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On the stability of the point reactor kinetics equations

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ABSTRACT

As the basic neutronic problem is unstable by nature, maintaining a reactor critical is a task that requires a lot of effort. This work presents and discusses some aspects related to the stability of the basic physics behind a nuclear reactor core based on the well-known point reactor kinetics equations. First, the linear non-feedback case is studied where differences between Lyapunov and BIBO stability are found. These differences are shown both numerically and analytically, and explained using a reactor physics based reasoning. Finally, a simple model is used to analyse the intrinsic stability of the point reactor equations when reactivity feedbacks are taken into account. A method for constructing conceptual stability design maps is proposed and a basic interpretation of the simple results obtained is given.

1. Introduction

The subject of the stability of a nuclear reactor is of particular concern in a conceptual engineering stage, as shown for example by the state-of-the-art generation IV design paths (Zanocco et al., 2003). The mechanisms of instability appearance are diverse, being the most common the coupling between neutron dynamics and the thermalhydraulics of the coolant loop (Theler, 2008; Doutta and Doshi, 2009). A frequency domain analysis can be made in order to study theoretical instabilities (Peng et al., 1984, 1986; Garea et al., 1999), but other linear and non-linear methods also exist (Lahey and Podowski, 1989). On the other hand, best estimate codes may be used to simulate reactor transients. Not only are these neutronic-thermalhydraulic coupled codes used nowadays to evaluate the plant state after postulated accidents (Salah and D'Auria, 2007, 2009) but also as a tool to assist the designer in the decision making process. In this paper we propose a method for analysing the stability of the point reactor kinetics equations, in order to grasp the general behaviour of the system and to generate some conceptual design maps. The development is based on approximations and general facts that are aimed at maintaining the mathematics fairly simple without entering into detailed and precise calculations.

Particularly, we address and study the stability of the well-known point reactor kinetics equations. This model might be a very crude simplification of a real reactor core, but it has two main advantages that make it suitable for our purposes. The first is that point kinetics equations maintain the underlying physics of the general transient neutron problem, and the second is that they have a reasonable and simple mathematical formulation. We ought to remark that the facts and results shown here only apply to the mathematical model and equations discussed and they are not directly applicable to real reactors.

2. The point kinetics equations

Every stability study presented in this work is based on the point reactor kinetics equations, whose derivation and particular analytical solutions may be found in the standard reactor physics literature. In particular, Duderstadt and Hamilton (1976) provide a rather intuitive approach to the equations while Henry (1975) shows how they can be rigorously obtained from the time-dependant diffusion equation. The resulting equations are

$$\begin{aligned} \frac{dn}{dt} &= \frac{\rho-\beta}{\Lambda}n + \sum_{i=1}^{I}\lambda_i c_i + q \\ \frac{dc_i}{dt} &= \frac{\beta_i}{\Lambda}n - \lambda_i c_i \end{aligned} \tag{1}$$

The variables and parameters that appear in these equations are the result of integrating the original partial derivatives formulation of the neutron conservation problem with respect to both the spacial and angular coordinates and to all available neutron energies. Nevertheless, these particular details are not of interest for the analysis that follows. The nomenclature used is the one that may usually be found in the literature and a brief description of each kinetic parameter is shown in Table 1. We take each one of these as a constant and *a priori* known parameter.

To complete the formulation of the problem, initial conditions $n(t=0)$ and $c_i(t=0)$ should be given. In the absence of an exter-

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Table 1
Description of the variables and parameters appearing in the point kinetics equations.

n	Neutron population [a]
I	Number of neutron precursor groups
c_i	i th precursor group concentration
ρ	Total reactivity $(k_{\text{eff}}-1)\cdot k_{\text{eff}}^{-1}$
β_i	i th precursor group fission yield
β	Total fraction of delayed neutrons $(\sum \beta_i)$
λ_i	i th precursor group decay constant
Λ	Average time between the emission of neutrons born in two consecutive generations [b]
q	Independent neutron source intensity

[a] All variables come from a weighted integral of the continuous diffusion equation. Only when the weight function is identically one, n is equal to the total number of neutrons.
[b] If the reactor is critical, Λ is the average lifetime of a single neutron.

nal neutron source, the equilibrium condition for a steady-state population $n^\star$ is

$$c_i(0) = \frac{\beta_i}{\lambda_i \Lambda} n^\star \tag{2}$$

Note that if $q = 0$, an homogeneous system of equations is obtained, and n and c_i may be scaled by an arbitrary factor without affecting the solution form. As the thermal power is proportional to the neutron flux, we may write the kinetics equations as a function of the power $P(t)$ instead of the neutron population in some sections of this paper.

3. Stability studies without reactivity feedback

Again, with the objective of looking at the physical background of the problem and not solving for a particular detailed solution, we take only one group of precursors instead of the traditional six groups. This approximation simplifies the associated mathematics and at the same time maintains the physical basis of the original problem.

Let us consider the point kinetics equations with one neutron precursor group and with no external source, written as a dynamical system with a state vector $\mathbf{x}$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{n} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} (\rho-\beta)/\Lambda \cdot n + \lambda c \\ \beta/\Lambda \cdot n - \lambda c \end{bmatrix} \tag{3}$$

There is a fixed point $\mathbf{x} = \mathbf{x}^\star$ when the derivatives vanish

$$0 = \frac{\rho^\star - \beta}{\Lambda} n^\star + \lambda c^\star$$
$$0 = \frac{\beta}{\Lambda} n^\star - \lambda c^\star$$

By summing both sides of these equations we obtain the necessary condition for the system to be in equilibrium

$$\rho^\star \cdot \frac{n^\star}{\Lambda} = 0 \tag{4}$$

As both $n^\star$ and Λ are positive, the reactivity must be zero for a fixed point to exist. This is the well-known result that a reactor is critical if and only if $k_{\text{eff}} = 1$ and, on the other hand, justifies the equilibrium condition shown in Eq. (2).

Let us now consider what happens around the fixed point $\mathbf{x}^\star$ re-writing the problem using the perturbation variables

$$\delta n = n - n^\star$$
$$\delta c = c - c^\star$$
$$\delta\rho = \rho - \rho^\star = \rho$$

Replacing in the system of Eq. (1) and taking into account that $\mathbf{x}^\star$ is a fixed point, we obtain

$$\begin{aligned} \delta\dot{n} &= \frac{n^\star}{\Lambda}\delta\rho + \frac{\delta\rho\,\delta n}{\Lambda} - \frac{\beta}{\Lambda}\delta n + \lambda\delta c \\ \delta\dot{c} &= \frac{\beta}{\Lambda}\delta n - \lambda\delta c \end{aligned} \tag{5}$$

The system (5) is mathematically equivalent to Eq. (1) with $q = 0$. There is only one explicit non-linearity in the crossed term $(\delta\rho\,\delta n)/\Lambda$. Thus, the point kinetics equations are linear only for a constant reactivity. In the following two sections we take the usual linear analysis approach of neglecting this second order term, while we will come back to study its influence in Sections 3.3 and 4.1.

3.1. Intrinsic linear stability

There are two different stability concepts that may be of interest for the nuclear reactor core designer. One is the so-called Lyapunov stability which refers to what happens when the vector state is perturbed near a fixed point in phase space. The other one is the bounded-input bounded-output (BIBO) stability concept that copes with perturbations on the external input of a given dynamical system. In order to proceed with the standard linear approach, we may neglect $\delta\rho\delta n$ with respect to the other terms—just for this section—and consider $\delta\rho(t)$ as an external control action. Thus, the equations are casted in a matrix based form as

$$\begin{bmatrix} \delta\dot{n} \\ \delta\dot{c} \end{bmatrix} = \begin{bmatrix} -\beta/\Lambda\lambda \\ \beta/\Lambda - \lambda \end{bmatrix} \begin{bmatrix} \delta n \\ \delta c \end{bmatrix} + \begin{bmatrix} n^\star/\Lambda \\ 0 \end{bmatrix} \delta\rho \tag{6}$$

To study the Lyapunov stability of the fixed point $\mathbf{x}^\star$ we may compute the eigenvalues of the system matrix. They are

$$\begin{aligned} s_2 &= -\frac{\beta}{\Lambda} - \lambda \\ s_1 &= 0 \end{aligned} \tag{7}$$

As there is one eigenvalue equal to zero, according to the Hartman–Grobman theorem (Solari et al., 1996), nothing can be said about the linear Lyapunov stability of the fixed point $\mathbf{x}^\star$ of the dynamical system (6). We will return to this subject on Section 3.3.

On the other hand, to study how the system responds to perturbations on the external input, we may compute the transfer function of the neutron population n with respect to the reactivity ρ. Laplace-transforming both Eq. (6) we get

$$\begin{aligned} &\mathcal{L}\{\delta\dot{n}\} = \mathcal{L}\left\{-\frac{\beta}{\Lambda}\delta n + \lambda\delta c + \frac{n^\star}{\Lambda}\delta\rho\right\} \\ &s\,\delta N(s) - \delta n(0) = -\frac{\beta}{\Lambda}N(s) + \lambda\,\delta C(s) + \frac{n^\star}{\Lambda}\delta R(s) \end{aligned} \tag{8}$$

$$\begin{aligned} &\mathcal{L}\{\delta\dot{c}\} = \mathcal{L}\left\{\frac{\beta}{\Lambda}\delta n + \lambda\,\delta c\right\} \\ &s\,\delta C(s) - \delta c(0) = \frac{\beta}{\Lambda}\delta N(s) - \lambda\,\delta C(s) \end{aligned} \tag{9}$$

To derive the open loop transfer function we set the initial conditions $\delta n(0) = \delta c(0) = 0$ and solve for $\delta C(s)$ in Eq. (9)

$$s\,\delta C(s) = \frac{\beta}{\Lambda}\delta N(s) - \lambda\,\delta C(s)$$
$$C(s) = \frac{\beta}{\Lambda(s+\lambda)}N(s)$$

By replacing $C(s)$ in Eq. (8) we arrive at

$$s\,\delta N(s) = -\frac{\beta}{\Lambda}\delta N(s) + \frac{\lambda\beta}{\Lambda(s+\lambda)}\delta N(s) + \frac{n^\star}{\Lambda}\delta R(s)$$

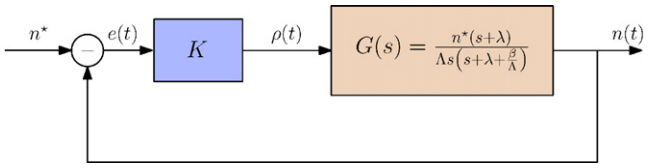

Fig. 1. Closed control loop with a proportional compensator action on the linearised point reactor equations with a group of precursors.

so we can solve for the linear transfer function $\delta N(s)/\delta R(s)$ between the reactivity and the neutron population

$$G(s) = \frac{\delta N(s)}{\delta R(s)} = \frac{n^\star}{\Lambda} \cdot \frac{(s+\lambda)}{s(s+\lambda+(\beta/\Lambda))} \tag{10}$$

whose poles are equal to the eigenvalues (7). Nevertheless, in this case we may conclude that the point kinetic equations are not BIBO-stable as the pole in the origin of the complex plane acts as an integrator of the excitation $\rho(t)$, modulated by the factor $n^\star/\Lambda$. For a positive reactivity the neutron population would monotonically increase in time, and vice versa.

The same conclusion may be reached by noting that the term $(n^\star\delta\rho)/\Lambda$ in the first equation of the system (5) has the functional form of an external neutron source—or sink if negative—that is proportional to the reactivity perturbation. Therefore, as there is at least one bounded external perturbation—namely a constant finite positive reactivity—that produces an unbounded output, the system is not BIBO-stable.

3.2. Linear stability with a control loop

In the previous section we studied the intrinsic stability of the point reactor equations without any feedback on the external reactivity. Without any of these feedback effects, the condition of criticality $\rho = 0$ would never be obtained as the probability of the matrix of the multigroup diffusion problem having an eigenvalue k_{eff} exactly equal to one is zero (Henry, 1975; Stacey, 2001). So we now turn out attention to study the linear stability of the point reactor equations with a closed control loop.

Consider the control scheme depicted in Fig. 1. The open loop transfer function is $KG(s)$ and the closed negative feedback loop transfer function is

$$G_{\text{cl}}(s) = \frac{KG(s)}{1+KG(s)}$$

In Fig. 2 we show the Nyquist plot that results when mapping the imaginary axis $s = i\omega$ excluding the origin using $s = re^{-i\theta}$ for $-\pi/2 < \theta < \pi/2$ with $r = 10^{-3}$ for the test function

$$KG(s) = \frac{(s+\lambda)}{s(s+\lambda+(\beta/\Lambda))}$$

with

$$\beta = 7.65 \times 10^{-3}$$
$$\lambda = 7.59 \times 10^{-2}\,\text{s}^{-1}$$
$$\Lambda = 1.76 \times 10^{-4}\,\text{s}$$

that approximately represent the kinetic parameters of an open pool research reactor operating at full power with an equilibrium core.

We see that the Nyquist plot does not circle the point (−1,0), and as $G(s)$ does not have any zeros in the right half plane, the system is stable for the proposed gain. Moreover, the system is stable for any positive K—provided the feedback loop is negative—as the gain effect is just to scale the diagrams shown in Fig. 2. Thus, even though the point reactor equations are not BIBO-stable, the simplest continuous compensator possible acting on the reactivity is able to render it stable (Lewins, 1978).

However, we must keep in mind that the unconditionally stability we found applies only to a linearised version of the point reactor equations that does not take into account any temperature, void or external reactivity feedback, and that the (continuous) controller is applied instantly and directly to ρ. In any real case, there may be feedbacks, non-linear effects (real control rods), discrete sampling times and phase lags that would reduce the stability margin, possibly rendering the system unstable.

3.3. Non-linear effects

In Section 3.1 we stated that nothing can be concluded about the Lyapunov stability of the fixed point $\mathbf{x} = \mathbf{x}^\star$ using the linear theory because the system matrix had a null eigenvalue. We now try to determine the Lyapunov stability by numerically solving the kinetics equations and interpreting the results using reactor physics theory.

To see if a small disturbance in phase space makes the system remain close and eventually converge back to the point $\mathbf{x}^\star$, we consider again the equations with $\rho = 0$:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{n} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} -\beta/\Lambda \cdot n + \lambda c \\ \beta/\Lambda \cdot n - \lambda c \end{bmatrix} \tag{11}$$

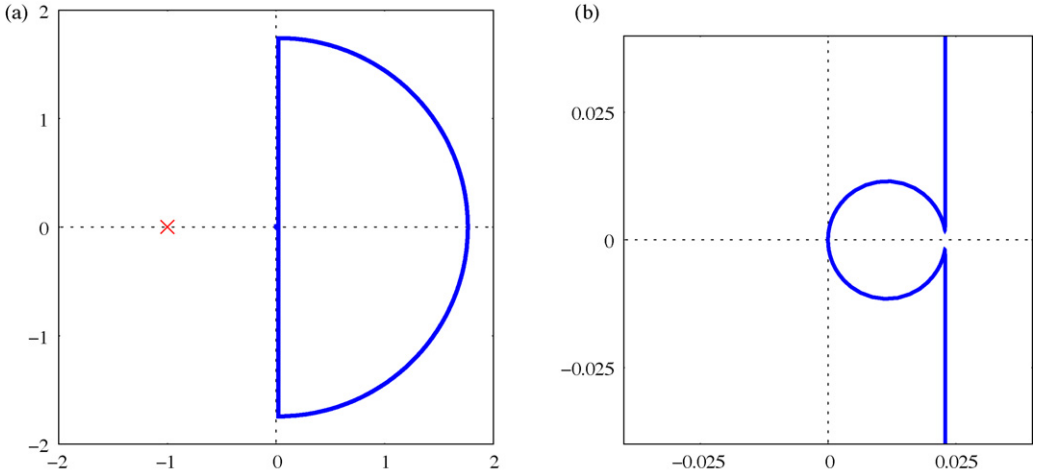

Fig. 2. Nyquist plot of the linearised kinetics equations transfer function $G(i\omega)$. The infinite closure was done excluding the origin making $s = re^{-i\theta}$ for $-\pi/2 < \theta < \pi/2$ with $r = 10^{-3}$.

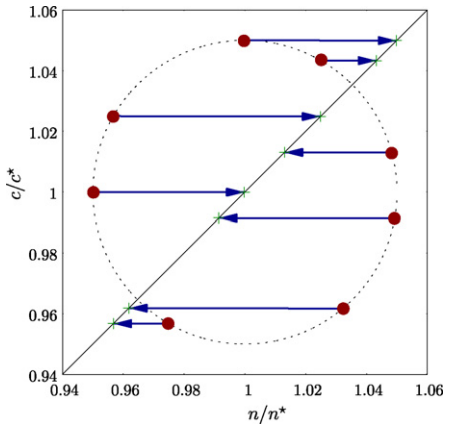

Fig. 3. Lyapunov stability. The straight line $c/c^\star = n/n^\star$ is the kernel of the system matrix, so any point that belongs to it is a fixed point. Initial conditions (spots) were taken on a circumference of a radius equal to 5% of the normalised variables. All cases converge to a new fixed point almost horizontally and, in particular, they never leave the shown square. Thus, the fixed point (1,1) is Lyapunov-stable.

that is a linear dynamical system with a singular matrix. For a given $n_0 \neq n^\star$, the vector

$$\mathbf{x}_0 = \begin{bmatrix} n_0 \\ \dfrac{\beta}{\Lambda\lambda} n_0 \end{bmatrix} \tag{12}$$

is also a fixed point of the system (11). In particular, we call $\mathbf{x}^\star = [n^\star c^\star]^T$ the design point, whose Lyapunov stability we want to address. We define the non-dimensional state vector as

$$\tilde{\mathbf{x}} = \begin{bmatrix} n/n^\star \\ c/c^\star \end{bmatrix}$$

By virtue of Eqs. (2) and (12), any point located in the first quadrant diagonal of the non-dimensional phase space is also a fixed point. This result is consistent with the fact that the matrix kernel has dimension one. To study the stability of the design point $\tilde{\mathbf{x}}^\star = [1\,1]^T$ we numerically integrate the differential equations using a few different initial conditions (marked as spots in Fig. 3) located on a circumference centred in $\mathbf{x}^\star$ with a non-dimensional radius equal to 5×10^{-2}. In all cases, the state vector $\mathbf{x}(t)$ evolves to a point located in the $c/c^\star = n/n^\star$ straight line (marked as crosses in Fig. 3).

It is apparent that the design point $\mathbf{x}^\star = [n^\star c^\star]^T$ is Lyapunov-stable, as we see that some perturbations $\delta(\epsilon)$ (in particular the circumference radius) do not make the resulting phase space trajectories leave a given environment of the fixed point (in particular the square enclosing Fig. 3). However, this stability is not asymptotic as the state vector does not go back to the design point but to another different fixed point.

Both this effect and the fact that the trajectories are almost horizontal in Fig. 3 may be explained with some reactor physics reasoning (Gho, 2005). When $\rho = 0$, summing Eq. (11) gives

$$\dot{n} + \dot{c} = 0$$

so the sum $n + c$ is always constant. Suppose we artificially introduce an extra deterministic neutron into a critical reactor core, that lives exactly a time Λ, gets instantly multiplied by a factor of $(1 - \beta)$ and that a fraction β of a delayed neutron appears exactly a time $1/\lambda$ after the fission occurred. For times much smaller than $1/\lambda$, the extra neutron transforms into successive fractions

$$\delta n(t + \Lambda) = (1 - \beta)\delta n(t)$$

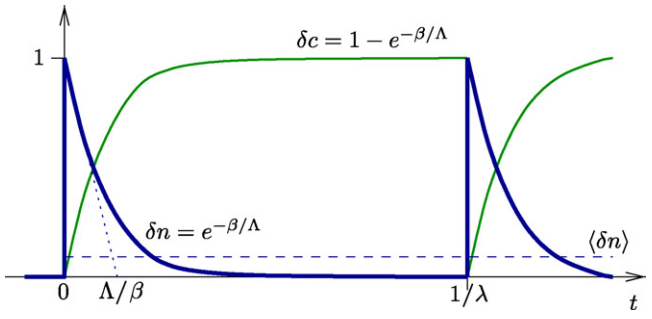

Fig. 4. Qualitative illustration of the dynamics of an extra deterministic neutron inserted at time $t = 0$ in a critical core.

We may estimate the time derivative of δn and integrate to get

$$\delta\dot{n}(t \ll 1/\lambda) \approx \frac{(1 - \beta)\delta n(t) - \delta n(t)}{\Lambda} = -\frac{\beta}{\Lambda}\delta n(t) \Rightarrow \delta n(t) = e^{-(\beta/\Lambda)t}$$

As at this time the precursor has not yet decayed and $\delta n + \delta c$ equals one, for we introduced only one single extra neutron, then

$$\delta\dot{c}(t \ll 1/\lambda) \approx \frac{\beta}{\Lambda}\delta n(t) \quad \Rightarrow \quad \delta c(t) = 1 - e^{-(\beta/\Lambda)t}$$

This is, for $t \ll 1/\lambda$ the extra neutron exponentially disappears and the precursor concentration rises, both with a time constant β/λ. If $\Lambda/\beta \ll 1/\lambda$, then for times $t \lesssim 1/\lambda$ the extra neutron may have virtually vanished. When $t = 1/\lambda$, the precursor decays, a new neutron appears and the whole process is repeated. Fig. 4 qualitatively shows this reasoning. The result is that the average increase of the neutron population is the mean value of $\delta n(t)$

$$\langle \delta n \rangle = \frac{1}{1/\lambda} \int_0^{1/\lambda} \exp\left(-\frac{\beta}{\Lambda} \cdot t\right) dt \tag{13}$$

that for $\lambda \ll \beta/\Lambda$—as usual for common kinetic parameters—gives

$$\langle \delta n \rangle \approx \frac{\lambda\Lambda}{\beta} \tag{14}$$

The system is still critical but reaches a new equilibrium point given by the new neutron population and the corresponding precursor concentration. It is interesting to note that even though one neutron is inserted, the population rises only by a small fraction $\lambda\Lambda/\beta$. The remainder of the extra neutron appears as "stored" into $(1 - \lambda\Lambda/\beta)$ precursors.

This reasoning also explains Fig. 3, where not only was the initial neutron population suddenly changed with respect to the equilibrium, but also the precursor concentration. Moreover, we can easily prove that the trajectories are always straight lines and for common kinetics parameters, essentially horizontal in the non-dimensional phase space. The neutron population derivative may be written using the chain rule as

$$\dot{n} = \frac{\partial n}{\partial c}\,\dot{c}$$

If we replace Eq. (11), we get

$$\frac{\partial n}{\partial c} = -1 \tag{15}$$

thus trajectories in phase space are straight lines. Furthermore, if we transform Eq. (15) into a non-dimensional expression, we obtain:

$$\frac{\partial \tilde{n}}{\partial \tilde{c}} = -\frac{c^\star}{n^\star} = -\frac{\beta}{\lambda\Lambda} \approx 0^-$$

which explains the results obtained numerically in Fig. 3. The minus sign and the fact that the diagonal is an equilibrium subspace prove that the design point $\mathbf{x}^\star$ is Lyapunov-stable.

4. Reactivity feedback analysis

The multiplication factor k_{eff} is a property of the core as a whole, and it depends on a lot of factors (such as the excess initial reactivity, control rods position, fuel burn-up, poisons distribution, and temperatures) in a rather complex way. In particular, it is a function of the thermal power itself so $\rho = \rho(n)$ and the point kinetics Eq. (3) are not linear anymore. Thus, the stability analysis cannot be addressed with the standard linear theory—at least not directly.

Feedback effects may be classified and analysed in a variety of ways—such as those proposed by Lewins (1978) or Duderstadt and Hamilton (1976) for example. One useful approach to compute the change in reactivity due to variations in the conditions of the core is the perturbation formula:

$$\rho = -\frac{\int_V \int_0^\infty \Phi^\dagger(\mathbf{r}, E) \cdot \left(\delta\Sigma_a + \delta\nu\Sigma_f\right) \cdot \Phi(\mathbf{r}, E)\, dE\, d\mathbf{r}}{\int_V \int_0^\infty \Phi^\dagger(\mathbf{r}, E) \cdot \nu\Sigma_f \cdot \Phi(\mathbf{r}, E)\, dE\, d\mathbf{r}}$$

that may be even further simplified when using a one-group approximation, as the adjoint flux equals the direct flux so

$$\rho = -\frac{\int_V \Phi^2(\mathbf{r}) \cdot (\delta\Sigma_a + \delta\nu\Sigma_f)\, d\mathbf{r}}{\int_V \Phi^2(\mathbf{r}) \cdot \nu\Sigma_f\, d\mathbf{r}}$$

However, to correctly compute the reactivity, one must know how the macroscopic cross-sections depend on the thermalhydraulic parameters or the position of the control rods. As stability is a conceptual design concern, this information may not be available at this engineering stage. Thus, we utilise the standard concept of feedback coefficients in this section.

The reactivity feedback coefficient due to the effect of the variation of a certain variable x is defined as

$$\alpha_x = \frac{\partial\rho}{\partial x} \tag{16}$$

so the change in the reactivity introduced by a variation δx is approximated by

$$\delta\rho_x \approx \alpha_x \cdot \delta x \tag{17}$$

Even though the feedback coefficient (16) is well defined, the actual coefficient varies with time and Eq. (17) applies only for small changes in x. Moreover, when there are many variables whose effect one wants to take into account, the usual practise is to sum up every single contribution to the total reactivity as a linear combination, neglecting correlations between these variables and other non-linear effects. Furthermore, reactivity coefficients play a major role in defining the design basis of a nuclear reactor design (Florido et al., 2007) and are extensively used in nuclear regulations and international safety guides (International Atomic Energy Agency, 1999, 2005).

4.1. Study case

In this section we show how a very simple time-domain analysis may be used to construct stability maps that could be used to help the core designer during the conceptual engineering of a nuclear reactor. Consider the zero-dimensional model of a research reactor shown in Fig. 5 where coolant enters at a constant rate $\dot{m}_{\text{cool}}$ with a fixed temperature T_{in}.

We take a single fuel temperature and a single coolant temperature—equal to the average of the coolant inlet and outlet temperatures—as lumped parameters for representing the thermal state of the core. Treating the reactivity feedback effects as separable with constant coefficients, the core kinetics equations with six

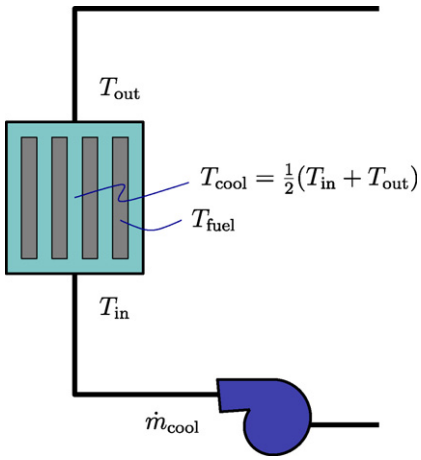

Fig. 5. Lumped parameter thermal model of a research reactor. Both the core inlet mass flow $\dot{m}_{\text{cool}}$ and temperature T_{in} are fixed.

groups of delayed neutron precursors may be written as

$$\dot{P} = \frac{\rho_{\text{ext}} + \alpha_{T_{\text{fuel}}}(T_{\text{fuel}} - T^\star_{\text{fuel}}) + \alpha_{T_{\text{cool}}}(T_{\text{cool}} - T^\star_{\text{cool}}) - \beta}{\Lambda} P + \sum_{i=1}^{6} \lambda_i c_i \tag{18}$$

$$\dot{c}_j = \frac{\beta_j}{\Lambda} P - \lambda_j c_j \qquad \text{for } j = 1 \ldots 6$$

where ρ_{ext} represents the external reactivity perturbations and the coolant temperature coefficient takes into account the change in reactivity due to both coolant temperature and density variations.

Using constant parameters for the lumped parameters thermal model, the coolant and fuel temperatures time derivatives may be written as

$$\dot{T}_{\text{fuel}} = \frac{1}{(mc)_{\text{fuel}}}\left[P - (hA)_{\text{core}} \cdot (T_{\text{fuel}} - T_{\text{cool}})\right] \tag{19}$$

$$\dot{T}_{\text{cool}} = \frac{1}{(mc)_{\text{cool}}}\left[(hA)_{\text{core}}(T_{\text{fuel}} - T_{\text{cool}}) - \dot{m}_{\text{cool}} \cdot c_{\text{cool}} \cdot 2(T_{\text{cool}} - T_{\text{in}})\right] \tag{20}$$

where $(mc)_{\text{fuel}}$ and $(mc)_{\text{cool}}$ are the total heat capacity of the fuel and the coolant inside the core respectively, c_{cool} is the coolant specific heat capacity and $(hA)_{\text{core}}$ is the product of the heat transfer coefficient and the exchange area between the fuel and the coolant. Eqs. (18)–(20) constitute a ninth order dynamical system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$$

with

$$\mathbf{x} = \begin{bmatrix} P \\ c_1 \\ \vdots \\ c_6 \\ T_{\text{cool}} \\ T_{\text{fuel}} \end{bmatrix}$$

For a fixed thermal power $P^\star$, the other components of the design point $\mathbf{x}^\star$ are

$$\begin{aligned} c^\star_j &= \frac{\beta_j}{\Lambda\lambda_j} P^\star \\ T^\star_{\text{cool}} &= \frac{P^\star + 2 \cdot \dot{m}_{\text{cool}} \cdot c_{\text{cool}} \cdot T_{\text{in}}}{2 \cdot \dot{m}_{\text{cool}} \cdot c_{\text{cool}}} \\ T^\star_{\text{fuel}} &= \frac{P^\star + (hA)_{\text{core}} \cdot T^\star_{\text{cool}}}{(hA)_{\text{core}}} \end{aligned} \tag{21}$$

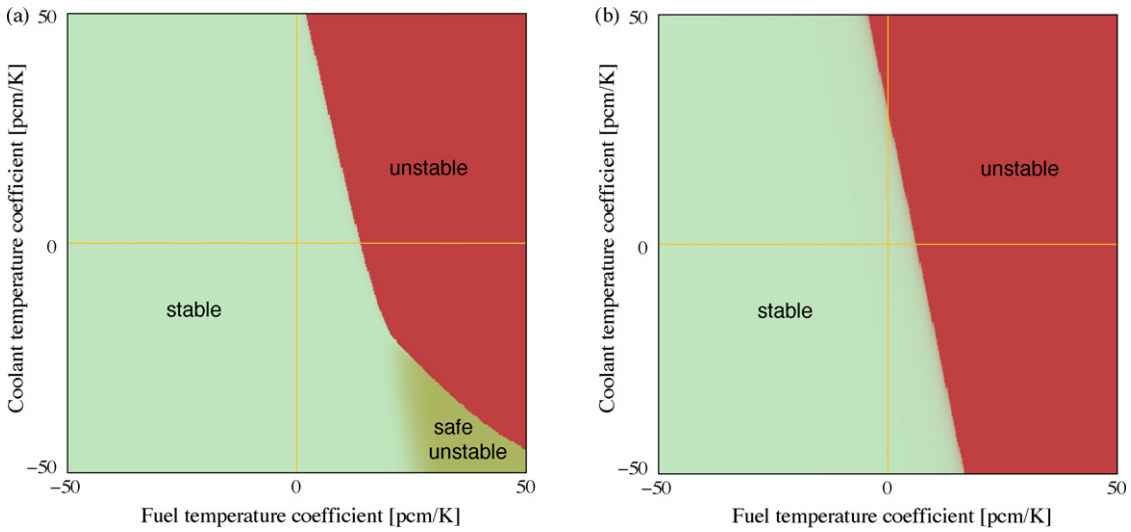

Fig. 6. Stability maps in the $\alpha_{T_{\text{cool}}}$ vs. $\alpha_{T_{\text{fuel}}}$ parameter plane. The perturbation in the Lyapunov study is a perturbation of 2 ◦ C in the inlet temperature of the coolant. The perturbation in the BIBO study is a positive 10 pcm external reactivity insertion.

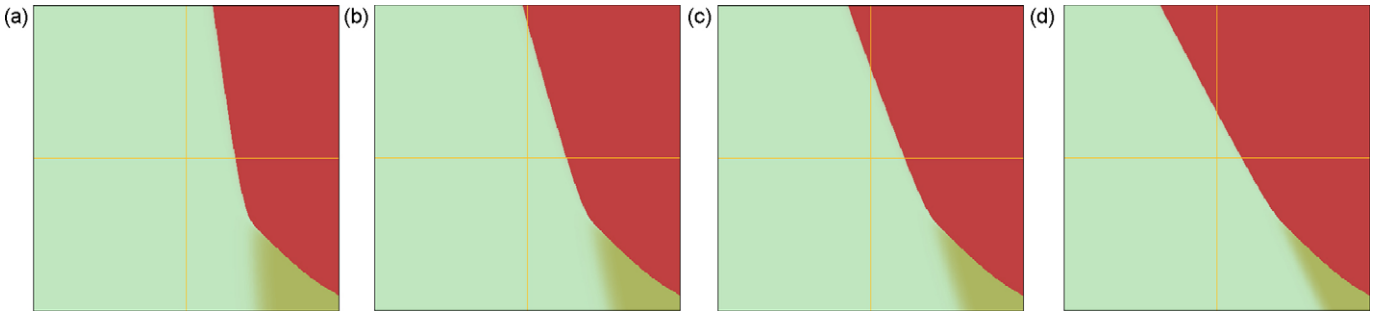

Fig. 7. Sensitivity analysis of the Lyapunov stability map with respect to coolant mass flow. (a) $2.00\dot{m}^{\star}_{\text{cool}}$, (b) $0.75\dot{m}^{\star}_{\text{cool}}$, (c) $0.50\dot{m}^{\star}_{\text{cool}}$, (d) $0.25.\dot{m}^{\star}_{\text{cool}}$.

4.2. Stability maps

When all the reactivity feedback coefficients are negative, it is clear that the reactor is stable. But when there is at least one positive coefficient, a question arises about the overall stability. Given the fact that the model shown in Fig. 5 has only two feedback coefficients, a two-dimensional stability map in the space $\alpha_{T_{\text{fuel}}} - \alpha_{T_{\text{cool}}}$ may be constructed as follows. We numerically integrate the point kinetics equations starting from a slightly perturbed equilibrium initial condition—either a perturbation in phase space or in the external reactivity—for a definite amount of time. We then repeat this process for different combinations of the two reactivity coefficients. A two-dimensional contour plot of the final thermal power as a function of both coefficients plays the role of a $\alpha_{T_{\text{fuel}}} - \alpha_{T_{\text{cool}}}$ stability map.

In Fig. 6, the resulting stability maps—in both Lyapunov and BIBO senses—for the study case are shown. In the first case, the initial perturbation corresponds to a step increase of 2 ◦ C of the nominal coolant inlet temperature at $t = 0$. In the BIBO case, the proposed perturbation is a positive external absolute reactivity insertion of 10^{-4} or 10 pcm.

The shape of both maps is essentially the same, showing that the fuel coefficient is the one that dominates the stability of the kinetics equations a as very positive coolant coefficient is needed in order to render the reactor unstable for negative fuel coefficients. The Lyapunov stability map shows a zone labelled “safe-unstable” where for very negative coolant coefficients, the thermal power goes to zero with the proposed perturbation. This behaviour may be defined unstable with respect to the Lyapunov definition, but nevertheless the power deviates to a safe value.

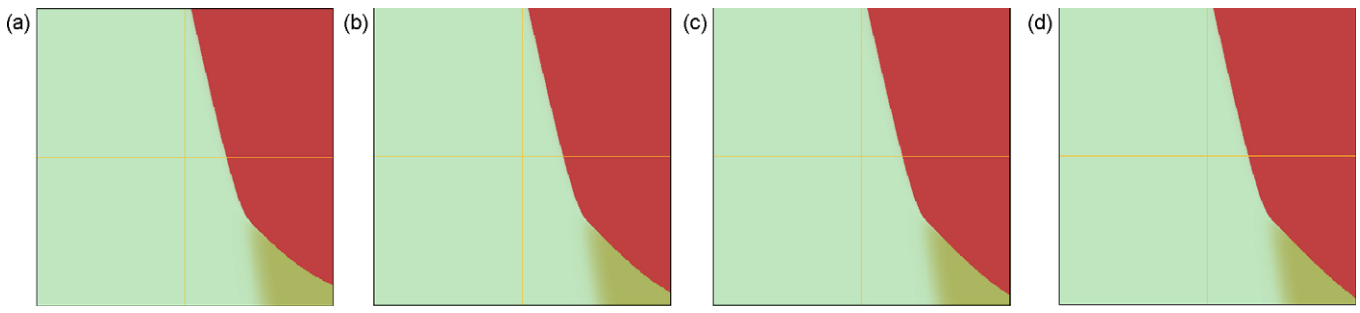

Fig. 8. Sensitivity analysis of the Lyapunov stability map with respect to fuel heat capacity. (a) $2.00\dot{m}^{\star}_{\text{fuel}}$, (b) $0.75\dot{m}^{\star}_{\text{fuel}}$, (c) $0.50\dot{m}^{\star}_{\text{fuel}}$, (d) $0.25\dot{m}^{\star}_{\text{fuel}}$.

The fact that the stability border shown does not pass through the origin has to do with the finite time adopted for the numerical integration. Had we provided a longer end time, the unstable zone would have moved to the left tangentially approaching the origin in both cases. As we already proved, there is a difference between Lyapunov stability and BIBO stability when both coefficients are zero. This also explains why the instability zone in Fig. 6(a) is slightly larger than the one in Fig. 6(b).

4.3. Sensitivity analysis

The proposed method of constructing these conceptual stability maps may be used to evaluate the impact of modifying a certain design parameter. For example, Fig. 7 shows how the stability limit changes as the coolant mass flow is reduced to different fractions of the design value $\dot{m}^{\star}_{\text{cool}}$ and Fig. 8 the influence of the fuel total calorific capacity $(mc)_{\text{fuel}}$. The coolant mass flow modifies the stability border and for high mass rates, the fuel temperature reactivity coefficient dominates. On the other hand, the fuel calorific heat capacity does not appreciably affect the stability zone in the proposed model.

These results are applicable only to the lumped parameter model and constant feedback coefficients proposed in Section 4.1, but even with the simple analysis shown, the designer may reach some interesting conclusions. First, that the stability topography does not depend heavily on the fuel calorific capacity. On the other hand, the coolant flow rate does change the stability border, but apparently not the safe-unstable zone. For high cooling flows, the coolant temperature is almost constant and the effect on the neutron kinetics is negligible. Therefore, if for some reason a core design has a positive coolant temperature reactivity coefficient, the reactor may be rendered stable by having a good cooling system thus driving the stability essentially with the usually negative fuel temperature coefficient.

5. Conclusions

The point reactor kinetics equations are a useful simplification of the time-dependant neutronic Boltzmann problem that allow studying the stability of a reactor core using dynamical systems theory. From a strictly mathematical point of view, we proved that the point reactor equations—in the absence of an external neutron source and without any feedback effects—are not BIBO-stable but are, nevertheless, Lyapunov-stable. Moreover, we were able to show that a simple proportional controller acting on the external reactivity makes the equations BIBO-stable. However, modifying directly the reactivity may not be possible and thus delays and other non-linearities may arise, changing the base problem. The result of unconditional linear stability applies only to the situation depicted in Fig. 1 as a pure theoretical analysis. It should be remarked again that all the studies and conclusions apply directly only to the mathematical models presented. The maps shown in Figs. 6–8 are just the results of the numerical integration of Eqs. (18), (20) and (19). The applicability of the methods and results to a real nuclear reactor is not straightforward. However, the basic physics of the feedback effects between the thermalhydraulics and the neutronics are retained.

This simple method—namely solving a dynamical system for different parameters and analysing the results in the time domain—may be extended to handle more complex cases. On the one hand, existing codes may be used to solve the governing equations thus avoiding a deep mathematical development, at least in early stages of the conceptual engineering. On the other hand, as each system to be solved is independent from the others, the process can be easily parallelised in modern multi-core computers and preliminary results may be obtained quickly. This kind of non-linear stability analysis is a powerful aid for nuclear reactor engineering, and the basic physics underlying the neutronic core design can be better understood by means of these tools.

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