq \emptyset$. □

Bernardes proved in [4, Theorem 2] that if X is an infinite dimensional separable Banach space then there exist a supercyclic homogeneous polynomial acting on X . Notice that the fact that $P|_{\mathbb{T}x_0 \oplus \text{Ker}(\varphi)}$ is transitive implies that in particular P is supercyclic (and also \mathbb{R}_+ -supercyclic). Moreover, such a polynomial may be also constructed in any separable infinite dimensional Fréchet space. Thus, we have given a simple proof and an extension of [4, Theorem 2].

Proposition 2.18. *For any infinite dimensional separable Fréchet space X , and every natural $m \geq 2$ there exists a homogeneous polynomial $P \in \mathcal{P}({}^m X; X)$ which is \mathbb{R}_+ -supercyclic.*

Remark 2.19. The study of the dynamics of homogeneous polynomials is closely related to the study of the iterations of holomorphic and meromorphic mappings on projective spaces. Each meromorphic mapping on \mathbb{P}^k has a lifting to \mathbb{C}^{k+1} which is a homogeneous polynomial. Moreover, a homogeneous polynomial on \mathbb{C}^{k+1} is supercyclic if and only if its associated mapping on the projective space has an orbit which is dense in \mathbb{P}^k . The first example of such a mapping was constructed by Lattes on \mathbb{P}^1 and this can be also done in \mathbb{P}^k , see for example [12]. Thus, we also know the existence of supercyclic homogeneous polynomials on finite dimensional spaces.

Bernardes in [4, Proposition 5] gave an example of homogeneous polynomial P with an irregular vector x_0 . In his example, $Orb_P(x_0)$ does not have accumulation points and moreover, the sequence $(\|P^n(x_0)\|)_n$ is an unbounded sequence of positive numbers whose only accumulation point is r_P . Next, we present an example of a norm one homogeneous polynomial (thus $r_P = 1$), with an irregular vector x such that $\overline{Orb_P(x)}$ is an affine hyperplane touching the sphere of radius $r_P = 1$, and thus $\overline{\{\|P^n(x)\| : n \in \mathbb{N}\}} = [r_P, \infty)$.

Example 2.20. Let P be the m -homogeneous polynomial defined in the Example 2.17. We set $m = 2$, $X = \ell_1$, $x_0 = e_1$, $\varphi = e'_1$, and $T : Ker(\varphi) \equiv \ell_1(\mathbb{N}_{\geq 2}) \rightarrow \ell_1(\mathbb{N}_{\geq 2})$ to be the hypercyclic operator $(1 + \epsilon)B$, where B is the backward shift on $\ell_1(\mathbb{N}_{\geq 2})$ and $0 < \epsilon < 1$.

Then $\|P\| = 1$ and $J_P = \mathbb{T}e_1 + Ker(e'_1)$.

Since $P|_{J_P}$ is transitive, there exists an orbit whose closure contains an affine hyperspace that meets the limit ball. In particular there is an irregular vector x such that $\overline{\{\|P^n(x)\| : n \in \mathbb{N}\}}$ is $[1, \infty) = [r_P, \infty)$.

Proof. By the previous example, it suffices to prove that $\|P\| = 1$.

Let x be in S_{ℓ_1} and suppose that $x = \lambda e_1 + \gamma$, where $\gamma \in Ker(e'_1)$ and $|\lambda| + \|\gamma\|_1 = 1$. Therefore

$$\|P(x)\|_1 = |\lambda^2| + |\lambda| \|(1 + \epsilon)B((\gamma_n))\|_1 \leq |\lambda|^2 + (1 + \epsilon)|\lambda| \|((\gamma_n)_n)\|_1,$$

and $\lambda + \|((\gamma_n)_n)\|_1 = 1$.

Thus, $\|P\|$ is less than or equal to the maximum of the function $f(s, t) = s^2 + (1 + \epsilon)st$ restricted to $s + t = 1, s \geq 0$ and $t \geq 0$. Since $0 < \epsilon < 1$, this maximum is exactly 1.

On the other hand, $P(e_1) = e_1$, and therefore $\|P\| = 1$. □

It is also possible to construct an analogous m -homogeneous polynomial on $X = \ell_p$, $1 < p < \infty$, provided $m > (1 + \epsilon)^p$.

In [21] it was shown that the polynomial defined as the backward shift to the power of m , acting on the (non-normable) Fréchet space $\mathbb{C}^{\mathbb{N}}$ is chaotic (see also [20]). On ℓ_p ($p < \infty$) or c_0 the dynamic of this polynomial is trivial since every orbit converges to 0, i.e. A_P is the whole space and the Julia set is empty. The next example analyzes this polynomial in the space of convergent sequences.

Example 2.21. Take $X = c$ the space of convergent sequences and consider the m -homogeneous polynomial defined as the backward shift to the power of m , that is, $P(x)_j = x_{j+1}^m$. Then $A_P = \{(a_j)_j : |\lim_j a_j| < 1\}$, $J_P = \{(a_j)_j : |\lim_j a_j| = 1\}$, $R_P = \{(a_j)_j : |\lim_j a_j| > 1\}$ and $P|_{J_P}$ is transitive.

Proof. Observe first that $\|P\| = 1 = r_P$.

Let $(a_n)_n \in c$ with $L = |\lim a_n| < 1$. Let $n_0 \in \mathbb{N}$ such that $|a_n| < \delta < 1$ for every $n \geq n_0$. Therefore $\|P^n((a_n)_n)\|_\infty \leq \delta^{m^n} \rightarrow 0$. This implies that $\{(a_j)_j : |\lim_j a_j| < 1\} \subseteq A_P$. Similarly $R_P \subseteq \{(a_j)_j : |\lim a_j| > 1\}$. Finally since $\{(a_j)_j : |\lim a_j| = 1\} \subseteq \partial A_P$ it follows that $J_P = \{(a_j)_j : |\lim a_j| = 1\}$.

Let us now show that $P|_{J_P}$ is transitive. Notice that $A := \{a + e^{i\theta}\mathbb{1} : a \in c_{00}, \theta \in [0, 2\pi]\}$ is dense in J_P , where $\mathbb{1}$ is the vector with $[\mathbb{1}]_i = 1$ for all i .

Let U, V be open sets and $v + e^{i\theta_2}\mathbb{1} \in V \cap A$. Let $\epsilon > 0$ with $B_\epsilon(v + e^{i\theta_2}) \subseteq V$. Since the map $\Phi : \mathbb{T} \rightarrow \mathbb{T}$, $\Phi(e^{i\theta}) = e^{mi\theta}$ induces a mixing dynamical system, there is $u + e^{i\theta_1}\mathbb{1} \in U \cap A$ such that the Φ -orbit of $e^{i\theta_1}$ is dense in \mathbb{T} . Suppose that $u = \sum_{k=1}^N u_k e_k$ and $v = \sum_{k=1}^N v_k e_k$.

In the following, denote by $a^{\frac{1}{n}}$, for each $0 \neq a \in \mathbb{C}$ and for each n , a $\frac{1}{n}$ -root of a so that $a^{\frac{1}{n}} \rightarrow 1$, as $n \rightarrow \infty$. Thus, there exists $n_0 > N$ such that for any $n \geq n_0$, the vector x_n belongs to U , where

$$x_n := \sum_{k=1}^N (u_k + e^{i\theta_1})e_k + \sum_{k=N+1}^n e^{i\theta_1}e_k + \sum_{k=n+1}^{N+n} (v_{k-n}e^{-i\theta_2} + 1)^{\frac{1}{m^n}} e^{i\theta_1}e_k + \sum_{k=N+n+1}^{\infty} e^{i\theta_1}e_k.$$

Therefore

$$\begin{aligned} \|P^n(x_n) - v - e^{i\theta_2}\mathbb{1}\|_\infty &= \left\| \sum_{k=1}^N (v_k e^{-i\theta_2} + 1) e^{i\theta_1 m^n} e_k + \sum_{k=N+1}^{\infty} e^{i\theta_1 m^n} e_k - v - e^{i\theta_2}\mathbb{1} \right\|_\infty \\ &= \left\| \sum_{k=1}^N v_k (e^{-i\theta_2} \Phi^n(e^{i\theta_1}) - 1) e_k + \sum_{k=1}^{\infty} (\Phi^n(e^{i\theta_1}) - e^{i\theta_2}) e_k \right\|_\infty \\ &\leq (\|v\|_\infty + 1) |\Phi^n(e^{i\theta_1}) - e^{i\theta_2}|. \end{aligned}$$

Thus, by the election of θ_1 , we can take $n \geq n_0$ such that $|\Phi^n(e^{i\theta_1}) - e^{i\theta_2}|$ is arbitrarily small, so that $P^n(x_n)$ belongs to V . \square

3. d -HYPERCYCLIC, WEAKLY HYPERCYCLIC AND Γ -SUPERCYCLIC HOMOGENEOUS POLYNOMIALS

No homogeneous polynomial on a Banach space can be hypercyclic, so it is natural to ask for the existence of weaker notions of hypercyclicity. In this section we will provide a simple and natural homogeneous polynomial that is at the same time d -hypercyclic, weakly hypercyclic and Γ -supercyclic for every unbounded or not bounded away from zero $\Gamma \subseteq \mathbb{C}$. Let us recall some basic definitions.

Let X be a Fréchet space and $(\rho_n)_n$ a fundamental system of seminorms. We will say that a set $A \subseteq X$ is d -dense for a sequence of positive numbers $d = (d_n)_n$ provided that for every $x \in X$ and every n , there exists $a \in A$ for which $\rho_n(x - a) < d_n$. We will say that a function $P : X \rightarrow X$ is d -hypercyclic if

there exist x such that $Orb_P(x)$ is d -dense in X . In this case, we will say that x is a d -hypercyclic vector for P .

This phenomenon was studied by N. Feldman in [13] in the context of linear operators on Banach spaces. There he proved that an operator on a Banach space is hypercyclic if and only if it is d -hypercyclic. With a similar proof it can be proven that, if X is a Fréchet space, then both notions are still equivalent for linear operators.

A function $P : X \rightarrow X$ is *weakly hypercyclic* if there exists $x \in X$ such that $Orb_P(x)$ is dense with respect to the weak topology of the space. Since the norm topology is stronger than the weak topology, it is clear that a hypercyclic function is automatically weakly hypercyclic while the converse is not necessarily true. In [10] the authors found the first weakly hypercyclic linear operator that is not hypercyclic.

We will also consider the weaker notion for hypercyclicity called Γ -supercyclicity. If $\Gamma \subseteq \mathbb{C}$, then we will say that a function $P : X \rightarrow X$ is Γ -supercyclic if there exists $x \in X$ (called Γ -supercyclic vector) such that $\Gamma \cdot Orb_P(x)$ is dense in the space. In the special case that $\Gamma = \mathbb{C}$ or Γ is a singleton we recover the notions of supercyclicity and hypercyclicity respectively. An important result concerning Γ -supercyclicity, for $\Gamma = \mathbb{T}$, was due to León Saavedra and Müller [18], where they proved that a linear operator is \mathbb{T} -supercyclic if and only if it is hypercyclic. The case when $\Gamma = \mathbb{R}_+$ is also called positive supercyclicity (see [22]). The concept of Γ -supercyclicity was recently introduced by Charpentier, Ernst and Menet in [11]. They proved that the subsets $\Gamma \subseteq \mathbb{C}$ such that Γ -supercyclicity is equivalent to hypercyclicity are exactly the subsets Γ which are bounded and such that $\Gamma \setminus \{0\}$ is bounded away from zero.

Every infinite dimensional separable Banach space supports supercyclic homogeneous polynomial (see [4] or Proposition 2.18), but no homogeneous polynomial can be hypercyclic. This means that there are \mathbb{C} -supercyclic homogeneous polynomials while no polynomial is $\{*\}$ -supercyclic. So it is natural to ask for which subsets $\Gamma \subseteq \mathbb{C}$ it is possible to have Γ -supercyclic homogeneous polynomials on Banach spaces.

3.1. Remarks on d -hypercyclicity. Notice that if x is a d -hypercyclic vector, then $x \in J_P$. Indeed, otherwise $P^n(x) \rightarrow \infty$ or $P^n(x) \rightarrow 0$ contradicting the d -density of $Orb_P(x)$.

If $R_P \neq \emptyset$, then it contains balls of arbitrary large radius, because if $B_\epsilon(y) \subseteq R_P$ then $B_{|t|\epsilon}(ty) \subseteq R_P$ for all $|t| > 1$. Thus we can find balls with arbitrary large radius not meeting J_P . Since a d -hypercyclic vector must belong to J_P , it follows that the polynomial is not d -hypercyclic, and then we have the following.

Remark 3.1. If $R_P \neq \emptyset$ then P is not d -hypercyclic.

Proposition 3.2. *Let X, Y be Fréchet spaces with fundamental system of seminorms $(q_n)_n$ and $(p_n)_n$ respectively. Let $F : X \rightarrow X$ and $G : Y \rightarrow Y$ be continuous functions such that G is a quasiconjugacy of F via a linear operator $\Phi : X \rightarrow Y$. If F is d -hypercyclic then G is \tilde{d} -hypercyclic for some $(\tilde{d}_n)_n$. In the*

Banach case we have that G is $(d\|\Phi\| + \epsilon)$ -hypercyclic for every $\epsilon > 0$.

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \downarrow \Phi & & \downarrow \Phi \\ Y & \xrightarrow{G} & Y \end{array}$$

Proof. Let x be a d -hypercyclic vector for F and $\epsilon > 0$. Let M_n and q_{k_n} such that $p_n(\Phi(x)) \leq M_n q_{k_n}(x)$ for every $x \in X$.

Let $y \in Y$ and take $x_0 \in X$ such that $p_n(\Phi(x_0) - y) < \epsilon$. Since x is a d -hypercyclic vector, there exists $l > 0$ such that $q_{k_n}(F^l(x) - x_0) < d_{k_n}$. Therefore

$$\begin{aligned} p_n(G^l \Phi(x) - y) &= p_n(\Phi F^l(x) - y) \leq p_n(\Phi F^l(x) - \Phi(x_0)) + p_n(\Phi(x_0) - y) \\ &\leq M_n d_{k_n} + \epsilon := \tilde{d}_n. \end{aligned}$$

□

3.2. Remarks on weak hypercyclicity. The orbits lying on R_P tend to infinity fast. This forces the orbit to be weakly closed and hence orbits on R_P are far from being weakly dense.

Proposition 3.3. *Let $x \notin J_P$. Then $Orb_P(x)$ is w -closed.*

For the proof we will need the following result, which may be found in [3, Proposition 10.1].

Proposition 3.4. *If there are some constants $k > 0$ and $C > 1$ such that $\|x_n\| \geq kC^n$, then $\{x_n : n \in \mathbb{N}\}$ is weakly closed.*

Proof of Proposition 3.3. If $x \in A_P$, $P^n(x) \rightarrow 0$ and therefore $\{P^n(x) : n \in \mathbb{N}\}$ is weakly closed. If $x \in R_P$, there is some $t > 1$ such that $\frac{x}{t} \in R_P$. Then $\|P^n(x)\| = t^{m^n} \|P^n(\frac{x}{t})\| \geq t^{m^n} r_P$. By the above proposition $Orb_P(x)$ is weakly closed. □

Corollary 3.5. *Let x be a weakly hypercyclic vector for a homogeneous polynomial. Then $x \in J_P$.*

It is known that homogeneous polynomials are not necessarily weak to weak continuous and therefore $(P, (X, \omega))$ is not a truly dynamical system. Nevertheless the property of being weakly hypercyclic is preserved under quasiconjugacy provided that the factor Φ is weak to weak continuous and Φ has weakly dense range.

Proposition 3.6. *Let X, Y be Fréchet spaces, $P \in \mathcal{P}(^m X; X)$, $Q \in \mathcal{P}(^m Y; Y)$ and $\Phi : X \rightarrow Y$ a weak to weak continuous map with weakly dense range such that the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{P} & X \\ \downarrow \Phi & & \downarrow \Phi \\ Y & \xrightarrow{Q} & Y \end{array}$$

If P is weakly hypercyclic then so it is Q .

Proof. Let $x \in X$ be a weakly hypercyclic vector for P . Since $\overline{Orb_P(x)}^\omega = X$ and Φ is weak to weak continuous with weakly dense range, we have that $Y = \overline{\Phi(Orb_P(x))}^\omega = \overline{Orb_Q(\Phi(x))}^\omega$. \square

3.3. Remarks and examples on Γ -supercyclicity. Recall that the set of $\Gamma \subseteq \mathbb{C}$ such that a Γ -supercyclic operator is automatically a hypercyclic operator are exactly the sets Γ such that Γ is bounded and bounded away from zero. In the same spirit, if Γ is bounded and bounded away from zero, then no homogeneous polynomial is Γ -supercyclic.

Proposition 3.7. *Let P be a Γ -supercyclic homogeneous polynomial and x a Γ -supercyclic vector. If Γ is bounded then $x \notin A_P$ and if $\Gamma \setminus \{0\}$ is bounded away from zero then $x \in A_P$. In particular no homogeneous polynomial is Γ -supercyclic if Γ is bounded and $\Gamma \setminus \{0\}$ is bounded away from zero.*

Proof. Suppose that $x \in A_P$ then $Orb_P(x)$ is bounded. Hence $\Gamma \cdot Orb_P(x)$ is bounded if Γ is bounded. Thus x is not a Γ -supercyclic vector.

Suppose that $x \notin A_P$ and that $\Gamma \setminus \{0\}$ is bounded away from zero. By Proposition 2.1, $\|P^n(x)\| \geq r_P$ for every n . Therefore $\Gamma \cdot Orb_P(x) \setminus \{0\}$ is bounded away from zero and again x cannot be a Γ -supercyclic vector. \square

A Γ -supercyclic vector is not necessarily in J_P . Indeed, in Examples 3.13, 3.15, and 3.14 we show Γ -supercyclic vectors belonging to A_P , J_P and R_P , respectively.

If the factor Φ is linear, then Γ -supercyclicity is preserved under quasiconjugacy.

Proposition 3.8. *Let P, Q be homogeneous polynomials such that Q is quasiconjugated to P under a linear factor Φ . If P is Γ -supercyclic, then Q is also Γ -supercyclic.*

Proof. If $\Gamma \cdot Orb_P(x)$ is dense then $\Gamma \cdot Orb_Q(\Phi(x)) = \Gamma \cdot \Phi(Orb_P(x)) = \Phi(\Gamma \cdot Orb_P(x))$ is also dense because Φ has dense range. \square

As in the linear case we can define the Γ -transitivity notion. For some examples it will be an useful tool to prove Γ -supercyclicity.

Definition 3.9. *Let X be a separable Fréchet space. We say that a polynomial $P \in \mathcal{P}({}^m X; X)$ is Γ -transitive, if for every open sets U and V , there exist $n \in \mathbb{N}$ and $\lambda \in \Gamma$ with*

$$\lambda P^n(U) \cap V \neq \emptyset.$$

It is easy to prove that Γ -transitivity implies Γ -supercyclicity. Moreover, in this case, the set of Γ -supercyclic vectors is residual. However the converse is false. Indeed, if X is Banach and Γ is bounded then no homogeneous polynomial is Γ -transitive.

Proposition 3.10. *Let X be a separable Fréchet space and $\Gamma \subseteq X$. If $P \in \mathcal{P}(^m X; X)$ is Γ -transitive then P is Γ -supercyclic.*

Proof. Let $\{V_i\}_{i \in \mathbb{N}}$ be a basis for X . For each $\gamma \in \Gamma$ and each k choose $\gamma^{\frac{1}{m^k}}$ to be a m^k -root of γ . Consider for all i ,

$$G_i := \bigcup_{k, \gamma \in \Gamma} \gamma^{\frac{1}{m^k}} P^{-k}(V_i).$$

Since P is Γ -transitive, G_i is a dense open set for each i . Therefore, $G = \bigcap_{i \in \mathbb{N}} G_i$ is non empty. Finally we notice that any point in G is a Γ -supercyclic vector. \square

We propose the following criterion to study Γ -transitivity.

Definition 3.11 (Γ -transitivity criterion). *We say that P satisfies the supercyclicity criterion for (λ_k) provided that there are dense sets X_0 and Y_0 and applications $S_k(x) : Y_0 \rightarrow X$ such that for all $x \in X_0$, $y \in Y_0$,*

- i) $S_k(x)(y) \rightarrow 0$ and*
- ii) $\lambda_k P^{n_k}(x + S_k(x)(y)) \rightarrow y$.*

Proposition 3.12. *Let $(X, (\rho_n)_n)$ be a separable Fréchet space, where $(\rho_n)_n$ is a fundamental system of nondecreasing seminorms, and let P be a homogeneous polynomial. Then P satisfies the Γ -transitivity criterion with respect to some $(\lambda_k) \subseteq \Gamma$ if and only if P is Γ -transitive.*

Proof. Suppose that P satisfies the Γ -transitivity criterion with respect to (λ_k) . Let U, V be open sets. Let $x \in U \cap X_0$ and $y \in V \cap Y_0$. Since $S_k(x)(y) \rightarrow 0$ there is some k with $x + S_k(x)(y) \in U$ for large k . Since $\lambda_k P^{n_k}(x + S_k(x)(y)) \rightarrow y$ there is some k such that $P^{n_k}(x + S_k(x)(y)) \in V$.

Suppose now that P is Γ -transitive. Let $(x_i)_{i \in \mathbb{N}}$ be a dense numerable sequence in X . Let $X_0 = Y_0 = \{x_i : i \in \mathbb{N}\}$. Fix $x, y \in X_0$. For each $k \in \mathbb{N}$ there is some $z_k \in \{w : \rho_k(x - w) < \frac{1}{k}\}$ such that for some $n_k > n_{k-1}$ and $\lambda_k \in \Gamma$ we have $\lambda_k P^{n_k}(z_k) \in \{w : \rho_k(y - w) < \frac{1}{k}\}$. We define $S_k(x)(y) = z_k - x$. Since $z_k \rightarrow x$, $S_k(x)(y) \rightarrow 0$ while $\lambda_k P^{n_k}(x + S_k(x)(y)) \rightarrow y$. \square

Example 3.13. Let X be c_0 and $P : c_0 \rightarrow c_0$ as $P(a)_j := a_{j+1}^m$. The polynomial P is Γ -supercyclic for every unbounded $\Gamma \subseteq \mathbb{C}$.

Proof. We will apply Criterion 3.11. Let $X_0 = Y_0 = c_{00}$ and let $x, y \in c_{00}$. We now define $S_k(x)(y)$. For $\lambda \in \mathbb{C}$ let $\lambda^{\frac{1}{m}}$ be any m -root of λ . Let $F : c_{00} \rightarrow c_{00}$ be defined as

$$F(a)_j := \begin{cases} \lambda^{\frac{1}{m}} a_{j-1} & \text{if } j \neq 1 \\ 0 & \text{if } j = 1, \end{cases}$$

and set $S_k(x)(y) = \frac{F^k(y)}{\lambda_k^{\frac{1}{m^k}}}$, where $\lambda_k \in \Gamma$ is chosen so that $\lambda_k^{\frac{1}{m^k}}$ tends to infinity.

Since $\|F(y)\| \leq \max\{1, \|y\|\}$ we get $S_k(x)(y) \rightarrow 0$. On the other hand, if $k > \max\{j : x_j \neq 0\}$ it follows that

$$\lambda_k P^k(x + S_k(x)(y)) = \lambda_k P^k S_k(x)(y) = P^k F^k(y) = y.$$

By Proposition 3.12, P is Γ -transitive and hence Γ -supercyclic. \square

The above polynomial cannot provide an example of a Γ -supercyclic homogeneous polynomial when Γ is a bounded set. Indeed, if Γ is bounded, then by Proposition 3.7 the Γ -supercyclic vector is not in A_P , but in this case $A_P = c_0$. The same happens if we consider this polynomial on ℓ_p , $1 \leq p < \infty$. However, in the space c of convergent sequences the situation is different.

Example 3.14. Let $X = c$ and $P : c \rightarrow c$ defined as $P(a)_j = (a_{j+1})^m$. Then P is D-supercyclic.

Proof. For $A \subseteq \mathbb{N}$ we denote by $\mathbb{1}_A$ the characteristic vector of A , i.e.,

$$\mathbb{1}_A(j) = \begin{cases} 1 & \text{if } j \in A; \\ 0 & \text{else.} \end{cases}$$

For each $\lambda = re^{i\theta} \in \mathbb{C}$ let $\lambda^{\frac{1}{n}}$ denote the number $r^{\frac{1}{n}} e^{i\theta/n}$.

We define $F : c \rightarrow c$ as

$$F(a)_j := \begin{cases} a_{j-1}^{\frac{1}{m}} & \text{if } j > 1; \\ 0 & \text{if } j = 1. \end{cases}$$

Clearly F is a well defined map and $P^k F^k = Id$, for every $k \in \mathbb{N}$.

Let $(\tilde{y}_n)_n$ be a dense sequence in c such that $(\tilde{y}_n)_n = (y_n)_n + (l_n \mathbb{1}_{\text{supp}(y_n)^c})_n$, where $(y_n)_n$ is a dense sequence in c_{00} , and $(l_n)_n$ is a sequence in \mathbb{C} . Without loss of generality, we may suppose that, for all n , $l_n \neq 0$ and $\text{supp}(y_n) = [1, m(y_n)]$, where $m(y_n) := \max\{j : [y_n]_j \neq 0\}$.

We can construct, by induction, a sequence $(n_k)_k \subseteq \mathbb{N}$ satisfying the following properties:

i) $n_k > m(y_{k-1}) + n_{k-1}$,

ii) for all $j \in n_k + \text{supp}(y_k)$, $\left| \frac{[F^{n_k}(y_k)]_j}{l_k^{m^{n_k}}} - 1 \right| < \frac{1}{k}$,

iii) for every $k > 0$, $\frac{l_k}{2^{m^{n_k}}} < 1$ and

iv) for all $j < k$, and all $i \in (n_k - n_j) + \text{supp}(y_k)$, $\left| l_j \left(\frac{[F^{n_k - n_j}(y_k)]_i}{l_k^{m^{n_j}}} - 1 \right) \right| < \frac{1}{k}$.

Note that it is possible to choose such a sequence $(n_k)_k$ because all four conditions will be satisfied once n_k is sufficiently large.

Let $\tilde{x} \in \ell_\infty$ be the vector defined as

$$\tilde{x} = \sum_{j=1}^{\infty} \frac{2F^{n_j}(y_j)}{l_j^{\frac{1}{m^{n_j}}}}.$$

The vector $\tilde{x} \notin c$ since there are gaps with zeros. By condition ii) the nonzero coordinates of \tilde{x} tend to 2. Therefore we fill the gaps to obtain a well defined vector in c , we define

$$x = \tilde{x} + 2\mathbb{1}_{\text{supp}(\tilde{x})^c}.$$

By condition i), the supports of the $F^{n_j}(y_j)$ are pairwise disjoint. Consequently,

$$(2) \quad P^k(\tilde{x}) = \sum_{j=1}^{\infty} P^k \left(\frac{2F^{n_j}(y_j)}{l_j^{\frac{1}{m^{n_j}}}} \right).$$

Also, by condition i), it follows that if $k > j$, then

$$(3) \quad P^{n_k}(F^{n_j}(y_j)) = 0.$$

We claim that x is a \mathbb{D} -supercyclic vector. Indeed, let k be in \mathbb{N} . By condition iii) each $\frac{l_k}{2^{m^{n_k}}} \in \mathbb{D}$ and thus,

$$(4) \quad \begin{aligned} \left\| \frac{l_k}{2^{m^{n_k}}} P^{n_k}(x) - \tilde{y}_k \right\| &= \left\| \frac{l_k}{2^{m^{n_k}}} P^{n_k}(\tilde{x} + 2\mathbb{1}_{\text{supp}(\tilde{x})^c}) - y_k - l_k \mathbb{1}_{\text{supp}(y_k)^c} \right\| \\ &= \left\| \frac{l_k}{2^{m^{n_k}}} P^{n_k}(\tilde{x}) + \frac{l_k}{2^{m^{n_k}}} 2^{m^{n_k}} P^{n_k}(\mathbb{1}_{\text{supp}(\tilde{x})^c}) - y_k - l_k \mathbb{1}_{\text{supp}(y_k)^c} \right\|. \end{aligned}$$

Let us observe that $\mathbb{N} = \text{supp}(P^{n_k}(\tilde{x} + \mathbb{1}_{\text{supp}(\tilde{x})^c})) = \text{supp}(P^{n_k}(\tilde{x})) \cup \text{supp}(P^{n_k}(\mathbb{1}_{\text{supp}(\tilde{x})^c}))$. and that $P^{n_k}(\mathbb{1}_{\text{supp}(\tilde{x})^c}) = \mathbb{1}_{\text{supp}(P^{n_k}(\tilde{x}))^c}$. Moreover, since $\text{supp}(y_k) \subseteq \text{supp}(P^{n_k}(\tilde{x}))$, we have also that

$$\mathbb{1}_{\text{supp}(P^{n_k}(\tilde{x})^c)} - \mathbb{1}_{\text{supp}(y_k)^c} = -\mathbb{1}_{(\text{supp}(P^{n_k}(\tilde{x})) \setminus \text{supp}(y_k))}.$$

Therefore, by equality (2), the above remarks and equation (4),

$$\begin{aligned}
 \left\| \frac{l_k}{2m^{n_k}} P^{n_k}(x) - \tilde{y}_k \right\| &= \left\| \frac{l_k}{2m^{n_k}} P^{n_k}(\tilde{x}) + l_k \mathbb{1}_{\text{supp}(P^{n_k}\tilde{x})^c} - y_k - l_k \mathbb{1}_{\text{supp}(y_k)^c} \right\| \\
 &= \left\| \frac{l_k}{2m^{n_k}} P^{n_k}(\tilde{x}) - y_k - l_k \mathbb{1}_{(\text{supp}(P^{n_k}(\tilde{x})) \setminus \text{supp}(y_k))} \right\| \\
 &= \left\| \frac{l_k}{2m^{n_k}} P^{n_k} \left(\sum_{j=1}^{\infty} \frac{2F^{n_j}(y_j)}{l_j^{\frac{1}{m^{n_j}}}} \right) - y_k - l_k \mathbb{1}_{(\text{supp}(P^{n_k}(\tilde{x})) \setminus \text{supp}(y_k))} \right\| \\
 &= \left\| \sum_{j=1}^{\infty} \frac{l_k}{2m^{n_k}} P^{n_k} \left(\frac{2F^{n_j}(y_j)}{l_j^{\frac{1}{m^{n_j}}}} \right) - y_k - l_k \mathbb{1}_{(\text{supp}(P^{n_k}(\tilde{x})) \setminus \text{supp}(y_k))} \right\| \\
 &= \left\| \sum_{j=1}^{\infty} \frac{l_k}{l_j^{\frac{m^{n_k}}{m^{n_j}}}} P^{n_k}(F^{n_j}(y_j)) - y_k - l_k \mathbb{1}_{(\text{supp}(P^{n_k}(\tilde{x})) \setminus \text{supp}(y_k))} \right\| \\
 &\leq \left\| \sum_{j < k} \frac{l_k}{l_j^{\frac{m^{n_k}}{m^{n_j}}}} P^{n_k} F^{n_j}(y_j) \right\| + \left\| \frac{l_k}{l_k^{\frac{m^{n_k}}{m^{n_k}}}} y_k - y_k \right\| \\
 &+ \left\| \sum_{j > k} \frac{l_k}{l_j^{\frac{m^{n_k}}{m^{n_j}}}} (F^{n_j - n_k}(y_j)) - l_k \mathbb{1}_{(\text{supp}(P^{n_k}(\tilde{x})) \setminus \text{supp}(y_k))} \right\|.
 \end{aligned}$$

By equality (3) the first term of the last expression is zero. Notice also that $\text{supp} \left(\sum_{j > k} F^{n_j - n_k}(y_j) \right) = \text{supp}(P^{n_k}(\tilde{x})) - \text{supp}(y_k)$. Therefore, by property iv),

$$\begin{aligned}
 \left\| \sum_{j > k} \frac{l_k}{l_j^{\frac{m^{n_k}}{m^{n_j}}}} (F^{n_j - n_k}(y_j)) - l_k \mathbb{1}_{(\text{supp}(P^{n_k}(\tilde{x})) \setminus \text{supp}(y_k))} \right\| &= \sup_{j > k} \sup_{i \in n_j - n_k + \text{supp}(y_j)} \left\| l_k \left(\frac{[F^{n_j - n_k}(y_j)]_i}{l_j^{\frac{m^{n_k}}{m^{n_j}}}} - 1 \right) \right\| \\
 &\leq \sup_{j > k} \frac{1}{j} = \frac{1}{k+1} \rightarrow 0.
 \end{aligned}$$

This implies that x is a \mathbb{D} -supercyclic vector. \square

In Example 2.21 we proved that the Julia set of the above polynomial is $J_P = \{x \in c : \lim x_j \in \mathbb{T}\}$. Since for each $\lambda \in \mathbb{D}$ and each $x \in J_P$, $\lim \lambda x_j \in \mathbb{D}$, it follows that if x is a \mathbb{D} -supercyclic vector then $x \in R_P$.

3.4. Main example. We proceed with the main example of our work. We prove in this subsection that the 2-homogeneous polynomial $P : \ell_p \rightarrow \ell_p$ defined as $P = e'_1 \cdot B$, where B is the backward shift operator, is weakly hypercyclic, d -hypercyclic for every $d > r_P$ and Γ -supercyclic for every $\Gamma \subset \mathbb{C}$ such that Γ is unbounded or not bounded away from zero. The proof is based on properties of its Julia set J_P .

Theorem 3.15. *Let $X = c_0$ or $X = \ell_p$, $1 \leq p < \infty$, and let $P : X \rightarrow X$ defined as*

$$P(x) = x_1 B(x),$$

where $B : X \rightarrow X$ is the backward shift. Then $P|_{J_P}$ is chaotic and, if $x_0 \in J_P$ is any vector such that $\text{Orb}_P(x_0)$ is dense in J_P then

- (1) x_0 is a weakly hypercyclic vector,
- (2) x_0 is a d -hypercyclic vector, for every $d > r_P$,
- (3) x_0 is a Γ -supercyclic vector, for every $\Gamma \subset \mathbb{C}$ such that 0 is an accumulation point of Γ .

Moreover, P is also Γ -supercyclic for every unbounded set $\Gamma \subset \mathbb{C}$.

We will prove the theorem for $X = \ell_p$, the proof for $X = c_0$ is similar. We will divide the proof in several steps. We will prove first that $P|_{J_P}$ is chaotic and then that the Julia set J_P is d -dense, weakly dense and that $\overline{\Gamma \cdot J_P} = \ell_p$ for every $d > r_P$ and every Γ with zero as an accumulation point. A dense orbit in J_P inherits all the mentioned properties.

Since Γ -supercyclic vectors must belong to A_P if Γ is bounded away from zero, the case when Γ is unbounded (but not necessarily with 0 as accumulation point) will be treated separately.

We start by computing the norm of P . The problem is equivalent to maximizing the function $f(x, y) = xy$ under the restrictions $x^p + y^p = 1, x \geq 0, y \geq 0$. Applying Lagrange multipliers it follows that $\|P\| = \frac{1}{2^{\frac{1}{p}}}$. Thus $r_P = 2^{\frac{2}{p}}$.

Notice that $P^n(x) = c_n(x)B^n(x)$, where

$$c_n(x) = x_1^{2^{n-1}} \cdot x_2^{2^{n-2}} \cdot \dots \cdot x_n;$$

equivalently $c_n(x)$ can be recurrently defined by the relations

$$\begin{cases} c_1(x) = x_1 \\ c_{n+1}(x) = c_n^2(x)x_{n+1}. \end{cases}$$

Since $c_{00} \subseteq A_P$, A_P is dense and $R_P = \emptyset$. By Proposition 2.13 J_P must be completely invariant. If $x \in J_P$ and $|t| > 1$ then $tx \notin A_P$. Since $R_P = \emptyset$, tx must belong to J_P . Thus, we may think J_P as an infinite union of half lines.

Let us first check that $J_P \neq \emptyset$. The following is a fixed vector for P

$$\bar{x} = 2^{\frac{1}{p}} \left(1, \frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{2}{p}}}, \frac{1}{2^{\frac{3}{p}}}, \dots \right).$$

Since \bar{x} is a fixed vector, then it must, by Proposition 2.5, belong to J_P . Also, $\bar{x} \in r_P S_X$ since

$$\|\bar{x}\|_p^p = 2 \sum_{i=0}^{\infty} \left(\frac{1}{2^{\frac{i}{p}}} \right)^{ip} = 2 \cdot \frac{1}{1 - \frac{1}{2}} = 4.$$

Proof that $P|_{J_P}$ is chaotic. Since J_P is closed and P -invariant, by Birkhoff's transitivity Theorem $P|_{J_P}$ is chaotic if and only if $P|_{J_P}$ is transitive and the periodic vectors of P are dense in J_P .

Let U, V be nonempty open sets intersecting J_P . Let $y \in V \cap J_P$, $x \in U \cap J_P$. Since $B^n(x) \rightarrow 0$ and $\|P^n(x)\| = |c_n(x)|\|B^n(x)\| \geq r_P$ we have that $c_n(x) \rightarrow \infty$.

We will perturb x to a vector \tilde{x} in such a way that $\tilde{x} \in U$ and for some n , $P^n(\tilde{x}) = y$. Notice that, since J_P is completely invariant, this implies that $\tilde{x} \in J_P$. Consider x^n the vector $x^n = \sum_{i=1}^n e'_i(x)e_i$.

Clearly $x^n \rightarrow x$ in ℓ_p . Since $c_n(x)$ reads only the first n coordinates, $c_n(x) = c_n(x^n)$. Consider $\tilde{x}^n = x^n + \frac{S^n(y)}{c_n(x)}$, where S denotes the forward shift. Since $\|S^n(y)\|_p = \|y\|_p$ and $c_n(x) \rightarrow \infty$, it follows that $\frac{S^n(y)}{c_n(x)} \rightarrow 0$ and that $\tilde{x}^n \rightarrow x$. Thus, for large n , $\tilde{x}^n \in U$. Also,

$$\begin{aligned} P^n(\tilde{x}^n) &= c_n(\tilde{x}^n)B^n\left(x^n + \frac{S^n(y)}{c_n(x)}\right) \\ &= c_n(x)\frac{y}{c_n(x)} = y. \end{aligned}$$

Therefore we may take $\tilde{x} = \tilde{x}^n$ for n sufficiently large.

Next we show that the periodic vectors are dense in J_P . Let $y \in J_P$ and $\epsilon > 0$. Since $y \in J_P$, there is some n_0 with $|c_n(y)| > 1$ for every $n \geq n_0$. Consider $y^n = \sum_{i=1}^n e'_i(y)e_i$. The vector

$$\tilde{y}^n = \sum_{i=0}^{\infty} \frac{S^{ni}(y^n)}{c_n(y)^i} = y^n + \sum_{i=1}^{\infty} \frac{S^{ni}(y^n)}{c_n(y)^i}$$

is n -periodic for P . Indeed

$$\begin{aligned} P^n(\tilde{y}^n) &= c_n(\tilde{y}^n)B^n\left(\sum_{i=0}^{\infty} \frac{S^{ni}(y^n)}{c_n(y)^i}\right) = c_n(y)\sum_{i=0}^{\infty} \frac{B^n S^{ni}(y^n)}{c_n(y)^i} \\ &= \sum_{i=1}^{\infty} \frac{S^{n(i-1)}(y^n)}{c_n(y)^{i-1}} = \tilde{y}^n. \end{aligned}$$

If $n \geq n_0$ it is also well defined, because

$$\|\tilde{y}^n\|_p \leq \sum_{i=0}^{\infty} \frac{\|S^{ni}(y^n)\|_p}{|c_n(y)|^i} \leq \sum_{i=0}^{\infty} \frac{\|y\|_p}{|c_n(y)|^i} < \infty.$$

Finally $\tilde{y}^n \rightarrow y$, since

$$\|\tilde{y}^n - y^n\|_p \leq \|y\|_p \sum_{i=1}^{\infty} \frac{1}{|c_n(y)|^i} \rightarrow 0.$$

□

Proof that J_P is d -dense, $d > r_P$. Let $x \in \ell_p$. Recall that the vector $\bar{x} = 2^{\frac{1}{p}}\left(1, \frac{1}{2^{\frac{1}{p}}}, \frac{1}{2^{\frac{2}{p}}}, \frac{1}{2^{\frac{3}{p}}}, \dots\right)$ belongs to J_P and has norm equal to r_P . Denoting $\text{sign}(0) = 1$ we define \tilde{x} as

$$\tilde{x}_i = \text{sign}(x_i) \max\{|x_i|, |\bar{x}_i|\}.$$

For each i we have that $|\tilde{x}_i| \geq |\bar{x}_i|$. This implies that for each n , $\|P^n(\tilde{x})\|_p \geq \|P^n(\bar{x})\|_p \geq r_P$ and hence the vector belongs to J_P . Also $\|x - \tilde{x}\|_p \leq \|\bar{x}\|_p = r_P$. \square

Proof that J_P is weakly dense. Let $U = U_{\{\epsilon, x_0, \varphi_1, \dots, \varphi_n\}}$ be a basic weakly open set.

Let $0 \neq y \in \bigcap_{i=1}^n \text{Ker}(\varphi_i)$. Let $M = \max\{\|\varphi_i\|\}$ and $\tilde{\epsilon} = \frac{\epsilon}{M}$. We consider for each $n \in \mathbb{N}$, $y^n = n \cdot y + x_0$. The vector y^n may fail to belong to J_P , however we can find a perturbation x^n of y^n so that $x^n \in J_P \cap U$. Noting again $\text{sign}(0) = 1$ and considering the fixed vector \bar{x} , we define

$$x_j^n := \text{sign}(y_j^n) \max\{|y_j^n|, \frac{\tilde{\epsilon}}{r_P} |\bar{x}_j|\}.$$

Clearly $\|x^n - y^n\|_p \leq \frac{\tilde{\epsilon}}{r_P} \|\bar{x}\|_p = \tilde{\epsilon}$. We claim that for large n , $x^n \in J_P$. Let k such that $y_k \neq 0$, so that $0 \neq x_k^n \rightarrow \infty$ as $n \rightarrow \infty$.

The vector

$$z^n = \frac{\tilde{\epsilon}}{r_P} (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{k-1}, x_k^n, \bar{x}_{k+1}, \dots)$$

satisfies that $|x_j^n| \geq |z_j^n|$ for every $j \geq 0$ and hence if z^n belongs to J_P so does x^n . But

$$P^k(z^n) = \left(\frac{\tilde{\epsilon}}{r_P}\right)^{2^k} \frac{c_k((\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{k-1}, x_k^n, \bar{x}_{k+1}, \dots))}{c_k(\bar{x})} P^k(\bar{x}) = \left(\frac{\tilde{\epsilon}}{r_P}\right)^{2^k} \frac{x_k^n}{\bar{x}_k} P^k(\bar{x}) = \left(\frac{\tilde{\epsilon}}{r_P}\right)^{2^k} \frac{x_k^n}{\bar{x}_k} \bar{x}.$$

Thus for big enough n we get that $P^k(z^n) = \lambda \bar{x}$ for some $|\lambda| \geq 1$ and therefore $P^k(z^n) \in J_P$ for large n . Since J_P is completely invariant this implies that $z^n \in J_P$.

Finally x^n belongs to U because, $|\varphi_i(x^n - x_0)| \leq \|\varphi_i\| \|x^n - y^n\|_p + |\varphi_i(y^n - x_0)| \leq \epsilon$. \square

Proof that $\Gamma \cdot J_P$ is dense for every Γ with zero as accumulation point. Let $x \in \ell_p$, $\epsilon > 0$ and $\gamma \in \Gamma$ such that $\frac{\epsilon}{r_P \gamma} > 1$. We consider

$$z = \frac{1}{\gamma} \text{sign}(x_j) \max\{|x_j|, \frac{\epsilon}{r_P} \bar{x}_j\}.$$

The vector z is in J_P since for each coordinate $|z_j| > |\bar{x}_j|$ and $\bar{x}_j \in J_P$. It is also immediate that $\|\gamma z - x\|_p < \epsilon$. \square

Proof that P is Γ supercyclic for every unbounded Γ . We will apply Criterion 3.11. Let $X_0 = Y_0$ be a dense set of vectors with nonzero coordinates. For $x, y \in X_0$ fixed we choose $(\lambda_n)_n \subset \Gamma$ such that $\lambda_n c_n(x) \rightarrow \infty$. Let x^n be the vector $x_j^n = \chi_{[1, n]}(j) x_j$, i.e. x^n is the truncation of x . Finally we define the inverses $F_n(x)$ as $F_n(x)(y) = x^n - x + \frac{S^n(y)}{c_n(x) \lambda_n}$, where S is the forward shift operator. It is clear that

$F_n(x)(y) \rightarrow 0$. Also

$$\begin{aligned} \lambda_n P^n(x + F_n(x)(y)) &= \lambda_n P^n\left(x^n + \frac{S^n(y)}{c_n(x)\lambda_n}\right) = \lambda_n c_n\left(x^n + \frac{S^n(y)}{c_n(x)\lambda_n}\right) B^n\left(x^n + \frac{S^n(y)}{c_n(x)\lambda_n}\right) \\ &= \lambda_n c_n(x) \frac{y}{c_n(x)\lambda_n} = y. \end{aligned}$$

□

We end the example by pointing some comments. The first to notice is that we can arrive to d -dense orbits of arbitrary small radius.

Remark 3.16. For every $d > 0$ there is some d -hypercyclic homogeneous polynomial.

Proof. Let P be the 2-homogeneous polynomial from the above theorem. By Proposition 2.8, λP is quasiconjugated to P for every $\lambda \neq 0$ under a linear isomorphism of norm $\frac{1}{|\lambda|}$. By Proposition 3.2, this implies that λP is d -hypercyclic for every $d > r_{\lambda P} = \frac{r_P}{|\lambda|}$. □

Remark 3.17. Proposition 2.15 says that tx with $|t| < 1$ is not an accumulation point of $Orb_P(x)$. The example from Theorem 3.15 shows that this does not happen for $|t| \geq 1$. Indeed, if x is a point whose P -orbit is dense in J_P , tx is in J_P for $|t| \geq 1$ and hence it is an accumulation point of the P -orbit of x .

Remark 3.18. The Julia set of the polynomial of Theorem 3.15 satisfies another nice property: the image of J_P under the backward shift is dense in ℓ_p . This implies that the dense orbit in J_P must satisfy that $\overline{\{B(P^n(x))\}} = \ell_p$. Thus, the family $\{c_n(\cdot)B^{n+1}(\cdot)\}$ is an universal family of homogeneous polynomials while $\{c_n(\cdot)B^n(\cdot)\} = \{P^n\}$ has only nowhere dense orbits.

Proof that $\overline{B(J_P)} = \ell_p$. Let $\epsilon > 0$ and $x \in \ell_p$. Let \bar{x} be a fixed vector that satisfies $\|\bar{x}\|_p = r_P$. Define $y \in \ell_p$ as $y_n = \text{sign}(x_n) \max\{|x_n|, \epsilon|\bar{x}_n|\}$. Thus $\|y - x\|_p \leq \epsilon r_P$ and $|y_n| \geq \epsilon|\bar{x}_n|$ for each n . We define $z = \frac{\epsilon_1}{\epsilon} + S(y)$. Then $B(z) = y$ and $z \in J_P$. Indeed, $P(z) = \frac{y}{\epsilon}$ which satisfies $|\frac{y_n}{\epsilon}| > |\bar{x}_n|$ for each coordinate n . Thus, $P(z)$ belongs to J_P and therefore z is also in J_P . □

Recall that a dynamical system (X, F) is said to be Devaney chaotic if it is transitive, the periodic vectors are dense and if it has *sensitive dependence on the initial conditions*. This last condition means that there exists a neighborhood U of 0 such that for every $x \in X$ and every neighborhood V of 0, there is some $n \in \mathbb{N}$ and $y \in x + V$ such that $F^n(y) \notin U + F^n(x)$. On metric spaces, sensitivity on the initial conditions is implied by the transitivity and the density of periodic vectors and, of course, no nonlinear homogeneous polynomial on a Banach space can have sensitivity on the initial conditions. But, if we consider the weak topology on the Banach space, we may have Devaney chaotic polynomials. Indeed, if P is the polynomial from Theorem 3.15, we already proved that $P|_{J_P}$ satisfies that it is transitive and that the periodic vectors are dense, and moreover we saw that J_P is weakly dense in ℓ_p . Thus, P satisfies the first two conditions of Devaney chaoticity. To see the last one, just take $U = \{x \in \ell_p : |x_1| < 1\}$.

Then for $x \in \ell_p$, V a weak neighborhood of 0, which we may suppose that, for some $\epsilon > 0$ and some n , $V \supset \{x \in \ell_p : |x_j| < \epsilon, j = 1, \dots, n\}$. define $y \in \ell_p$ as,

$$y_k = \begin{cases} \epsilon/2 & \text{if } k \leq n, x_k = 0, \\ x_k & \text{if } k \leq n, x_k \neq 0, \\ c & \text{if } k = n + 1, \\ 0 & \text{if } k > n + 1. \end{cases}$$

Then $y \in x + V$ for every c and

$$e'_1(P^n x - P^n y) = e'_1(c_n(x)B^n(x) - c_n(y)ce_1) = c_n(x)x_{n+1} - c_n(y)c.$$

Since $c_n(y) \neq 0$ and it does not depend on c , we can find c such that $|e'_1(P^n x - P^n y)| > 1$ and thus $P^n(y) \notin U + P^n(x)$. Therefore $P = e'_1 \cdot B$ is Devaney chaotic on (ℓ_p, w) .

Example 3.19. Using the polynomial P defined in Theorem 3.15, it is possible to exhibit an example of a homogeneous polynomial whose Julia set is not completely invariant: let $X = \mathbb{C} \oplus_\infty c_0(c_0)$ and define $Q \in \mathcal{P}^2(X)$ as

$$Q(\lambda, (y^j)_{j \in \mathbb{N}}) = (0, P(\lambda x), P(y^1), P(y^2), \dots),$$

where $x \in c_0$ is a vector in the Julia set of P . Then, for $|\lambda| > 1$, the vector $(\lambda, (0)_j)$ is in R_Q because if $|t| > 1$,

$$\|Q^n(t, (y^j)_{j \in \mathbb{N}})\|_X \geq \|P^n(tx)\|_\infty \rightarrow \infty.$$

On the other hand, the vector $Q(\lambda, (0)_j) = (0, P(\lambda x), 0, 0, \dots)$ belongs to J_Q . Indeed, we may approximate $P(\lambda x)$ by a vector $z \in c_0$ with only finite nonzero coordinates. Then the vector $(0, z, 0, 0, \dots)$ approximates $(0, P(\lambda x), 0, 0, \dots)$ in X and clearly, $(0, z, 0, 0, \dots)$ is in A_Q .

3.5. Numerically hypercyclicity. The notion of numerically hypercyclicity was recently introduced by Kim, Peris and Song [16]. A function F is said to be numerically hypercyclic provided that there are vectors $x \in S_X, x^* \in S_{X^*}$, with $x^*(x) = 1$ and such that its numerical orbit $Norb_F(x, x^*) := \{x^*(F^n(x)) : n \in \mathbb{N}_0\}$ is dense in \mathbb{C} . In [16, 17] the authors proved that every infinite dimensional and separable Banach space supports a homogeneous polynomial of degree $d \geq 1$. In view of the Example 3.15 one may expect that the homogeneous polynomial $P(x) = x_1 B(x)$ is numerically hypercyclic in ℓ_p or c_0 . However since the limit radius of the polynomial is $r_P = 2^{\frac{2}{p}} > 1$, for $X = \ell_p$ and $r_P = 1$ for $X = c_0$, $\|P^n(x)\| \leq r_P$ for every $x \in S_X$ and every n , and thus the numerical orbits can never be dense.

The lack of numerically dense orbits is only due to the fact that $B_X \subseteq r_P B_X$. Thus, by considering a multiple of the polynomial and exploiting that homogeneous polynomials behave well under conjugation via a linear isomorphism we can easily construct a numerically hypercyclic homogeneous polynomial.

Example 3.20 (A numerically hypercyclic homogeneous polynomial). Let $X = \ell_p$ and $P(x) = x_1 B(x)$. Let $x_0 \in J_P$ such that the orbit of x_0 under P is dense in J_P is hypercyclic. Consider $\tilde{P} = \|x_0\|P$. By Proposition 2.9 applied to $\Phi(x) = \frac{1}{\|x_0\|}x$ it follows that $J_{\tilde{P}} = \|x_0\|J_P$ and that $\frac{x_0}{\|x_0\|}$ is hypercyclic for

$\tilde{P}|_{J_{\tilde{P}}}$. Now, by Example 3.15 and Proposition 3.6, $\frac{x_0}{\|x_0\|}$ is a weakly hypercyclic vector for \tilde{P} . Thus, for any $x^* \in S_{X^*}$ it satisfies that $\overline{x^*(\tilde{P}^n(x))} = \mathbb{C}$.

Remark 3.21. There are several notions of numerical range for nonlinear mappings on Banach spaces, which coincide for linear operators (see for example [1, Chapter 11]). The numerical range in the sense of Bauer of a mapping F is the set

$$\{x^*(F(x)) : x^*(x) = \|x\|^2, \|x\| = \|x^*\|\},$$

i.e. the norm of x and x^* is allowed to be different from 1. Inspired in this notion of numerical range, a polynomial would be numerically hypercyclic if there are vectors $x \in X, x^* \in X^*$, with $x^*(x) = \|x\|^2, \|x\| = \|x^*\|$ and such that its numerical orbit $Norb_F(x, x^*) := \{x^*(F^n(x)) : n \in \mathbb{N}_0\}$ is dense in \mathbb{C} . With this modified definition, the polynomial of Theorem 3.15 would also be numerically hypercyclic.

4. EXISTENCE OF d -HYPERCYCLIC, WEAKLY HYPERCYCLIC AND Γ -SUPERCYCLIC HOMOGENEOUS POLYNOMIALS ON ARBITRARY FRÉCHET SPACES

In this section we will prove our main Theorem 4.5, which states that on any separable and infinite dimensional Fréchet space there exists a d -hypercyclic, weakly hypercyclic and Γ -supercyclic homogeneous polynomial for every subset Γ which is unbounded or not bounded away from zero.

The proof will follow some of the ideas of Bonet and Peris [7] to prove the existence of hypercyclic operators on arbitrary Fréchet spaces. We will look for a polynomial P acting on ℓ_1 satisfying all the mentioned properties and at the same time we want to find on each Fréchet space X a polynomial Q being quasiconjugated to P .

The polynomial P on ℓ_1 will be of the type $e'_1(x)B_w(x)$, where B_w is a weighted backward shift. The main difference with Bonet and Peris proof is that not every weight works. Indeed, if the weights are too small then the Julia set can be empty. This forces us to adapt their main Lemma [7, Lemma 2] to our requirements.

Theorem 4.1. *Let $X = \ell_1$ and P the homogeneous polynomial $P = e'_1 \cdot B_w \in \mathcal{P}({}^2\ell_1; \ell_1)$, where B_w is the weighted backward shift defined as $[B_w(x)]_i = \frac{1}{(i+1)^4}x_{i+1}$. Then P is weakly hypercyclic, d -hypercyclic and Γ -supercyclic for every $\Gamma \subset \mathbb{C}$ such that Γ is unbounded or zero is an accumulation point of Γ .*

The strategy of the proof is the same that we used in Example 3.15. We will prove that $P|_{J_P}$ is hypercyclic and that J_P is weakly dense, d -dense and $\overline{\Gamma J_P} = \ell_1$ for Γ that accumulates at zero. Again we will analyze the case when Γ is unbounded separately.

Note that $c_{00} \subseteq \bigcup_n Ker P^n$ so that $R_P = \emptyset$ and by Proposition 2.13, J_P is completely invariant. The fact that $R_P = \emptyset$ implies also that if $x \in J_P$ then all the ray $\{tx : |t| \geq 1\} \subseteq J_P$.

The iterations of a vector x are

$$P^n(x) = (x_1^{2^{n-1}} \dots x_n) \cdot \left(\frac{1}{1^{2^{n-1}}} \frac{1}{(2!^4)^{2^{n-2}}} \dots \frac{1}{((n-1)!^4)^2} \frac{1}{n!^4} \right) \cdot B_\omega^n(x).$$

So if we define $\Omega \in \mathbb{C}^{\mathbb{N}}$ as $(\Omega)_n = \frac{1}{(n!)^4}$, and $c_n : \mathbb{C}^{\mathbb{N}} \rightarrow \mathbb{C}$ as $c_n(z) = \prod_{i=1}^n z_i^{2^{n-i}}$, then

$$P^n(x) = c_n(x)c_n(\Omega)B_\omega^n(x).$$

We must verify that the Julia is not empty. A typical member of J_P must satisfy that the first coordinates are large in comparison to the decreasing rate of the tail and the weights $\frac{1}{n!^4}$.

Lemma 4.2. *Let $j, l \in \mathbb{N}$ and $k_j > 1$ such that for all $n \in \mathbb{N}$,*

$$k_j \left(2^{2^{\frac{n+1}{2}}} \right)^{\sqrt{2}-1} \geq (n+1)!^4 (j+n-1)!^l.$$

Then for every $N \geq k_j 2^{2^{\frac{1}{2}}}$ the vector

$$x = \left(N, \frac{1}{j!^l}, \frac{1}{(j+1)!^l}, \dots \right)$$

belongs to J_P .

Proof. We are going to prove by induction that for all n , $c_n(x)c_n(\Omega) \geq k_j 2^{2^{\frac{n}{2}}}$. By definition of N , $c_1(x)c_1(\Omega) = N \geq k_j 2^{2^{\frac{1}{2}}}$. Suppose that our claim is true for some $n \geq 1$, then,

$$\begin{aligned} c_{n+1}(x)c_{n+1}(\Omega) &= \frac{c_n(x)^2 c_n(\Omega)^2}{(n+1)!^4 (j+n-1)!^l} \geq \frac{k_j^2 2^{2^{\frac{n}{2}+1}}}{(n+1)!^4 (j+n-1)!^l} \\ &= k_j 2^{2^{\frac{n+1}{2}}} \frac{k_j \left(2^{2^{\frac{n+1}{2}}} \right)^{\sqrt{2}-1}}{(n+1)!^4 (j+n-1)!^l} \geq k_j 2^{2^{\frac{n+1}{2}}}. \end{aligned}$$

Thus, since $\|B_\omega^n(x)\| \geq |B_\omega^n(x)_1| = \frac{1}{(n+1)!^4 (j+n-1)!^l}$, it follows that

$$\|P^n(x)\| \geq \frac{k_j 2^{2^{\frac{n+1}{2}}}}{(n+1)!^4 (j+n-1)!^l} \rightarrow \infty,$$

as $n \rightarrow \infty$, and therefore x is in J_P . □

Proof that $P|_{J_P}$ is hypercyclic. We prove that $P|_{J_P}$ is transitive. By Birkhoff's transitivity Theorem it follows that there exists a dense orbit in J_P .

Inspired in the hypercyclicity criterion for linear operators we will find two J_P -dense sets X, Y and applications $S_{n,x} : Y \rightarrow \ell_1$ such that the following properties are satisfied:

- i) for all $x \in X, y \in Y, S_{n,x}(y) \rightarrow 0$,
- ii) if $x^n = \sum_{i=1}^n e'_i(x)e_i$, then $x^n + S_{n,x}(y) \in J_P$ and

iii) for all $x \in X$, $y \in Y$, $P^n(x^n + S_{n,x}(y)) \rightarrow y$.

If those properties are satisfied then clearly $P|_{J_P}$ is transitive.

We define X as $X := \{x \in J_P : tx \in J_P \text{ for some } |t| < 1\}$. Since R_P is the empty set, if $x \in J_P$ then $tx \in J_P$ for all $|t| > 1$. This implies that X is dense in J_P .

For $y \in J_P$ we define $y^{(n)} \in \ell_1$ as

$$y_j^{(n)} := \begin{cases} y_j & \text{if } j \leq n; \\ \frac{1}{j!} & \text{if } j > n. \end{cases}$$

We define $Y = \{y^{(n)} : y^{(n)} \in J_P \text{ and } y \in X\} \subset X \subset J_P$. We defined Y in such a way because of two reasons. We want the formal inverse S_ω^n of B_ω^n to be defined in Y for all n , but at the same time we want to have a (good) control of the decreasing rate of the tail.

Note that

$$(5) \quad S_\omega^k(y^{(n)})_j = \begin{cases} 0 & \text{if } j \leq k; \\ \frac{j!^4}{(j-k)!^4} y_{j-k} & \text{if } k < j \leq k+n; \\ \frac{j!^4}{(j-k)!^4 (j-k)!} & \text{if } j > k+n. \end{cases}$$

Thus $S_\omega^n(y^{(k)}) \in \ell_1$ for all k .

In order to show that Y is dense in J_P , it suffices to see that Y is dense in X . Since $y^{(n)} \rightarrow y$, it suffices to show that if $y \in X$ then for large n , $y^{(n)} \in J_P$. We will exploit the fact that $c_n(y)$ grows very fast when $y \in X$. Let $|t| > 1$ with $\frac{y}{t} \in J_P$. We have that there is some constant $C > 0$ such that for every n ,

$$(6) \quad |c_n(y)c_n(\Omega)| = |t|^{2n-1} \left| c_n\left(\frac{y}{t}\right) c_n(\Omega) \right| \geq C|t|^{2n}.$$

Since P is completely invariant it suffices to check that $P^n(y^{(n)}) = c_n(y)c_n(\Omega)\left(\frac{1}{n!}, \frac{1}{(n+1)!}, \dots\right)$ belongs to J_P . The coordinates of this vector are, for sufficiently large n , greater than the coordinates of $\left(\frac{c_n(y)c_n(\Omega)}{n!}, \frac{1}{(n+1)!}, \dots\right)$. Thus, it suffices to show that this new vector is in J_P for large n . This is a straightforward application of Lemma 4.2 because $\frac{t^{2n}}{2(n+1)!^5} \rightarrow \infty$. Therefore, it follows that $y^{(n)} \in J_P$ for large n and thus Y is dense in J_P .

We are now able to define the inverse application $S_{n,x}(y)$. For $y \in Y$, $x \in X$, let $S_{n,x}(y) := \frac{S_\omega^n(y)}{c_n(x)c_n(\Omega)}$. Observe that c_n reads only the first n coordinates, so we get

$$\begin{aligned} P^n\left(x^n + \frac{S_\omega^n(y)}{c_n(x)c_n(\Omega)}\right) &= c_n(x)c_n(\Omega)B_\omega^n\left(x^n + \frac{S_\omega^n(y)}{c_n(x)c_n(\Omega)}\right) \\ &= y. \end{aligned}$$

Since $y \in J_P$ and J_P is completely invariant it follows that $x^n + \frac{S_\omega^n(y)}{c_n(x)c_n(\Omega)} \in J_P$. So we have that properties ii) and iii) of our criterion are satisfied. It remains to show that $\frac{S_\omega^n(y)}{c_n(x)c_n(\Omega)} \rightarrow 0$, but recall

that for $x \in X$, $c_n(x)c_n(\omega)|t|^{-2^n} \rightarrow \infty$ for some $|t| > 1$. At the same time we have a good control of the decreasing rate of the tail of y .

Take n such that $y = y^{(n)}$. Recall that by (5),

$$\begin{aligned} \|S_\omega^k(z^{(n)})\| &= \sum_{j=k+1}^{k+n} \frac{j!^4}{(j-k)!^4} |z_{j-k}| + \sum_{j>k+n} \frac{j!^4}{(j-k)!^4(j-k)!} \\ &\leq \|z\| \frac{(k+n)!^4}{n!^4} + \sum_{j>0} \frac{(j+k+n)!^4}{(j+n)!^5}. \end{aligned}$$

Note that $\sum_{j>0} \frac{(j+k+n)!^4}{(j+n)!^5}$ is a convergent series. Let $|t| > 1$ such that $\frac{x}{t} \in J_P$. Then by (6), we have

$$(7) \quad \left\| \frac{S_\omega^k(y)}{c_k(x)c_k(\Omega)} \right\| \leq \frac{\|z\| \frac{(k+n)!^4}{n!^4} + \sum_{j>0} \frac{(j+k+n)!^4}{(j+n)!^5}}{Ct^{2^k}} \rightarrow 0,$$

This proves that $P|_{J_P}$ is transitive. \square

Proof that J_P is d -dense. Let $x \in \ell_1$. Let N such that the vector $\bar{x} = (N, \frac{1}{2!}, \frac{1}{3!}, \dots) \in J_P$. We define \tilde{x} as

$$\tilde{x}_i = \text{sign}(x_i) \max\{|x_i|, |\bar{x}_i|\},$$

with $\text{sign}(0) = 1$. For each i we have that $|\tilde{x}_i| \geq |\bar{x}_i|$. This implies that for each n , $\|P^n(\tilde{x})\| \geq \|P^n(\bar{x})\| \geq r_P$ and hence the vector belongs to J_P . Also $\|x - \tilde{x}\| \leq \|\bar{x}\|$.

Therefore J_P is d -dense for every $d > \|\bar{x}\|$. \square

Proof that J_P is weakly dense. Let $U = U_{\{\epsilon, x_0, \varphi_1, \dots, \varphi_n\}}$ be a basic weakly open set.

Let $0 \neq y \in \bigcap_{i=1}^n \text{Ker}(\varphi_i)$. Let $M = \max\{\|\varphi_i\|\}$ and $\tilde{\epsilon} = \frac{\epsilon}{M}$. We consider for each $n \in \mathbb{N}$, $y^n = n \cdot y + x_0$. The vector y^n may fail to belong to J_P , however we can find a perturbation x^n so that $x^n \in J_P \cap U$. Denoting $\text{sign}(0) = 1$ and \bar{x} as in the previous proof, we define

$$x_j^n := \text{sign}(y_j^n) \max\{|y_j^n|, \frac{\tilde{\epsilon}}{\|\bar{x}\|} |\bar{x}_j|\}.$$

Clearly $\|x^n - y^n\| \leq \frac{\tilde{\epsilon}}{\|\bar{x}\|} \|\bar{x}\| = \tilde{\epsilon}$. We claim that for large n , $x^n \in J_P$. Let k such that $y_k \neq 0$. Note that this implies that $0 \neq x_k^n \rightarrow \infty$ as $n \rightarrow \infty$.

The vector

$$z^n = \frac{\tilde{\epsilon}}{\|\bar{x}\|} (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{k-1}, x_k^n, \bar{x}_{k+1}, \dots)$$

satisfies that $|x_j^n| \geq |z_j^n|$ for every $j \geq 0$ and hence if z^n belongs to J_P so does x^n . But

$$P^k(z^n) = \left(\frac{\tilde{\epsilon}}{\|\bar{x}\|} \right)^{2^k} \frac{c_k((\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{k-1}, x_k^n, \bar{x}_{k+1}, \dots))}{c_k(\bar{x})} P^k(\bar{x}) = \left(\frac{\tilde{\epsilon}}{\|\bar{x}\|} \right)^{2^k} \frac{x_k^n}{\bar{x}_k} P^k(\bar{x}).$$

Since $P^k(\bar{x}) \in J_P$ we have that $P^k(z^n) \in J_P$ for sufficiently large n . Thus, the complete invariance of J_P implies that $z^n \in J_P$.

Finally x^n belongs to U because, $|\varphi_i(x^n - x_0)| \leq \|\varphi_i\|\tilde{\epsilon} + \varphi(y^n - x_0) < \epsilon$. \square

Proof that $\Gamma \cdot J_P$ is dense for every Γ with zero as accumulation point. Let $x \in \ell_1$, $\epsilon > 0$ and $\gamma \in \Gamma$ such that $\frac{\epsilon}{\|\bar{x}\|^\gamma} > 1$. We consider

$$z = \frac{1}{\gamma} \text{sign}(x_j) \max\{|x_j|, \frac{\epsilon}{\|\bar{x}\|} \bar{x}_j\}.$$

The vector z is in J_P since for each coordinate $|z_j| > |\bar{x}_j|$ and $\bar{x}_j \in J_P$. It is also immediate that $\|\gamma z - x\|_1 < \epsilon$. \square

Proof that P is Γ supercyclic for every unbounded Γ . It suffices to prove that P is Γ -transitive for every unbounded Γ . Let U and V be open sets. We want to show that for some $\gamma \in \Gamma$ and $n \in \mathbb{N}$, $\gamma P^n(U) \cap V \neq \emptyset$. Since $V \cap J_P$ and $U \cap J_P$ may be both the empty set, we will skip the Julia set J_P . However our approach will be very similar to the one that we used to prove that $P|_{J_P}$ was transitive.

The inverses S_ω^k are not defined on ℓ_1 . However, by perturbing the tail of the sequence as we did in (5), we have that the set $Y := \{y \in \ell_1 : S_\omega^k(y) \in \ell_1 \text{ for every } k\}$ is dense in ℓ_1 .

Let $x \in U$ such that $x_j \neq 0$ for every j , and $y \in Y \cap V$ and consider $x^n := \sum_{j \leq n} x_j e_j$. Since Γ is unbounded we can choose $(\gamma_n)_n \subseteq \Gamma$ with $\frac{S_\omega^n(y)}{c_n(x)c_n(\Omega)\gamma_n} \rightarrow 0$. Therefore,

$$x^n + \frac{S_\omega^n(y)}{c_n(x)c_n(\Omega)\gamma_n} \rightarrow x$$

and since c_n reads only the first n coordinates,

$$\gamma_n P^n \left(x^n + \frac{S_\omega^n(y)}{c_n(x)c_n(\Omega)\gamma_n} \right) = \gamma_n c_n(x^n) c_n(\Omega) B_w^n \left(\frac{S_\omega^n(y)}{c_n(x)c_n(\Omega)\gamma_n} \right) = y.$$

\square

To prove the existence Theorem 4.5, we will use a version of the well known Lemma [7, Lemma 2] due to Bonet and Peris. This Lemma was a key ingredient to prove the existence of hypercyclic operators on arbitrary separable infinite dimensional Fréchet spaces. The main difference of this version is that we provide a control of the asymptotic behavior of the sequence $\alpha(n) = x_n^*(x_n)$.

Lemma 4.3. *Let X be an infinite dimensional separable Fréchet space not isomorphic to $\mathbb{C}^{\mathbb{N}}$ and let $\alpha(n)$ be a sequence such that $n\alpha(n) \rightarrow 0$. Then, there are sequences $(x_n)_n$ in X and $(x_n^*)_n$ in X^* such that*

- (1) $x_n \rightarrow 0$ and $\text{span}\{x_n\}$ is dense in X ,
- (2) $\{x_n^*\}$ is equicontinuous and
- (3) $x_n^*(x_k) = \alpha(n)\delta_{n,k}$.

The proof of the above Lemma follows [7], together with a careful choice of the elements $(x_k)_k$.

Proof. Let $(\rho_n)_n$ be a non-decreasing fundamental system of seminorms generating the topology on X . Since X is not isomorphic to $\mathbb{C}^{\mathbb{N}}$, there is a dense subspace $M \subseteq X$ and a continuous norm $\|\cdot\|$ defined on M . Without loss of generality we may assume that $\rho_1|_M = \|\cdot\|$.

Consider $(z_n)_n \subset S_M$, a linearly independent sequence in the sphere of M such that its span is dense in M . Let $(\beta_n)_n$ be a nondecreasing sequence such that, $\beta_1 > 1$, $\beta_n \rightarrow \infty$ and such that $n\alpha(n)\beta_n \rightarrow 0$.

We claim that, by adding new seminorms to our fundamental system $\{\rho_n\}_n$ and new vectors $(w_n)_n \subseteq S_M$ to the sequence $(z_n)_n$, we may assume that $\rho_k(z_j) \leq \beta_k$ for all $k \geq j$.

Indeed, we will construct a subsequence $(n_l)_l$ of natural numbers and consider $(\psi_n)_n$ the non-decreasing fundamental system for X given by $\psi_n = \rho_l$ for $n_{l-1} \leq n < n_l$, and a new sequence of linearly independent vectors $(w_n)_n$, satisfying that $z_{l+1} = w_{n_l}$. This pair (ψ_n, w_n) will satisfy $\psi_k(w_j) \leq \beta_k$ for all $k \geq j$. Since $\text{span}\{z_n : n \in \mathbb{N}\}$ is already dense, it will follow that $\text{span}\{w_n : n \in \mathbb{N}\}$ is also dense and since the ψ_n 's are the ρ_n 's but repeated, the systems $\{\psi_n\}$ and $\{\rho_n\}$ will generate the same topological space. We construct the pairs (ψ_n, w_n) inductively. Put $\psi_1 = \rho_1, w_1 = z_1$ and recall that $\rho_1(z_1) = 1 < \beta_1$. Since $\beta_k \rightarrow \infty$, there exists n_1 such that $\rho_2(z_1) < \beta_{n_1}$ and $\rho_2(z_2) < \beta_{n_1}$. We define ψ_{n_1} as ρ_2 , $w_{n_1} = z_2$ and for $k < n_1$ we define ψ_k as ρ_1 . Since ρ_2 and ρ_1 are continuous, there exist $w_2, \dots, w_{n_1-1} \in S_M$ near z_1 and linearly independent to $\{z_n : n \in \mathbb{N}\}$ such that $\rho_2(w_i) < \beta_{n_1}$ for all $2 \leq i \leq n_1 - 1$ and such that $\psi_k(w_j) = \rho_1(w_j) = 1 < \beta_1 \leq \beta_k$ for all $2 \leq j \leq k \leq n_1$. Suppose now that we have constructed $\psi_1, \dots, \psi_{n_l}$ non-decreasing seminorms and vectors $w_1, \dots, w_{n_l} \in S_M$ such that the set $\{w_i : i \leq n_l\} \cup \{z_n : n \in \mathbb{N}\}$ is linearly independent, $w_{n_l} = z_{l+1}$, each ψ_k is some of the ρ_n 's, $\psi_{n_l} = \rho_{l+1}$ and $\psi_k(w_j) < \beta_k$ for all $j \leq k \leq n_l$. We proceed as we did in the first step. There exists n_{l+1} such that $\rho_{l+2}(w_j) < \beta_{n_{l+1}}$ for all $j \leq n_l$ and $\rho_{l+2}(z_{l+2}) < \beta_{n_{l+1}}$. We put for $n_l < k < n_{l+1}$, $\psi_k = \rho_{l+1} = \psi_{n_l}$, $\psi_{n_{l+1}} = \rho_{l+2}$ and $w_{n_{l+1}} = z_{l+2}$. Notice that for $n_l < k < n_{l+1}$ and $j \leq n_l$ we have that $\psi_k(w_j) = \psi_{n_l}(w_j) < \beta_{n_l} \leq \beta_k$. If $k = n_{l+1}$, then this last relation also holds by the definition of n_{l+1} . Finally we choose for $n_l + 1 \leq j < n_{l+1}$ vectors $w_j \in S_M$ close to z_1 such that $\psi_{n_{l+1}}(w_j) < \beta_{n_{l+1}}$, such that, for $j \leq k < n_{l+1}$, $\psi_k(w_j) = \psi_{n_l}(w_j) < \beta_{n_l}$ and such that $\{w_j : j \leq n_{l+1}\} \cup \{z_n : n \in \mathbb{N}\}$ is linearly independent. The sequences $(\psi_k)_k, (w_j)_j$ constructed satisfy all the desired properties, and we have proved the claim.

Now we choose by Hahn-Banach elements $(x_k^*)_k$ in M^* such that $x_k^*(z_j) = \delta_{kj}$ for all $k > j$ and that $\|x_k^*\| = 1$.

Following a Gram-Schmidt argument we consider $y_n = z_n - \sum_{k < n} x_k^*(z_n)z_k$. It follows that (y_n, x_n^*) is a biorthogonal sequence in M . Observe also that $\rho_n(y_n) \leq n\beta_n$. We extend each x_n^* to X (which we keep notating x_n^*). Since x_n^* has norm 1 in M , this means that $|x_n^*(x)| \leq \rho_1(x)$ in X and hence the x_n^* are equicontinuous. Finally, the vectors x_n will be $x_n = \alpha(n)y_n$. Then, for every seminorm ρ_j and $n > j$,

$$\rho_j(x_n) = \rho_j(\alpha(n)y_n) \leq \alpha(n)\rho_n(y_n) \leq n\beta_n\alpha(n) \rightarrow 0.$$

□

Theorem 4.4. *Let $P \in \mathcal{P}({}^m\ell_1; \ell_1)$ be the polynomial $(e'_1)B_{\frac{1}{n^4}}$ and let X be any infinite dimensional and separable Fréchet space. Then there exists $Q \in \mathcal{P}({}^2X; X)$ and $\Phi \in \mathcal{L}(\ell_1; X)$ such that Q is a quasiconjugacy of P under Φ . In the case that X is Banach we have also that $Q|_{J_Q}$ is a quasiconjugacy of $P|_{J_P}$.*

Proof. Let $(x_n)_n \subseteq X$ and (x_n^*) be the sequences given by the above lemma applied to the sequence $(\alpha_n)_n$ defined by $\alpha_n = \frac{1}{(n-1)^2}$ for $n \geq 2$ and $\alpha_1 = 1$. Let $Q \in \mathcal{P}({}^2X; X)$ be defined as

$$Q(x) = x_1^*(x) \sum_{n=1}^{\infty} \frac{x_{n+1}^*(x)x_n}{n^2}.$$

Since the x_n^* are equicontinuous and $x_n \rightarrow 0$ the polynomial is well defined.

Let $\Phi : \ell_1 \rightarrow X$ defined as $\Phi((a_n)_n) = \sum a_n x_n$. Again, since $x_n \rightarrow 0$ and $(a_n)_n \in \ell_1$ the operator is well defined.

We see now that Q is a quasiconjugacy of P under Φ . Since $\text{span}(x_n)$ is dense and it is contained in the range of Φ , we have that Φ has dense range.

Let $a \in \ell_1$. We have that

$$\Phi(P(a)) = \Phi(a_1 B_{\frac{1}{n^4}}(a)) = a_1 \sum_{n=1}^{\infty} \frac{a_{n+1} x_n}{n^4}$$

while

$$\begin{aligned} Q(\Phi(a)) &= Q\left(\sum_{n=1}^{\infty} a_n x_n\right) = x_1^*\left(\sum_{n=1}^{\infty} a_n x_n\right) \sum_{n=1}^{\infty} x_{n+1}^*(a_{n+1} x_{n+1}) \frac{x_n}{n^2} \\ &= a_1 \sum_{n=1}^{\infty} \frac{a_{n+1} x_n}{n^4}. \end{aligned}$$

A little more difficult is to prove that, in the Banach space case, $Q|_{J_Q}$ is a quasifactor of $P|_{J_P}$. We need to show two things. First that $P(J_P) \subseteq J_Q$ and second that $\overline{P(J_P)} = J_Q$. The first assertion is easy to prove. Let $a \in J_P$. Note that this implies that, for all n , $\sup_k \frac{|P^n(a)_k|}{(k-1)^2} > 1$. So we can choose n_k satisfying $|P^{n_k}(a)_{n_k}| > (n_k - 1)^2$. Since the x_n^* are equicontinuous it follows that $\|x_n^*\| \leq C$ for some constant C . Therefore,

$$C\|Q^{n_k}\Phi(a)\| = C\|\Phi P^{n_k}(a)\| \geq |x_{n_k}^*(\Phi P^{n_k}(a))| > 1.$$

Since R_Q is the empty set, $\Phi(a) \in J_Q$.

We will show now that $P(J_P)$ is dense in J_Q . Let $x \in J_Q$. We want to find for every $\epsilon > 0$, $y \in \Phi(J_P) \cap B_\epsilon(x)$. Let $T = \sum_{n=1}^{\infty} \frac{x_{n+1}^*(x)x_n}{n^2}$ so that we can write $Q^n(x)$ as $d_n(x)T^n(x)$ in the usual way, where $d_n(x)$ is a continuous function depending only on $\{x_1^*(x), \dots, x_1^*(T^n(x))\}$. Notice that

$c_n(a)c_n(\Omega) = d_n(\Phi(a))$. Indeed, $\Phi B_{\frac{1}{n^4}} = T\Phi$, and we have that

$$(8) \quad \begin{aligned} c_n(a)c_n(\Omega)T^n(\Phi(a)) &= \Phi(c_n(a)c_n(\Omega)B_{\frac{1}{n^4}}^n(a)) = \Phi(P^n(a)) \\ &= Q^n(\Phi(a)) = d_n(\Phi(a))T^n(\Phi(a)). \end{aligned}$$

We conclude that $c_n(a)c_n(\Omega) = d_n(\Phi(a))$ for every $a \in \ell_1$. Without loss of generality we may suppose that there is some $t > 1$ for which $|d_n(x)| > t^{2^n}$. Therefore we can find, for each $n \in \mathbb{N}$, $x^n \in \text{span}\{x_k : k \in \mathbb{N}\}$ such that $\|x - x^n\| < \frac{\epsilon}{2}$ and $|d_n(x^n)| > \frac{t^{2^n}}{2}$. Note that if $m_n = \max\{j : x_j^*(x^n) \neq 0\}$, then $n \leq m_n$. Let N such that $\sum_{j \geq N} \frac{1}{j!} \leq \frac{\epsilon}{2}$ and $\|x_n\| < 1$ for every $n > N$. If each $x^n = \sum b_j^n x_j$ we consider the vector $a^n \in \ell_1$

$$a_j^n = \begin{cases} b_j^n & \text{if } j \leq n; \\ \text{sg}(b_j^n) \max\{|b_j^n|, \frac{1}{(N-n+j)!}\} & \text{if } j > n \end{cases}.$$

We claim that $a^n \in J_P$ belongs to J_P for large n . It suffices to show that $P^n(a^n) \in J_P$ for large n . Since $\Phi(a^n)$, satisfies that $|d_n(\Phi(a^n))| = |d_n(x^n)| = |c_n(a^n)c_n(\Omega)| \geq \frac{t^{2^n}}{2}$. We notice that $P^n(a^n) = c_n(a^n)c_n(\Omega)B_{\frac{1}{j^4}}^n(a^n)$ has all its coordinates, with modulus greater than or equal to the coordinates of

$$|c_n(a^n)c_n(\Omega)| \left(\frac{1}{N!n!^4}, \frac{2!^4}{(N+1)!(n+1)!^4}, \frac{3!^4}{(N+2)!(n+2)!^4}, \dots \right)$$

which are also greater than or equal to the coordinates of

$$\left(\frac{|c_n(a^n)c_n(\Omega)|}{n!^5}, \frac{1}{n+1!^5}, \frac{1}{n+2!^5}, \dots \right)$$

provided that $n > N$. Then by Lemma 4.2, it follows that if k_n satisfies that

$$k_n \left(2^{2^{\frac{j+1}{2}}} \right)^{\sqrt{2}-1} \geq (j+1)!^4(j+n-1)!^5 \text{ for every } j$$

then for every $L \geq k_n 2^{2^{\frac{1}{2}}}$ the vector $(L, \frac{1}{n!^5}, \frac{1}{(n+1)!^5}, \dots) \in J_P$. Note that we may take $k_n = (2n)!^5$. Thus, since $\frac{t^{2^n}}{2k_n} \rightarrow \infty$, for large enough n , we have $\frac{|c_n(a^n)c_n(\Omega)|}{n!^5} \geq k_n 2^{2^{\frac{1}{2}}}$ and therefore the vector $P^n(a^n)$ (and hence a^n) is eventually in J_P . Consequently, $\Phi(a^n) \in J_Q$.

Finally we have that

$$\begin{aligned} \|\Phi(a^n) - x\| &\leq \|x - x^n\| + \|\Phi(a^n) - x^n\| \\ &\leq \frac{\epsilon}{2} + \sum_{j=n+1}^{m_n} (b_j^n - a_j^n)\|x_j\| + \sum_{j=m_n+1}^{\infty} \frac{1}{j!}\|x_j\| \\ &\leq \frac{\epsilon}{2} + \left\| \left(\frac{1}{j!} \right)_{j \geq N} \right\|_{\ell_1} \leq \epsilon. \end{aligned}$$

□

Theorem 4.5. *Let X be an infinite dimensional separable Fréchet space, then there exists a non linear homogeneous polynomial that is at the same time weakly hypercyclic, d -hypercyclic and Γ -supercyclic for each $\Gamma \subset \mathbb{C}$ which is unbounded or has zero as an accumulation point.*

Proof. By Theorem 4.4 there exists $Q \in \mathcal{P}(^2X; X)$ a quasiconjugacy of $P = e'_1(x)B_{\frac{1}{n^4}}$ under a linear factor Φ . This polynomial is, by Theorem 4.1 weakly hypercyclic, d -hypercyclic and Γ -supercyclic for every unbounded or not bounded away from zero Γ . Since Φ is a linear operator, it follows by Propositions 3.2, 3.6 and 3.8 that Q satisfies the required properties. \square

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