

# OPTIMAL SHIFT INVARIANT SPACES AND THEIR PARSEVAL FRAME GENERATORS

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ABSTRACT. Given a set of functions  $\mathcal{F} = \{f_1, \dots, f_m\} \subset L^2(\mathbb{R}^d)$ , we study the problem of finding the shift-invariant space  $V$  with  $n$  generators  $\{\varphi_1, \dots, \varphi_n\}$  that is “closest” to the functions of  $\mathcal{F}$  in the sense that

$$V = \operatorname{argmin}_{V' \in \mathcal{V}_n} \sum_{i=1}^m w_i \|f_i - P_{V'} f_i\|^2,$$

where  $w_i$ s are positive weights, and  $\mathcal{V}_n$  is the set of all shift-invariant spaces that can be generated by  $n$  or less generators. The Eckart-Young Theorem uses the singular value decomposition to provide a solution to a related problem in finite dimension. We transform the problem under study into an uncountable set of finite dimensional problems each of which can be solved using an extension of the Eckart-Young Theorem. We prove that the finite dimensional solutions can be patched together and transformed to obtain the optimal shift-invariant space solution to the original problem, and we produce a Parseval frame for the optimal space. A typical application is the problem of finding a shift-invariant space model that describes a given class of signals or images (e.g., the class of chest X-Rays), from the observation of a set of  $m$  signals or images  $f_1, \dots, f_m$ , which may be theoretical samples, or experimental data.

## 1. INTRODUCTION

In many signal and image processing applications, images and signals are assumed to belong to some shift-invariant space of the form:

$$\mathcal{S}(\Phi) := \operatorname{closure}_{L_2} \operatorname{span}\{\varphi_i(x - k) : i = 1, \dots, n, k \in \mathbb{Z}^d\} \quad (1.1)$$

where  $\Phi = \{\varphi_1, \dots, \varphi_n\}$  is a set of functions in  $L^2(\mathbb{R}^d)$ . The functions  $\varphi_1, \varphi_2, \dots, \varphi_n$  are called a *set of generators* for the space  $\mathcal{S} = \mathcal{S}(\Phi) = \mathcal{S}(\varphi_1, \dots, \varphi_n)$  and any such space  $\mathcal{S}$  is called a *finitely generated shift-invariant space (FSIS)* (see e.g., [BDR94a, AG01]). For example, if  $n = 1$ ,  $d = 1$  and  $\phi(x) = \operatorname{sinc}(x)$ , then the underlying space is the space of band-limited functions (often used in communications).

Finitely generated shift-invariant spaces, can have different sets of generators. The *length* of an FSIS  $\mathcal{S}$  is,

$$\mathcal{L}(\mathcal{S}) = \min\{\ell \in \mathbb{N} : \exists \varphi_1, \dots, \varphi_\ell \in \mathcal{S} \text{ with } \mathcal{S} = \mathcal{S}(\varphi_1, \dots, \varphi_\ell)\}$$

We will denote by  $\mathcal{V}_n$  the set of all shift-invariant invariant spaces with length less than or equal to  $n$ . That is, an element in  $\mathcal{V}_n$  is a shift-invariant space that has a set of  $s$  generators with  $s \leq n$ .

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In most applications, the shift-invariant space chosen to describe the underlying class of signals is not derived from experimental data - for example many signal processing applications assume “band-limitedness” of the signal, which has theoretical advantages, but generally does not necessarily reflect the underlying class of signals accurately. Furthermore, in applications, the a priori hypothesis that the class of signals belongs to a shift-invariant space with a known number of generators, may not be satisfied. For example, the class of functions from which the data is drawn may not be a shift-invariant space. Another example is when the shift-invariant space hypothesis is correct but the assumptions about the number of generators is wrong. A third example is when the a priori hypothesis is correct but the data is corrupted by noise. In addition, for computational considerations, a shift-invariant space of length  $m$  could be modeled by a shift-invariant model space with length  $n$  much smaller than  $m$ . For example, in Learning Theory, the problem of reducing the number of generators for a subspace of a reproducing kernel Hilbert space is also important for improving the efficiency and sparsity of learning algorithms (see [SZ04]). In order to model classes of signals or images by FSIS in realistic cases, or to model a very large data set by a computationally manageable shift-invariant space, we consider the following problem:

**Problem 1.** Given a large set of experimental data  $\mathcal{F} = \{f_1, f_2, \dots, f_m\} \subset L^2(\mathbb{R}^d)$ , we wish to determine a shift-invariant space  $V \in \mathcal{V}_n$  (where typically  $n$  is chosen to be small compared to  $m$ ) that models the signals in “some” best way. For this purpose, we consider the following least squares problem:

$$V = \operatorname{argmin}_{V' \in \mathcal{V}_n} \sum_{i=1}^m w_i \|f_i - P_{V'} f_i\|^2 \quad (1.2)$$

where  $w_i$  are positive weights and where  $P_{V'}$  is the orthogonal projection on  $V'$ .

A space  $V$  satisfying (1.2) will be said to *solve Problem 1* for  $(\mathcal{F}, w, n)$ .

The weights  $w_i$  can be chosen to normalize or to reflect our confidence about the data. For example we can choose  $w_i = \|f_i\|^{-2}$  to place the data on a sphere or we can choose a small weight  $w_i$  for a given  $f_i$  if —due to noise or other factors— our confidence about the accuracy of  $f_i$  is low. The goal is to see if we can perform operations on the observed data  $\mathcal{F} = \{f_1, f_2, \dots, f_m\}$  to construct (if it exists) a shift-invariant space  $\mathcal{S}(\Phi)$  whose length doesn’t exceed a small number  $n$ , that minimizes the error with our data  $\mathcal{F}$ .

Problem 1 can be viewed as non-linear infinite dimensional constrained minimization problem. It may also be viewed in light of the recent learning theory developed in [BCDV05, CS02, SZ03], and estimates of model fit in terms of noise and approximation space may be derived. Beside the fundamental question of existence of an optimal space, it will be important for applications to have a way to construct the generators of the optimal space if it exists, and to estimate the error  $\mathcal{E}(\mathcal{F}, w, n) = \sum_{i=1}^m w_i \|f_i - P_V f_i\|^2$  where  $V \in \mathcal{V}_n$  is an optimal space for  $\mathcal{F}, w$  and  $n$ .

Typical applications involve large data sets (for example consider the problem of finding a shift-invariant space model for the collection of chest X-rays using data collected by a hospital during the last 10 years). The space  $\mathcal{S}(\mathcal{F})$  generated by a set of experimental data contains all the data as possible signals, but it is too large to be an appropriate model for use in applications. A space with a “small” number

of generators is more suitable, since if the space is chosen correctly, it would reduce noise, and would give a computationally manageable model for a given application.

Least squares problems of the form above in *finite dimensional* spaces can be solved using the singular value decomposition (SVD). Shift-invariant spaces are infinite dimensional and the SVD cannot be applied directly. However, due to the special structure of shift-invariant spaces, the Fourier transform converts Problem 1 into finite dimensional least square problems at each frequency as will be discussed in Section 4.1.

## 2. MAIN THEOREMS

In this paper we will sometimes deal with the standard Hilbert space  $\mathbb{C}^N$ . Elements of this vector space are column vectors with  $N$  coordinates. We will use the notation  $A^t$  and  $A^*$  to denote the transpose and the conjugate transpose respectively of a complex matrix  $A$ . We will say that a vector  $y \in \mathbb{C}^N$  is a left eigenvector of the matrix  $A$  associated to the eigenvalue  $\lambda$ , if  $y^t A = \lambda y^t$ .

For clarity in the exposition, we will consider the unweighted case ( $w_i = 1, i = 1, \dots, m$ ). The general case can be derived by simply applying the results of the unweighted case to the set of normalized observations  $\mathcal{F} = \{f_1/w_1^2, \dots, f_m/w_m^2\}$ .

The first theorem establishes the existence of an optimal space  $V$ . It also establishes that  $V$  can always be chosen to be a subspace of the shift-invariant space  $\mathcal{S}(\mathcal{F})$  generated by the totality of the data. This optimal space  $V$  may not be unique. However, under additional assumptions that are often satisfied in practice, there is only one optimal space  $V$ , as stated in Theorem 2.4.

**Theorem 2.1.** *Let  $\mathcal{F} = \{f_1, \dots, f_m\}$  be a set of functions in  $L^2(\mathbb{R}^d)$ . Then*

- (1) *There exists  $V \in \mathcal{V}_n$  such that*

$$\sum_{i=1}^m \|f_i - P_V f_i\|^2 \leq \sum_{i=1}^m \|f_i - P_{V'} f_i\|^2, \quad \forall V' \in \mathcal{V}_n \quad (2.1)$$

- (2) *The optimal shift-invariant space  $V$  in (2.1) can be chosen such that  $V \subset \mathcal{S}(\mathcal{F})$ .*

*Remarks.*

- (i) Although we do not make the assumption that  $n \leq m$ , if  $n > m$ , then  $\mathcal{S}(\mathcal{F})$  is an optimal space that belongs to  $\mathcal{V}_n$ . Thus, we will always assume that  $n \leq m$  for the remainder of this paper.
- (ii) In practice it will often be the case that  $n$  is chosen (or found) to be much smaller than  $m$ .

We still need to explicitly find an optimal space  $V$  and estimate the error

$$\mathcal{E}(\mathcal{F}, n) = \min_{V' \in \mathcal{V}_n} \sum_{i=1}^m \|f_i - P_{V'} f_i\|^2. \quad (2.2)$$

To compute the error  $\mathcal{E}(\mathcal{F}, n)$  we need to consider the Gramian matrix  $G_{\mathcal{F}}$  of  $\mathcal{F} = \{f_1, \dots, f_m\}$ . Specifically, the *Gramian*  $G_{\Phi}$  of a set of functions  $\Phi = \{\varphi_1, \dots, \varphi_n\}$  with elements in  $L^2(\mathbb{R}^d)$  is defined to be the  $n \times n$  matrix of  $\mathbb{Z}^d$ -periodic functions

$$[G_{\Phi}(\omega)]_{i,j} = \sum_{k \in \mathbb{Z}^d} \widehat{\varphi}_i(\omega + k) \overline{\widehat{\varphi}_j(\omega + k)}, \quad \omega \in \mathbb{R}^d, \quad (2.3)$$

where  $\widehat{\varphi}_i$  denotes the Fourier transform of  $\varphi_i$ , and where  $\overline{\widehat{\varphi}_i}$  denotes the complex conjugate of  $\widehat{\varphi}_i$ . It is known that  $G_\Phi$  is  $\mathbb{Z}^d$ -periodic non-negative and self-adjoint for almost every  $\omega$ . In this paper, we use the following definition for the Fourier transform of a function  $\phi \in L^2(\mathbb{R}^d)$ :

$$\widehat{\phi}(\omega) := \int_{\mathbb{R}^d} \phi(x) e^{-i2\pi x^t \omega} dx, \quad \omega \in \mathbb{R}^d, \quad (2.4)$$

where  $dx$  denotes Lebesgue measure on  $\mathbb{R}^d$ .

If  $V = \{v_1, \dots, v_n\}$  is a set of vectors in a Hilbert space  $\mathcal{H}$ , we will denote by  $\mathfrak{G}(V) = \mathfrak{G}(v_1, \dots, v_n)$  the matrix

$$[\mathfrak{G}(V)]_{i,j} = \langle v_i, v_j \rangle_{\mathcal{H}} \quad i, j = 1, \dots, n. \quad (2.5)$$

Our next theorem produces a generator for an optimal space  $V$  and provides a formula for the exact value of the error, but we first recall the definition and some properties of frames used in its statement (see for example [CC97, HL00, HLW02]).

**Definition 2.2.** Let  $\mathcal{H}$  be a Hilbert space and  $\{u_i\}_{i \in I}$  a countable subset of  $\mathcal{H}$ . The set  $\{u_i\}_{i \in I}$  is said to form a *frame* for  $\mathcal{H}$  if there exist  $q, Q > 0$  such that

$$q\|f\|^2 \leq \sum_{i \in I} |\langle f, u_i \rangle|^2 \leq Q\|f\|^2, \quad \forall f \in \mathcal{H}.$$

If  $q = Q$ , then  $\{u_i\}_{i \in I}$  is called a *tight frame*, and it is called a *Parseval frame* if  $q = Q = 1$ .

If  $\{u_i\}_{i \in I}$  is a Parseval frame for a subspace  $W$  of a Hilbert space  $\mathcal{H}$ , and if  $a \in \mathcal{H}$ , then the orthogonal projection of  $a$  onto  $W$  is given by:

$$P_W(a) = \sum_{i \in I} \langle a, u_i \rangle u_i. \quad (2.6)$$

Thus, a Parseval frames acts as if it were an orthonormal basis of  $W$ , even though it may not be one.

**Theorem 2.3.** Under the same assumptions as in Theorem 2.1, let  $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \dots \geq \lambda_m(\omega)$  be the eigenvalues of the Gramian  $G_{\mathcal{F}}(\omega)$ . Then

- (1) The eigenvalues  $\lambda_i(\omega)$ ,  $1 \leq i \leq m$  are  $\mathbb{Z}^d$ -periodic, measurable functions in  $L^2([0, 1]^d)$  and

$$\mathcal{E}(\mathcal{F}, n) = \sum_{i=n+1}^m \int_{[0, 1]^d} \lambda_i(\omega) d\omega. \quad (2.7)$$

- (2) Let  $E_i := \{\omega : \lambda_i(\omega) \neq 0\}$ , and define  $\tilde{\sigma}_i(\omega) = \lambda_i^{-1/2}(\omega)$  on  $E_i$  and  $\tilde{\sigma}_i(\omega) = 0$  on  $E_i^c$ . Then, there exists a choice of measurable left eigenvectors  $y_1(\omega), \dots, y_n(\omega)$  with  $y_i = (y_{i1}, \dots, y_{im})^t$ ,  $i = 1, \dots, n$ , associated with the first  $n$  largest eigenvalues of  $G_{\mathcal{F}}(\omega)$  such that the functions defined by

$$\widehat{\varphi}_i(\omega) = \tilde{\sigma}_i(\omega) \sum_{j=1}^m y_{ij}(\omega) \widehat{f}_j(\omega), \quad i = 1, \dots, n, \quad \omega \in \mathbb{R}^d$$

are in  $L^2(\mathbb{R}^d)$ . Furthermore, the corresponding set of functions  $\Phi = \{\varphi_1, \dots, \varphi_n\}$  is a generator for an optimal space  $V$  and the set  $\{\varphi_i(\cdot - k), k \in \mathbb{Z}^d, i = 1, \dots, n\}$  is a Parseval frame for  $V$ .

The following example shows that the optimal space  $V$  does not need to be unique. Let  $m = 2$ ,  $n = 1$ , and let  $f_1, f_2$  be two orthonormal functions. For this situation,  $G_{\mathcal{F}}(\omega)$  is the  $2 \times 2$  identity matrix for almost all  $\omega \in \mathbb{R}^d$ . It follows that any function  $\varphi = c_1 f_1 + c_2 f_2$  with  $c = (c_1, c_2)$  a unit vector in  $\mathbb{R}^2$  generates an optimal space and  $\mathcal{E}(\mathcal{F}, 1) = 1$ . Obviously, in this particular case there are infinitely many optimal spaces. However, under some mild assumptions, there exists a unique optimal space  $V$  as described in the following theorem:

**Theorem 2.4.** *Let  $\mathcal{F} = \{f_1, \dots, f_m\}$  functions in  $L^2(\mathbb{R}^d)$  be given. If  $\lambda_n(\omega) > \lambda_{n+1}(\omega)$  for almost all  $\omega$ , then the optimal space  $V$  in (2.1) is unique. In this case,  $n \leq r_{\min} = \min_{\omega \in [0,1]^d} \text{rank } G_{\mathcal{F}}(\omega)$  and the set  $\{\varphi(\cdot - k), k \in \mathbb{Z}^d, i = 1, \dots, n\}$  in part (2) of Theorem 2.3 is an orthonormal basis for  $V$ .*

*Remarks.*

- (i) In case that  $n = \mathcal{L}(\mathcal{S}(f_1, \dots, f_m))$ , Theorem 2.3 gives a proof of the known result that every FSIS has a set of generators forming a Parseval frame.
- (ii) It will be clear from the proofs of Theorems 2.1 and 2.3 that the optimal space  $V$  can be decomposed as  $V = S(\varphi_1) \oplus \dots \oplus S(\varphi_\ell)$  where  $\ell = \mathcal{L}(V)$ , the direct sum is orthogonal and each  $\varphi_i$  is a Parseval frame generator of  $S(\varphi_i)$ .
- (iii) Theorem 2.4 can only be used when  $n < m$ . When  $n = m$  then  $\mathcal{S}(\mathcal{F})$  is an optimal space and it is the unique optimal space if and only if  $\mathcal{L}(\mathcal{S}(\mathcal{F})) = m$ .
- (iv) Obviously, if  $n = m$  then the error between the model and the observation is null. However, by plotting the error in (2.7) in terms of the number of generators, an optimal number  $n$  may be heuristically derived. Alternatively, one may choose  $n$  so that a cost functional (depending on the error and on  $n$ ) is optimized as in other dimension reduction schemes.

### 3. PRELIMINARIES ON FINITELY GENERATED SHIFT-INVARIANT SPACES

In this section we state some known results about finitely generated shift-invariant spaces that we will need later. See for example ([Hel64, BDR94b, BDR94a, RS95, Bow00].)

We need first to introduce some definitions.

Given  $f \in L^2(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$  the *fiber* of  $f$  at  $x$  is the sequence  $\Gamma_x f = \{f(x+k) : k \in \mathbb{Z}^d\}$ .

If  $V$  is a FSIS (recall Definition (1.1)) and  $\omega \in [0, 1]^d$  we set  $V_\omega = \{\Gamma_\omega \hat{f}; f \in V\}$  the *fiber space* associated to  $V$  and  $\omega$ .

If  $\mathcal{M}$  is a closed subspace of a Hilbert space  $\mathcal{H}$ , throughout this article we will denote by  $P_{\mathcal{M}}$  the orthogonal projection operator in  $\mathcal{H}$  onto  $\mathcal{M}$ .

With this notation we have:

**Lemma 3.1.** *If  $f \in L^2(\mathbb{R}^d)$ , then*

- (1) *The sequence  $(\Gamma_\omega \hat{f})_k = (\hat{f}(\omega + k))$  is a well-defined sequence in  $\ell_2(\mathbb{Z}^d)$  a.e.  $\omega \in \mathbb{R}^d$ ; and*
- (2)  *$\|\Gamma_\omega \hat{f}\|_{\ell_2}$  is a measurable function of  $\omega$  and  $\|f\|^2 = \|\hat{f}\|^2 = \int_{[0,1]^d} \|\Gamma_\omega \hat{f}\|_{\ell_2}^2 d\omega$ .*

**Lemma 3.2.** *Let  $V$  be a FSIS in  $L^2(\mathbb{R}^d)$ . Then we have:*

- i.  *$V_\omega$  is a closed subspace of  $\ell_2(\mathbb{Z}^d)$  for almost all  $\omega \in [0, 1]^d$ .*

- ii.  $V = \{f \in L^2(\mathbb{R}^d) : \Gamma_\omega \hat{f} \in V_\omega \text{ for almost all } \omega \in [0, 1]^d\}$ .
- iii. For each  $f \in L^2(\mathbb{R}^d)$  we have that  $\|\Gamma_\omega(\widehat{P_V f})\|_{\ell_2}$  is a measurable function of the variable  $\omega$  and  $\Gamma_\omega(\widehat{P_V f}) = \Gamma_\omega P_{\hat{V}} \hat{f} = P_{V_\omega}(\Gamma_\omega \hat{f})$ .
- iv. Let  $\varphi_1, \dots, \varphi_r \in L^2(\mathbb{R}^d)$ . We have that
  - a.  $\{\varphi_1, \dots, \varphi_r\}$  is a set of generators of  $V$ , if and only if the fibers  $\Gamma_\omega \hat{\varphi}_1, \dots, \Gamma_\omega \hat{\varphi}_r$  span  $V_\omega$  for almost all  $\omega \in [0, 1]^d$
  - b. the integer translates of  $\varphi_1, \dots, \varphi_r$  are a frame of  $V$ , if and only if  $\Gamma_\omega \hat{\varphi}_1, \dots, \Gamma_\omega \hat{\varphi}_r$  are a frame of  $V_\omega$  with the same frame bounds, for almost all  $\omega \in [0, 1]^d$ .

**Lemma 3.3.** Let  $\mathcal{F} = \{f_1, \dots, f_m\}$  be functions in  $L^2(\mathbb{R}^d)$  and let  $A(\omega)$  be the infinite matrix  $A_{kj}(\omega) = \left(\Gamma_\omega \hat{f}_j\right)(k) = \hat{f}_j(\omega + k)$ ,  $j = 1, \dots, m$ ,  $k \in \mathbb{Z}^d$ , and  $\omega \in \mathbb{R}^d$ . Then  $G_{\mathcal{F}}(\omega) = A^t(\omega) \overline{A(\omega)}$ , and  $\text{rank } G_{\mathcal{F}}(\omega) = \text{rank } A(\omega) = \text{rank } A^*(\omega)$ , a.e.  $\omega \in \mathbb{R}^d$ . In particular,  $G_{\mathcal{F}}(\omega) = \mathfrak{G}(\Gamma_\omega \hat{f}_1, \dots, \Gamma_\omega \hat{f}_m)$ .

#### 4. PROOFS

To prove the theorems in Section 2, we proceed in several steps. First we reduce the optimization problem into an uncountable set of finite dimensional problems in the Hilbert space  $\mathcal{H} = \ell_2(\mathbb{Z}^d)$ . We then apply the Eckart-Young Theorem to prove that the reduced problems have solutions. Finally, we construct the generators of the optimal space patching together the solutions of the reduced problems to obtain the solution to the original problem.

**4.1. Reduction.** In this section, we reduce Problem 1 to a set of finite dimensional problems. To see this let us first consider the following :

**Problem 2.** Let  $H$  be a Hilbert space,  $n, m$  positive integers and  $A = \{a_1, \dots, a_m\}$  a set of vectors in  $\mathcal{H}$ . We want to find a closed subspace  $S$  of  $\mathcal{H}$  with  $\dim(S) \leq n$  that satisfies

$$\sum_{i=1}^m \|a_i - P_S a_i\|^2 \leq \sum_{i=1}^m \|a_i - P_{S'} a_i\|^2 \quad (4.1)$$

for every subspace  $S' \subset \mathcal{H}$  with  $\dim(S') \leq n$ .

If such an  $S$  exists, we say that  $S$  solves Problem 2 for the data  $(A, n)$ .

If  $B = \{b_1, \dots, b_r\}$  is a set of vectors from  $\mathcal{H}$  with  $S = \text{span}(B)$  we will say that the vectors in  $B$  solve Problem 2 for the data  $(A, n)$ . The error for Problem 2 is

$$\mathfrak{E}(A, n) = \min_{\dim(S') \leq n} \sum_{i=1}^m \|a_i - P_{S'} a_i\|^2.$$

Note that in Problem 2 we take the minimum over all subspaces of dimension less than  $n$ , while in Problem 1 the minimization is taken over a particular class of infinite dimensional subspaces, so the two problems are essentially different.

In the next section we state and prove an extension of the Eckart-Young theorem. We conclude from this extension that Problem 2 always has a solution for any set of data  $(A, n)$  in an arbitrary Hilbert space. That is, given  $A$  and  $n$  there always exists a subspace  $S$  with  $\dim(S) \leq n$  satisfying (4.1). We will also see that a solution  $S$  can be chosen in such a way that  $S \subset \text{span}(A)$  when  $n \leq \dim(\text{span}(A))$ .

Before proving these results let us see how Problem 2 helps our original question.

Let  $\mathcal{F} = \{f_1, \dots, f_m\} \subset L^2(\mathbb{R}^d)$ . We want to find out if there exists  $V \in \mathcal{V}_n$  such that  $V$  minimizes  $\sum_{i=1}^m \|f_i - P_V f_i\|^2$ . Using Lemma 3.1 we obtain that for any  $V \in \mathcal{V}_n$ ,

$$\begin{aligned} \sum_{i=1}^m \|f_i - P_V f_i\|^2 &= \sum_{i=1}^m \int_{[0,1]^d} \|\Gamma_\omega \hat{f}_i - \Gamma_\omega \widehat{P_V f_i}\|_{\ell_2}^2 d\omega \\ &= \int_{[0,1]^d} \sum_{i=1}^m \|\Gamma_\omega \hat{f}_i - \Gamma_\omega \widehat{P_V f_i}\|_{\ell_2}^2 d\omega. \end{aligned} \quad (4.2)$$

By Lemma 3.2(iii),  $\Gamma_\omega \widehat{P_V f_i} = P_{V_\omega} \Gamma_\omega \hat{f}_i$ . So from (4.2) we conclude that,

$$\sum_{i=1}^m \|f_i - P_V f_i\|^2 = \int_{[0,1]^d} \sum_{i=1}^m \|\Gamma_\omega \hat{f}_i - P_{V_\omega} \Gamma_\omega \hat{f}_i\|_{\ell_2}^2 d\omega. \quad (4.3)$$

The sum inside the integral on the right hand-side of (4.3) is of the same type than the sum that is involved in Problem 2 in the case that  $\mathcal{H} = \ell_2(\mathbb{Z}^d)$  and  $S = V_\omega$ . Since we are assuming that Problem 2 always has a solution, we know that for almost each  $\omega \in [0,1]^d$  there exists a subspace  $S_\omega \subset \ell_2(\mathbb{Z}^d)$  that solves Problem 2 for the data  $(\mathcal{F}_\omega, n)$  where  $\mathcal{F}_\omega = \{\Gamma_\omega \hat{f}_1, \dots, \Gamma_\omega \hat{f}_m\}$ . Note that the subspace  $S_\omega$  does not need to be related with the fiber space of any FSIS. If the function  $\omega \mapsto \sum_{i=1}^m \|\Gamma_\omega \hat{f}_i - P_{S_\omega} \Gamma_\omega \hat{f}_i\|_{\ell_2}^2$  were a measurable function of  $\omega$  then we would have

$$\int_{[0,1]^d} \sum_{i=1}^m \|\Gamma_\omega \hat{f}_i - P_{S_\omega} \Gamma_\omega \hat{f}_i\|_{\ell_2}^2 d\omega \leq \sum_{i=1}^m \|f_i - P_{V'} f_i\|^2 \quad (4.4)$$

for every  $V' \in \mathcal{V}_n$ .

Therefore, in case that there exists a FSIS  $V \in \mathcal{V}_n$  such that  $V_\omega = S_\omega$  a.e.  $\omega \in [0,1]^d$ , then by Lemmas 3.1 and 3.2 the above function would be measurable and  $V$  necessarily will be a solution to Problem 1, since

$$\begin{aligned} \sum_{i=1}^m \|f_i - P_V f_i\|^2 &= \int_{[0,1]^d} \sum_{i=1}^m \|\Gamma_\omega \hat{f}_i - P_{S_\omega} \Gamma_\omega \hat{f}_i\|_{\ell_2}^2 d\omega \leq \\ &= \int_{[0,1]^d} \sum_{i=1}^m \|\Gamma_\omega \hat{f}_i - P_{V_\omega} \Gamma_\omega \hat{f}_i\|_{\ell_2}^2 d\omega = \sum_{i=1}^m \|f_i - P_{V'} f_i\|^2 \end{aligned} \quad (4.5)$$

for every  $V' \in \mathcal{V}_n$ .

We will see later that such a FSIS indeed exists. More precisely we will construct a set of generators such that its integer translates form a frame of the optimal FSIS. We will do that by patching together the fibers of the generators of each of the optimal subspaces  $S_\omega$ .

**4.2. Solution to Problem 2.** We now prove that Problem 2 always has a solution.

**Theorem 4.1.** *Let  $\mathcal{H}$  be an infinite dimensional Hilbert space,  $\mathcal{F} = \{f_1, \dots, f_m\} \subset \mathcal{H}$ ,  $\mathcal{X} = \text{span}\{f_1, \dots, f_m\}$ ,  $\lambda_1 \geq \dots \geq \lambda_m$  the eigenvalues of the matrix  $\mathfrak{G}(\mathcal{F})$  defined as in (2.5) and  $y_1, \dots, y_m \in \mathbb{C}^m$ , with  $y_i = (y_{i1}, \dots, y_{im})^t$  orthonormal left eigenvectors associated to the eigenvalues  $\lambda_1, \dots, \lambda_m$ . Let  $r = \dim \mathcal{X} = \text{rank } \mathfrak{G}(\mathcal{F})$ .*

*Define the vectors  $q_1, \dots, q_n \in \mathcal{H}$  by*

$$q_i = \tilde{\sigma}_i \sum_{j=1}^m y_{ij} f_j, \quad i = 1, \dots, n \quad (4.6)$$

where  $\tilde{\sigma}_i = \lambda_i^{-1/2}$  if  $\lambda_i \neq 0$ , and  $\tilde{\sigma}_i = 0$  otherwise. Then  $\{q_1, \dots, q_n\}$  is a Parseval frame of  $W = \text{span}\{q_1, \dots, q_n\}$  and the subspace  $W$  is optimal in the sense that

$$\mathfrak{E}(\mathcal{F}, n) = \sum_{i=1}^m \|f_i - P_W f_i\|^2 \leq \sum_{i=1}^m \|f_i - P_{W'} f_i\|^2, \quad \forall \text{ subspace } W', \dim W' \leq n.$$

Furthermore we have the following formula for the error

$$\mathfrak{E}(\mathcal{F}, n) = \sum_{i=n+1}^m \lambda_i. \quad (4.7)$$

*Remark.* If  $r$  is small (i.e.  $r \leq n$ ) then all the vectors  $q_{r+1}, \dots, q_n$  are null and  $\{q_1, \dots, q_r\}$  is an orthonormal set.

One could also choose  $q_{r+1}, \dots, q_n$  to be any orthonormal set in the orthogonal complement of  $\mathcal{X}$  and so obtain an orthonormal set of  $n$  elements and the formula for the error would still hold.

If  $\mathcal{H}$  is finite dimensional and  $n \leq r$ , then Theorem 4.1 is a consequence of the Eckart-Young theorem (see appendix). To prove Theorem 4.1 we will reduce it to the finite dimensional case and then use the Eckart-Young result.

We first need the following Lemma:

**Lemma 4.2.** *Let  $\mathcal{H}$  be a Hilbert space,  $\{f_1, \dots, f_m\} \subset \mathcal{H}$ ,  $\mathcal{X} = \text{span}\{f_1, \dots, f_m\}$ . Assume that there exists  $M \subset \mathcal{H}$  with  $\dim M \leq n$  such that*

$$\sum_{i=1}^m \|f_i - P_M f_i\|^2 \leq \sum_{i=1}^m \|f_i - P_{M'} f_i\|^2$$

*for any subspace  $M' \subset \mathcal{H}$  with  $\dim M' \leq n$ , then there exists  $W \subset \mathcal{X}$ , with  $\dim W \leq n$ , such that*

$$\sum_{i=1}^m \|f_i - P_W f_i\|^2 = \sum_{i=1}^m \|f_i - P_M f_i\|^2.$$

*Proof.* Define the subspace  $W = P_{\mathcal{X}} M$  as the orthogonal projection of  $M$  onto  $\mathcal{X}$ . By construction,  $W \subset \mathcal{X}$ , and  $\dim W \leq n$ .

Let  $f \in \mathcal{X}$ , then we have

$$\begin{aligned} \|f - P_W f\|^2 &= \inf\{\|f - g\|^2 : g \in W\} \\ &\leq \|f - P_{\mathcal{X}} P_M f\|^2 \\ &= \|P_{\mathcal{X}} f - P_{\mathcal{X}} P_M f\|^2 \\ &\leq \|f - P_M f\|^2. \end{aligned}$$

□

This Lemma shows that, in a possibly infinite dimensional Hilbert space  $\mathcal{H}$ , the problem of finding a finite dimensional subspace  $M \subset \mathcal{H}$  with  $\dim M \leq n$  that “best approximates”  $m$  vectors  $\{f_1, \dots, f_m\}$ , can always be reduced to a search in the finite dimensional space  $\mathcal{X} = \text{span}\{f_1, \dots, f_m\}$ .

We now prove Theorem 4.1.



*Proof.* (of Theorem 4.1). Let  $\tau : \mathcal{X} \longrightarrow \mathbb{C}^m$  be an isometric isomorphism. Set  $b_i = \tau(f_i)$ , and let  $B$  be the matrix having the vectors  $b_i$  as columns. So,  $r = \dim \mathcal{X} = \text{rank}(B)$  and  $B^t \overline{B}$  coincides with  $\mathfrak{G}(\mathcal{F}) = \{ \langle f_i, f_j \rangle_{\mathcal{H}} \}_{i,j}$ .

Choose orthonormal left eigenvectors  $y_1, \dots, y_m \in \mathbb{C}^m$ , with  $y_i = (y_{i1}, \dots, y_{im})^t$  associated to the eigenvalues  $\lambda_1 \geq \dots \geq \lambda_m$  of  $B^t \overline{B}$ , and define the vectors

$$u_i = \tilde{\sigma}_i \sum_{j=1}^m y_{ij} b_j, \quad i = 1, \dots, n \quad (4.8)$$

where as before  $\tilde{\sigma}_i = \lambda_i^{-1/2}$  if  $\lambda_i \neq 0$ , and  $\tilde{\sigma}_i = 0$  otherwise.

Then, if  $n \leq r$  by Theorem 4.6 in appendix, the subspace  $M \subset \mathbb{C}^m$ ,  $M = \text{span} \{u_1, \dots, u_n\}$  satisfies:

$$\sum_{i=1}^m \|b_i - P_M b_i\|^2 \leq \sum_{i=1}^m \|b_i - P_{M'} b_i\|^2 \quad (4.9)$$

for every subspace  $M' \subset \mathbb{C}^m$  with  $\dim M' \leq n$ .

If however,  $n \geq r$  then the left side of (4.9) is 0 and therefore the inequality is also satisfied.

Setting  $W = \tau^{-1}(M)$  and noting that  $\tau^{-1} P_M = P_W$  we have from (4.9)

$$\mathfrak{E}(B, n) = \mathfrak{E}(\mathcal{F}, n) = \sum_{i=1}^m \|f_i - P_W f_i\|^2 \leq \sum_{i=1}^m \|f_i - P_{W'} f_i\|^2 \quad (4.10)$$

for every subspace  $W' \subset \mathcal{H}$  with  $\dim W' \leq n$ . So,  $W \subset \mathcal{H}$  is optimal for  $(\mathcal{F}, n)$  and  $q_i = \tau^{-1}(u_i)$ ,  $i = 1, \dots, n$  is a Parseval frame for  $W$ . Furthermore, the formula (4.7) also holds.  $\square$

*Remarks.*

- (i) If  $n > m$  the optimal space  $W$  is not unique since any space  $W'$  of dimension  $n$  containing  $\text{span}\{a_1, \dots, a_m\}$  will be optimal. The same argument also shows that the space  $W$  is not unique if  $n > r = \dim \mathcal{X}$ .
- (ii) If  $n \leq r$ , the vectors  $u_i$  and  $y_i$  are related by  $\sqrt{\lambda_i} u_i = A y_i$  as described in the Appendix.

**4.3. Solution to Problem 1.** In order to solve Problem 1, we need the following technical proposition concerning the measurability of the eigenvalues and the existence of measurable eigenvectors of a non-negative matrix with measurable entries (cf. [RS95, Lemma 2.3.5].)

**Lemma 4.3.** *Let  $G = G(\omega)$  be an  $m \times m$  self-adjoint matrix of measurable functions defined on a measurable subset  $E \subset \mathbb{R}^d$  with eigenvalues  $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \dots \geq \lambda_m(\omega)$ . Then the eigenvalues  $\lambda_i$ ,  $i = 1, \dots, m$ , are measurable on  $E$  and there exists an  $m \times m$  matrix of measurable functions  $U = U(\omega)$  on  $E$  such that  $U(\omega)U^*(\omega) = I$  a.e.  $\omega \in E$  and such that*

$$G(\omega) = U(\omega)\Lambda(\omega)U^*(\omega) \quad \text{a.e. } \omega \in E \quad (4.11)$$

where  $\Lambda(\omega) := \text{diag}(\lambda_1(\omega), \dots, \lambda_m(\omega))$ .

*Proof.* (of Theorems 2.1 and 2.3)

In what follows we will apply Theorem 4.1 to find the solution to Problem 1. As before, let  $\mathcal{F} = \{f_1, \dots, f_m\} \subset L^2(\mathbb{R}^d)$  and for  $\omega \in [0, 1]^d$  let  $G_{\mathcal{F}}(\omega)$  be the

associated Gramian matrix with eigenvalues  $\lambda_1(\omega) \geq \dots \geq \lambda_m(\omega) \geq 0$ . Let  $U(\omega)$  be a measurable  $m \times m$  matrix as in Lemma 4.3. Since  $G_{\mathcal{F}}(\omega)$  is  $\mathbb{Z}^d$ -periodic on  $\mathbb{R}^d$ , we can choose  $U(\omega)$  to be  $\mathbb{Z}^d$ -periodic as well. Let  $U_i(\omega)$  denote the  $i$ -th row of  $U(\omega)$ . Multiplying (4.11) on the left by  $U^*(\omega)$  shows that  $y_i(\omega) := U_i(\omega)^*$  is a left-eigenvector of  $G(\omega)$  with eigenvalue  $\lambda_i(\omega)$  for  $i = 1, \dots, m$ . Furthermore, the left eigenvectors  $y_i(\omega) = (y_{i1}(\omega), \dots, y_{im}(\omega))^t$ ,  $i = 1, \dots, m$ , form an orthonormal basis of  $\mathbb{C}^m$ .

For each fixed  $\omega \in [0, 1]^d$ , we consider Problem 2 in the space  $\ell_2(\mathbb{Z}^d)$  for the data  $(\mathcal{F}_\omega, n)$  with  $\mathcal{F}_\omega = \{\Gamma_\omega \hat{f}_1, \dots, \Gamma_\omega \hat{f}_m\}$ . Define  $q_1(\omega), \dots, q_n(\omega) \in \ell_2(\mathbb{Z}^d)$  by

$$q_i(\omega) = \tilde{\sigma}_i(\omega) \sum_{j=1}^m y_{ij}(\omega) \Gamma_\omega \hat{f}_j, \quad i = 1, \dots, n \quad (4.12)$$

where  $\tilde{\sigma}_i(\omega) = \lambda_i^{-1/2}(\omega)$  if  $\lambda_i(\omega) \neq 0$ , and  $\tilde{\sigma}_i(\omega) = 0$  otherwise. Since  $G_{\mathcal{F}}(\omega) = \mathfrak{G}(\mathcal{F}_\omega)$  (see Lemma 3.3), Theorem 4.1 shows that the space  $S_\omega := \text{span} \{q_1(\omega), \dots, q_n(\omega)\}$  optimizes Problem 2. Moreover, the vectors  $\{q_1(\omega), \dots, q_n(\omega)\}$  form a Parseval frame for  $S_\omega$  and we have the following formula for the error

$$\mathfrak{E}(\mathcal{F}_\omega, n) = \sum_{i=n+1}^m \lambda_i(\omega). \quad (4.13)$$

Define now the functions  $h_i : \mathbb{R}^d \rightarrow \mathbb{C}$ ,  $i = 1, \dots, m$ ,

$$h_i(\omega) = \tilde{\sigma}_i(\omega) \sum_{j=1}^m y_{ij}(\omega) \hat{f}_j(\omega). \quad (4.14)$$

Since  $\tilde{\sigma}_i$  and  $y_i$  are measurable functions of  $\omega$ , then  $h_i$  is also measurable. Moreover,  $h_i$  is in  $L^2(\mathbb{R}^d)$  as the following simple argument shows. Since

$$|h_i(\omega)|^2 = h_i(\omega) \overline{h_i(\omega)} = \tilde{\sigma}_i(\omega)^2 \sum_{j,s=1}^m y_{ij}(\omega) \hat{f}_j(\omega) \overline{\hat{f}_s(\omega)} \overline{y_{is}(\omega)}$$

we have (using that if  $y_i$  is a left eigenvector of the self-adjoint matrix  $G_{\mathcal{F}}$ , then  $\overline{y_i}$  is a right eigenvector for that matrix associated to the same eigenvalue),

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} |h_i(\omega + k)|^2 &= \tilde{\sigma}_i(\omega)^2 \sum_{j=1}^m y_{ij}(\omega) \sum_{s=1}^m [G_{\mathcal{F}}(\omega)]_{js} \overline{y_{is}(\omega)} \\ &= \tilde{\sigma}_i(\omega)^2 \lambda_i(\omega) \sum_{j=1}^m y_{ij}(\omega) \overline{y_{ij}(\omega)} = \tilde{\sigma}_i(\omega)^2 \lambda_i(\omega). \end{aligned} \quad (4.15)$$

If  $\lambda_i(\omega) \neq 0$  then the product in (4.15) is one, otherwise it is zero. That is  $\sum_{k \in \mathbb{Z}^d} |h_i(\omega + k)|^2 = \mathbf{1}_{\{\omega : \lambda_i(\omega) > 0\}}$  and by Lemma 3.1,  $\|h_i\| \leq 1$ .

Now define functions  $\varphi_1, \dots, \varphi_n$  in  $L^2(\mathbb{R}^d)$  by:

$$\hat{\varphi}_i(\omega) = h_i(\omega), \quad i = 1, \dots, n$$

and let  $V = S(\varphi_1, \dots, \varphi_n)$ . The space  $V$  is a shift invariant space of length no bigger than  $n$ . So  $V \in \mathcal{V}_n$ . Furthermore, by Lemma 3.2 (iv-a), the space  $V_\omega$  is spanned by  $\Gamma_\omega \hat{\varphi}_i$ ,  $i = 1, \dots, n$ .

Since  $(\Gamma_\omega \hat{\varphi}_i)(k) = h_i(\omega + k) = q_i(\omega)(k)$ ,  $k \in \mathbb{Z}^d$ ,  $i = 1, \dots, n$  a.e., then  $V_\omega = S_\omega$  (the optimal space for the data  $(\mathcal{F}_\omega, n)$ ) in  $\ell_2(\mathbb{Z}^d)$ .

By equation (4.5) and the comment before,  $V$  is optimal, that is  $V$  solves Problem 1 for the data  $(\mathcal{F}, n)$ .

Now, since  $\{\Gamma_\omega \hat{\varphi}_1, \dots, \Gamma_\omega \hat{\varphi}_n\}$  is a Parseval frame of  $S_\omega$  for a.e.  $\omega \in [0, 1]^d$  then by Lemma 3.2 (iv-b) the integer translates of  $\varphi_1, \dots, \varphi_n$  form a Parseval frame of  $V$ . On the other hand, Formula (4.3) says that

$$\mathcal{E}(\mathcal{F}, n) = \int_{[0, 1]^d} \mathfrak{E}(\mathcal{F}_\omega, n) d\omega. \quad (4.16)$$

Thus using (4.13) we have that  $\mathcal{E}(\mathcal{F}, n) = \sum_{i=n+1}^m \int_{[0, 1]^d} \lambda_i(\omega) d\omega$ .  $\square$

*Proof.* (of Theorem 2.4).

Under the hypothesis of Theorem 2.4, Theorem 4.1 and the Remark after it guarantee the uniqueness of the optimal spaces  $S_\omega$  associated to the data  $(\mathcal{F}_\omega, n)$  for almost all  $\omega$ . Since these fiber spaces characterize the optimal space  $V$ , then the theorem follows.  $\square$

## APPENDIX

**4.4. Best linear approximation and the SVD.** Here we review the singular value decomposition (SVD) of a matrix and its relation to finite dimensional least-squares problems. For an overview see [Ste93], and for a very detailed treatment see for example [HJ85].

We start with the following proposition.

**Proposition 4.4 (SVD).** *Let  $A = [a_1, a_2, \dots, a_m]$  be the matrix with columns  $a_i \in \mathbb{C}^N$ ,  $m \leq N$ . Let  $r := \dim \text{span} \{a_1, \dots, a_m\}$ . Then there are  $m$  numbers  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > \lambda_{r+1} = \dots = \lambda_m = 0$ , an orthonormal collection of  $m$  (column) vectors  $y_1, \dots, y_m \in \mathbb{C}^m$ , and an orthonormal collection of  $m$  (column) vectors  $u_1, \dots, u_m \in \mathbb{C}^N$  such that*

$$A = \sum_{k=1}^m \sqrt{\lambda_k} u_k y_k^* = U \Lambda^{1/2} Y^* \quad (4.17)$$

Where  $U \in \mathbb{C}^{N \times m}$  is the matrix  $U = [u_1, \dots, u_m]$ ,  $\Lambda^{1/2} = \text{diag}(\lambda_1^{1/2}, \dots, \lambda_m^{1/2})$ , and  $Y = \{y_1, \dots, y_m\} \in \mathbb{C}^{m \times m}$  with  $U^* U = I_m = Y^* Y = Y Y^*$ .

The representation of  $A$  given in (4.17) is called the singular value decomposition (SVD) of  $A$ .

The SVD of a matrix  $A$  can be obtained as follows. Consider the matrix  $A^* A \in \mathbb{C}^{m \times m}$ . Since  $A^* A$  is self-adjoint and positive semi-definite, its eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$  are nonnegative and the associated eigenvectors  $y_1, \dots, y_m$  can be chosen to form an orthonormal basis of  $\mathbb{C}^m$ . Note that the rank  $r$  of  $A$  corresponds to the largest index  $i$  such that  $\lambda_i > 0$ . The left singular vectors  $u_1, \dots, u_r$  can then be obtained from

$$\sqrt{\lambda_i} u_i = A y_i, \quad \text{that is } u_i = \lambda_i^{-1/2} \sum_{j=1}^m y_{ij} a_j. \quad (1 \leq i \leq r).$$

Here  $y_i = (y_{i1}, \dots, y_{im})^t$ . The remaining left singular vectors  $u_{r+1}, \dots, u_m$  can be chosen to be any orthonormal collection of  $m - r$  vectors in  $\mathbb{C}^N$  that are perpendicular to  $\text{span} \{a_1, \dots, a_m\}$ . One may then readily verify that (4.17) holds.

The *Frobenius norm* of a matrix  $X = [x_1, \dots, x_m] \in \mathbb{C}^{N \times m}$  is  $\|X\|_F = \text{tr}(X^* X)$ , where  $\text{tr}$  denote the trace of a matrix.

Now, the following approximation theorem of Schmidt (cf. [Sch07]) and later rediscovered by Eckart and Young ([EY36]) shows that the SVD can be used to find the subspace of dimension  $n$  that is closest to a given finite numbers of vectors.

**Theorem 4.5.** *Let  $\{a_1, \dots, a_m\}$ , be a set of vectors in  $\mathbb{C}^N$ , such that  $r = \dim(\text{span}\{a_1, \dots, a_m\})$ , and suppose  $A = [a_1, \dots, a_m]$ , has SVD  $A = U\Lambda^{1/2}Y^*$  with  $0 < n \leq r$ . Then  $A_n := \sum_{j=1}^n \sqrt{\lambda_j} u_j y_j^*$  satisfies*

$$\|A - A_n\|_F = \min_{\text{rank } B \leq n} \|A - B\|_F = \left( \sum_{j=n+1}^r \lambda_j \right)^{1/2}.$$

If  $\lambda_{n+1} \neq \lambda_n$ , then  $A_n$  is the unique such matrix of rank at most  $n$ .

Equivalently,

**Theorem 4.6.** *Let  $\{a_1, \dots, a_m\}$ , be a set of vectors in  $\mathbb{C}^N$  such that  $r = \dim(\text{span}\{a_1, \dots, a_m\})$ , and suppose  $A = [a_1, \dots, a_m]$ , has SVD  $A = U\Lambda^{1/2}Y^*$  and that  $0 < n \leq r$ . If  $W = \text{span}\{u_1, \dots, u_n\}$ , then*

$$\{P_W a_1, \dots, P_W a_m\} = \sum_{i=1}^n \sqrt{\lambda_i} u_i y_i^* = A_n$$

and

$$\sum_{i=1}^m \|a_i - P_W a_i\|_2^2 \leq \sum_{i=1}^m \|a_i - P_M a_i\|_2^2, \quad \forall M, \dim M \leq n, \quad (4.18)$$

and the space  $W$  is unique if  $\lambda_{n+1} \neq \lambda_n$ .

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#### REFERENCES

- [AG01] A. Aldroubi and K-H. Gröchenig, *Non-uniform sampling in shift-invariant space*, Siam Review **43** (2001), no. 4, 585–620.
- [BCDV05] P. Binev, A. Cohen, W. Dahmen, and R. DeVore, *Universal algorithms for learning theory part I : piecewise constant functions*, Journal of Machine Learning Research, **6** (2005), 1297–1321.
- [BD03] R. Balan and I. Daubechies, *Optimal stochastic encoding and approximation schemes using Weyl-Heisenberg sets*, Advances in Gabor analysis, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2003, pp. 259–320.
- [Bow00] M. Bownik, *The structure of shift-invariant subspaces of  $L^2(\mathbb{R}^n)$* , Journal of Functional Analysis **177** (2000), 282–309.
- [CC97] P. Cazassa and O. Christensen, *Perturbation of operators and applications to frame theory*, J. Fourier Anal. Appl. **3** (1997), 543–557.
- [CS02] F. Cucker and S. Smale, *On the mathematical foundations of learning*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 1, 1–49 (electronic).
- [BDR94a] C. de Boor, R. DeVore, and A. Ron, *Approximation from shift-invariant subspaces of  $L_2(\mathbb{R}^d)$* , Trans. Amer. Math. Soc. **341** (1994), 787–806.
- [BDR94b] ———, *The structure of finitely generated shift-invariant subspaces of  $L_2(\mathbb{R}^d)$* , J. Funct. Anal. **119** (1994), 37–78.

- [EY36] C. Eckart and G. Young, *The approximation of one matrix by another of lower rank*, Psychometrika **1** (1936), 211 – 218.
- [HL00] D. Han, and D. Larson, *Frames, bases and group representations*, Mem. Amer. Math. Soc. **147** (2000), no. 697, x+94 pp.
- [Hel64] H. Helson, *Lectures on invariant subspaces*, Academic Press, London, 1964.
- [HLW02] E. Hernández, D. Labate, and G. Weiss, *A unified characterization of reproducing systems generated by a finite family. II*, J. Geom. Anal. **12** (2002), 615–662.
- [HJ85] R. Horn and C. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.
- [RS95] A. Ron, Z. Shen, *Frames and stable bases for shift invariant subspaces of  $L_2(\mathbb{R}^d)$* , Canadian Journal of Mathematics, **47** (1995), 1051–1094.
- [SZ03] S. Smale and D-X. Zhou, *Estimating the approximation error in learning theory*, Anal. Appl. (Singap.) **1** (2003), 17–41.
- [SZ04] ———, *Shannon sampling and function reconstruction from point values*, Bull. Amer. Math. Soc. **41** (2004), 279–305.
- [Sch07] E. Schmidt, *Zur theorie der linearen und nichtlinearen integralgleichungen. i teil. entwicklung willkürlichen funktionen nach system vorgeschriebener*, Math. Ann. **63** (1907), 433–476.
- [Ste93] G. W. Stewart, *On the early history of the singular value decomposition*, SIAM Rev. **35** (1993), 551–566.

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