
Bouligand-Severi tangents in MV-algebras

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Abstract. In their recent seminal paper published in the Annals of Pure and Applied Logic, Dubuc and Poveda call an MV-algebra A *strongly semisimple* if all principal quotients of A are semisimple. All boolean algebras are strongly semisimple, and so are all finitely presented MV-algebras. We show that for any 1-generator MV-algebra, semisimplicity is equivalent to strong semisimplicity. Further, a semisimple 2-generator MV-algebra A is strongly semisimple iff its maximal spectral space $\mu(A) \subseteq [0, 1]^2$ does not have any rational Bouligand-Severi tangents at its rational points. In general, when A is finitely generated and $\mu(A) \subseteq [0, 1]^n$ has a Bouligand-Severi tangent then A is not strongly semisimple. An MV-algebra A is strongly semisimple iff so is every two-generator subalgebra of A .

1. Introduction

We refer to [4] and [8] for background on MV-algebras. Following Dubuc and Poveda [5], we say that an MV-algebra A is *strongly semisimple* if for every principal ideal I of A the quotient A/I is semisimple. Since $\{0\}$ is a principal ideal of A , every strongly semisimple MV-algebra is semisimple. The definition of “logically complete” MV-algebras in [1] is a variant of this notion, where one further assumes $I \neq \{0\}$. The paper [7] is devoted to the frame-theoretic variant of strongly semisimple MV-algebras, called “Yosida frame”. All these papers, along with the results of the present paper, show that strong semisimplicity is a very interesting purely algebraic counterpart of the simplicial, topological, and differential structure of MV-algebras. Further, from the logical viewpoint, 4.3 in [9] shows that strongly semisimple MV-algebras coincide with Lindenbaum algebras of theories Θ in infinite-valued Łukasiewicz logic having the following property: for any formula ψ , the set of syntactic consequences of $\Theta \cup \{\psi\}$ coincides with the set of (Bolzano-Tarski) semantic consequences of $\Theta \cup \{\psi\}$.

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From a classical result by Hay [6] and Wójcicki [14] (also see 4.6.7 in [4] and 1.6 in [8]), it follows that every finitely presented MV-algebra is strongly semisimple. Trivially, all hyperarchimedean MV-algebras, whence in particular all boolean algebras, are strongly semisimple, and so are all simple and all finite MV-algebras, (see 3.5 and 3.6.5 in [4]).

For any real-valued function g we will write $Zg = g^{-1}(0)$ for its zeroset.

Our paper is devoted to n -generator strongly semisimple MV-algebras. When $n = 1$ strong semisimplicity is equivalent to semisimplicity (Theorem 5.1). To deal with the general case, we first recall that the free n -generator MV-algebra is the MV-algebra $\mathcal{M}([0, 1]^n)$ of all McNaughton functions $f: [0, 1]^n \rightarrow [0, 1]$, with pointwise operations of negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$. See 9.1.5 in [4].

For any nonempty closed set $X \subseteq [0, 1]^n$ we let $\mathcal{M}(X)$ denote the MV-algebra of restrictions to X of the functions in $\mathcal{M}([0, 1]^n)$, in symbols,

$$\mathcal{M}(X) = \{f \upharpoonright X \mid f \in \mathcal{M}([0, 1]^n)\}.$$

By 3.6.7 in [4], $\mathcal{M}(X)$ is a semisimple MV-algebra—actually, up to isomorphism, $\mathcal{M}(X)$ is the most general possible n -generator semisimple MV-algebra A : to see this, pick generators $\{a_1, \dots, a_n\}$ of A . Let $\pi_i: [0, 1]^n \rightarrow [0, 1]$ be the projection functions in the free MV-algebra $\mathcal{M}([0, 1]^n)$ for $i = 1, \dots, n$. Then the assignment that maps $\pi_i \mapsto a_i$ for each $i = 1, \dots, n$, uniquely extends to a homomorphism $\eta_a: \mathcal{M}([0, 1]^n) \rightarrow A$ of the free n -generator MV-algebra onto A . Let $\mathfrak{h}_a = \ker(\eta_a)$ be the kernel of this homomorphism and

$$(1.1) \quad \mathcal{Z}_a = \bigcap \{Zf \mid f \in \mathfrak{h}_a\}$$

the intersection of the zerosets of the McNaughton functions in \mathfrak{h}_a . Then

$$(1.2) \quad A \cong \mathcal{M}(\mathcal{Z}_a).$$

A point $x \in \mathbb{R}^n$ is said to be *rational* if so are all its coordinates. By a *rational vector* we mean a nonzero vector $w \in \mathbb{R}^n$ such that the line $\mathbb{R}w \subseteq \mathbb{R}^n$ contains at least two rational points. An MV-algebra A is strongly semisimple iff so is every two-generator subalgebra of A (Proposition 4.1). A 2-generator MV-algebra $A = \mathcal{M}(X)$, with nonempty closed $X \subseteq [0, 1]^2$, is strongly semisimple iff X has no rational outgoing Bouligand-Severi tangent vector at any of its rational points, [2, 12, 10]. See Theorem 3.1. As proved in Theorem 2.3, for any closed $X \subseteq [0, 1]^n$, having such a tangent is a sufficient condition for $\mathcal{M}(X)$ not to be strongly semisimple.

Notation: Following p.33 in [4] or p.21 in [8], for $k \in \mathbb{N}$, $k \cdot g$ stands for k -fold pointwise truncated addition of g .

2. Strong semisimplicity and Bouligand-Severi tangents

Severi (see §53, p.59 and p.392 of [11], as well as §1, p.99 of [12]) and independently, Bouligand (p.32 in [2]) called a half-line $H \subseteq \mathbb{R}^n$ *tangent* to a set $X \subseteq \mathbb{R}^n$ at an

accumulation point x of X if for all $\epsilon, \delta > 0$ there is $y \in X$ other than x such that $\|y - x\| < \epsilon$, and the angle between H and the half-line through y originating at x is $< \delta$.

Here as usual, $\|v\|$ is the length of the vector $v \in \mathbb{R}^n$.

On §2, p.100 and §4, p.102 of [12] Severi noted that for any accumulation point x of a closed set X there is a half-line H tangent to X at x .

Today (see, e.g., p.16 in [3], or p.1376 in [10]), Bouligand-Severi tangents are routinely introduced as follows:

Definition 2.1. Let x be an element of a closed subset X of \mathbb{R}^n , and u a unit vector in \mathbb{R}^n . We then say that u is a *Bouligand-Severi tangent (unit) vector to X at x* if X contains a sequence x_0, x_1, \dots of elements, all different from x , such that

$$\lim_{i \rightarrow \infty} x_i = x \quad \text{and} \quad \lim_{i \rightarrow \infty} (x_i - x) / \|x_i - x\| = u.$$

Observe that x is an accumulation point of X . We further say that u is *outgoing* if for some $\lambda > 0$ the segment $\text{conv}(x, x + \lambda u)$ intersects X only at x .

Already Severi noted that his definition of tangent half-line $H = x + \mathbb{R}_{\geq 0}u$ is equivalent to Definition 2.1. More precisely:

Proposition 2.2. (§5, p.103 of [12]). *For any nonempty closed subset X of \mathbb{R}^n , point $x \in X$, and unit vector $u \in \mathbb{R}^n$ the following conditions are equivalent:*

- (i) *For all $\epsilon, \delta > 0$, the cone $\text{cone}_{x,u,\epsilon,\delta}$ with apex x , axis parallel to u , vertex angle 2δ and height ϵ contains infinitely many points of X .*
- (ii) *u is a Bouligand-Severi tangent vector to X at x .*

When $n = 1$, $\text{cone}_{x,u,\epsilon,\delta}$ is the segment $\text{conv}(x, x + \epsilon u)$. When $n = 2$, $\text{cone}_{x,u,\epsilon,\delta}$ is the isosceles triangle $\text{conv}(x, a, b)$ with vertex x , basis $\text{conv}(a, b)$, height equal to ϵ (and parallel to u), and vertex angle $\widehat{axb} = 2\delta$.

The next two results provide geometric necessary and sufficient conditions on X for the semisimple MV-algebra $\mathcal{M}(X)$ to be strongly semisimple. These conditions are stated in terms of the non-existence of Bouligand-Severi tangent vectors having certain rationality properties.

Theorem 2.3. *Let X be a nonempty closed set in $[0, 1]^n$. Suppose X has a Bouligand-Severi rational outgoing tangent vector u at some rational point $x \in X$. Then $\mathcal{M}(X)$ is not strongly semisimple.*

Proof. Since u is outgoing, let $\lambda > 0$ satisfy $X \cap \text{conv}(x, x + \lambda u) = \{x\}$. Without loss of generality $x + \lambda u \in \mathbb{Q}^n$. By Definition 2.1, our hypothesis yields a sequence w_1, w_2, \dots of distinct points of X , all distinct from x , accumulating at x , at strictly decreasing distances from x , in such a way that the sequence of unit vectors u_i given by $(w_i - x) / \|w_i - x\|$ tends to u as i tends to ∞ . Let $y = x + \lambda u$. Since $X \cap \text{conv}(x, y) = \{x\}$, no point w_i lies on the segment $\text{conv}(x, y)$, and we can

further assume that the sequence of angles $\widehat{w_i x y}$ is strictly decreasing and tends to zero as i tends to ∞ .

Since both points x and y are rational, then by 2.10 in [8] for some $g \in \mathcal{M}([0, 1]^n)$ the zero set

$$Zg = \{z \in [0, 1]^n \mid g(z) = 0\}$$

coincides with the segment $\text{conv}(x, y)$. Thus,

$$\frac{\partial g(x)}{\partial(u)} = 0.$$

Let J be the ideal of $\mathcal{M}([0, 1]^n)$ generated by g ,

$$J = \{f \in \mathcal{M}([0, 1]^n) \mid f \leq k \cdot g \text{ for some } k = 0, 1, 2, \dots\}.$$

Then for each $f \in J$,

$$\frac{\partial f(x)}{\partial(u)} = 0.$$

Since the directional derivatives of f at x are continuous, (meaning that the map $t \mapsto \partial f(x)/\partial t$ is continuous) it follows that

$$(2.1) \quad \lim_{t \rightarrow u} \frac{\partial f(x)}{\partial(t)} = \frac{\partial f(x)}{\partial(u)} = 0.$$

Let $g^! = g \upharpoonright X$ and

$$J^! = \{f^! \in \mathcal{M}(X) \mid f^! \leq k \cdot g^! \text{ for some } k = 0, 1, 2, \dots\}$$

be the ideal of $\mathcal{M}(X)$ generated by $g^!$. A moment's reflection shows that

$$(2.2) \quad J^! = \{l \upharpoonright X \mid l \in J\}.$$

One inclusion is trivial. For the converse inclusion, if $f \upharpoonright X \leq (k \cdot g) \upharpoonright X$ then letting $l = f \wedge k \cdot g$ we get $l \leq k \cdot g$. So $l \in J$ and $l \upharpoonright X = f \upharpoonright X$, whence $f \upharpoonright X$ is extendible to some $l \in J$.

For any $f \in \mathcal{M}([0, 1]^n)$, the piecewise linearity of f ensures that for all large i the value of the incremental ratio $(f(w_i) - f(x))/\|w_i - x\|$ coincides with the directional derivative $\partial f(x)/\partial u_i$ along the unit vector $u_i = (w_i - x)/\|w_i - x\|$. Thus in particular, if $f \upharpoonright X = f^! \in J^!$, from (2.1)-(2.2) it follows that

$$\lim_{i \rightarrow \infty} \frac{f^!(w_i) - f^!(x)}{\|w_i - x\|} = 0.$$

Since x is rational, again by 2.10 in [8] there is $j \in \mathcal{M}([0, 1]^n)$ with $Zj = \{x\}$. For some $\omega > 0$ we have $\partial j(x)/\partial(u) = \omega$, whence

$$\lim_{i \rightarrow \infty} \frac{j^!(w_i) - j^!(x)}{\|w_i - x\|} = \omega.$$

Therefore, $j^! \notin J^!$. Since $Zg \cap X = \{x\}$, recalling 4.19 in [8] we see that the only maximal ideal of $\mathcal{M}(X)$ containing $J^!$ is the set of all functions in $\mathcal{M}(X)$ that vanish at x . Thus, $j^!$ belongs to all maximal ideals of $\mathcal{M}(X)$ containing $J^!$. By 3.6.6 in [4], $\mathcal{M}(X)$ is not strongly semisimple: specifically, $j^!/J^!$ is infinitesimal in the principal quotient $\mathcal{M}(X)/J^!$. \square

3. A partial converse of Theorem 2.3

Theorem 3.1. *Let $X \subseteq [0, 1]^n$ be a nonempty closed set. Suppose the MV-algebra $\mathcal{M}(X)$ is not strongly semisimple.*

- (i) *Then X has a Bouligand-Severi tangent vector u at some point $x \in X$ satisfying the following nonalignment condition: there is a sequence of distinct $w_i \in X$, all distinct from x such that*

$$\lim_{i \rightarrow \infty} w_i = x, \quad \lim_{i \rightarrow \infty} \frac{w_i - x}{\|w_i - x\|} = u, \quad w_i \notin \text{conv}(x, x + u) \text{ for all } i.$$

- (ii) *In particular, if $n = 2$, then X has a Bouligand-Severi outgoing rational tangent vector u at some rational point $x \in X$.*

Proof. (i) The hypothesis yields a function $g \in \mathcal{M}([0, 1]^n)$, with its restriction $g' = g \upharpoonright X \in \mathcal{M}(X)$, in such a way that the principal ideal J' of $\mathcal{M}(X)$ generated by g' ,

$$J' = \{l' \in \mathcal{M}(X) \mid l' \leq k \cdot g' \text{ for some } k = 1, 2, \dots\}$$

is strictly contained in the intersection I of all maximal ideals of $\mathcal{M}(X)$ containing J' . Thus for some $j \in \mathcal{M}([0, 1]^n)$ letting $j' = j \upharpoonright X$ we have $j' \in I \setminus J'$. By 3.6.6 in [4] and 4.19 in [8],

$$(3.1) \quad j' = 0 \text{ on } Zg', \text{ i.e., } X \cap Zj \supseteq X \cap Zg$$

and

$$(3.2) \quad \forall m = 0, 1, \dots \exists z_m \in X, \quad j'(z_m) > m \cdot g'(z_m).$$

There is a sequence of integers $0 < m_0 < m_1 < \dots$ and a subsequence y_0, y_1, \dots of $\{z_i, z_2, \dots\}$ such that $y_i \neq y_l$ for $i \neq l$ and

$$(3.3) \quad \forall t = 0, 1, \dots, \quad j'(y_t) > m_t \cdot g'(y_t).$$

The compactness of X yields an accumulation point $x \in X$ of the y_t . Without loss of generality (taking a subsequence, if necessary) we can further assume

$$(3.4) \quad \|y_0 - x\| > \|y_1 - x\| > \dots, \text{ whence } \lim_{i \rightarrow \infty} y_i = x.$$

By (3.3), for all t , $j'(y_t) > 0$. Then by (3.1), $g'(y_t) > 0$. For each $i = 0, 1, \dots$, letting the unit vector $u_i \in \mathbb{R}^n$ be defined by $u_i = (y_i - x)/\|y_i - x\|$, we obtain a sequence of (possibly repeated) unit vectors $u_i \in \mathbb{R}^n$. Since the boundary of the unit ball in \mathbb{R}^n is compact, some unit vector $u \in \mathbb{R}^n$ satisfies

$$\forall \epsilon > 0 \text{ there are infinitely many } i \text{ such that } \|u_i - u\| < \epsilon.$$

Some subsequence w_0, w_1, \dots of the y_i will satisfy the condition

$$(3.5) \quad \forall \epsilon, \delta > 0 \text{ there is } k \text{ such that for all } i > k, \quad w_i \in \text{cone}_{x, u, \epsilon, \delta}.$$

Correspondingly, the sequence v_0, v_1, \dots given by $v_k = (w_k - x)/\|w_k - x\|$ will satisfy

$$(3.6) \quad \lim_{i \rightarrow \infty} v_i = u.$$

We have just proved that u is a Bouligand-Severi tangent to X at x .

To complete the proof of (i) we prepare:

Fact 1. $g'(x) = 0$.

Otherwise, from the continuity of g , for some real $\rho > 0$ and suitably small $\epsilon > 0$, we have the inequality $g(z) > \rho$ for all z in the open ball $B_{x, \epsilon}$ of radius ϵ centered at x . By (3.5), $B_{x, \epsilon}$ contains infinitely many w_i . There is a fixed integer $\bar{m} > 0$ such that $1 = \bar{m} \cdot g' \geq j'$ for all these w_i , which contradicts (3.3).

Fact 2. $j'(x) = 0$.

This immediately follows from (3.1) and Fact 1.

Fact 3. $\partial g(x)/\partial u = 0$.

By way of contradiction, suppose $\partial g(x)/\partial u = \theta > 0$. In view of the continuity of the map $t \mapsto \partial g(x)/\partial t$, let $\delta > 0$ be such that $\partial g(x)/\partial r > \theta/2$, for any unit vector r such that $\widehat{ru} < \delta$. Since by Fact 2 $j(x) = 0$ and both g and j are piecewise linear, there is an $\epsilon > 0$ together with an integer $\bar{k} > 0$ such that $\bar{k} \cdot g \geq j$ over the cone $C = \text{cone}_{x, u, \epsilon, \delta}$. By (3.5), C contains infinitely many w_i , in contradiction with (3.3).

To conclude the proof of the nonalignment condition in (i), it is sufficient to settle the following:

Fact 4. *There is $\lambda > 0$ such that for all large i the segment $\text{conv}(x, x + \lambda u)$ contains no w_i .*

For otherwise, from Fact 3, $\partial g(x)/\partial(u) = 0$, whence the piecewise linearity of g ensures that g vanishes on infinitely many w_i of $\text{conv}(x, x + \lambda u)$ arbitrarily near x . Any such w_i belongs to X , whence by (3.1), $j(w_i) = 0$, in contradiction with (3.3).

The proof of (i) is now complete.

(ii) Let H^\pm be the two closed half-spaces of \mathbb{R}^2 determined by the line passing through x and $x + u$. By (3.5), infinitely many w_i lie in the same closed half-space, say, H^+ . Without loss of generality, $H^+ \cap \text{int}([0, 1]^2) \neq \emptyset$. Let u^\perp be the orthogonal vector to u such that $x + u^\perp \in H^+$.

Fact 5. *For all small $\epsilon > 0$,*

$$\frac{\partial g(x + \epsilon u)}{\partial u^\perp} > 0.$$

By way of contradiction, assume $\partial g(x + \epsilon u)/\partial u^\perp = 0$. Since g is piecewise linear, by Facts 1 and 3, for suitably small $\eta, \omega > 0$, the function g vanishes over the triangle $T = \text{conv}(x, x + \eta u, x + \eta u + \omega u^\perp)$. By (3.5), T contains infinitely many w_i . By (3.1), $g(w_i) = j(w_i) = 0$ against (3.3).

Fact 6.

$$\frac{\partial j(x)}{\partial u} > 0.$$

Otherwise, $\partial j(x)/\partial u = 0$. Fact 5 yields a fixed integer \bar{h} such that, on a suitably small triangle of the form $T = \text{conv}(x, x + \epsilon u, x + \epsilon u + \omega u^\perp)$, we have $\bar{h} \cdot g \geq j$. By (3.5), T contains infinitely many w_i , again contradicting (3.3).

We now prove a strong form of Fact 4, showing that u is an *outgoing* tangent vector:

Fact 7. *For some $\lambda > 0$ the segment $\text{conv}(x, x + \lambda u)$ intersects X only at x .*

Otherwise, from Facts 1 and 3 it follows that g vanishes on infinitely many points of $X \cap \text{conv}(x, x + \lambda u)$ converging to x . By (3.1), j vanishes on all these points. Since j is piecewise linear, $\partial j(x)/\partial u = 0$, against Fact 6.

By a *rational* line in \mathbb{R}^n we mean a line passing through at least two distinct rational points.

Fact 8. *x is a rational point, and u is a rational vector.*

As a matter of fact, Facts 6 and 2 yield a rational line L through x . On the other hand, Facts 3 and 5 show that the line passing through x and $x + u$ is rational and different from L . Thus x is rational, whence so is the vector u .

We conclude that X has u as a Bouligand-Severi *outgoing rational* tangent vector at the rational point x . \square

Figure 1 is a sketch of the functions g and j in the foregoing proof.

Recalling Theorem 2.3 we now obtain:

Corollary 3.2. *Let $X \subseteq [0, 1]^2$ be a nonempty closed set. Then $\mathcal{M}(X)$ is not strongly semisimple iff X has a Bouligand-Severi outgoing rational tangent vector u at some rational point $x \in X$.*

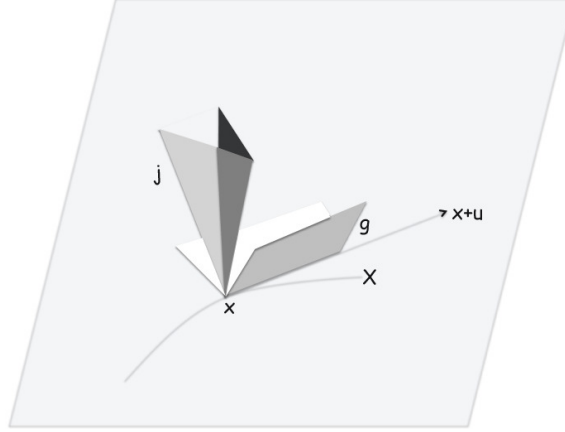


Figure 1: A Bouligand-Severi outgoing tangent vector u to X at x , and two functions g and j . The restriction $g|_X$ generates a principal ideal J' of $\mathcal{M}(X)$. The restriction $j|_X$ does not belong to J' , but belongs to the only maximal ideal I' of $\mathcal{M}(X)$ containing J' , namely the set of all functions in $\mathcal{M}(X)$ vanishing at x . So the principal quotient $\mathcal{M}(X)/J'$ is not semisimple.

Examples. The above corollary provides many examples of two-generator strongly semisimple MV-algebras:

- (i) Let $\kappa \in [0, 1]$ be irrational. Let W be the arc of parabola $\{(x, y) \in [0, 1]^2 \mid y = \kappa x^2\}$. Then $\mathcal{M}(W)$ is strongly semisimple—for want of rational points in W . One can similarly construct two-generator strongly semisimple MV-algebras of the form $\mathcal{M}(V)$, by letting V be a closed subset of $[0, 1]^2$ without rational points, or else, without outgoing rational tangents.
- (ii) Following [13], let $Q \subseteq [0, 1]^2$ be a *polyhedron* in $[0, 1]^2$, i.e., a finite union of m -simplexes ($m = 0, 1, 2$) in $[0, 1]^2$. Then Q does not have any outgoing Bouligand-Severi tangent, whence $\mathcal{M}(Q)$ is strongly semisimple.
- (iii) (Generalizing (ii)). Let A be a two-generator subalgebra of a semisimple tensor product (see §9.4 in [8]) of the form $[0, 1] \otimes D$, where D is a finitely presented MV-algebra. Using Lemma 3.6 and Theorem 6.3 in [8], one sees that A is isomorphic to an MV-algebra of the form $\mathcal{M}(Q)$ for some polyhedron $Q \subseteq [0, 1]^2$. Thus A is strongly semisimple.

4. The general case

The central role of finitely generated, and especially of 2-generator strongly semisimple MV-algebras among all strongly semisimple MV-algebras, is shown by the following result:

Proposition 4.1. *For any MV-algebra A the following conditions are equivalent:*

- (i) *A is strongly semisimple;*
- (ii) *A is the direct limit of a direct system $\mathcal{S} = \{A_i, \phi_{ij}\}$ of finitely generated strongly semisimple algebras A_i , where all the homomorphisms $\phi_{ij}: A_i \rightarrow A_j$ are embeddings;*
- (iii) *Each 2-generator subalgebra of A is strongly semisimple.*

Proof. Recall that an MV-algebra is semisimple iff it has no infinitesimals. For any MV-algebras C, D and embedding $\phi: C \rightarrow D$, letting for any $y \in C$, $\langle y \rangle_C$ denote the ideal generated by y in C , we first make the following elementary observations:

- (I) For each $c \in C$, the map $\bar{\phi}: C/\langle c \rangle_C \rightarrow D/\langle \phi(c) \rangle_D$ defined by $x/\langle c \rangle_C \mapsto \phi(x)/\langle \phi(c) \rangle_D$ is an embedding. This immediately follows by observing that $\phi(\langle c \rangle_C) = \langle \phi(c) \rangle_D \cap \phi(C)$.
- (II) $c \in C$ is an infinitesimal of C iff $\phi(c)$ is an infinitesimal of D .
- (III) If D is strongly semisimple then so is C . As a matter of fact, for any $c \in C$, the map $\bar{\phi}: C/\langle c \rangle_C \rightarrow D/\langle \phi(c) \rangle_D$ of (I) is an embedding. By hypothesis, $D/\langle \phi(c) \rangle_D$ is semisimple, whence so is $C/\langle c \rangle_C$ by (II).

We are now ready to prove the proposition:

(i) \Rightarrow (ii). Let $\mathcal{A} = \{A_i \subseteq A \mid A_i \text{ is a finitely generated subalgebra of } A\}$, and let $\phi_{ij}: A_i \rightarrow A_j$ be the inclusion map whenever $A_i \subseteq A_j$. Then \mathcal{A} together the homomorphisms ϕ_{ij} is a direct system of MV-algebras, having A as its direct limit. By (III), each A_i is strongly semisimple.

(ii) \Rightarrow (i). Let $\mathcal{S} = \{A_i, \phi_{ij}\}$ be a directed system of strongly semisimple MV-algebras, indexed by the directed partially ordered set I , where each ϕ_{ij} is an embedding of A_i into A_j . Let A be the direct limit of \mathcal{S} with the telescopic maps $\phi_{i\infty}: A_i \rightarrow A$. Each $\phi_{i\infty}$ is an embedding. Suppose that A is not strongly semisimple, (absurdum hypothesis), and let $g \in A$ be such that $A/\langle g \rangle_A$ is not semisimple. Then there is an element $e \in A$ such that $e/\langle g \rangle_A$ is an infinitesimal of $A/\langle g \rangle_A$. Since the partial order of the index set I is directed, for some $i \in I$ there are $g_i, e_i \in A_i$ with $\phi_{i\infty}(g_i) = g$ and $\phi_{i\infty}(e_i) = e$. The map $\bar{\phi}_{i\infty}: A_i/\langle g_i \rangle_{A_i} \rightarrow A/\langle g \rangle_A$ of (I) is an embedding. By (II), $e_i/\langle g_i \rangle_{A_i}$ is an infinitesimal element of $A_i/\langle g_i \rangle_{A_i}$, against the hypothesis that A_i is strongly semisimple.

(i) \Rightarrow (iii). Immediate from (III).

(iii) \Rightarrow (i). If A is not strongly semisimple there are elements $g, e \in A$ such that $e/\langle g \rangle_A$ is an infinitesimal in $A/\langle g \rangle_A$. Let $B \subseteq A$ be the subalgebra of A generated by g and e . By (I)-(II) $e/\langle g \rangle_B$ is an infinitesimal element of $B/\langle g \rangle_B$, and B is not strongly semisimple. \square

5. Coda: one-generator MV-algebras

The following result is an easy consequence of Theorem 3.1. We include the elementary proof because it provides a technique to deal with strong semisimplicity independently of Bouligand-Severi tangents.

Theorem 5.1. *Every one-generator semisimple MV-algebra A is strongly semisimple.*

Proof. As in (1.1)-(1.2), let $X \subseteq [0, 1]$ be a nonempty closed set such that $A \cong \mathcal{M}(X)$. For some $g \in \mathcal{M}([0, 1])$ let J be the principal ideal of $\mathcal{M}([0, 1])$ generated by g , and J' be the principal ideal of $\mathcal{M}(X)$ generated by $g' = g \upharpoonright X$.

The short argument immediately following (2.2) shows that $J' = \{l \upharpoonright X \mid l \in J\}$. For every $f \in \mathcal{M}([0, 1])$, letting $f' = f \upharpoonright X$ we must prove: if f' belongs to all maximal ideals of $\mathcal{M}(X)$ to which g' belongs, then f' belongs to J' . By 3.6.6 in [4] and 4.19 in [8], this amounts to proving

$$(5.1) \quad \text{if } f = 0 \text{ on } Zg \cap X \text{ then } f \upharpoonright X \in J'.$$

Let Δ be a triangulation of $[0, 1]$ such that f and g are linear over every simplex of Δ . The existence of Δ follows from the piecewise linearity of f and g , [13]. In view of the compactness of X and $[0, 1]$, it is sufficient to settle the following

Claim. Suppose $f \in \mathcal{M}([0, 1])$ vanishes over $Zg \cap X$. Then for all $x \in X$ there is an open neighbourhood $\mathcal{N}_x \ni x$ in $[0, 1]$ together with an integer $m_x \geq 0$ such that $m_x \cdot g \geq f$ on $\mathcal{N}_x \cap X$.

We proceed by cases:

Case 1: $g(x) > 0$. Then for some integer r and open neighbourhood $\mathcal{N}_x \ni x$ we have $g > 1/r$ over \mathcal{N}_x . Letting $m_x = r$ we have $1 = m_x \cdot g \geq f$ over \mathcal{N}_x , whence a fortiori, $m_x \cdot g \geq f$ over $\mathcal{N}_x \cap X$.

Case 2: $g(x) = 0$. Since f vanishes over $Zg \cap X$, then $f(x) = 0$. Let T be a 1-simplex of Δ such that $x \in T$. Let T_x be the smallest face of T containing x .

Subcase 2.1: $T_x = T$. Then $x \in \text{int}(T)$. Since g is linear over T then g vanishes over T . By our hypotheses on f and Δ , f vanishes over T , whence and $0 = g \geq f = 0$ on T . Letting $\mathcal{N}_x = \text{int}(T)$ and $m_x = 1$, we get $m_x \cdot g \geq f$ over \mathcal{N}_x whence a fortiori, the inequality holds over $\mathcal{N}_x \cap X$.

Subcase 2.2: $T_x = \{x\}$. Then $T = \text{conv}(x, y)$ for some $y \neq x$. Without loss of generality, $y > x$. We will exhibit a right open neighbourhood $\mathcal{R}_x \ni x$ and an integer $r_x \geq 0$ such that $r_x \cdot g \geq f$ on $\mathcal{R}_x \cap X$. The same argument yields a left neighbourhood $\mathcal{L}_x \ni x$ and an integer $l_x \geq 0$ such that $l_x \cdot g \geq f$ on $\mathcal{L}_x \cap X$. One then takes $\mathcal{N}_x = \mathcal{R}_x \cup \mathcal{L}_x$ and $m_x = \max(r_x, l_x)$.

Subsubcase 2.2.1: If both g and f vanish at y , then they vanish over T (because they are linear over T). Upon defining $\mathcal{R}_x = \text{int}(T) \cup \{x\}$ and $r_x = 1$ we get $r_x \cdot g \geq f$ over \mathcal{R}_x , whence in particular, over $\mathcal{R}_x \cap X$.

Subsubcase 2.2.2: If both g and f are > 0 at y then for all suitably large m we have $m \cdot g \geq f$ on T because $f(x) = 0$ and both f and g are linear on T . Letting r_x the smallest such m and $\mathcal{R}_x = \text{int}(T) \cup \{x\}$ we have the desired inequality over \mathcal{R}_x and a fortiori over $\mathcal{R}_x \cap X$.

Subsubcase 2.2.3: $g(y) = 0, f(y) > 0$. By our hypotheses on Δ , g is linear over T and hence $g = 0$ over T . It follows that $X \cap T = \{x\}$: for otherwise, our assumption $Zf \cap X \supseteq Zg \cap X$ together with the linearity of f over T would imply $f(y) = 0$, against our current hypothesis. Letting $\mathcal{R}_x = \text{int}(T) \cup \{x\}$ and $r_x = 1$ we have $r_x \cdot g \geq f$ over $\mathcal{R}_x \cap X$. \square

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