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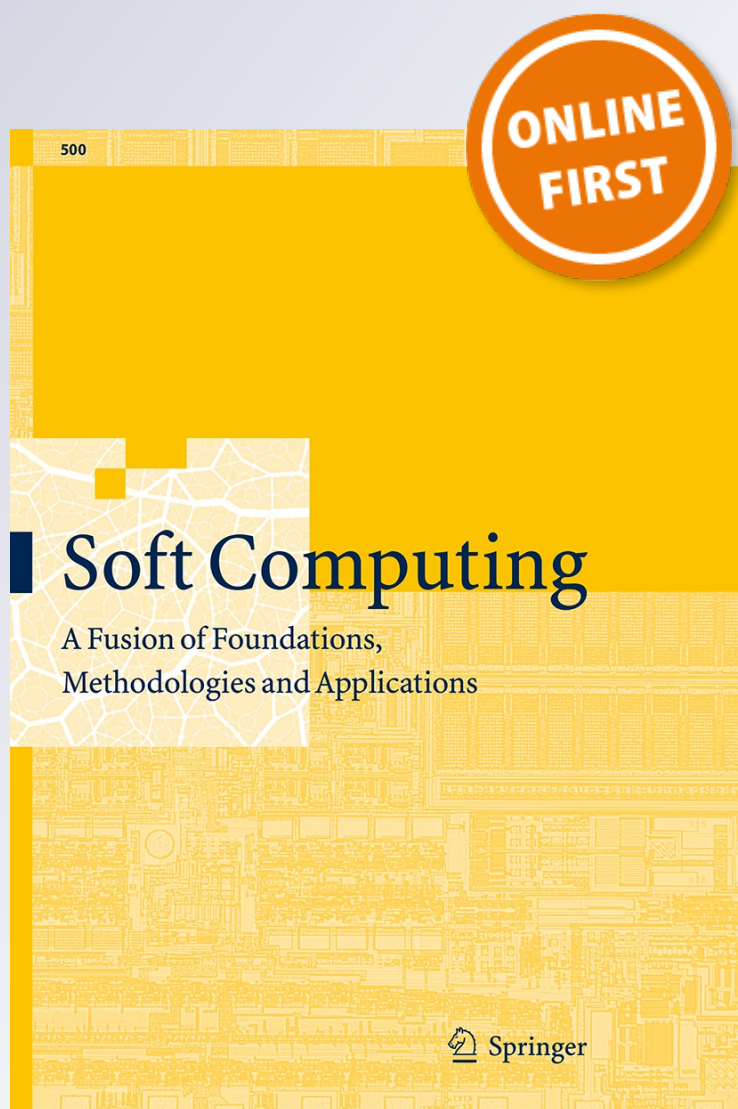
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On products of posets and coproducts of KM-algebras

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Abstract The main goal of this work is to give an explicit computation of some coproducts in the variety of KM-algebras introduced by the Chişinău group headed by Kuznetsov during the decade of the 1970s. This is done using a Priestley style duality.

Keywords Heyting algebras with operators · KM-algebras · Coproducts · Esakia duality · Products

1 Introduction and basic results

Consider the category whose objects are finite posets and whose morphisms are those p-morphisms which preserve the strict order. Write lpFP for this category. Let us write lpFP_2 for the full subcategory of lpFP whose objects have height at most 2. In this work an explicit description of the categorical product in lpFP between a root system and another arbitrary object is given. An explicit construction of the categorical product in lpFP_2 is also given. Apart from the interest in its own of the description of these products, this allows us to achieve the aforementioned goal.

In what follows we introduce some basic notions and results that we shall use in the rest of this work.

Let X be a poset and $Y \subseteq X$. We say that Y is an upset of X if $x \geq y$ and $y \in Y$ imply that $x \in Y$. We say that Y is a

downset of X if $x \leq y$ and $y \in Y$ imply that $x \in Y$. We write Y^c for the complement set of Y and Y_M for the set of maximal elements of Y . If $Z \subseteq X$, we also define $Y \setminus Z := Y \cap Z^c$.

Let \mathbb{N} be the set of natural numbers (starting with 1). Then for every $n \in \mathbb{N}$ we define an increasing sequence of sets $\{X_n\}_{n \geq 1}$ as follows:

$$\begin{aligned} X_1 &= X_M, \\ X_{n+1} &= X_n \cup (X_n^c)_M. \end{aligned}$$

Then for $n \geq 2$ we define the sets \hat{X}_n by

$$\begin{aligned} \hat{X}_1 &= X_1, \\ \hat{X}_n &= (X_{n-1}^c)_M = X_n \setminus X_{n-1}. \end{aligned}$$

We say that the poset X has height n if $X = X_n$ and n is the minimum natural number with this property. Note that for every $n \in \mathbb{N}$, X_n is an upset and $X_n = \bigcup_{i=1}^n \hat{X}_i$, so we can equivalently say that the poset X has height less than or equal to n if and only if $X = \bigcup_{i=1}^n \hat{X}_i$.

In this work we shall use the word height of an element x in a poset X of height n , to indicate the i such that $x \in \hat{X}_i$. Compare this use with that given in Def. 7.1. of Harzheim (2005). Note that for an element x in a poset X of height n we have that the height of x is smaller or equal than n .

Let $f : X \rightarrow Y$ be a morphism of posets. We say that f is a p -morphism if given $x \in X$ and $y \in Y$ such that $f(x) \leq y$ then there exists $z \in X$ such that $x \leq z$ and $f(z) = y$. We say that f preserves levels if $\hat{X}_i \subseteq f^{-1}(\hat{Y}_i)$. We say that f is a strict morphism if for every $x, y \in X$, $x < y$ implies that $f(x) < f(y)$ (where $<$ is the strict order associated to the order \leq).

Now we define the following categories: The following elementary lemma (Lemma 2.2 of Castiglioni and San Martín 2011) will be useful for this work:

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Category	Objects	Morphisms
FP	Finite posets	Morphisms of posets
IFP	Finite posets	Morphisms of posets which preserve levels
pFP	Finite posets	Morphisms of posets which are p -morphisms
lpFP	Finite posets	Morphisms in IFP \cap pFP

Lemma 1 *Let X be a poset. Then:*

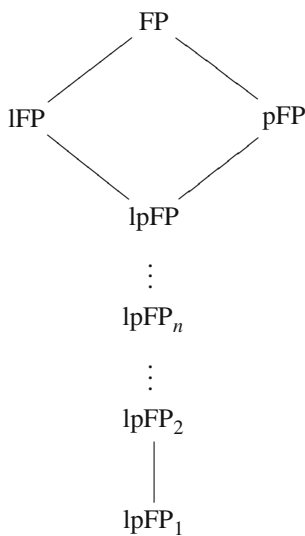
- (a) *If $i \neq j$, then $\hat{X}_i \cap \hat{X}_j = \emptyset$.*
- (b) *If $x \leq y$, $x \in \hat{X}_i$ and $y \in \hat{X}_j$ then $j \leq i$.*
- (c) *If $i \geq 2$ and $x \in \hat{X}_i$ then there is $y \in \hat{X}_{i-1}$ such that $x < y$.*

Let $f \in \text{FP}$. It is immediate that if $f \in \text{IFP}$, then f is a strict morphism. On the other hand, if f is a p -morphism and a strict morphism, it follows from straightforward computations based on Lemma 1 that $f \in \text{IFP}$. Therefore, we obtain the following:

Remark 1 Let f be a morphism in FP. Then, f is a morphism in lpFP if and only if f is a strict morphism.

For every n we define lpFP_n as the full subcategory of lpFP whose objects have height less than or equal to n .

We establish the relationship among the above mentioned categories in the following diagram:



In Sect. 2 we give an explicit description of the categorical product in IFP. We also give an explicit description of the categorical product in lpFP between a root system and another arbitrary element, and an explicit construction for the categorical product in lpFP_2 . Finally, in Sect. 3 we apply the results of the previous section to get some coproducts in the category of Heyting algebras with successor.

2 Computing products

For $X, Y \in \text{FP}$, let us define $X \times_L Y := \bigcup_{i=1}^{\infty} (\hat{X}_i \times \hat{Y}_i)$. In particular, X and Y have height less than or equal to n for some n . Thus, $X \times_L Y = \bigcup_{i=1}^n (\hat{X}_i \times \hat{Y}_i)$. This set is a poset with the order induced by the usual one in the Cartesian product $X \times Y$. If k is the minimum between the heights of X and Y , we also have that $X \times_L Y = \bigcup_{i=1}^k (\hat{X}_i \times \hat{Y}_i)$. We shall see that $X \times_L Y$ is the categorical product in IFP.

Lemma 2 *Let $X, Y \in \text{FP}$ and $Z = X \times_L Y$. Then $\hat{Z}_i = \hat{X}_i \times \hat{Y}_i$ for each i .*

Proof For $i = 1$ the property is immediate. Suppose that the property holds for every $i \leq m$ for some m . Let us prove it for $i = m + 1$. Let $(x, y) \in \hat{X}_{m+1} \times \hat{Y}_{m+1}$. Suppose that $(x, y) \in Z_m$, so $(x, y) \in (\hat{X}_i \times \hat{Y}_i)$ for some $i \leq m$, which is a contradiction by (a) of Lemma 1. Then $(x, y) \notin Z_m$. Let $(x, y) \leq (z, w)$, with $(z, w) \notin Z_m$. Thus, $x \leq z$ and $y \leq w$. Note that $(z, w) \in \hat{X}_i \times \hat{Y}_i$ for some i , so by (b) of Lemma 1 we have that $i \leq m + 1$. Taking into account that $(z, w) \notin Z_m$, we obtain that $i \geq m + 1$. Then we have that $i = m + 1$. Hence, $x = z$ and $y = w$. Thus, $\hat{X}_{m+1} \times \hat{Y}_{m+1} \subseteq \hat{Z}_{m+1}$. Conversely, let $(x, y) \in \hat{Z}_{m+1}$. In particular, $(x, y) \in \hat{X}_i \times \hat{Y}_i$ for some i . Using that $\hat{X}_i \times \hat{Y}_i \subseteq \hat{Z}_i$ we have that $(x, y) \in \hat{Z}_i$. Hence, by (a) of Lemma 1 we have that $i = m + 1$. Therefore, we obtain that $\hat{Z}_{m+1} \subseteq \hat{X}_{m+1} \times \hat{Y}_{m+1}$. \square

Let $X, Y \in \text{FP}$. We define the maps $\pi_X : X \times_L Y \rightarrow X$ as $\pi_X(x, y) = x$, and $\pi_Y : X \times_L Y \rightarrow Y$ as $\pi_Y(x, y) = y$.

Proposition 1 *Let $X, Y \in \text{FP}$. Then π_X and π_Y are morphisms in lpFP.*

Proof It is immediate that π_X is monotone. In order to prove that π_X is a p -morphism, let $(x, y) \in \bigcup_{i=1}^n (\hat{X}_i \times \hat{Y}_i)$ and $z \in X$ such that $x = \pi_X(x, y) \leq z$. In particular, $(x, y) \in \hat{X}_i \times \hat{Y}_i$ for some i , and $z \in \hat{X}_j$ for some j . Taking into account (b) of Lemma 1 we have that $j \leq i$. Suppose that $i = 1$, so $x \in X_M$ and $x = z$. Thus, $(x, y) \leq (z, y) \in X_M \times Y_M$ and $z = \pi_X(z, y)$. Suppose now that $i > 1$. It follows from (c) of Lemma 1 that there exists $w \in \hat{Y}_j$ such that $y \leq w$. Hence, $(x, y) \leq (z, w) \in \hat{X}_j \times \hat{Y}_j$ and $z = \pi_X(z, w)$. In consequence, π_X is a p -morphism. It is immediate that π_X is a morphism in IFP. Therefore, π_X is a morphism in lpFP. Analogously we can prove that π_Y is a morphism in lpFP. \square

Theorem 3 *Let $X, Y \in \text{IFP}$. Then $X \times_L Y$ is the categorical product in IFP.*

Proof Let $f : Z \rightarrow X \in \text{IFP}$, $g : Z \rightarrow Y \in \text{IFP}$. We define $h : Z \rightarrow X \times_L Y$ by $h(z) = (f(z), g(z))$. The well definition of h follows from that $f, g \in \text{IFP}$. It is immediate that h is monotone. Using that $f, g \in \text{IFP}$ and Lemma 2, we

obtain that $\hat{Z}_i \subseteq f^{-1}(\hat{X}_i) \cap g^{-1}(\hat{Y}_i) = h^{-1}(\hat{X}_i \times \hat{Y}_i) = h^{-1}(X \times_L Y_i)$ for every i . Hence, $h \in \text{IFP}$. Hence, taking into account Proposition 1 we conclude that $X \times_L Y$ is the categorical product in IFP. \square

Remark 2 Since the coproduct of finite Heyting algebras is not in general a finite algebra, it follows from Esakia duality (1974) that pFP is not necessarily closed under finite products.

If X is a poset and $Y \subseteq X$, we define $[Y] := \{y \leq x \text{ for some } y \in Y\}$ and $[Y] := \{y \geq x \text{ for some } y \in Y\}$. If $Y = \{y\}$, we write $[y]$ in place of $\{[y]\}$ and $[y]$ in place of $\{[y]\}$.

Definition 1 A poset X is said to be a root system if for every $x \in X$ it holds that $[x]$ is a chain.

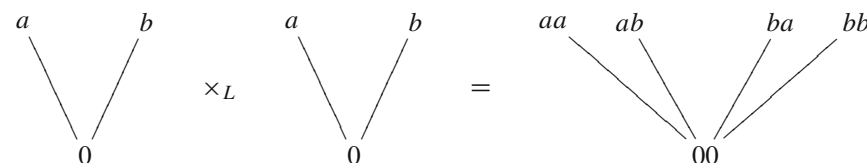
Corollary 4 Let $X, Y \in \text{lpFP}$ and assume that either X or Y is a root system. Then $X \times_L Y$ is the categorical product in lpFP.

Proof Consider $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ morphisms in lpFP. Define h as in the proof of Theorem 3. Suppose that Y is a root system. In order to prove that $X \times_L Y$ is the categorical product in lpFP, we need to show that h is a p -morphism. Let $h(z) = (f(z), g(z)) \leq (t, s)$. It follows from the fact that f and g are p -morphisms that there exist $u, v \in Z$ such that $z \leq u, z \leq v, f(u) = t$ and $g(v) = s$. In particular, $g(z) \leq g(u)$ and $g(z) \leq s$. Using that Y is a root system we obtain that $g(u) \leq s$ or $s \leq g(u)$. Besides there exists i such that $(t, s) \in \hat{X}_i \times \hat{Y}_i$. Thus, we have that $u, v \in \hat{Z}_i$ and $g(u), s \in \hat{Y}_i$. Hence $g(u) = s$. Thus, $z \leq u$ and $h(z) \leq (t, s) = (f(u), g(u)) = h(u)$. Therefore we conclude that h is a p -morphism. \square

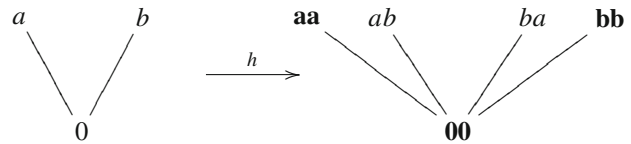
Remark 3 Let $X, Y \in \text{lpFP}_n$, and assume that either X or Y is a root system. Then $X \times_L Y$ is the categorical product in lpFP_n .

However, it is not in general the case that $X \times_L Y$ is the categorical product in lpFP, as the following example shows.

Example 1 Let $X = Y = \{0, a, b\}$, with $0 < a, 0 < b$ and a and b incomparable. Then we have that



Let now be $Z = X$ and $f, g : Z \rightarrow X$ the identity. Hence, the map $h : Z \rightarrow X \times_L X$ given by $h(z) = (z, z)$ is not a p -morphism, as it follows from the following diagram:



In the rest of this section we shall develop some tools in order to obtain a description of the categorical product in lpFP_2 .

Definition 2 Let $P \in \text{FP}$. For every $p \in P$, let $ht_P(p)$ be the height of p in P . A subposet Q of P is called rooted if it satisfies the following conditions:

1. For every $q \in Q$, $ht_Q(q) = ht_P(q)$.
2. The poset Q has minimum, i.e. there exist $q \in Q$ such that $Q = [q]_P$.

If Q is a rooted subposet of P , we write $m(Q)$ for the minimum of Q . We write $R(P)$ to indicate the poset which elements are rooted subposets of P , where the order is given by

$$Q_1 \leq Q_2 \text{ if and only if } [m(Q_2)]_{Q_1} = Q_2, \text{ i.e. } Q_2 \text{ is an upset in } Q_1.$$

Definition 3 Let $X, Y \in \text{FP}$. We write $R^*(X \times_L Y)$ for the subposet of $R(X \times_L Y)$ whose elements Q satisfy the following condition: $\pi_X(Q)$ is an upset in X and $\pi_Y(Q)$ is an upset in Y .

Let k be the minimum between the heights of $X, Y \in \text{FP}$. Straightforward computations show that $R^*(X \times_L Y)$ is a finite poset of height k .

Proposition 2 Let $X, Y \in \text{lpFP}$. The map $m : R^*(X \times_L Y) \rightarrow X \times_L Y$ is a surjective morphism in lpFP. Moreover, if X or Y are root systems, then m is an isomorphism in lpFP.

Proof It is clear that m is an surjective morphism in lpFP. Assume that Y is a root system. In order to prove that m is an injective map, let $Q_1, Q_2 \in R^*(X \times_L Y)$ such that $(x_1, y_1) = \min(Q_1) = \min(Q_2) = (x_2, y_2)$, and let $(x, y) \in Q_1$. Then $(x_2, y_2) = (x_1, x_2) \leq (x, y)$ and $\pi_1(x_2, y_2) \leq x$. Then there

exists $z \in Y$ such that $(x, z) \in Q_2$. But $y_2 \leq y$ and $y_2 \leq z$, so using that Y is a root system we have that $y \leq z$ or $z \leq y$. As the heights of y and z are equals, we obtain that $y = z$. Thus, $(x, y) \in Q_2$. Hence, $Q_1 \subseteq Q_2$. In the same way can be proved the converse inclusion. Thus, m is an injective map.

Assume again that Y is a root system. In order to prove that m is an isomorphism in lpFP , it suffices to show that if $Q_1, Q_2 \in R^*(X \times_L Y)$ are such that $m(Q_1) \leq m(Q_2)$, then $Q_1 \subseteq Q_2$. Let $Q_1, Q_2 \in R^*(X \times_L Y)$ such that $m(Q_1) \leq m(Q_2)$. Put $(x_1, y_1) = m(Q_1)$ and $(x_2, y_2) = m(Q_2)$. Let $(x, y) \in Q_2$. Then, $x_1 \leq x$ and $y_1 \leq y$. Thus, there exist $z \in Y$ and $w \in X$ such that $(x, z) \in Q_1$ and $(w, y) \in Q_1$. As Y is a root system, $x \leq w$ or $w \leq x$. But $(x, y) \in \hat{X}_i \times \hat{Y}_i$ for some i and $w \in \hat{X}_i$ (because $(w, y) \in Q_1$). Hence, $x = w$ and $(x, y) \in Q_1$. Thus, $Q_2 \subseteq [m(Q_2)]_{Q_1}$. Conversely, let $(x, y) \in [m(Q_2)]_{Q_1}$. So we obtain that $x_2 \leq x$, $y_2 \leq y$ and $(x, y) \in Q_1$. Then there exist $z \in Y$ and $w \in X$ such that $(x, z), (w, y) \in Q_2$. Using that $(x, y) \in \hat{X}_i \times \hat{Y}_i$ for some i , we have that $(x, z) \in \hat{X}_i \times \hat{Y}_i$. As $y_2 \leq x$, $y_2 \leq z$ and Y is a root system, we have that $x \leq z$ or $z \leq x$. Taking into account that $x, z \in \hat{X}_i$ we have that $x = z$, so $(x, y) \in Q_2$. Thus, $[m(Q_2)]_{Q_1} \subseteq Q_2$. Therefore we conclude that $[m(Q_2)]_{Q_1} \subseteq Q_2$, i.e. $Q_1 \subseteq Q_2$, which was our aim. \square

Lemma 5 Let $X, Y \in \text{lpFP}$. The maps $p_X : R^*(X \times_L Y) \rightarrow X$ and $p_Y : R^*(X \times_L Y) \rightarrow Y$ given by $p_X(Q) = \pi_X(m(Q))$ and $p_Y(Q) = \pi_Y(m(Q))$ are morphisms in lpFP .

Proof It is immediate that p_X is monotone. In order to prove that p_X is p -morphism, let $x \in X$ and $Q \in R^*(X \times_L Y)$ such that $p_X(Q) \leq x$. Then there exists $y \in Y$ such that $(x, y) \in Q$. Defining $T = [(x, y)]_Q$, we have that $Q \leq T$ and $p_X(T) = x$. Hence, p_X is p -morphism. Straightforward computation show that p_X preserves $<$. Hence, it follows from Remark 1 that p_X is a morphism in lpFP . In a similar way can be proved that p_Y is a morphism in lpFP . \square

Proposition 3 Let $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ be two morphisms in lpFP_2 , and let $h : Z \rightarrow R^*(X \times_L Y)$ be the map given by $h(z) = \{(f(w), g(w)) : z \leq w\}$. Then h is a morphism in lpFP_2 .

Proof Is immediate that h is monotone. We shall prove that h is a p -morphism. Let $z \in Z$ and $Q \in R^*(X \times_L Y)$ such that $h(z) \leq Q$, i.e. $[m(Q)]_{h(z)} = Q$. In particular, there exists $w \geq z$ such that $m(Q) = (f(w), g(w))$. If $z = w$ we have that $h(z) = h(w) = Q$. Let $z < w$. We shall prove that $h(w) = Q$. In order to prove that $h(w) \subseteq Q$, let $t \in Z$ such that $w \leq t$. Using that $z < w$ we obtain that $z \in \hat{Z}_2$ and $w \in \hat{Z}_1$, so $t = w$. Taking into account that $m(Q) = (f(w), g(w)) = (f(t), g(t))$, we have that $(f(t), g(t)) \in Q$. Hence, $h(w) \subseteq Q$. Conversely, let $(x, y) \in Q$, so there exists $t \in Z$ such that $z \leq t$ and $(x, y) = (f(t), g(t))$. Using that $(f(w), g(w)) \leq (f(t), g(t))$ and that $w \in \hat{Z}_1$,

we obtain that $(f(t), g(t)) = (f(w), g(w))$. Thus, $(x, y) \in h(w)$. Thus, $h(w) = Q$. Finally we shall prove that h is a strict morphism. Let $z < w$ and suppose that $h(z) = h(w)$. Thus, $(f(z), g(z)) = (f(w), g(w))$. It means that z and w have the same height, which is a contradiction. Therefore, it follows from Remark 1 that h is a morphism in lpFP_2 . \square

Lemma 6 Let $\bar{h} : Z \rightarrow R^*(X \times_L Y)$ a morphism in lpFP_2 such that $p_X(\bar{h}(z)) = f(z)$ and $p_Y(\bar{h}(z)) = g(z)$, i.e. $m(\bar{h}(z)) = (f(z), g(z))$. Then $\bar{h} = h$.

Proof Let $(x, y) \in h(z)$. Thus, $(x, y) = (f(w), g(w))$ for some $w \geq z$. Taking into account that \bar{h} is monotone, we have that $\bar{h}(w) \geq \bar{h}(z)$. So, $[(f(w), g(w))]_{\bar{h}(z)} = \bar{h}(w)$. Hence, $(f(w), g(w)) \in \bar{h}(z)$ and then $h(z) \subseteq \bar{h}(z)$. Conversely, let $(x, y) \in \bar{h}(z)$. If $z \in \hat{Z}_1$, we have that $\bar{h}(z) = h(z)$. Let $z \in \hat{Z}_2$. If $(x, y) \in \hat{X}_2 \times \hat{Y}_2$, using that $(f(z), g(z)) \leq (x, y)$ we obtain that $(x, y) = (f(z), g(z)) \in h(z)$. If $(x, y) \in X_M \times Y_M$, then $\bar{h}(z) \leq \{(x, y)\}$. It follows from that \bar{h} is p -morphism that there exists $w \geq z$ such that $\bar{h}(w) = \{(x, y)\}$. Hence, $(x, y) = (f(w), g(w)) \in h(z)$. Therefore, $h(z) = \bar{h}(z)$. \square

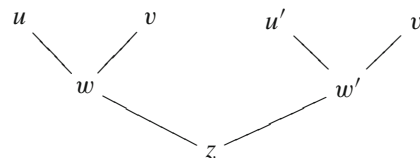
Then we obtain the following:

Theorem 7 Let $X, Y \in \text{lpFP}_2$. Then $R^*(X \times_L Y)$ is the categorical product in lpFP_2 .

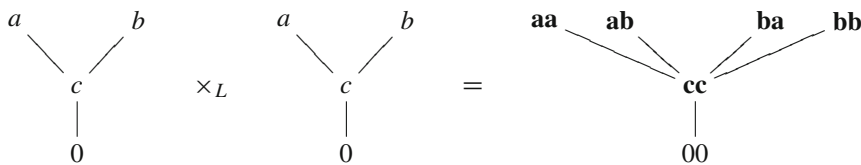
Remark 4 Similar arguments to those developed in this work allow us to prove versions of Theorem 3, Corollary 4, Remark 3 and Theorem 7 for posets of finite height, removing the finiteness condition.

Let us remark that the construction developed above for the categorical product in lpFP_2 does not work for lpFP , as the following simple example shows.

Example 2 Let $X = Y = \{0, a, b, c\}$, with $0 < c < a, b$ and a and b incomparable. Let Z be the following poset:

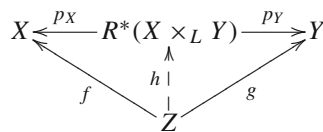


Define $f, g : Z \rightarrow X$ as $f(u) = f(u') = g(u) = g(u') = a$, $f(v) = f(v') = g(v) = g(v') = b$, $f(w) = f(w') = g(w) = g(w') = c$ and $f(z) = g(z) = 0$. Let us now compute $X \times_L X$ and take $Q = \{aa, bb, ab, ba, cc\} \in R^*(X \times_L Y)$.



Let h be as in Proposition 3. Since $h(z) = X \times_L X$, we have that $h(z) \leq Q$. Since $Q \neq h(w)$ and $Q \neq h(w')$, we obtain that $h : Z \rightarrow R^*(X \times_L Y)$ is not a p -morphism.

Let X, Y and Z be finite posets, $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ p -morphisms which preserve levels. A straightforward computation shows that the $h : Z \rightarrow R^*(X \times_L Y)$ defined above makes the following diagram commute:



It may be seen that if there exists a p -morphism, \bar{h} , in IFP, making previous diagram (with h replaced by \bar{h}) commute, then $\bar{h} = h$.

Therefore, we can conclude that $R^*(X \times_L Y)$ is not the categorical product in lpFP .

Let n be a natural number. Although we can not give a reasonable presentation of the categorical product for posets in lpFP_n , only based in particular computations we strongly suspect that this is also the case for these categories that finite products do exist. So we left the following open problem: does the category lpFP_n have finite products?

3 KM-algebras

In this section we shall apply the constructions given for the categorical product between two posets in lpFP , with one of them a root system, and for the categorical product of two posets in lpFP_2 , in order to obtain a description of some finite coproducts of finite algebras in certain varieties.

During the decade of the 1970s the Chisinau group headed by Kuznetsov introduced the notion of Δ -pseudoBoolean algebra. This class of algebras was later named KM-algebras by Esakia. In this article we shall name them KM-algebras, after Esakia. The historical remarks about these algebras can be found in Muravitsky and Logic (2014).

A KM-algebra is a Heyting algebra endowed with a unary map which satisfies certain identities. This unary map is called successor by Caicedo and Cignoli (2001). They considered it as an example of an implicit compatible operation on Heyting algebras. The compatibility of the successor was first proved by Simonova (1990) in Proposition 1. In this arti-

cle, we shall use a capital S to denote the successor in place of Δ .

Let H be a Heyting algebra and $f : H \rightarrow H$ a function. Recall that f is said to be compatible with a congruence θ in H if $(f(x), f(y)) \in \theta$ whenever $(x, y) \in \theta$. We say that f is a compatible function of H provided it is compatible with all the congruences of H .

A set $E(f)$ of equations in the signature of Heyting algebras augmented with a unary function symbol f is said to define an implicit operation if for any Heyting algebra H there is at most one function $f_H : H \rightarrow H$. The function f is an implicit compatible operation provided all f_H are compatible.

If it is possible, the successor, S , is defined on Heyting algebras by the following set of equations:

- (S1) $x \leq S(x)$,
- (S2) $S(x) \leq y \vee (y \rightarrow x)$,
- (S3) $S(x) \rightarrow x = x$.

The successor is an implicit compatible operation. It was proved in Esakia (2006) that in an equivalent way the successor function can be defined as $S(x) = \min\{y : y \rightarrow x \leq x\}$. A KM-algebra is a Heyting algebra endowed with its successor function, when it exists.

Let X be a Priestley space (1970). We say that X is an Esakia space if for every clopen U in X we have that $(U]$ is clopen. Recall that Esakia duality (1974) establishes a dual equivalence between the category \mathcal{H} of Heyting algebras and Heyting algebra morphisms, and the category \mathcal{E} of Esakia spaces and continuous p -morphisms,

$$\mathbf{X} : \mathcal{H} \rightleftharpoons \mathcal{E}^{op} : \mathbf{D}$$

Here, $\mathbf{X}(H)$ is the set of prime filters of the Heyting algebra H and $\mathbf{D}(X)$ is the set of clopen upsets of the Esakia space X . The unit and counit of the adjunction are given by $\varphi_H(x) = \{P \in \mathbf{X}(H) : x \in P\}$ and $\epsilon_X(x) = \{U \in \mathbf{D}(X) : x \in U\}$, respectively.

Write \mathcal{SH} for the category whose objects are KM-algebras and whose morphisms are the Heyting algebra morphisms that commute with the successor function. An Esakia space X is an S -space if for every $U \in \mathbf{D}(X)$ the set $U \cup (U^c)_M$ is clopen. Note that X is an S -space if and only if it is an Esakia space such that for every clopen downset V the set V_M is clopen. Let X and Y be S -spaces. A continuous p -morphism

$g : X \rightarrow Y$ (or an Esakia morphism) is an S -morphism if for every clopen downset V in Y it holds that $g^{-1}(V_M) = [g^{-1}(V)]_M$. We shall write \mathcal{SE} for the category whose objects are S -spaces and whose morphisms are S -morphisms. From Lemma 4 of Muravitsky (1988) we deduce a representation theorem for KM-algebras. In Castiglioni et al. (2010), it is shown that there is actually a dual categorical equivalence between \mathcal{SH} and \mathcal{SE} . If X is an S -space then in $\mathbf{D}(X)$ the successor function takes the form

$$S(U) = U \cup (U^c)_M.$$

We define the height of an S -space as the height of its underlying poset (when it exists). For a natural number n we write \mathcal{SE}_n for the full subcategory of \mathcal{SE} whose objects are S -spaces of height less than or equal to n .

We say that a KM-algebra H has height n if $S^{(n)}(0) = 1$ and n is the minimum natural number with this property. We write \mathcal{SH}_n for the class of KM-algebras of height less than or equal to n . This class is a variety with the defining equations of KM-algebras together with the additional equation

$$S^{(n)}(0) = 1.$$

Note that \mathcal{SH}_1 is just the variety of Boolean algebras, and that we have that

$$\mathcal{SH}_1 \subseteq \mathcal{SH}_2 \subseteq \dots \mathcal{SH}_n \subseteq \dots$$

The previous chain of inclusions is proper because the Heyting chain of $n + 2$ elements, L_{n+2} , is such that $L_{n+2} \in \mathcal{SH}_{n+1}$ and $L_{n+2} \notin \mathcal{SH}_n$. The notion of height of a KM-algebra was introduced by Simonova (1990).

We also write \mathcal{SH}_n for the category of KM-algebras of height less than or equal to n . We have that there exists a dual categorical equivalence between \mathcal{SH}_n and \mathcal{SE}_n (Thm 1.3 of Castiglioni and San Martín 2011). Moreover, if H is a KM algebra of height n then $\mathbf{X}(H)$ is a poset of height n . If X is a poset of height n , then the set of upsets of X is a KM-algebra of height n (Prop. 2.3 of Castiglioni and San Martín 2011).

Remark 5 Let $f : X \rightarrow Y$ be a morphism in \mathcal{E} with Y a finite poset. A moment's reflection shows that the Proposition 1.5 of Castiglioni and San Martín (2011) can be given in a more general way: f is a morphism in \mathcal{SE} if and only if f is a strict morphism.

Remark 6 (i) Let H be a KM-algebra. It was proved in Esakia (2006) that for every $x, y \in H$ we have that $S(x \wedge y) = S(x) \wedge S(y)$. In particular, S is a monotone function.

(ii) Let H be a KM-algebra and $x \in H$. It follows from (S1) and (S3) that $S(x) = x$ if and only if $x = 1$.

- (iii) It was observed in Muravitsky (1990) and also on p. 87 of Kuznetsov and Muravitsky (1986) that the successor exists in every finite Heyting algebra.
- (iv) It follows from (ii), (iii) and equation (S1) that in finite Heyting algebras there is $n \in \mathbb{N}$ such that $S^{(n)}(0) = 1$. See also Kuznetsov and Muravitsky (1986) and Muravitsky (1990).
- (v) It follows from the dual categorical equivalence between \mathcal{SH}_n and \mathcal{SE}_n and the previous remarks that lpFP is a full subcategory of \mathcal{SE} , and lpFP_n is a full subcategory of \mathcal{SE}_n for every n . Moreover, lpFP is dually equivalent to the full subcategory of \mathcal{SH} whose objects are finite KM-algebras, and lpFP_n is equivalent to the full subcategory of \mathcal{SH}_n whose objects are finite KM-algebras.

Prelinear Heyting algebras were considered by Horn (1969) as an intermediate step between the classical calculus and intuitionistic one and they were studied also by Monteiro (1980), Martínez and Priestley (1998) and others. This is the subvariety of Heyting algebras generated by the class of totally ordered Heyting algebras and can be axiomatized by the usual equations for Heyting algebras plus the prelinearity law $(x \rightarrow y) \vee (y \rightarrow x) = 1$. In Balbes and Dwinger (1974, ch. IX) and Monteiro (1980) there are characterizations for prelinear Heyting algebras. Horn (1969) showed (although it was in fact proved before by Monteiro (1980)) that prelinear Heyting algebras can be characterized among Heyting algebras in terms of the prime filters. More precisely, a Heyting algebra H is prelinear if and only if $\mathbf{X}(H)$ (with the inclusion) is a root system.

Proposition 4 Let X and Y finite posets in \mathcal{SE} , and assume that either X or Y is a root system. Then $X \times_L Y$ is the categorical product in \mathcal{SE} . Moreover, if $X, Y \in \mathcal{SE}_n$ then $X \times_L Y$ is the categorical product in \mathcal{SE}_n .

Proof Let $f : Z \rightarrow X \in \mathcal{SE}$ and $g : Z \rightarrow Y \in \mathcal{SE}$. Define h as in Theorem 3. Taking into account the proof of Corollary 4, and Remark 5, we only need to prove that h is a continuous map. Let $(x, y) \in X \times_L Y$. Using the equality $h^{-1}(\{(x, y)\}) = f^{-1}(\{x\}) \cap g^{-1}(\{y\})$ and that f, g are continuous maps, we conclude that h is a continuous map. \square

Let $f : Z \rightarrow X$ and $g : Y \rightarrow Z$ be morphisms in \mathcal{SE}_2 . Let h be as in Proposition 3, and let $Q \in R^*(X \times_L Y)$. For every $(x, y) \in X \times_L Y$ we define $\Sigma_{(x,y)} = \{w \in Z : (f(w), g(w)) = (x, y)\}$. Then we define $I_Q = \bigcap_{(x,y) \in Q} (\Sigma_{(x,y)})$ and $J_Q = \bigcap_{(x,y) \notin Q} (\Sigma_{(x,y)})^c$. In what follows we shall give a lemma in order to give a description of the categorical product in \mathcal{SE}_2 .

Lemma 8 Let $z \in Z$. Then

- (a) $Q \subseteq h(z)$ if and only if $z \in I_Q$.

- (b) $h(z) \subseteq Q$ if and only if $z \in J_Q$.
- (c) I_Q and J_Q are clopens in Z .

Proof Straightforward computations show (a) and (b). In order to prove (c), let $(x, y) \in X \times_L Y$. First note that $\Sigma_{(x,y)} = f^{-1}(\{x\}) \cap g^{-1}(\{y\})$. As X, Y are finite and f, g are continuous maps, we have that $f^{-1}(\{x\})$ and $g^{-1}(\{y\})$ are clopens in Z . Thus, $\Sigma_{(x,y)}$ is clopen in Z . As Z is an Esakia space we obtain that $(\Sigma_{(x,y)})$ is clopen. Therefore, I_Q and J_Q are clopens in Z . \square

Proposition 5 *Let X, Y be finite posets in \mathcal{SE}_2 . Then $R^*(X \times_L Y)$ is the categorical product in \mathcal{SE}_2 .*

Proof Let $f : Z \rightarrow X$ and $g : Y \rightarrow Z$ be morphisms in \mathcal{SE}_2 . Let h be defined as in Proposition 3. Taking into account the proof of Proposition 3, we only need to prove that h is a continuous map. Let $Q \in R^*(X \times_L Y)$. It follows from Lemma 8 that $h^{-1}(\{Q\}) = I_Q \cap J_Q$, which is clopen. Therefore, h is a morphism in \mathcal{SE}_2 . \square

Corollary 9 *The coproduct of finite algebras in the variety \mathcal{SH}_2 is finite.*

As a consequence of the previous corollary we can deduce that free algebras on finite generators in the variety \mathcal{SH}_2 are finite. In order to obtain this result, we shall depict the dual space of the free algebra in one generator in the variety \mathcal{SH}_2 .

We start by giving the definition of Solovay algebras introduced in Esakia and Grigolia (2008), and we study its relation with \mathcal{SH}_2 .

Definition 4 A KM-algebra is a Solovay algebra if it satisfies the additional equation:

$$(S(x) \rightarrow S(y)) \vee (S(y) \rightarrow x) = 1.$$

We write \mathcal{S}_{Sol} for the variety of Solovay algebras.

Lemma 10 *Let H be a KM-algebra. The following conditions are equivalent:*

- (a) $H \in \mathcal{SH}_2$.
- (b) The identity $y \vee (y \rightarrow S(0)) = 1$ is valid in H .
- (c) The identity $y \vee (y \rightarrow S(x)) = 1$ is valid in H .
- (d) The identity $S(y) \vee (S(y) \rightarrow S(x)) = 1$ is valid in H .

Proof Let $H \in \mathcal{SH}_2$ and $x, y \in H$. Then $1 = S^{(2)}(0) \leq y \vee (y \rightarrow S(0))$, so $1 = y \vee (y \rightarrow S(0))$. Suppose now that $1 = y \vee (y \rightarrow S(0))$ for every $x, y \in H$. Taking into account that S is a monotone function, we obtain that $S(0) \leq S(x)$. Thus, $1 = y \vee (y \rightarrow S(0)) \leq y \vee (y \rightarrow S(x))$. Hence, $y \vee (y \rightarrow S(x)) = 1$. Suppose that $y \vee (y \rightarrow S(x)) = 1$ for every $x, y \in H$. In particular, it follows from a simple substitution that $S(y) \vee (S(y) \rightarrow S(x)) = 1$. Finally, suppose

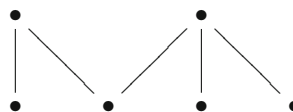
that $S(y) \vee (S(y) \rightarrow S(x)) = 1$ for every $x, y \in H$. Put $x = 0$ and $y = S(0)$. Then we have that $1 = S^2(0) \vee (S^2(0) \rightarrow S(0)) = S^2(0) \vee S(0) = S^2(0)$. Therefore, $H \in \mathcal{SH}_2$. \square

The following Corollary is a straightforward consequence of previous Lemma. However, for the sake of completeness and clearness we give a proof of it.

Corollary 11 *\mathcal{SH}_2 is a proper subvariety of \mathcal{S}_{Sol} .*

Proof Let $H \in \mathcal{SH}_2$. Then by Lemma 10 we have that $1 = S(x) \vee (S(x) \rightarrow S(y)) \leq S(y) \vee (S(y) \rightarrow x) \vee (S(x) \rightarrow S(y)) = (S(x) \rightarrow S(y)) \vee (S(y) \rightarrow x)$. Then \mathcal{SH}_2 is a subvariety of \mathcal{S}_{Sol} . Moreover, it is a proper subvariety of \mathcal{S}_{Sol} , since, as it can be seen from the dual characterization of the free algebra in one generator depicted in Solovay (1976), this algebra does not belong to \mathcal{SH}_2 . \square

The following poset is a dual characterization of the free algebra on one generator in the variety \mathcal{SH}_2 :



This fact can be shown by taking the first two layers from the dual characterization of the free algebra in one generator in \mathcal{S}_{Sol} . Therefore, the previous description simply follows from Solovay (1976).

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