Mediterr. J. Math. 12 (2015), 187–204
DOI 10.1007/s00009-014-0399-9
0378-620X/15/010187-18
published online May 16, 2014

Mediterranean Journal of Mathematics

Solvable Models for Kodaira Surfaces

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To the memory of Sergio Console

Abstract. In this work, we study families of compact spaces which are of the form $G/\Lambda_{k,i}$ for G the oscillator group and $\Lambda_{k,i} < G$ a lattice. The solvmanifolds $G/\Lambda_{k,i}$ are not pairwise diffeomorphic and one has the coverings $G \to M_{k,0} \to M_{k,\pi} \to M_{k,\pi/2}$ for $k \in \mathbb{Z}$. We compute their cohomologies and minimal models. Each manifold $M_{k,0}$ is diffeomorphic to a Kodaira–Thurston manifold, i.e., a compact quotient $S^1 \times \mathrm{H}_3(\mathbb{R})/\Gamma_k$ where Γ_k is a lattice of the real three-dimensional Heisenberg group $\mathrm{H}_3(\mathbb{R})$. Furthermore, any $M_{k,0}$ provides an example of a solvmanifold whose cohomology does not depend on the Lie algebra only. We explain some geometrical aspects of those compact spaces, to show how to distinguish them (by invariant complex, symplectic and metric structures). For instance, no invariant symplectic structure of G can be induced to the any quotient.

Mathematics Subject Classification (2010). 53C50, 53C30, 22E25, 57S25.

Keywords. solvmanifolds, solvable Lie group, Heisenberg group.

1. Introduction

A solvmanifold M is a compact homogeneous space of a solvable Lie group, that is $M = G/\Gamma$ where G is a connected and simply connected solvable Lie group and Γ is a lattice in G (that is, a co-compact discrete subgroup of G).

This work is devoted to the explanation of several topological and geometrical properties of some families of solvmanifolds in dimension four. Among the considered examples we have four-dimensional solvmanifolds which constitute models for Kodaira surfaces [8]. In fact both the primary and the secondary Kodaira manifolds can be realized as quotients of a fixed solvable (non-nilpotent) Lie group G of dimension four by different lattices. This Lie group is known as the oscillator group and it is an example of *almost nilpotent* solvable Lie group (see Sect. 2.1).

This work was supported by Secyt-UNR, ANCyT and CONICET, by MIUR and GNSAGA of INdAM.

Birkhäuser

By fixing the *oscillator group*, which is the semidirect product $G = \mathbb{R} \ltimes_\rho \mathrm{H}_3(\mathbb{R})$ of the (real) three-dimensional Heisenberg group $\mathrm{H}_3(\mathbb{R})$ and $\mathbb{R}$, we start determining three families of lattices $\Lambda_{k,i}$ in the solvable Lie group G.

We prove that all subgroups of the families $\Lambda_{k,i}$ are not pairwise isomorphic, hence they determine infinitely many non-diffeomorphic solvmanifolds, for $k \in \mathbb{N}$:

$$\begin{aligned} M_{k,0} &= G/\Lambda_{k,0}, \\ M_{k,\pi} &= G/\Lambda_{k,\pi}, \\ M_{k,\pi/2} &= G/\Lambda_{k,\pi/2}. \end{aligned}$$

By Theorem 2.1, we shall see that $M_{k,0} = G/\Lambda_{k,0}$ is diffeomorphic to $S^1 \times \mathrm{H}_3(\mathbb{R})/\Gamma_k$, which gives rise to a *Kodaira–Thurston manifold.*

Moreover, for any fixed k, we have the finite coverings

$$\begin{aligned} p_\pi : M_{k,0} \to M_{k,\pi}\,, \\ p_{\pi/2} : M_{k,0} \to M_{k,\pi/2}\,, \end{aligned}$$

which are 2 and 4 sheeted, respectively, and so

$$G \longrightarrow M_{k,0} \longrightarrow M_{k,\pi} \longrightarrow M_{k,\pi/2}.$$

In Sect. 3, we compute the cohomology of the minimal model of all solvmanifolds in the above families.

Theorem 1.1. *The Betti numbers b_i of the solvmanifolds $M_{k,*}$ are given by*

	b_0	b_1	b_2
$M_{k,0}$	1	3	4
$M_{k,\pi}$	1	1	0
$M_{k,\pi/2}$	1	1	0

(clearly $b_3 = b_1$ and $b_4 = b_0$, by Poincaré duality). A minimal model of $M_{k,0}$ is given by

$$\mathcal{M}_{k,0} = (\Lambda(x_1, y_1, z_1, t_1), d)$$

where the index denotes the degree (hence all generators have degree one) and the only non vanishing differential is given by $dz_1 = -x_1 y_1$.

A minimal model of $M_{k,\pi}$ and $M_{k,\pi/2}$ is given by

$$\mathcal{M}_{k,\pi} = \mathcal{M}_{k,\pi/2} = (\Lambda(t_1, w_3), d = 0)$$

where the index denotes the degree, cf. [18,19].

Observe that any $M_{k,0}$ gives an (low dimensional) example of solvmanifold whose de Rham cohomology does not agree with the invariant one, i.e., the cohomology of the Chevalley–Eilenberg complex on the solvable Lie algebra (unlike the case of nilmanifolds and solvmanifolds in the completely solvable case [10] and, more generally, for which the Mostow condition holds [2,7,22], cf. Sect. 3).

Actually, it turns out that $M_{k,0}$ has the same cohomology as a nilmanifold, namely the Kodaira–Thurston manifold $S^1 \times \mathrm{H}_3(\mathbb{R})/\Gamma_k$, cf. Sect. 3. Moreover, passing from $M_{k,0}$ to the covered manifolds $M_{k,\pi}$ and $M_{k,\pi/2}$, the cohomology changes. So the cohomology depends on the lattice and not on the solvable Lie algebra only.

Finally, we see some geometrical features which distinguish these spaces. Recall that a left-invariant geometric structure on a Lie group G (complex, symplectic, metric structure) is determined by its value at the Lie algebra level $\mathfrak{g}$. Hence, it can be induced to the compact space $\Gamma\backslash G$ for $\Gamma < G$ a lattice, so that $\Gamma\backslash G$ is diffeomorphic to G/Γ.

Furthermore, both spaces G and $\Gamma\backslash G$ are locally equivalent via the considered geometric structure, that is projection map $p : G \to \Gamma\backslash G$ is a local diffeomorphism preserving the structure. In this way, the Lie group G acts locally but not necessarily transitively preserving the geometric structure.

Thus, the spaces $M_{k,0}$ provide examples of solvmanifolds which admits symplectic structures but which are not induced by left-invariant symplectic structures on G, see Theorem 4.1.

Moreover, for any k the space $M_{k,0}$ covers $M_{k,\pi}$ and $M_{k,\pi/2}$ which do not admit any symplectic structure, since their second Betti number vanishes.

It is known that if a given nilmanifold N/Γ admits a symplectic structure, then it admits an N-invariant one. Hence, we provide low-dimensional examples which show that this is not true for solvmanifolds.

The different classes of complex structure on G induce complex structures on G. However, while $\mathbb{R} \times \mathrm{H}_3(\mathbb{R})$ admits abelian complex structures, G does not admit anyone of this type.

There exists a left-invariant Lorentzian metric on G and $\mathbb{R}\times\mathrm{H}_3(\mathbb{R})$ which makes these spaces isometric [3]. Once one induces this metric to the quotients $M_{k,i}$ one can verify that these spaces can be distinguished by their geometry: the isometry groups are different so as their periodic geodesics. Moreover, any compact space $M_{k,*}$ is a naturally reductive space, hence geodesics are projections of one-parameter subgroups.

2. Solvmanifolds in Dimension Four

Unlike the special case of nilmanifolds (i.e., compact quotients of nilpotent Lie groups by a lattice), there is no simple criterion for the existence of a lattice in a connected and simply-connected solvable Lie group. A necessary condition is that the connected and simply-connected solvable Lie group is unimodular [14, Lemma 6.2].

Lattices determine the topology of solvmanifolds since they are Eilenberg–MacLane spaces of type $K(\pi, 1)$ (i.e., all homotopy groups vanish, besides the first) with finitely generated torsion-free fundamental group. Actually lattices associated to solvmanifolds yield their diffeomorphism class as the following theorem states.

Theorem 2.1. [22, Theorem 3.6] *Let G_i/Γ_i be solvmanifolds for $i \in \{1, 2\}$ and let $\varphi : \Gamma_1 \to \Gamma_2$ denote an isomorphism. Then there exists a diffeomorphism $\Phi : G_1 \to G_2$ such that*

(i) $\Phi|_{\Gamma_1} = \varphi$,
(ii) $\Phi(p\gamma) = \Phi(p)\varphi(\gamma)$, *for any* $\gamma \in \Gamma_1$ *and any* $p \in G_1$.

As a consequence two solvmanifolds with isomorphic fundamental groups are diffeomorphic.

Recall that if the action of the group Γ on the topological space Y is properly discontinuous, then there is a differentiable structure on Y/Γ (resp. $\Gamma\backslash G$) such that $Y \to Y/\Gamma$ (resp. $G \to \Gamma\backslash G$) is a normal covering. The lattice Γ is the Deck transformation group of the covering and if Y is simply connected then Γ is isomorphic to $\pi_1(Y/\Gamma)$ [9, Proposition 1.40]. See the following examples in dimension four.

Nilmanifolds. The differentiable manifold $\mathbb{R}^3$ when equipped with the canonical differentiable structure and multiplication map given by

$$(x, y, z)(x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\right)$$

gives rise to the Heisenberg Lie group of dimension three $\mathrm{H}_3(\mathbb{R})$.

Let N denote the trivial extension of $\mathrm{H}_3(\mathbb{R})$, namely $N = \mathbb{R} \times \mathrm{H}_3(\mathbb{R})$, and for every $k \in \mathbb{N}$ consider Λ_k the following lattice in N:

$$\Lambda_k = 2\pi\mathbb{Z} \times \Gamma_k < N \quad \text{where} \quad \Gamma_k = \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2k}\mathbb{Z} < \mathrm{H}_3(\mathbb{R}) \tag{1}$$

for Γ_k a lattice in $\mathrm{H}_3(\mathbb{R})$.

Each discrete subgroup Λ_k acts properly discontinuous on the simply connected space $\mathbb{R} \times \mathrm{H}_3(\mathbb{R})$ giving rise to the nilmanifolds N/Λ_k.

Solvmanifolds. Consider the Lie group homomorphism $\rho : \mathbb{R} \to \mathrm{Aut}(\mathrm{H}_3(\mathbb{R}))$ which on vectors $(v, z) \in \mathbb{R}^2 \oplus \mathbb{R}$ has the form

$$\rho(t) = \begin{pmatrix} R(t) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{where} \quad R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}. \tag{2}$$

On the smooth manifold $\mathbb{R}^4$ consider the algebraic structure resulting from the semidirect product of $\mathbb{R}$ and $\mathrm{H}_3(\mathbb{R})$, via ρ. Thus, the multiplication is given by

$$(t, v, z) \cdot (t', v', z') = \left(t + t', v + R(t)v', z + z' + \frac{1}{2}v^{\mathrm{T}} JR(t)v'\right) \tag{3}$$

with J and $R(t)$ as above.

Let G denote the simply connected Lie group $G = \mathbb{R} \ltimes_\rho \mathrm{H}_3(\mathbb{R})$. The Lie group G is known as the *oscillator group.*

Every lattice $\Gamma_k < \mathrm{H}_3(\mathbb{R})$ is invariant under the subgroups generated by $\rho(2\pi)$, $\rho(\pi)$ and $\rho(\frac{\pi}{2})$, ($\rho : \mathbb{R} \to Aut(\mathrm{H}_3(\mathbb{R})$ as in (2)). Consequently, we have three families of lattices in $G = \mathbb{R} \ltimes_\rho \mathrm{H}_3(\mathbb{R})$:

$$\begin{aligned} \Lambda_{k,0} &= 2\pi\mathbb{Z} \ltimes \Gamma_k < G \\ \Lambda_{k,\pi} &= \pi\mathbb{Z} \ltimes \Gamma_k < G \\ \Lambda_{k,\pi/2} &= \frac{\pi}{2}\mathbb{Z} \ltimes \Gamma_k < G. \end{aligned} \tag{4}$$

so that $\Lambda_{k,0} \triangleleft \Lambda_{k,\pi} \triangleleft \Lambda_{k,\pi/2}$, which induce the solvmanifolds

$$\begin{aligned} M_{k,0} &= \Lambda_{k,0}\backslash G \simeq G/\Lambda_{k,0}\,, \\ M_{k,\pi} &= \Lambda_{k,\pi}\backslash G \simeq G/\Lambda_{k,\pi}\,, \\ M_{k,\pi/2} &= \Lambda_{k,\pi/2}\backslash G \simeq G/\Lambda_{k,\pi/2}. \end{aligned} \tag{5}$$

Lemma 2.2. *Let $k \in \mathbb{N}$. Then*

1. $Z_k = 2\pi\mathbb{Z} \times 0 \times 0 \times \frac{1}{2k}\mathbb{Z}$ *is the center of* $\Lambda_{k,i}$ *for* $i = 0, \pi, \pi/2$ *and*
2. $C = 0 \times 0 \times 0 \times \mathbb{Z}$ *is the commutator of* $\Lambda_{k,0}$.

Proof. Fix $k \in \mathbb{N}$ and take $i = 0, \pi, \pi/2$. By a simple computation we see that for each k the set Z_k is contained in the center of G, and then it is contained in the center of $\Lambda_{k,i}$.

Now, let (θ, a, b, c) be in the center of $\Lambda_{k,i}$, then

$$\begin{aligned} &(\theta, a, b, c)(0, 1, 0, 0) = (0, 1, 0, 0)(\theta, a, b, c), \\ &\quad\left(\theta, a + \cos\theta, b - \sin\theta, c - \frac{1}{2}(a\sin\theta + b\cos\theta)\right) = \left(\theta, 1 + a, b, c + \frac{1}{2}b\right), \end{aligned}$$

It follows that $\theta = 2l\pi$ and $b = 0$.

Also from $(2l\pi, a, 0, c)(0, 0, 1, 0) = (0, 0, 1, 0)(2l\pi, a, 0, c)$ we get $a = 0$. Then $(\theta, a, b, c) \in Z_k$

Now we prove that C is the commutator of $\Lambda_{k,0}$. By computing we see

$$\begin{aligned} &(2l\pi, a, b, c)(2l'\pi, a', b', c')(2l\pi, a, b, c)^{-1}(2l'\pi, a', b', c')^{-1} \\ &\quad = (0, 0, 0, ab' - a'b) \in C \end{aligned}$$

Since C is a subgroup, we have that the commutator is contained in C. But taking $(2l\pi, a, b, c) = (0, x, 0, 0)$ and $(2l'\pi, a', b', c') = (0, 0, 1, 0)$ for any $x \in \mathbb{Z}$, it follows that the element of C given by $(0, 0, 0, x)$ belongs to the commutator and this completes the proof. □

Proposition 2.3. *The groups $\Lambda_{k,i}$, $k \in \mathbb{N}$, $i = 0, \pi/2, \pi$, are pairwise not isomorphic.*

Proof. First we observe that if $\varphi : \Lambda_{p,j} \to \Lambda_{k,i}$ is an isomorphism then $\varphi((2l+1)\pi, a, b, c) \neq (2l'\pi, a', b', c')$ for $l, l' \in \mathbb{Z}$. Otherwise, we get

$$\begin{aligned} &\varphi((4l+2)\pi, 0, 0, z) = \varphi((2l+1)\pi, a, b, c)^2 = (2l'\pi, a', b', c')^2 \\ &\quad = (4l'\pi, 2a', 2b', z') \in Z_k \end{aligned}$$

which is the center of $\Lambda_{k,i}$ by Lemma 2.2; it follows that $a' = b' = 0$. So $\varphi((2l+1)\pi, a, b, c) \in Z_k$ and $((2l+1)\pi, a, b, c) \in Z_p$, which is a contradiction. Considering φ^{-1} we get also that $\varphi(2l\pi, a, b, c) \neq ((2l'+1)\pi, a', b', c')$ for $l, l' \in \mathbb{Z}$. We conclude that $\Lambda_{k,0}$ is isomorphic neither to $\Lambda_{p,\pi}$ nor to $\Lambda_{p,\pi/2}$.

If there is an isomorphism $\varphi_1 : \Lambda_{k,\pi/2} \to \Lambda_{p,\pi}$ with $\varphi_1(\pi/2, 0, 0, 0) = (l\pi, a, b, c)$, then:

$$\varphi_1(\pi/2, 0, 0, 0)^2 = (l\pi, a, b, c)^2 \Rightarrow \varphi_1(\pi, 0, 0, 0) = (2l\pi, x, y, z),$$

and we show that this cannot happen.

Suppose that $\varphi_2 : \Lambda_{k,0} \to \Lambda_{p,0}$ is an isomorphism with $p < k$. By Lemma 2.2, $\varphi_2(C) = C$.

$$\varphi_2(0,0,0,1) = \varphi_2\left(0,0,0,\frac{1}{2k}\right)^{2k} = \left(2\pi a, b, c, \frac{d}{2p}\right)^{2k} = \left(4\pi k a, 2kb, 2kc, \frac{e}{2p}\right),$$

for some $a, b, c, d, e \in \mathbb{Z}$. Then $(2ka, 2kb, 2kc, \frac{e}{2k}) \in C$ and $a = b = c = 0$.

$$\varphi_2\left(0,0,0,\frac{p}{k}\right) = \varphi_2\left(0,0,0,\frac{1}{2k}\right)^{2p} = \left(0,0,0,\frac{d}{2p}\right)^{2p} = (0,0,0,d) \in C,$$

So we have $(0,0,0,\frac{p}{k}) \in C$. Absurd.

Let $\varphi_3 : \Lambda_{k,\pi} \to \Lambda_{p,\pi}$ be an isomorphism. By the remark at the beginning of the proof, we conclude that the restriction of φ_3 to $\Lambda_{k,0}$ is an isomorphism from $\Lambda_{k,0}$ to $\Lambda_{p,0}$ which contradicts the last paragraph.

Now, let $\varphi_4 : \Lambda_{k,\pi/2} \to \Lambda_{p,\pi/2}$ be an isomorphism. If $\varphi_4((2l+1)\pi/2, a, b, c) = (l'\pi, a', b', c')$, then $\varphi_4((2l+1)\pi/2, a, b, c)^2 = (l'\pi, a', b', c')^2$ and follows that $\varphi_4((2l+1)\pi, x, y, z) = (2l'\pi, 0, 0, z')$ which also contradicts the remark. Then $\varphi_4(\Lambda_{k,\pi}) = \Lambda_{p,\pi}$. □

Theorem 2.1. *The subgroups $\Lambda_{k,i}$ are the only lattices of G of the form $L_1 \times L_2 \times L_3 \times L_4$ where $L_i \subset \mathbb{R}$ is a subgroup for every $i = 1, 2, 3, 4$.*

Proof. Let $L = L_1 \times L_2 \times L_3 \times L_4$ be a lattice of G, then it is easy to see that L_i is a discrete subgroup of $\mathbb{R}$ for $i = 1, 2, 3, 4$. Then there are $p, q, r, s \in \mathbb{R}_{\geq 0}$ such that

$$L = p\mathbb{Z} \times q\mathbb{Z} \times r\mathbb{Z} \times s\mathbb{Z}$$

Let $m \in \mathbb{Z}$, since $(0, q, 0, 0) \in L$ and $(0, 0, rm, 0) \in L$ then $(0, q, rm, \frac{qr}{2}m) \in L$ and $\frac{qr}{2}m \in L_4$. It follows that $s = \frac{qr}{2k}$ for some $k \in \mathbb{N}$.

On the other hand, since $(p, 0, 0, 0) \in L$ and $(0, q, 0, 0) \in L$ then $(p, q\cos p, -q\sin p, 0) \in L$ and $\cos p \in \mathbb{Z}$. It follows that $p = \frac{\pi}{2}l$ for some non negative integer l.

We conclude that:

$$L = \frac{\pi}{2}l\mathbb{Z} \times q\mathbb{Z} \times r\mathbb{Z} \times \frac{qr}{2k}\mathbb{Z}$$

for $q, r \in \mathbb{R}_{\geq 0}$, $k \in \mathbb{N}$ and l a non negative integer.

If $l = 0$, then $G/L \cong \mathbb{R} \times \mathrm{H}_3(\mathbb{R})/L'$ which is not compact for $L' \subset \mathrm{H}_3(\mathbb{R})$. If $r = 0$ then $L \cap \mathrm{H}_3(\mathbb{R}) = q\mathbb{Z} \times 0 \times 0$ which is not a lattice in $\mathrm{H}_3(\mathbb{R})$. Analogously, if $q = 0$, the same follows.

Therefore, l, r, q are non zero real numbers, moreover, $l \in \mathbb{N}$.

Now we consider four cases $l \equiv 0, 1, 2, 3 \pmod 4$.

- $l \equiv 0 \pmod 4$

  A set of the form $L = 2\pi l\mathbb{Z} \times q\mathbb{Z} \times r\mathbb{Z} \times \frac{qr}{2k}\mathbb{Z}$ is a lattice of G and it is isomorphic to $\Lambda_{k,0}$ via the isomorphism:

$$\gamma_1 : \Lambda_{k,0} \to 2\pi l\mathbb{Z} \times q\mathbb{Z} \times r\mathbb{Z} \times \tfrac{qr}{2k}\mathbb{Z}$$
$$\gamma_1(t, x, y, z) = (lt, qx, ry, qrz)$$

- $l \equiv 2 \pmod 4$

  A set of the form $L = (2l+1)\pi\mathbb{Z} \times q\mathbb{Z} \times r\mathbb{Z} \times \frac{qr}{2k}\mathbb{Z}$ is a lattice of G which is isomorphic to $\Lambda_{k,\pi}$ via the isomorphism:

$$\gamma_2 : \Lambda_{k,\pi} \to (2l+1)\pi\mathbb{Z} \times q\mathbb{Z} \times r\mathbb{Z} \times \tfrac{qr}{2k}\mathbb{Z}$$
$$\gamma_2(t,x,y,z) = ((2l+1)t, qx, ry, qrz)$$

- $l \equiv 1 \pmod 4$

  Let $L = (4l+1)\frac{\pi}{2}\mathbb{Z} \times q\mathbb{Z} \times r\mathbb{Z} \times \frac{qr}{2k}\mathbb{Z}$ be a subgroup of G, then
- $((4l+1)\frac{\pi}{2}, 0,0,0)(0,-q,0,0) = ((4l+1)\frac{\pi}{2}, 0, q, 0) \in L$ and
- $((4l+1)\frac{\pi}{2}, 0,0,0)(0,0,r,0) = ((4l+1)\frac{\pi}{2}, r, 0, 0) \in L$.

  Thus, we deduce $r|q$ and $q|r$, so $q = r$.

  A set of the form $L = (4l+1)\frac{\pi}{2}\mathbb{Z} \times q\mathbb{Z} \times q\mathbb{Z} \times \frac{q^2}{2k}\mathbb{Z}$ is a lattice of G and it is isomorphic to $\Lambda_{k,\pi/2}$ via the isomorphism:

$$\gamma_3 : \Lambda_{k,\pi/2} \to (4l+1)\tfrac{\pi}{2}\mathbb{Z} \times q\mathbb{Z} \times q\mathbb{Z} \times \tfrac{q^2}{2k}\mathbb{Z}$$
$$\gamma_3(t,x,y,z) = ((4l+1)t, qx, qy, q^2 z)$$

- $l \equiv 3 \pmod 4$

  Let $L = (4l+3)\frac{\pi}{2}\mathbb{Z} \times q\mathbb{Z} \times r\mathbb{Z} \times \frac{qr}{2k}\mathbb{Z}$ be a subgroup of G, as before
- $((4l+3)\frac{\pi}{2}, 0,0,0)(0,q,0,0) = ((4l+3)\frac{\pi}{2}, 0, q, 0) \in L$ and
- $((4l+3)\frac{\pi}{2}, 0,0,0)(0,0,-r,0) = ((4l+3)\frac{\pi}{2}, r, 0, 0) \in L$

  which implies $q = r$.

  The set $L = (4l+3)\frac{\pi}{2}\mathbb{Z} \times q\mathbb{Z} \times q\mathbb{Z} \times \frac{q^2}{2k}\mathbb{Z}$ is a lattice of G which is isomorphic to $\Lambda_{k,\pi/2}$ via the isomorphism:

$$\gamma_4 : \Lambda_{k,\pi/2} \to (4l+1)\tfrac{\pi}{2}\mathbb{Z} \times q\mathbb{Z} \times q\mathbb{Z} \times \tfrac{q^2}{2k}\mathbb{Z}$$
$$\gamma_4(t,x,y,z) = ((4l+3)t, -qx, -qy, -q^2 z)$$

□

Remark 1. Notice that there exist lattices in G which are not of the form $L_1 \times L_2 \times L_3 \times L_4$. For instance, let L be the next one

$$L = \left\{ \left(2l\pi, 2x, 2y, \frac{1}{2}z\right) : l,x,y,z \in \mathbb{Z} \right\} \cup \left\{ \left((2l+1)\pi, 2x+1, 2y+1, \frac{1}{2}z\right) : l,x,y,z \in \mathbb{Z} \right\}.$$

It contains the lattice $2\pi\mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z} \times 2\mathbb{Z}$. But this lattice is isomorphic to $\Lambda_{2,\pi}$ via the isomorphism $(t,x,y,z) \to (t, \frac{x-y}{2}, \frac{x+y}{2}, \frac{z}{2})$. We conjecture that every lattice of G is isomorphic to one of the family $\Lambda_{k,i}$ as above.

2.1. The Mostow Bundle and Almost Nilpotent Lie Groups

Let $M = G/\Gamma$ be a solvmanifold that is not a nilmanifold. Let N be the nilradical of G, i.e., the largest connected nilpotent normal subgroup of G.

Then $\Gamma_N := \Gamma \cap N$ is a lattice in N, $\Gamma N = N\Gamma$ is closed in G and $G/(N\Gamma) =: \mathbb{T}^k$ is a torus. Thus, we have the so-called *Mostow fibration*:

$$N/\Gamma_N = (N\Gamma)/\Gamma \hookrightarrow G/\Gamma \longrightarrow G/(N\Gamma) = \mathbb{T}^k$$

Most of the rich structure of solvmanifolds is encoded in this bundle.

The fundamental group Γ of M can be represented as an extension of a torsion-free nilpotent group Λ of rank $n-k$ by a free abelian group of rank k where $1 \leq k \leq 4$:

$$0 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \mathbb{Z}^k \longrightarrow 0 \tag{6}$$

The classification of solvmanifolds of dimension four reduces to the classification of the groups Γ as the group extensions above (see [8]).

Example 1. The subgroup $\Gamma_k < \mathrm{H}_3(\mathbb{R})$ can be expressed as a non-split group extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_k \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

thus the lattice of N given by $\Lambda_k = \Gamma_k \times \mathbb{Z}$ is a nilpotent group of rank four.

A connected and simply-connected solvable Lie group G with nilradical N is called *almost nilpotent* if its nilradical has codimension one. In this case, G can be written as a semidirect product $G = \mathbb{R} \ltimes_\mu N$. In addition, if N is abelian, i.e., $N = \mathbb{R}^n$, then G is called *almost abelian*.

Let $G = \mathbb{R} \ltimes_\mu N$ be an almost nilpotent Lie group. Since N has codimension one in G, we can consider μ as a one-parameter group $\mathbb{R} \to \mathrm{Aut}(N)$. Observe that $\mathrm{d}\mu =: \phi$ is one-parameter subgroup of the automorphism group of the Lie algebra $\mathfrak{n}$ of N.

Example 2. Let us consider the 3-dimensional solvable Lie group $\mathbb{R} \ltimes \mathbb{R}^2$ with structure equations

$$\begin{cases} de^1 = 0, \\ de^2 = 2\pi e^{13}, \\ de^3 = -2\pi e^{12}. \end{cases}$$

It is a non-completely solvable Lie group which admits a compact quotient and the uniform discrete subgroup is of the form $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^2$ (see [18, Theorem 1.9] and [13]). Indeed, the Lie group $\mathbb{R} \ltimes \mathbb{R}^2$ is the group of matrices

$$\begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) & 0 & x \\ -\sin(2\pi t) & \cos(2\pi t) & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the lattice Γ generated by 1 in $\mathbb{R}$ and the standard lattice $\mathbb{Z}^2$. The semidirect product is relative to the one-parameter subgroup

$$t \mapsto \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) & 0 \\ -\sin(2\pi t) & \cos(2\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The solvable Lie group $\mathbb{R} \ltimes \mathbb{R}^2$ is almost abelian.

Example 3. The oscillator group $G = \mathbb{R} \ltimes_\rho \mathrm{H}_3(\mathbb{R})$ is almost nilpotent but the nilpotent Lie group $N = \mathbb{R} \times \mathrm{H}_3(\mathbb{R})$ is almost abelian.

Moreover, since Λ_k in (1) is isomorphic to $\Lambda_{k,0}$ in (4) by Theorem 2.1 the compact manifolds $G/\Lambda_{k,0}$ and N/Λ_k are diffeomorphic.

3. Cohomology and Minimal Models in Dimension Four

In this section, we shall study the cohomology and minimal models of the compact solvmanifolds $M_{k,i}$.

Let $M = G/\Gamma$ be a solvmanifold. If the algebraic closures $\mathcal{A}(\mathrm{Ad}_G(G))$ and $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ are equal, one says that G and Γ satisfy the *Mostow condition*. In this case, the de Rham cohomology $H^*(M)$ of the compact solvmanifold $M = G/\Gamma$ can be computed by the Chevalley–Eilenberg cohomology $H^*(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G (see [15] and [22, Corollary 7.29]); indeed, one has the isomorphism $H^*(M) \cong H^*(\mathfrak{g})$. A special case is provided by nilmanifolds (Nomizu's Theorem, [16]) and more generally if G is *completely solvable* [10], i.e., all the linear operators $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$, $X \in \mathfrak{g}$ have only real eigenvalues.

In Remark 2, we show examples where the Mostow condition does not hold and, however, the de Rahm cohomology of the compact solvmanifolds G/Γ coincides with the Chevalley–Eilenberg cohomology $H^*(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G. These examples are provided by the solvmanifolds $M_{k,i} = G/\Lambda_{k,i}$.

Actually to compute the cohomology of $M_{k,0}$ one can remark that it is diffeomorphic to $S^1 \times \mathrm{H}_3(\mathbb{R})/\Gamma_k$, the *Kodaira–Thurston manifold*. Hence, we can easily write down its cohomology classes, in terms of the ones of S^1 and $\mathrm{H}_3(\mathbb{R})/\Gamma_k$.

By Nomizu's Theorem, the cohomology of $\mathrm{H}_3(\mathbb{R})/\Gamma_k$ is given by the Chevalley–Eilenberg cohomology $H^*(\mathfrak{h}_3)$ of the Lie algebra $\mathfrak{h}_3$ of $\mathrm{H}_3(\mathbb{R})$. By the structure equations

$$d\alpha = 0, \qquad d\beta = 0, \qquad d\gamma = -\alpha\beta$$

it follows that

- $H^1(\mathrm{H}_3(\mathbb{R})/\Gamma_k)$ is generated by α, β,
- $H^2(\mathrm{H}_3(\mathbb{R})/\Gamma_k)$ is generated by $\alpha\gamma, \beta\gamma$ and
- $H^3(\mathrm{H}_3(\mathbb{R})/\Gamma_k)$ is generated by $\alpha\beta\gamma$.

Let τ be the generator of $H^1(S^1) \cong \mathbb{R}^*$.

Proposition 3.1. *The de Rham cohomology classes of $M_{k,0} = G/\Lambda_{k,0}$ are given by:*

- $H^1(M_{k,0}) \cong \mathbb{R}^3$ *is generated by τ, α, β.*
- $H^2(M_{k,0}) \cong \mathbb{R}^4$ *is generated by $\tau\alpha, \tau\beta, \alpha\gamma, \beta\gamma$.*
- $H^3(M_{k,0}) \cong \mathbb{R}^4$ *is generated by $\tau\alpha\gamma, \tau\beta\gamma, \alpha\beta\gamma$.*
- $H^4(M_{k,0}) \cong \mathbb{R}^4$ *is generated by $\tau\alpha\beta\gamma$.*

A minimal model of $M_{k,0}$ is given by

$$\mathcal{M}_{k,0} = (\Lambda(x_1, y_1, z_1, t_1), d)$$

where the index denotes the degree (hence all generators have degree one) and the only non vanishing differential is given by $dz_1 = -x_1y_1$. It suffices to send t_1 to τ, x_1 to α (and so on) to have a quasi isomorphism $\mathcal{M}_{k,0} \to \Lambda M_{k,0}$, where $\Lambda M_{k,0}$ denotes the de Rham algebra of $M_{k,0}$.

This result can be also obtained by applying the method in [18,19] for the Koszul–Sullivan model of the Mostow fibration.

In order to compute the cohomologies of $M_{k,\pi}$ and $M_{k,\pi/2}$ recall that there are the 2-sheeted and 4-sheeted coverings $p_\pi : M_{k,0} \to M_{k,\pi}$ and $p_{\pi/2} : M_{k,0} \to M_{k,\pi/2}$.

In general, if $q : X \to \tilde{X}$ is a finite sheeted covering defined by the action of a group Φ on X, then the cohomologies of $\tilde{X}$ are given by the invariants by the action of the finite group Φ), i.e.,

$$H^*(\tilde{X}) \cong H^*(X)^\Phi \,,$$

(see e.g. [9, Proposition 3G,1]).

Now, the (nontrivial part of the) action of $\Lambda_{k,\pi}/\Lambda_{k,0}$ is given by $\alpha \mapsto -\alpha$ and $\beta \mapsto -\beta$.

The (nontrivial part of the) action of $\Lambda_{k,\pi/2}/\Lambda_{k,0}$ is given by $\alpha \mapsto -\beta$ and $\beta \mapsto \alpha$. Computing the invariants, one easily sees that the de Rham cohomology of $M_{k,\pi/2}$ is the same as the one of $M_{k,\pi}$.

Proposition 3.2. *The cohomology of $M_{k,\pi}$ and $M_{k,\pi/2}$ is*

- $H^1(M_{k,\pi}) \cong \mathbb{R}$ *is generated by* τ*.*
- $H^2(M_{k,\pi})$ *is trivial* (*there is no invariant 2-form*)*.*
- $H^3(M_{k,\pi}) \cong \mathbb{R}$ *is generated by* $\alpha\beta\gamma$*.*
- $H^4(M_{k,\pi}) \cong \mathbb{R}$ *is generated by* $\tau\alpha\beta\gamma$*.*

A minimal model of $M_{k,\pi}$ and $M_{k,\pi/2}$ is given by

$$\mathcal{M}_{k,\pi} = \mathcal{M}_{k,\pi/2} = (\Lambda(t_1, w_3), d = 0)$$

where the index denotes the degree, cf. [19, Example 3.2]*. A quasi isomorphism $\mathcal{M}_{k,\pi} \to \Lambda M_{k,\pi}$ is given by $t_1 \mapsto \tau$ and $w_3 \mapsto \alpha\beta\gamma$.*

Remark 2. The cohomologies of the Chevalley–Eilenberg complexes of the Lie algebras $\mathfrak{g}$ of the oscillator group G and of the nilpotent Lie algebra $\mathfrak{t} \times \mathfrak{h}_3$ of $S^1 \times \mathrm{H}_3(\mathbb{R})$ are given by:

	b_0	b_1	b_2	b_3	b_4
$\mathfrak{g}$	1	1	0	1	1
$\mathfrak{t} \times \mathfrak{h}_3$	1	3	4	3	1

Hence, $M_{k,0}$ has the same cohomology as a nilmanifold, namely the Kodaira–Thurston manifold $S^1 \times \mathrm{H}_3(\mathbb{R})/\Gamma_k$. Thus, any $M_{k,0}$ gives an (low dimensional) example of solvmanifold whose cohomology does not agree with the invariant one, i.e., the cohomology of the Chevalley–Eilenberg complex on the solvable Lie algebra $\mathfrak{g}$. On the other hand, the de Rham cohomologies of $M_{k,\pi}$ and $M_{k,\pi/2}$ are isomorphic to the cohomology of the corresponding solvable Lie algebra $\mathfrak{g}$, although the Mostow condition does not hold. See also [1].

Remark 3. In general, if the Mostow condition does not hold, as far as we know two techniques can be applied: the modification of the solvable Lie group [2,7] and the cited Koszul–Sullivan models of fibrations in the almost nilpotent case [18,19]. As for the first method, one knows by Borel density theorem (see e.g. [22, Theorem 5.5]) that there exists a compact torus $\mathbb{T}_{\rm cpt}$ such that $\mathbb{T}_{\rm cpt}\mathcal{A}(\mathrm{Ad}_G(\Gamma)) = \mathcal{A}(\mathrm{Ad}_G(G))$. Then one shows (see [2]) that there exists a subgroup $\tilde{\Gamma}$ of finite index in Γ and a simply connected normal subgroup $\tilde{G}$ (the "modified solvable Lie group") of $\mathbb{T}_{\rm cpt} \ltimes G$ such that $\mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{\Gamma})) = \mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{G}))$. Therefore, $\tilde{G}/\tilde{\Gamma}$ is diffeomorphic to $G/\tilde{\Gamma}$ and $H^*(G/\tilde{\Gamma}) \cong H^*(\tilde{\mathfrak{g}})$, where $\tilde{\mathfrak{g}}$ is the Lie algebra of $\tilde{G}$. In the case of the families $M_{k,0}$, $M_{k,\pi}$ and $M_{k,\pi/2}$ one can see that $\tilde{\Gamma}_k = \Lambda_{k,0}$ (for any k) and that $\tilde{G}_k$ is $S^1 \times \mathrm{H}_3(\mathbb{R})$.

4. Geometry to Distinguish

Here we study some geometrical features of the solvmanifolds above, as models of compact spaces provided with complex, symplectic structures or metric tensors. The aim was to distinguish these spaces also by their geometry.

Fact. Let G denote a Lie group admitting a lattice Γ such that $\Gamma\backslash G$ is a compact space. In this situation a left-invariant geometric element on G (say complex, symplectic or metric structure on G) which is determined at the Lie algebra level, induces a (sometimes called) *invariant structure* on $\Gamma\backslash G$. By this construction in general the Lie group G does not act on $\Gamma\backslash G$ transitively leaving the structure invariant, but locally. That is the compact manifold $\Gamma\backslash G$ is a locally homogeneous manifold with respect to the induced geometric structure above (see the examples below).

We shall see that with respect to complex or symplectic structures the manifolds G and N are different. However, there is a Lorentzian metric on G and N which makes these spaces isometric. However, the quotients spaces provided with these metrics can be distinguished.

4.1. Complex Structures

Recall that a left-invariant complex structure on a Lie group G corresponds to an endomorphism

on the Lie algebra $J : \mathfrak{g} \to \mathfrak{g}$ such that $J^2 = -Id$ and $N_J(x, y) \equiv 0$ where N_J denotes the Nijenhuis tensor which is given by

$$N_J(x, y) = [Jx, Jy] - [x, y] - J[Jx, y] - J[x, Jy] \qquad \text{for all } x, y \in \mathfrak{g}.$$

An invariant complex structure on G induces a complex atlas on G giving rise to a complex manifold. Due to results of Ue [24], a complex surface S is diffeomorphic to a $\mathbb{T}^2$ bundle over $\mathbb{T}^2$ if and only if S is a complex torus, Kodaira surface or hyperelliptic surface. Moreover, Hasegawa [8] proved the following:

Theorem 4.1. *A complex surface is diffeomorphic to a four-dimensional solvmanifold if and only if it is one of the following surfaces: complex torus, hyperelliptic surface, Inou surface of type S^0, primary Kodaira surface, secondary Kodaira surface, Inoue surface of type $S^{\pm}$. And every complex structure on each of these complex surfaces is invariant.*

The invariance of the complex structure concerns the algebraic structure of the group covering the manifold. Thus, for the so-called Kodaira–Thurston manifold (above Kodaira surface of type I), we have two solvable groups which covers this space. On the one hand the oscillator group G and on the other hand the nilpotent Lie group $\mathbb{R} \times \mathrm{H}_3(\mathbb{R})$.

Let us explain this. As a real manifold, both N and the oscillator group G are diffeomorphic to $\mathbb{R}^4$ together with its canonical differentiable structure. Denote $v = (x, y) \in \mathbb{R}^2$ and for each $(t_1, v_1, z_1) \in \mathbb{R}^4$ consider the following differentiable functions of M:

$$L^N_{(t_1,v_1,z_1)}(t_2, v_2, z_2) = \left(t_1 + t_2, v_1 + v_2, z_1 + z_2 + \frac{1}{2}v_1^t J v_2\right) \tag{7}$$

$$L^G_{(t_1,v_1,z_1)}(t_2, v_2, z_2) = \left(t_1 + t_2, v_1 + R(t_1)v_2, z_1 + z_2 + \frac{1}{2}v_1^t J R(t_1)v_2\right) \tag{8}$$

where J and $R(t)$ are the linear maps on $\mathbb{R}^2$ given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \qquad t \in \mathbb{R}. \tag{9}$$

The maps $L^N_{(t_1,v_1,z_1)}$ and $L^G_{(t_1,v_1,z_1)}$ are diffeomorphisms of $\mathbb{R}^4$: in fact on the basis $\{\partial_t, \partial_x, \partial_y, \partial_z\}$ of $T\mathbb{R}^4$ one has the following differentials

$$L^N_{(t_1,x_1,y_1,z_1)*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2}y_1 & \frac{1}{2}x_1 & 1 \end{pmatrix}$$

$$L^G_{(t_1,x_1,y_1,z_1)*} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos t_1 & -\sin t_1 & 0 \\ 0 & \sin t_1 & \cos t_1 & 0 \\ 0 & \mu & \nu & 1 \end{pmatrix} \quad \text{with} \quad \begin{aligned} \mu &= \tfrac{1}{2}(x_1 \sin t_1 - y_1 \cos t_1), \\ \nu &= \tfrac{1}{2}(x_1 \cos t_1 + y_1 \sin t_1). \end{aligned}$$

Now take any almost complex structure J on $T_0\mathbb{R}^4$ and translate it to every $T_p\mathbb{R}^4$ by mean of the L^N_q and L^G_q. This construction gives rise to a *left-invariant complex structure* on N and G, respectively.

Let now Γ be a lattice of N or of G. Since the complex structure is left invariant, one induces it to the quotient $\Gamma\backslash N$ and $\Gamma\backslash G$ in such way that $J_{\gamma x}X = J_x X$ for every $\gamma \in \Gamma$.

The complex structure on the compact space $\Gamma\backslash N$ or $\Gamma\backslash G$ is said to be *invariant* if it is induced by a left-invariant complex structure on G. However, the compact manifold $\Gamma\backslash G$ or $\Gamma\backslash N$ is not necessarily homogeneous.

It is known that in the corresponding nilpotent Lie group $\mathbb{R} \times \mathrm{H}_3(\mathbb{R})$ there is only one integrable almost complex structure up to equivalence but on G there are two non-equivalent ones.

Invariant complex structures on N. Let $\mathbb{R} \times \mathfrak{h}_3$ denote the Lie algebra of $\mathbb{R} \times \mathrm{H}_3(\mathbb{R})$ which is the Lie algebra generated by the following basis of left-invariant vector fields:

$$\tilde{T} = \frac{\partial}{\partial t} \qquad \tilde{X} = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z} \quad \tilde{Y} = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z} \qquad \tilde{Z} = \frac{\partial}{\partial z}$$

Every complex structure here is equivalent to (see [5])

$$J\tilde{X} = \tilde{Y} \qquad J\tilde{Z} = \tilde{T} \qquad J^2 = -1 \tag{10}$$

which is obtained by translating via the maps L_p^N the almost complex structure on $T_0\mathbb{R}^4$ given by

$$J\partial_x = \partial_y \qquad J\partial_z = \partial_t \qquad J^2 = -1. \tag{11}$$

For every k the lattice Λ_k acts properly discontinuous on (N, J) giving rise to the compact complex manifold $\Lambda_k\backslash N$, a primary Kodaira–Thurston manifold.

Notice that each invariant complex structure J on $N = \mathbb{R} \times \mathrm{H}_3(\mathbb{R})$ is abelian, that is it satisfies

$$[Ju, Jv] = [u, v] \qquad \text{for all } u, v.$$

Equivalently by complexifying the tangent space $T_p^{\mathbb{C}}\mathbb{R}^4$ the eigenspaces corresponding to the eigenvalues i and $-i$ are abelian subalgebras.

Invariant complex structures on G. On the oscillator group any invariant complex structure is equivalent [20] to one $J_\pm$ obtained by translating via the maps L_p^G the almost complex structures on $T_0\mathbb{R}^4$ given by

$$J_\pm\partial_x = \partial_y \qquad J_\pm\partial_z = \pm\partial_t \qquad J^2 = -1. \tag{12}$$

For the left-invariant vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{T}$ on G:

$$\tilde{T} = \frac{\partial}{\partial t} \qquad \tilde{X} = \cos(t)\frac{\partial}{\partial x} - \sin(t)\frac{\partial}{\partial y} - \frac{1}{2}(x\cos(t) + y\sin(t))\frac{\partial}{\partial z}$$
$$\tilde{Z} = \frac{\partial}{\partial z} \qquad \tilde{Y} = \sin(t)\frac{\partial}{\partial x} + \cos(t)\frac{\partial}{\partial y} + \frac{1}{2}(x\cos(t) - y\sin(t))\frac{\partial}{\partial z}$$

one gets the complex structure $J_\pm\tilde{X} = \tilde{Y} J_\pm\tilde{Z} = \pm\tilde{T}$. Nevertheless $J_\pm$ cannot be abelian.

Now each lattice $\Lambda_{k,i}$ $i = 0, \pi, \pi/2$ acts free and properly discontinuous on the complex manifold $(G, J_\pm)$ giving the compact complex spaces $M_{k,i}$. Moreover, one has the next covering as complex spaces

$$G \longrightarrow M_{k,0} \longrightarrow M_{k,\pi} \longrightarrow M_{k,\pi/2}. \tag{13}$$

showing that the primary Kodaira surface $M_{k,0}$ covers both secondary Kodaira surfaces $M_{k,i}$ for $i = \pi, \pi/2$. [8]. And, moreover, the secondary Kodaira surface $M_{k,\pi}$ covers the secondary Kodaira surface $M_{k,\pi/2}$ which shows a relationship among certain secondary Kodaira surfaces.

4.2. Symplectic Geometry

As above we shall say that $\Gamma\backslash G$ admits an *invariant* symplectic structure if an invariant symplectic structure ω on G is induced to $\Gamma\backslash G$. This is possible since

$$\omega_{gx}(U, V)) = \omega_x(U, V) \qquad \text{for all } g, x \in G.$$

It is known that if a given nilmanifold $\Gamma\backslash N$ admits a symplectic structure, then it admits an N-invariant one. This follows by Nomizu's Theorem, since any de Rham cohomology class has an invariant representative, and it is more in general true for solvmanifolds for which the Mostow condition holds (see [6]).

As already proved the oscillator group G does not admit any invariant symplectic structure [12,21]. But the Kodaira–Thurston manifold $M_{k,0}$ was the first example constructed in order to provide an example of a compact manifold admitting a symplectic structure but not Kähler structures [23].

Theorem 4.1. *The compact spaces $M_{k,0}$ are symplectic.*

In fact every left-invariant symplectic structure on $\mathbb{R} \times \mathrm{H}_3(\mathbb{R})$ can be induced to the quotients $M_{k,0}$. However, no symplectic structure on $M_{k,0}$ is induced by the oscillator group G.

Notice that G does not satisfies the Mostow condition. The example above shows that this is not true for any solvable Lie group.

Moreover, $M_{k,0}$ covers $M_{k,\pi}$ and $M_{k,\pi/2}$ which do not admit any symplectic structure, since their second Betti number vanishes.

Remark 4. [11, Remarks 2 and 3] Another example of a non-symplectic manifold finitely covered by a symplectic manifold is constructed but in higher dimension. The examples above provide the first examples (known to us) of this situation in dimension four.

By following usual computations one proves the next result. Let $\mathrm{H}_{2n+1}(\mathbb{R})$ denote the Heisenberg Lie algebra of dimension $2n + 1$.

Proposition 4.2. *The trivial extension $\mathbb{R} \times \mathrm{H}_{2n+1}(\mathbb{R})$ admits an invariant symplectic structure if and only if $n = 1$.*

Therefore for any lattice Λ no compact space $\Lambda\backslash G$ admits an invariant symplectic structure.

4.3. Lorentzian Metric

Explained information of this part can be found in [3,4]. Let $\mathbb{R}^4$ with the following Lorentzian metric

$$g = \mathrm{d}t\left(\mathrm{d}z + \frac{1}{2}y\mathrm{d}x - \frac{1}{2}x\mathrm{d}y\right) + \mathrm{d}x^2 + \mathrm{d}y^2 \tag{14}$$

where (t, x, y, z) are usual coordinates for $\mathbb{R}^4$. It is not hard to see that both Lie groups N and G act simply and transitively by isometries on $(\mathbb{R}^4, g)$. Translate this Lorentzian metric to both G and N.

As a consequence (N, g) is isometric to (G, g) (see [3] for more details). While the metric g is left and right invariant on G, the metric g is only left invariant on N.

The metric g on N (14) can be induced to the quotient spaces $\Lambda_k\backslash N$. In fact, for every $\gamma \in \Lambda_k$ one has:

$$\begin{aligned} g(Z_{\gamma x}, Y_{\gamma x})_{\gamma x} &= g(dp_{\gamma x}(Z), \mathrm{d}p_{\gamma x}(Y))_{p(\gamma x)} \\ &= g(dp_x(Z), \mathrm{d}p_x(Y))_{p(x)} = g(Z_x, Y_x)_x \end{aligned}$$

thus the canonical projection $p : N \to \Lambda_k\backslash N$ is a local isometry.

Analogously one induces the metric of G to the quotients $M_{k,i}$. The solvable Lie group $G = \mathbb{R} \ltimes \mathrm{H}_3(\mathbb{R})$ acts by isometries on each of the compact spaces $M_{k,i}$ for $k \in \mathbb{N}$ and $i = 0, \pi, \pi/2$. As a consequence the Heisenberg Lie group $\mathrm{H}_3(\mathbb{R}) < G$ also acts on each of the compact spaces $M_{k,i}$ for $k \in \mathbb{N}$ and $i = 0, \pi, \pi/2$. Both actions of G and $\mathrm{H}_3(\mathbb{R})$ on the respective compact spaces are locally faithful.

Let $\mathcal{N}_G(\Lambda_{k,s})$ denote the normalizer of $\Lambda_{k,s}$. Consider the following groups

$$\widetilde{\mathrm{F}}(M_{k,i}) \simeq \mathcal{N}_G(\Lambda_{k,s})/\{h \in \mathcal{N}_G(\Lambda_{k,s}) \,:\, h = (2\pi s, 0, r) : s \in \mathbb{Z}, r \in \mathbb{R}\} \quad (15)$$

where

1. $\mathcal{N}_G(\Lambda_{k,0}) = \frac{\pi}{2}\mathbb{Z} \ltimes (\frac{1}{2k}\mathbb{Z} \times \frac{1}{2k}\mathbb{Z} \times \mathbb{R})$,
2. $\mathcal{N}_G(\Lambda_{k,\pi}) = \frac{\pi}{2}\mathbb{Z} \ltimes (\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z} \times \mathbb{R})$,
3. Set $\mathcal{W} = \{(m,n) \in \mathbb{Z}^2 : m \equiv n \pmod 2\}$ then

$$\mathcal{N}_G(\Lambda_{k,\frac{\pi}{2}}) = \begin{cases} \frac{\pi}{2}\mathbb{Z} \ltimes (\mathcal{W} \times \mathbb{R}) & \text{for } k = 1, \\ \frac{\pi}{2}\mathbb{Z} \ltimes (\frac{1}{2}\mathcal{W} \times \mathbb{R}) & \text{for } k \geq 2. \end{cases}$$

are normalizers in G of the lattices $\Lambda_{k,i}$ and

$$\widetilde{\mathrm{L}}(M_{k,s}) \simeq G/\left\{h \in G \,/\, h = (2\pi s, 0, z) : s \in \mathbb{Z}, z \in \frac{1}{2k}\mathbb{Z}\right\}. \quad (16)$$

In [4] it was proved the following theorem which shows that the compact spaces $M_{k,i}$ can be distinguished by their isometry groups relative to the induced Lorentzian metric g.

Theorem 4.2. *Let $M_{k,s}$ denote the solvmanifolds of dimension four as in (5) equipped with the naturally reductive metric induced by the bi-invariant metric of G given by g (14). Then the isometry group of $M_{k,s}$ is given by*

$$\mathrm{Iso}(M_{k,s}) = \widetilde{\mathrm{F}}(M_{k,i}) \cdot \widetilde{\mathrm{L}}(M_{k,s})$$

where $\widetilde{\mathrm{F}}(M_{k,i})$ is the group in (15) and $\widetilde{\mathrm{L}}(M_{k,s})$ is the group in (16).

Moreover,

- *$\widetilde{\mathrm{L}}(M_{k,s})$ is a normal subgroup and*
- *$\widetilde{\mathcal{N}}(M_{k,s}) \cap \widetilde{L}(M_{k,s}) = \{\tau_Z \circ \widetilde{\chi}_\gamma,\ \text{where } Z := (0,0,0,z)\, z \in \mathbb{R},\ \gamma \in \Lambda_{k,s}\}$.*

Since g is bi-invariant on G, the compact spaces $M_{k,s}$ are naturally reductive spaces and the geodesics starting at $p(e)$ are precisely the projections of the geodesics of G through the identity element e (see [17, Ch. 11]). Any other geodesic of G is the translation on the left of a geodesic through e, giving rise to any geodesic on the quotient.

Moreover, notice that since g is bi-invariant the compact spaces $M_{k,i}$ are homogeneous Lorentzian spaces.

If G/K is a naturally reductive pseudo-Riemannian space then every closed geodesic in G/K is periodic.

Let α denote a curve on G and $\bar{\alpha}$ its projection on $M_{k,i}$. If $X_e = \sum_{i=0}^{3} a_i X_i(e) \in T_eG$, then the geodesic α through e with initial condition $\alpha'(0) = X_e$ is the integral curve of the left-invariant vector field $X = \sum_{i=0}^{3} a_i X_i$. Then we should have $\alpha'(s) = X_{\alpha(s)}$. Thus, the geodesic through $e = (0,0,0,0)$ with initial condition X_e satisfies:

$$\begin{aligned} t(s) &= a_0 s, \\ x(s) &= \frac{a_1}{a_0} \sin a_0 s + \frac{a_2}{a_0} \cos a_0 s - \frac{a_2}{a_0}, \\ y(s) &= -\frac{a_1}{a_0} \cos a_0 s + \frac{a_2}{a_0} \sin a_0 s + \frac{a_1}{a_0}, \\ z(s) &= \frac{1}{2}\left[\left(\frac{a_1^2}{a_0} + \frac{a_2^2}{a_0} + 2a_3\right) s - \left(\frac{a_2^2}{a_0^2} + \frac{a_1^2}{a_0^2}\right) \sin a_0 s\right]. \end{aligned}$$

If $a_0 = 0$, it is easy to see that $\alpha(s) = (0, a_1 s, a_2 s, a_3 s)$ is the corresponding geodesic.

Theorem 4.3. *Let $M_{k,i}$ denote the solvmanifolds above.*

- *Every null geodesic is periodic on $M_{k,i}$ for $i = 0, \pi, \pi/2$.*
- *There are closed and non closed time-like and space-like geodesics on $M_{k,i}$ for $i = 0, \pi, \pi/2$.*

Indeed a geodesic α on G through e with tangent vector $X = \sum_{i=0}^{3} a_i X_i$, for X_i left invariant gives rise to a closed geodesic on $M_{k,0}$ if and only if there exists $T \in \mathbb{R}$ such that $\alpha(T) \in \Lambda_{k,i}$, $i = 2\pi, \pi, \pi/2$, which

- for $a_0 \neq 0$ gives the following condition

$$\begin{aligned} a_0 T &\in i\mathbb{Z} \\ a_0^{-1}(R(a_0 T)J - J)(a_1, a_2)^t &\in \mathbb{Z} \times \mathbb{Z} \\ \left(\frac{a_1^2 + a_2^2}{2a_0} + a_3\right) T - \frac{a_1^2 + a_2^2}{a_0^2} \sin(a_0 T) &\in \frac{1}{2k}\mathbb{Z}. \end{aligned} \tag{17}$$

- For $a_0 = 0$ notice the geodesic $\bar{\alpha}$ is closed if there exists $T \in \mathbb{R}$ such that

$$\begin{aligned} (a_1 T, a_2 T)^t &\in \mathbb{Z} \times \mathbb{Z} \\ a_3 T &\in \frac{1}{2k}\mathbb{Z} \end{aligned} \tag{18}$$

It is clear that for $a_0 \neq 0$ and $a_0 T = 2\pi$, one has $R(a_0 T) = Id$ and $\sin(a_0 T) = 0$ so that null geodesics (those s.t. $a_1^2 + a_2^2 + 2a_0 a_3 = 0$) satisfy the conditions above (17). Thus, $T = 2\pi/a_0$ shows the first statement.

But for $i = \pi$ there are periodic null geodesics with smaller period than $2\pi/a_0$. In fact if $a_0 T = \pi$ then $R(a_0 T) = -Id$ and $\sin(a_0 T) = 0$. Then $(a_1, a_2) \in \frac{a_0}{2}\mathbb{Z} \times \mathbb{Z}$ so that together with $a_1^2 + a_2^2 + 2a_0 a_3 = 0$ one gets null geodesics of period π/a_0.

Remark 5. In view of the coverings $M_{k,0} \to M_{k,\pi} \to M_{k,\pi/2}$ a periodic geodesic on $M_{k,0}$ projects to periodic geodesics on $M_{k,\pi}$ and $M_{k,\pi/2}$. As explained above on the compact spaces $M_{k,\pi}$ and $M_{k,\pi/2}$ it is possible to find other periodic geodesics, which can be distinguished by their periods.

Acknowledgments

The authors deeply thank A. Fino for useful discussions and comments on the paper. The second author is grateful to the Department of Mathematics of the University of Torino, for the kindly hospitality during her stay at Torino, where part of the present work was done.

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Received: December 3, 2013.
Accepted: February 7, 2014.