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Generalized limited packings of some graphs with a limited number of P_4-partners ☆

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ABSTRACT

By using modular decomposition and handling certain graph operations such as join and union, we show that the Generalized Limited Packing Problem—NP-complete in general—can be solved in polynomial time in some graph classes with a limited number of P_4-partners; specifically P_4-tidy graphs, which contain cographs and P_4-sparse graphs. In particular, we describe an algorithm to compute the associated numbers in polynomial time within these graph classes. In this way, we generalize some of the previous results on the subject. We also make some progress on the study of the computational complexity of the Generalized Multiple Domination Problem in graphs.

1. Introduction

The notion of 2-packing in graphs, introduced by Meir and Moon in [11], was generalized to k-limited packing for a positive integer k by Gallant et al. [5]. These concepts are good graph models for many utility location problems in operations research.

In this paper we consider a problem, already introduced in [2], that models several such scenarios as the location of obnoxious facilities—e.g. garbage dumps—in a city. In these scenarios, no neighborhood should be close to too many of such facilities, nor should the facilities themselves be too close together. If a graph G models the scenario, we consider a subset $\mathcal{A}$ of its vertex set representing the possible locations for the facilities. We are interested in a maximum sized subset of these facilities we can build subject to the condition that, for every vertex v, the number of facilities located inside its closed neighborhood does not exceed k_v, a positive integer representing the capacity of v. We call this problem the Generalized Limited Packing Problem (GLP). The concept of k-limited packing introduced in [5] is a special case of ours, where $k_v = k$ for every vertex v and $\mathcal{A}$ is the whole vertex set of G. It is known that GLP is NP-complete, even for split graphs when $k_v = k$ for every vertex v and $\mathcal{A}$ is the whole vertex set [3].

The main purpose of the present work is to show that GLP is polynomial time solvable in P_4-tidy graphs—a non-perfect graph class with a limited number of P_4-partners that generalizes cographs and P_4-sparse graphs—for any capacity vector and any subset of allowed vertices. In this way, the result in [3] regarding uniform capacity vectors and $\mathcal{A}$ being the whole vertex set can be seen as a corollary of this more general result. We also connect our results with a related problem

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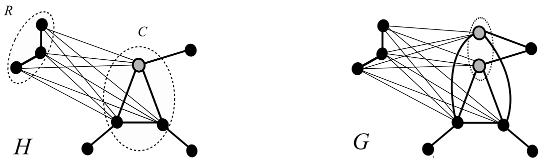

Fig. 1. A quasi-spider G obtained from a thin spider H.

in graphs—the Generalized Multiple Domination Problem [9]—and make some progress on the study of its computational complexity.

Throughout this work, graphs are simple and $V(G)$ and $E(G)$ denote, respectively, the vertex and edge sets of the graph G. For other definitions and notation not defined here, the reader is referred to [12].

For $v \in V(G)$, $N_G[v]$ denotes the *closed neighborhood* and $N_G(v)$ the *(open) neighborhood* of v. Two vertices u and w are *false twins* in G if $N_G(u) = N_G(w)$ and *true twins* in G if $N_G[u] = N_G[w]$. Given a graph G and $R \subseteq V(G)$, $G[R]$ denotes the subgraph *induced* by the vertices in R, i.e. $V(G[R]) = R$ and $E(G[R])$ is the subset of $E(G)$ consisting of those edges with both endpoints belonging to R. In particular, $G - v$ denotes the induced subgraph $G[V(G) - \{v\}]$. For $n \in \mathbb{N}$, S_n, C_n, K_n and P_n denote respectively a graph without edges, a cycle, a complete graph and a path on n vertices.

Given two graphs G^1 and G^2, with $V(G^1) \cap V(G^2) = \emptyset$, the *(disjoint) union* of G^1 and G^2, denoted by $G^1 \cup G^2$, is the graph with $V(G) = V(G^1) \cup V(G^2)$ and $E(G) = E(G^1) \cup E(G^2)$. The *join* of G^1 and G^2, denoted by $G^1 \vee G^2$, is the graph with $V(G) = V(G^1) \cup V(G^2)$ and $E(G) = E(G^1) \cup E(G^2) \cup \{ij : i \in V(G^1), j \in V(G^2)\}$.

Let G and G' be two graphs such that $V(G) \cap V(G') = \emptyset$ and $v \in V(G)$. The graph obtained by *replacing* v by G' is the graph whose vertex set is $(V(G) - \{v\}) \cup V(G')$ and whose edge set is $E(G - v) \cup E(G') \cup \{ij : i \in V(G'), j \in N_G(v)\}$.

Let G be an arbitrary graph. A set M of vertices is a *module* if every vertex in $V(G) - M$ is either adjacent to all the vertices in M or to none of them. Hence, a module M of G is also a module of the complement graph of G. The empty set, the singletons and $V(G)$ are the trivial modules of G. A module M is a *strong* module if, for any other module A, $M \cap A = \emptyset$ or one module is contained into the other. Any graph distinct from K_1, K_2 and S_2 and having only trivial modules is a *prime* graph.

There are several methods for decomposing the structure of a graph. A consequence of a decomposition process is the simplification of NP-hard problems, for instance. In this paper we will use *modular decomposition* [4], a form of decomposition of a graph G that associates with G a unique modular decomposition tree $T(G)$. The leaves of $T(G)$ are the vertices of G and the nodes of $T(G)$ are modules of G. The modular decomposition tree can be recursively defined as follows: the root of the tree corresponds to the entire graph; if the graph is not connected, the root is called *parallel* and its children are the modular decomposition trees of its components; if the complement graph of G is not connected, the root is called *series* and its children are the decomposition trees of the components of $\overline{G}$; if G and $\overline{G}$ are both connected, the root is called *neighborhood* and its children are the modular decomposition trees of the graphs induced by its maximal strong modules. In this last case, a representative graph is associated with the node; this graph has one vertex representing each maximal strong module and two vertices are adjacent if and only if there is an edge in G with one extreme in each module. The representative graph of a neighborhood module is a prime graph. The efficient construction of the modular decomposition tree had been extensively studied. In [1] and [10], independent linear-time algorithms are provided.

A *spider* graph, introduced in [7], is a graph whose vertex set can be partitioned into S, C and R, where $S = \{s_1, \dots, s_r\}$ is a stable set; $C = \{c_1, \dots, c_r\}$ is a clique, $r \geq 2$; s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); R, called the *head*, is allowed to be empty and if it is not, all vertices in R are adjacent to all vertices in C and non-adjacent to all vertices in S. A *leg* of the spider is any edge with one endpoint in S and the other in C. It is straightforward to see that the complement of a thin spider graph is a thick spider graph, and vice-versa. The triple (S, C, R) is called the *(spider) partition* and can be found in linear time [8]. If the head R is empty or contains one vertex, then a spider with thin (thick) legs is called an *urchin* (*starfish*). Urchins and starfish are prime graphs.

A graph is a *quasi-spider* graph (see Fig. 1) if it can be obtained from a spider graph with partition (S, C, R) by replacing at most one vertex of $S \cup C$ by a graph with two vertices. Observe that every spider graph is also a quasi-spider graph.

A *partner* of an induced P_4 in a graph G is a vertex $v \in V(G) - V(P_4)$ such that the subgraph induced by $V(P_4) \cup \{v\}$ has at least two induced P_4's.

A graph is a P_4-*tidy* graph if every induced P_4 has at most one partner (see [6]). It is not difficult to prove that the family of P_4-tidy graphs is hereditary and self-complementary.

We will base our results on the following theorems. Notice that the result in Theorem 1 is not stated literally in [6]; nevertheless it can be derived from that paper:

Theorem 1 *(Characterization of P_4-tidy graphs). A graph G is a P_4-tidy graph if and only if exactly one of the following conditions holds:*

(1) *G or the complement of the graph G is not connected and each one of its components is a P_4-tidy graph;*
(2) *G is a quasi-spider (S, C, R) and $G[R]$ is a P_4-tidy graph;*
(3) *G is isomorphic to P_5, $\overline{P_5}$, C_5 or K_1, where $\overline{P_5}$ denotes the complement of the graph P_5.*

Theorem 2. *(See [6].) Let G be a prime P_4-tidy graph. Then, G is isomorphic to C_5, P_5, $\overline{P_5}$, a starfish or an urchin, where $\overline{P_5}$ denotes the complement of the graph P_5.*

Theorem 3. *(See [6].) Let G be a P_4-tidy graph and M a neighborhood module of G. If the representative graph of M is a prime spider $H = (S, C, R)$ (starfish or urchin) then $G[M]$ is obtained from H by replacing at most one vertex of $S \cup C$ by a K_2 or an S_2, and replacing R by the subgraph induced by a module.*

In this paper, $\mathbb{Z}_+$ denotes the set of nonnegative integer numbers.

For a set H, $\mathbb{Z}_+^H$ indicates the space consisting of vectors of nonnegative integer numbers of dimension $|H|$ with entries indexed by elements of H.

The all ones vector is denoted by $\mathbf{1}$.

A *shift* of a vector $\mathbf{a}$ is a vector of the form $\mathbf{a} - r\mathbf{1}$, for $r \in \mathbb{N}$.

For a vector $\mathbf{k} = (k_v) \in \mathbb{Z}_+^{V(G)}$, k_v denotes the capacity of $v \in V(G)$.

For a graph G, $\mathcal{A} \subseteq V(G)$ and $\mathbf{k} = (k_v) \in \mathbb{Z}_+^{V(G)}$, $B \subseteq V(G)$ is a $(\mathbf{k}, \mathcal{A})$*-limited packing* of G if $B \subseteq \mathcal{A}$ and $|N_G[v] \cap B| \le k_v$, for every $v \in V(G)$. The size of a $(\mathbf{k}, \mathcal{A})$-limited packing of G of maximum cardinality is denoted by $L_{\mathbf{k},\mathcal{A}}(G)$.

Let us introduce GLP formally [2].

The Generalized Limited Packing problem (GLP) is formulated as

INSTANCE: A graph G, a vector $\mathbf{k} \in \mathbb{Z}_+^{V(G)}$, $\mathcal{A} \subseteq V(G)$ and $\alpha \in \mathbb{N}$.
QUESTION: Does G contain a $(\mathbf{k}, \mathcal{A})$-limited packing of size at least α?

In order to simplify the presentation, an instance of GLP will be denoted by $(G, \mathbf{k}, \mathcal{A})$. Clearly, if $k_v \ge |N_G[v]|$ or $k_v \ge |\mathcal{A}|$ for every $v \in V(G)$ then $L_{\mathbf{k},\mathcal{A}}(G) = |\mathcal{A}|$. Also, given $(G, \mathbf{k}, \mathcal{A})$ and an induced subgraph G' of G, when we talk about $(\mathbf{k}, \mathcal{A})$-limited packings of G', we mean that $\mathbf{k}$ represents its projection onto $\mathbb{Z}^{V(G')}$ and $\mathcal{A}$ represents the set $\mathcal{A} \cap V(G')$.

Remark 4. Let $(G, \mathbf{k}, \mathcal{A})$ be an instance of GLP such that $k_v = 0$ for a certain $v \in V(G)$. Notice that solving $(G, \mathbf{k}, \mathcal{A})$ is equivalent to solving $(G, \mathbf{k}, \mathcal{A} - N_G[v])$. Therefore, we assume that whenever such an instance is given, we will answer the corresponding question for the instance $(G, \mathbf{k}, \mathcal{A} - N_G[v])$.

For a set $X \subseteq V(G)$ and a vector $\mathbf{k} = (k_v) \in \mathbb{Z}_+^{V(G)}$, we denote

$$m_X := \min\{k_v : v \in X\}.$$

For instances of GLP given by complete graphs, the packing parameter $L_{\mathbf{k},\mathcal{A}}(G)$ is easily calculated in the following way, for every vector $\mathbf{k}$ and $\mathcal{A}$:

Remark 5. For every instance $(G, \mathbf{k}, \mathcal{A})$ of GLP where G is a complete graph, we have $L_{\mathbf{k},\mathcal{A}}(G) = \min\{|\mathcal{A}|, m_{V(G)}\}$.

2. Union, join and generalized limited packings

In this section, we show how to handle generalized limited packings and some graph operations such as union and join. We write V^i to denote $V(G^i)$ for $i \in \{1, 2\}$.

The first proposition is straightforward.

Proposition 6. *Let $(G^1 \cup G^2, \mathbf{k}, \mathcal{A})$ be an instance of GLP. Then,*

$$L_{\mathbf{k},\mathcal{A}}\big(G^1 \cup G^2\big) = L_{\mathbf{k},\mathcal{A}}\big(G^1\big) + L_{\mathbf{k},\mathcal{A}}\big(G^2\big).$$

We also have:

Proposition 7. *Let $(G^1 \vee G^2, \mathbf{k}, \mathcal{A})$ be an instance of GLP. Then,*

$$L_{\mathbf{k},\mathcal{A}}\big(G^1 \vee G^2\big) = \max_{s,r \in \mathbb{Z}_+, r \le m_{V^1}, s \le m_{V^2}} \big\{s + r : r \le L_{\mathbf{k}-s\mathbf{1},\mathcal{A}}\big(G^2\big), s \le L_{\mathbf{k}-r\mathbf{1},\mathcal{A}}\big(G^1\big)\big\}.$$

Proof. Let $h := \max_{s,r \in \mathbb{Z}_+, r \le m_{V^1}, s \le m_{V^2}} \{s + r : r \le L_{\mathbf{k}-s\mathbf{1},\mathcal{A}}(G^2), s \le L_{\mathbf{k}-r\mathbf{1},\mathcal{A}}(G^1)\}$.

Let B be a $(\mathbf{k}, \mathcal{A})$-limited packing of $G^1 \vee G^2$ and let $s := |B \cap V^1|$ and $r := |B \cap V^2|$. Given any $v \in V^1$, it follows that $k_v \ge |N_G[v] \cap B| = |N_G[v] \cap B \cap V^1| + r$ and then $|N_G[v] \cap B \cap V^1| \le k_v - r$. Thus $r \le k_v$ and moreover, $B \cap V^1$ is a $(\mathbf{k} - r\mathbf{1}, \mathcal{A})$-limited packing of G^1, implying that $s \le L_{\mathbf{k}-r\mathbf{1},\mathcal{A}}(G^1)$. Similarly, $r \le L_{\mathbf{k}-s\mathbf{1},\mathcal{A}}(G^2)$. Therefore, since $|B| = s + r$, $L_{\mathbf{k},\mathcal{A}}(G) \le h$.

In order to prove the other inequality, take $s, r \in \mathbb{Z}_+$ such that $r \leq m_{V^1}$, $r \leq L_{\mathbf{k}-s\mathbf{1},\mathcal{A}}(G^2)$, $s \leq m_{V^2}$, $s \leq L_{\mathbf{k}-r\mathbf{1},\mathcal{A}}(G^1)$. Let B_1 be a $(\mathbf{k}-r\mathbf{1}, \mathcal{A})$-limited packing of G^1 of size s and let B_2 be a $(\mathbf{k}-s\mathbf{1}, \mathcal{A})$-limited packing of G^2 of size r and define $B := B_1 \cup B_2$. Given $v \in V^1$, we have $|N_G[v] \cap B| = |N_G[v] \cap B_1| + |N_G[v] \cap B_2| \leq k_v - r + r = k_v$. In the same way, given $v \in V^2$ we have $|N_G[v] \cap B| \leq s + k_v - s = k_v$. Then B is a $(\mathbf{k}, \mathcal{A})$-limited packing of $G^1 \vee G^2$ of size $s + r$. As a result, $L_{\mathbf{k},\mathcal{A}}(G^1 \vee G^2) \geq h$. $\square$

Notice that each of the two propositions shown above reduces the computation of $L_{\mathbf{k},\mathcal{A}}(G)$ (with $G = G_1 \cup G_2$ or $G = G_1 \vee G_2$) to the computation on two disjoint subgraphs of G. This computation only needs to be done for the vector $\mathbf{k}$ and a finite number of shifts of $\mathbf{k}$.

3. Generalized limited packings of quasi-spider graphs

The results in this section will be used later to construct a polynomial time algorithm for P_4-tidy graphs.

We begin our study analyzing generalized limited packings of spider graphs, although not giving the exact value of the packing parameter for a general spider.

From now on, we will consider instances $(H, \mathbf{k}, \mathcal{A})$ of GLP where H is a spider graph with partition (S, C, R), $S = \{s_1, \ldots, s_r\}$ and $C = \{c_1, \ldots, c_r\}$. We denote $\mathcal{A}_S := \mathcal{A} \cap S$, $\mathcal{A}_C := \mathcal{A} \cap C$ and $\mathcal{A}_R := \mathcal{A} \cap R$.

3.1. Thin spider graphs

We will show that, for solving GLP on an instance $(H, \mathbf{k}, \mathcal{A})$, it is enough to solve GLP on the instance given by $H[R \cup C]$ and appropriate capacities and allowed vertices.

First we can establish:

Proposition 8. *For every instance $(H, \mathbf{k}, \mathcal{A})$ of GLP with H thin, there exists a maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H containing $\mathcal{A}_S$.*

Proof. Let B be a maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H with the maximum possible number of vertices in $\mathcal{A}_S$. If $\mathcal{A}_S - B \neq \emptyset$, let $s_j \in \mathcal{A}_S - B$. By Remark 4, $k_{s_j} \geq 1$ and $k_{c_j} \geq 1$, and since B is maximum, $|N_H[v] \cap B| = k_v$ for $v = c_j$ or $v = s_j$. Thus, there exists $w \in (R \cup C) \cap B$ and it follows that $B' = (B - \{w\}) \cup \{s_j\}$ is a maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H such that $|B' \cap \mathcal{A}_S| > |B \cap \mathcal{A}_S|$. $\square$

As an immediate corollary we obtain:

Corollary 9. *Let $(H, \mathbf{k}, \mathcal{A})$ be an instance of GLP where H is thin. If there exists $c_i \in C$ such that $k_{c_i} = 0$ (or $k_{c_i} = 1$ and $s_i \in \mathcal{A}_S$), then $L_{\mathbf{k},\mathcal{A}}(H) = |\mathcal{A}_S|$.*

The remaining instances concerning thin spider graphs are covered by the next proposition:

Proposition 10. *Let $(H, \mathbf{k}, \mathcal{A})$ be an instance of GLP where H is thin and $k_{c_i} \geq 2$ for each $c_i \in C$ for which $s_i \in \mathcal{A}_S$. Defining $\hat{\mathcal{A}}$ and $\hat{\mathbf{k}}$ by*

$$\hat{\mathcal{A}} := \mathcal{A} - \{c_i \in C : s_i \in \mathcal{A}_S \wedge k_{s_i} = 1\} \text{ and } \hat{k}_v := k_v - |N_H[v] \cap \mathcal{A}_S| \text{ if } v \in R \cup C,$$

it follows that, if $\hat{B}$ is a maximum $(\hat{\mathbf{k}}, \hat{\mathcal{A}})$-limited packing of $H[R \cup C]$, then $\hat{B} \cup \mathcal{A}_S$ is a maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H.

Proof. Let $\hat{B}$ be a maximum $(\hat{\mathbf{k}}, \hat{\mathcal{A}})$-limited packing of $H[R \cup C]$ and consider $\hat{B} \cup \mathcal{A}_S$. From the definitions of $\hat{\mathbf{k}}$ and $\hat{\mathcal{A}}$, $\hat{B} \cup \mathcal{A}_S$ is a $(\mathbf{k}, \mathcal{A})$-limited packing of H, thus it remains to be proved that it is maximum.

Take a maximum $(\mathbf{k}, \mathcal{A})$-limited packing $\hat{T} \cup \mathcal{A}_S$ of H, with $\hat{T} \subseteq R \cup C$. Let us show that $\hat{T}$ is a $(\hat{\mathbf{k}}, \hat{\mathcal{A}})$-limited packing of $H[R \cup C]$. Indeed, for any $c_i \in C$, $k_{c_i} \geq |N_H[c_i] \cap (\hat{T} \cup \mathcal{A}_S)| = |N_{H[R\cup C]}[c_i] \cap \hat{T}| + |\{s_i\} \cap \mathcal{A}_S|$ for each $c_i \in C$, thus $\hat{k}_{c_i} \geq |N_{H[R\cup C]}[c_i] \cap \hat{T}|$ for each $c_i \in C$. For each $v \in R$, the corresponding inequality trivially holds.

Since $\hat{B}$ is maximum, $|\hat{T}| \leq |\hat{B}|$. The maximality of $\hat{T} \cup \mathcal{A}_S$ implies that $|\hat{T} \cup \mathcal{A}_S| = |\hat{B} \cup \mathcal{A}_S|$ and therefore, $\hat{B} \cup \mathcal{A}_S$ is maximum. $\square$

3.2. Thick spider graphs

Let us start by noting that if there exists $c_i \in C$ with $k_{c_i} = 0$, we can use Remark 4 to conclude that $L_{\mathbf{k},\mathcal{A}}(H) = 1$ when $s_i \in \mathcal{A}_S$ or $L_{\mathbf{k},\mathcal{A}}(H) = 0$ when $s_i \notin \mathcal{A}_S$.

Recalling that $m_C = \min\{k_{c_i} : c_i \in C\}$, it is straightforward to see that $L_{\mathbf{k},\mathcal{A}}(H) \leq m_C + 1$, for every $\mathbf{k}$ and $\mathcal{A}$, whenever H is thick.

We also denote:

$$C^1 := \{c_i \in C : k_{c_i} = m_C\} \quad \text{and} \quad S^1 := \left\{s_i \in \mathcal{A}_S : c_i \in C^1\right\}.$$

Clearly, $|C^1| \geq |S^1|$.

Instances considered in the following proposition are easily solved:

Proposition 11. *Let $(H, \mathbf{k}, \mathcal{A})$ be an instance of GLP where H is thick. Then,*

(1) *if $|S^1| \geq m_C + 2$, then $L_{\mathbf{k},\mathcal{A}}(H) = m_C$;*
(2) *if $|S^1| \leq m_C + 1 \leq |\mathcal{A}_S|$, then*

$$L_{\mathbf{k},\mathcal{A}}(H) = \begin{cases} m_C & \text{if } |C^1| > |S^1| \\ m_C + 1 & \text{if } |C^1| = |S^1|. \end{cases}$$

Proof. (1) Any subset of S^1 with m_C elements is a $(\mathbf{k}, \mathcal{A})$-limited packing of H, thus $L_{\mathbf{k},\mathcal{A}}(H) \geq m_C$. Let B be a $(\mathbf{k}, \mathcal{A})$-limited packing of H with $|B| = m_C + 1$. By hypothesis, there exists $c_j \in C^1$ such that $s_j \in S^1 - B$. Thus $|N_H[c_j] \cap B| = |B| > m_C = k_{c_j}$, leading to a contradiction. Therefore $L_{\mathbf{k},\mathcal{A}}(H) = m_C$.

(2) When $|C^1| > |S^1|$, any subset B of $\mathcal{A}_S$ with m_C elements is a $(\mathbf{k}, \mathcal{A})$-limited packing of H. To see that B is maximum, notice that there exists $c_j \in C^1$ such that $s_j \notin \mathcal{A}_S$. Then, for any $(\mathbf{k}, \mathcal{A})$-limited packing B' of H, we have $m_C = k_{c_j} \geq |N_H[c_j] \cap B'| = |B'|$, concluding that $L_{\mathbf{k},\mathcal{A}}(H) = m_C$. When $|C^1| = |S^1|$, any subset B of $\mathcal{A}_S$ with $S^1 \subseteq B$ and $|B| = m_C + 1$ is a maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H. Since $L_{\mathbf{k},\mathcal{A}}(H) \leq m_C + 1$, the statement holds. $\square$

Next, in Propositions 12 and 16, we complete the study of instances $(H, \mathbf{k}, \mathcal{A})$ where H is thick and $|\mathcal{A}_S| \leq m_C$.

Proposition 12. *Let $(H, \mathbf{k}, \mathcal{A})$ be an instance of GLP with H thick and $|\mathcal{A}_S| \leq m_C$. There exists a maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H containing $\mathcal{A}_S$.*

Proof. Let B be a maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H. If $\mathcal{A}_S \subseteq B$, we are done. If not, take $s_i \in \mathcal{A}_S - B$ and notice that there exists $v \in N_H[s_i]$ such that $|N_H[v] \cap B| = k_v$. Since $|\mathcal{A}_S| \leq m_C$ and $s_i \in \mathcal{A}_S - B$, $|B \cap \mathcal{A}_S| \leq m_C - 1$, thus it follows that there exists $w \in (N_H[v] \cap B) - \mathcal{A}_S$. It turns out that $B' := (B - \{w\}) \cup \{s_i\}$ is another maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H, since for any $x \in V(H)$, $|N_H[x] \cap B'| \leq |N_H[x] \cap B|$. Clearly, $|B' \cap \mathcal{A}_S| > |B \cap \mathcal{A}_S|$. $\square$

To complete the study of the remaining instances given by thick spiders H, it is enough to consider the subgraph $H[R \cup C]$. First observe the following straightforward fact:

Remark 13. Let $(H, \mathbf{k}, \mathcal{A})$ be an instance of GLP with H thick and F be a $(\mathbf{k}, \mathcal{A})$-limited packing of $H[R \cup C]$. For every $c_j \in \mathcal{A}_C - F$ and $c_i \in \mathcal{A}_C \cap F$, $(F - \{c_i\}) \cup \{c_j\}$ is a $(\mathbf{k}, \mathcal{A})$-limited packing of $H[R \cup C]$.

Definition 14. Let $(H, \mathbf{k}, \mathcal{A})$ with H thick be an instance of GLP that has a maximum $(\mathbf{k}, \mathcal{A})$-limited packing containing $\mathcal{A}_S$. We define $\widetilde{\mathbf{k}} \in \mathbb{Z}_+^{V(H[R\cup C])}$ in the following way:

$$\widetilde{k}_v := \begin{cases} k_v - |\mathcal{A}_S - \{s_i\}| & \text{if } v = c_i \text{ for } c_i \in C \\ k_v & \text{otherwise.} \end{cases}$$

From the way $\widetilde{\mathbf{k}}$ was defined, it is clear that

Remark 15. Let $(H, \mathbf{k}, \mathcal{A})$ be an instance of GLP with H thick and $F \subseteq R \cup C$. If $F \cup \mathcal{A}_S$ is a $(\mathbf{k}, \mathcal{A})$-limited packing of H, then F is a $(\widetilde{\mathbf{k}}, \mathcal{A})$-limited packing of $H[R \cup C]$.

We have:

Proposition 16. *Let $(H, \mathbf{k}, \mathcal{A})$ be an instance of GLP where H is thick and $|\mathcal{A}_S| \leq m_C$. There exists $\widetilde{\mathcal{A}}_C \subseteq \mathcal{A}_C$ such that*

$$L_{\mathbf{k},\mathcal{A}}(H) = L_{\widetilde{\mathbf{k}},\mathcal{A}_R \cup \widetilde{\mathcal{A}}_C}\big(H[R \cup C]\big) + |\mathcal{A}_S|,$$

where $\widetilde{\mathbf{k}}$ is given in Definition 14 and $\widetilde{\mathcal{A}}_C$ can be obtained in linear time.

Proof. We will analyze two cases:

- *Case 1:* $|\mathcal{A}_C| \leq m_S - 1$.
  Set $\widetilde{\mathcal{A}}_C = \mathcal{A}_C$ and consider T, a maximum $(\widetilde{\mathbf{k}}, \mathcal{A})$-limited packing of $H[R \cup C]$. We will show that $T \cup \mathcal{A}_S$ is a maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H. For every $s_i \in S$, we have $|N_H[s_i] \cap (T \cup \mathcal{A}_S)| = |(C - \{c_i\}) \cap T| + |\{s_i\} \cap \mathcal{A}_S| \leq |\mathcal{A}_C| + 1 \leq m_S - 1 + 1 \leq k_{s_i}$. To see that $T \cup \mathcal{A}_S$ is maximum, let B be a $(\mathbf{k}, \mathcal{A})$-limited packing of H. By Proposition 12, we may assume that B contains A_S. It is easy to see that $B - \mathcal{A}_S$ is a $(\widetilde{\mathbf{k}}, \mathcal{A})$-limited packing of $H[R \cup C]$ and $|B| \leq |B - \mathcal{A}_S| + |\mathcal{A}_S| \leq |T| + |\mathcal{A}_S|$.
- *Case 2:* $|\mathcal{A}_C| \geq m_S$. We consider the sets

$$\mathcal{D} := \{s_i \in \mathcal{A}_S : k_{s_i} = m_S\} \quad \text{and} \quad \mathcal{F} := \{s_i \in S : k_{s_i} - |s_i \cap \mathcal{A}_S| = m_S\}.$$

  - *Subcase 2.1:* There exists $s_i \in \mathcal{D}$ such that $c_i \notin \mathcal{A}_C$ or ($|\mathcal{D}| \geq m_S + 1$ and $c_i \in \mathcal{A}_C$ for every $s_i \in \mathcal{D}$).
    By Proposition 12, there is a maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H containing $\mathcal{A}_S$, say $F \cup \mathcal{A}_S$, with $F \subseteq R \cup C$ and $L_{\mathbf{k},\mathcal{A}}(H) = |F| + |\mathcal{A}_S|$. Each of the two hypotheses implies $|F \cap \mathcal{A}_C| \leq m_S - 1$.
    By Remark 15, F is a $(\widetilde{\mathbf{k}}, \mathcal{A})$-limited packing of $H[R \cup C]$. Take any $\widetilde{\mathcal{A}}_C \subseteq \mathcal{A}_C$ with $|\widetilde{\mathcal{A}}_C| = m_S - 1$. By Remark 13, there exists a $(\widetilde{\mathbf{k}}, \mathcal{A}_R \cup \widetilde{\mathcal{A}}_C)$-limited packing $\widetilde{F}$ of $H[R \cup C]$ such that $|\widetilde{F}| = |F|$. The result follows from Case 1.
  - *Subcase 2.2:* $|\mathcal{D}| \leq m_S$ and $c_i \in \mathcal{A}_C$ for every $s_i \in \mathcal{D}$.
    - $|\mathcal{D}| \geq 1$.
      Take $\widetilde{\mathcal{A}}_C \subseteq \mathcal{A}_C$ with $\{c_j \in \mathcal{A}_C : s_j \in \mathcal{D}\} \subseteq \widetilde{\mathcal{A}}_C$ and $|\widetilde{\mathcal{A}}_C| = m_S$. Given a $(\widetilde{\mathbf{k}}, \mathcal{A}_R \cup \widetilde{\mathcal{A}}_C)$-limited packing T' of $H[R \cup C]$, there exists a $(\widetilde{\mathbf{k}}, \mathcal{A}_R \cup \widetilde{\mathcal{A}}_C)$-limited packing T of $H[R \cup C]$ such that $|T| = |T'|$ and $T \cup \mathcal{A}_S$ is a $(\mathbf{k}, \mathcal{A})$-limited packing of H (if $|T' \cap C| \leq m_S - 1$, take $T = T'$; otherwise, Remark 13 allows us to choose T satisfying $\{c_i \in \mathcal{A}_C : s_i \in \mathcal{D}\} \subseteq T$). Thus $L_{\widetilde{\mathbf{k}},\mathcal{A}_R \cup \widetilde{\mathcal{A}}_C}(H[R \cup C]) + |\mathcal{A}_S| \leq L_{\mathbf{k},\mathcal{A}}(H)$. In order to prove the reverse inequality, let B be a maximum $(\mathbf{k}, \mathcal{A})$-limited packing of H such that $\mathcal{A}_S \subseteq B$. By Remark 15, $B - \mathcal{A}_S$ is a $(\widetilde{\mathbf{k}}, \mathcal{A})$-limited packing of $H[R \cup C]$. In this case, we have $|B \cap \mathcal{A}_C| \leq m_S$. By Remark 13, there exists a $(\widetilde{\mathbf{k}}, \mathcal{A}_R \cup \widetilde{\mathcal{A}}_C)$-limited packing $\widetilde{T}$ of $H[R \cup C]$ with $|\widetilde{T}| = |B - \mathcal{A}_S|$. Therefore $L_{\mathbf{k},\mathcal{A}}(H) = |B| = |B - \mathcal{A}_S| + |\mathcal{A}_S| \leq L_{\widetilde{\mathbf{k}},\mathcal{A}_R \cup \widetilde{\mathcal{A}}_C}(H[R \cup C]) + |\mathcal{A}_S|$.
    - $\mathcal{D} = \emptyset$.
      When $|\mathcal{F}| \geq m_S + 2$, the statement holds for any $\widetilde{\mathcal{A}}_C \subseteq \mathcal{A}_C$ with $|\widetilde{\mathcal{A}}_C| = m_S$ (since for every $(\mathbf{k}, \mathcal{A})$-limited packing B of H, $|B \cap \mathcal{A}_C| \leq m_S$).
      When $|\mathcal{F}| \leq m_S + 1$, do the following: if $|\mathcal{A}_C| \geq m_S + 1$ and $c_j \in \mathcal{A}_C$ for all j such that $s_j \in \mathcal{F}$, take $\widetilde{\mathcal{A}}_C \subseteq \mathcal{A}_C$ such that $\{c_j \in \mathcal{A}_C : s_j \in \mathcal{F}\} \subseteq \widetilde{\mathcal{A}}_C$ and $|\widetilde{\mathcal{A}}_C| = m_S + 1$; otherwise, take $\widetilde{\mathcal{A}}_C = \mathcal{A}_C$. In both cases, the result clearly follows. $\square$

Let us recall that a quasi-spider graph H that is not a spider is obtained from a spider graph by replacing at most one vertex of exactly one leg by a graph with two vertices. Observe that these two vertices are twins in H. The next straightforward proposition shows how to obtain the objective involved in GLP on instances given by graphs with twins.

Proposition 17. *Let $(G, \mathbf{k}, \mathcal{A})$ be an instance of GLP where G is any graph and let u and w be twins in G. Then, assuming without loss of generality that $k_u \geq k_w$, we have,*

i. $L_{\mathbf{k},\mathcal{A}}(G - u) \leq L_{\mathbf{k},\mathcal{A}}(G) \leq L_{\mathbf{k},\mathcal{A}}(G - u) + 1$.
ii. *If $u \notin \mathcal{A}$ then $L_{\mathbf{k},\mathcal{A}}(G) = L_{\mathbf{k},\mathcal{A}}(G - u)$.*
iii. *If $w \notin \mathcal{A}$ and $k_u = k_w$, then $L_{\mathbf{k},\mathcal{A}}(G) = L_{\mathbf{k},\mathcal{A}}(G - w)$.*
iv. *If $u \in \mathcal{A}$ then $L_{\mathbf{k},\mathcal{A}}(G) = L_{\mathbf{k},\mathcal{A}}(G - u) + 1$ if and only if:*
   - $L_{\mathbf{k}^*,\mathcal{A}}(G - u) = L_{\mathbf{k},\mathcal{A}}(G - u)$ *when u and w are false twins, $k_u > k_w$ and $\mathbf{k}^*$ is defined by $k^*_v := k_v - 1$ if $v \in N_G(w)$ and $k^*_v := k_v$ otherwise;*
   - $L_{\mathbf{k}',\mathcal{A}-\{u,w\}}(G - u) \geq L_{\mathbf{k},\mathcal{A}}(G - u) - 1$ *when u and w are false twins, $w \in \mathcal{A}$, $k_u = k_w$ and $\mathbf{k}'$ is defined by $k'_v := k_v - 2$ if $v \in N_{G-u}(w)$, $k'_w := k_w - 1$ and $k'_v := k_v$ otherwise;*
   - $L_{\mathbf{k}'',\mathcal{A}}(G - u) = L_{\mathbf{k},\mathcal{A}}(G - u)$, *when u and w are true twins and $\mathbf{k}''$ is defined by $k''_v := k_v - 1$ if $v \in N_{G-u}[w]$ and $k''_v := k_v$ otherwise.*

Going back to P_4-tidy graphs, the results achieved so far allow us to state the following theorem, which generalizes that in [3]:

Theorem 18. *GLP can be solved in polynomial time for P_4-tidy graphs.*

Sketch of the proof. Let $(G, \mathbf{k}, \mathcal{A})$ be an instance of GLP where G is a P_4-tidy graph.

Consider $T(G)$, the modular decomposition tree of G. The results up to now give a dynamic programming algorithm to recursively compute $L_{\mathbf{k},\mathcal{A}}(G)$ from $T(G)$.

Traverse $T(G)$ down-top, computing the packing parameters for each strong module that occurs in it. For each node of $T(G)$ representing a module H of G, you have to follow the scheme below in order to tackle the next level of $T(G)$. In the worst case, you have to dynamically compute $L_{\mathbf{k}-r\mathbf{1},\mathcal{A}}(H)$ for each r such that $0 \le r \le \min\{m_{V(H)}, |\mathcal{A} \cap V(H)|\}$.

Notice that for any subgraph H of G and shift $\mathbf{c}$ of $\mathbf{k}$ such that $\min\{c_v : v \in V(H)\} \ge |\mathcal{A} \cap V(H)|$, we have $L_{\mathbf{c},\mathcal{A}}(H) = |\mathcal{A} \cap V(H)|$.

Scheme:

(1) It is straightforward to compute $L_{\mathbf{k},\mathcal{A}}(H)$ when H is a single vertex, P_5, C_5 or $\overline{P_5}$.
(2) If H is a thin spider (S, C, R) with $R = \emptyset$, then $L_{\mathbf{k},\mathcal{A}}(H)$ can be calculated using Corollary 9 or Proposition 10. In the latter case, use Remark 5 for the calculation of $L_{\hat{\mathbf{k}},\mathcal{A}_C}(H[C])$.
(3) If H is a thick spider (S, C, R) with $R = \emptyset$, then $L_{\mathbf{k},\mathcal{A}}(H)$ can be calculated using Proposition 11 or 16. In the latter case, use Remark 5 for the calculation of $L_{\tilde{\mathbf{k}},\mathcal{A}_C}(H[C])$.
(4) If H is a parallel node, then use Proposition 6 to obtain the value of $L_{\mathbf{k},\mathcal{A}}(H)$.
(5) If H is a series node, then use Proposition 7 to obtain the value of $L_{\mathbf{k},\mathcal{A}}(H)$.
Moreover, both Proposition 6 and Proposition 7 reduce the computation of $L_{\mathbf{k},\mathcal{A}}(H)$ to the computation on disjoint subgraphs of H, as already stated. Thus, there are at most $|V(H)|$ occurrences of such reduction steps.
(6) If H is a spider (S, C, R) with $R \ne \emptyset$, then apply Corollary 9, Proposition 10, Proposition 11 or Proposition 16 where, in order to evaluate the packing parameter for $H[R \cup C]$, use Proposition 7 with $G_1 = H[R]$ and $G_2 = H[C]$. Recursively, compute the packing parameter for the P_4-tidy graph $H[R]$ and, for $H[C]$, recall Remark 5.
(7) If H is a quasi-spider that is not a spider, it was obtained from a spider graph with partition (S, C, R) where at most one vertex of $S \cup C$ was replaced by a graph on two vertices, let us call these vertices u and w with $k_u \ge k_w$. Observe that u and w are twins in H and $H - u$ $(H - w)$ is a P_4-tidy spider. Apply Proposition 17, taking into account that $\mathbf{k}^*$, $\mathbf{k}'$ and $\mathbf{k}''$ are shifts of the vector $\mathbf{k}$ when restricted to the set R and that the set $\mathcal{A}_R$ is unchanged. Calculate the packing parameters for $H - u$ $(H - w)$ according to cases (2), (3) or (6) above. In this way, $L_{\mathbf{k},\mathcal{A}}(H)$ can be obtained linearly.

4. Connection with dominating sets

To end this paper, we connect our findings on limited packings with certain dominating sets in graphs.

Given a graph G, $\mathcal{R} \subseteq V(G)$ and $\mathbf{r} = (r_v) \in \mathbb{Z}_+^{V(G)}$, $D \subseteq V(G)$ is an $(\mathbf{r}, \mathcal{R})$*-tuple dominating set* of G if $\mathcal{R} \subseteq D$ and $|N_G[v] \cap D| \ge r_v$, for every $v \in V(G)$ [9]. Notice that G has $(\mathbf{r}, \mathcal{R})$-tuple sets if and only if $r_v \le |N_G[v]|$ for every v.

The Generalized Multiple Domination problem (GMD) is formulated as

INSTANCE: A graph G, a vector $\mathbf{r} \in \mathbb{Z}_+^{V(G)}$, $\mathcal{R} \subseteq V(G)$ and $\alpha \in \mathbb{Z}_+$.
QUESTION: Does G contain a $(\mathbf{r}, \mathcal{R})$-tuple dominating set of size at most α?

GMD was proved to be NP-complete, by proving the NP-completeness for the instances corresponding to $r_v = r$ for every $v \in V(G)$ and each $r \in \mathbb{N}$ and $\mathcal{R} = \emptyset$ [9].

The following result shows that an optimal solution of GLP can be obtained linearly from an optimal solution of GMD and vice versa.

Proposition 19. *(See [2].) Let $(G, \mathbf{k}, \mathcal{A})$ be an instance of GLP and $B \subseteq V(G)$. Then, B is a $(\mathbf{k}, \mathcal{A})$-limited packing of G if and only if $V(G) - B$ is an $(\mathbf{r}, V(G) - \mathcal{A})$-tuple dominating set of G, where $r_v := \max\{0, |N_G[v]| - k_v\}$, for $v \in V(G)$.*

Due to the relationship between GLP and GMD shown above, we are able to obtain new polynomial time solvable instances of GMD, as a corollary of Theorem 18:

Corollary 20. *GMD can be solved in polynomial time for P_4-tidy graphs.*

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