Pairs of projections: geodesics, Fredholm and compact pairs

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Abstract

A pair (P,Q) of orthogonal projections in a Hilbert space $\mathcal{H}$ is called a Fredholm pair if

$$QP : R(P) \to R(Q)$$

is a Fredholm operator. Let $\mathcal{F}$ be the set of all Fredholm pairs. A pair is called compact if $P-Q$ is compact. Let $\mathcal{C}$ be the set of all compact pairs. Clearly $\mathcal{C} \subset \mathcal{F}$ properly. In this paper it is shown that both sets are differentiable manifolds, whose connected components are parametrized by the Fredholm index. In the process, pairs P,Q that can be joined by a geodesic (or equivalently, a minimal geodesic) of the Grassmannian of $\mathcal{H}$ are characterized: this happens if and only if

$$\dim(R(P)\cap N(Q)) = \dim(R(Q)\cap N(P)).$$

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1 introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators in $\mathcal{H}$. Let $\mathcal{B}_s(\mathcal{H})$ be the space of selfadjoint operators, and $\mathcal{P}$ the set of orthogonal projections.

There are several remarkable precendents on the subject of pairs of projections: two small chapters in the book by T. Kato [13], the papers by J. Dixmier [11], P.R. Halmos [12], C. Davis [9]. More recently, the subject has been treated by [7], [5], [1] and [6]. These works deal with several related problems: when are two projections unitarilly equivalent, when is an operator the difference of two projections, when the difference of two projections is an invertible operator, etc.

On the other hand the space of projections, or Grassmannian of $\mathcal{H}$, has been subject to a differential geometric study [14], [8]. As a differentiable manifold with a natural connection and a non regular Finsler metric, it is well behaved: for instance, short curves (of the metric) and geodesics (of the connection) have been characterized.

This paper is an attempt to study geometric aspects of pairs of projections. We start by giving a result characterizing projections which can be joined by a short curve (or equivalently, by a geodesic). Next we consider space of Fredholm and compact pairs of projections. Fredholm pairs of projections were defined in [5] (see also [1]): (P,Q) is a Fredholm pair if

$$QP|_{R(Q)} : R(Q) \to R(P)$$

is a Fredholm operator (here $R(P), R(Q)$ denote the ranges of P and Q). The index $i(P,Q)$ is defined as the index of this operator.

We say that (P,Q) is compact if $P-Q$ is compact. A simple argument shows that compact pairs are particular cases of Fredholm pairs, and the inclusion is strict. We denote by

$$\mathcal{F} = \{ \text{ Fredholm pairs } \} \quad \text{and} \quad \mathcal{C} = \{ \text{ compact pairs } \}$$

We show that both sets are C^∞ submanifolds of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$. The space $\mathcal{F}$ is in fact an open subset of $\mathcal{P} \times \mathcal{P}$, and thus a complemented submanifold. The space $\mathcal{C}$ is a non complemented submanifold.

The connected components are characterized. Note that two pairs in either class, that can be joined by a continuous path inside the class, lie in the same connected component of $\mathcal{P} \times \mathcal{P}$. We show that this happens if and only if they have the same index. To establish this result, the geodesics of $\mathcal{P}$ play an important role.

In [8] it was shown that $\mathcal{P}$ is a C$^\infty$ the Banach-Lie manifold, a submanifold of $\mathcal{B}_s(\mathcal{H})$, and a homogeneous space of the unitary group $\mathcal{U}(\mathcal{H})$, by means of the action

$$U \cdot P = UPU^*,$$

for $U \in \mathcal{U}(\mathcal{H}), P \in \mathcal{P}$. If $T \in \mathcal{B}(\mathcal{H})$, $N(T)$ and $R(T)$ denote the kernel and the range of T, respectively. It is well known that the connected components of $\mathcal{P}$ are parametrized by the dimensions of these subspaces. Thus an obvious characterization of the connected components of $\mathcal{F}$ and $\mathcal{C}$ can be obtained, in terms of the index and the nullities and ranks of the elements of the pair.

The contents of the paper are the following. Section 2 contains preliminary results, mainly by the above cited authors. These results are organized in three remarks. Section 3 contains the characterization of pairs P, Q that can be joined by a geodesic, minimal or not. In Section 4 we examine the local structure of $\mathcal{F}$. Section 5 contains the characterization of the connected components of $\mathcal{F}$ ($n \geq 0$). Section 6 contains the study of $\mathcal{C}$, whose connected components are analogously chracterized.

2 Preliminary results

In this section we collect basic facts on $\mathcal{P}$ [14], [8], on pairs of projections [12], and on Fredholm pairs [5] (see also [1]).

Remark 2.1. [14], [8]

1. $\mathcal{P}$ is a complemented submanifold of $\mathcal{B}(\mathcal{H})$. Its tangent space $(T\mathcal{P})_P$ at P is given by

$$(T\mathcal{P})_P = \{Y = iXP - iPX : X \in \mathcal{B}_s(\mathcal{H})\},$$

which consists of selfadjoint operators Y which are co-diagonal with respect to P (i.e. $PYP = (1-P)Y(1-P) = 0$). A natural projection $E_p : \mathcal{B}(\mathcal{H})_h \to (T\mathcal{P})_P$ is given by

$$E_P(X) = \text{co-diagonal part of } X = PX(1-P) + (1-P)XP.$$

This maps induce a linear connection in $\mathcal{P}$: if $X(t)$ is a tangent field along a curve $\gamma(t) \in \mathcal{P}$,

$$\frac{DX}{dt} = E_\gamma(\dot{X}).$$

2. For any $P \in \mathcal{P}$, the map
$$\pi_P : \mathcal{U}(\mathcal{H}) \to \mathcal{P}, \quad \pi_P(U) = UPU^*,$$
whose range is the connected component of P in $\mathcal{P}$, is a C^∞ submersion. In particular, it has C^∞ local cross sections.

3. If $P_0, P_1 \in \mathcal{P}$ with $\|P_0 - P_1\| < 1$, there exists a unique selfadjoint operator Z, with $\|Z\| < \pi/2$, which is co-diagonal with respect to P_0, such that
$$P_1 = e^{iZ} P_0 e^{-iZ}.$$
The curve $\delta(t) = e^{itZ} P_0 e^{-itZ}$ is the unique geodesic of $\mathcal{P}$ joining P_0 and P_1 (up to reparametrization).

4. If one defines a Finsler metric in $\mathcal{P}$, by endowing each tangent space with the usual norm in $\mathcal{B}(\mathcal{H})$, then the above geodesic has minimal length, among all rectifiable curves in $\mathcal{P}$ joining the same endpoints. We point that this Finsler metric is non smooth, much less regular.

The results in [14] were presented for symmetries rather than projections. A symmetry is a selfadjoint unitary operator $\epsilon \in \mathcal{B}(\mathcal{H})$: $\epsilon = \epsilon^*$, $\epsilon^2 = 1$. The mapping
$$P \leftrightarrow \epsilon_P = 2P - 1$$
establishes the equivalence of this alternate formulation. Morever, a geodesic in $\mathcal{P}$ has a simpler form in terms of symmetries: Z is co-diagonal with respect to P if and only if $Z\epsilon_p = -\epsilon_p Z$. Thus the geodesic $\delta(t)$ transforms to
$$\epsilon_{\delta(t)} = e^{itZ} \epsilon_0 e^{-itZ} = e^{2itZ} \epsilon_0 = \epsilon_0 e^{-2itZ}.$$
Note that ϵ_δ is a curve of unitaries. It is a folklore result that curves of unitaries which are minimal for the Finsler metric given by the operator norm are of the form $\mu(t) = Ue^{itX}$, for $X^* = X$, and remain minimal as long as $|t| \leq \pi/\|Z\|$ (see for instance [4], [3]). This implies that ϵ_δ is not only minimal among symmetries (i.e. projections), but also among unitaries, for $|t| \leq 1$ if $\|Z\| \leq \pi/2$.

Given two projections P, Q, consider the following subspaces
$$\mathcal{H}_{11} = R(P) \cap R(Q),\ \mathcal{H}_{10} = R(P) \cap N(Q),\ \mathcal{H}_{01} = N(P) \cap R(Q),\ \mathcal{H}_{00} = N(P) \cap N(Q)$$
and $\mathcal{H}_0$ the orthogonal of all the former. Apparently, $\mathcal{H}_{ii}$ reduce P and Q: both projections act as zero or the identity in these subspaces, and thus are unitarilly equivalent there. Also it is apparent that
$$P(\mathcal{H}_{10}) \subset \mathcal{H}_{10},\ P(\mathcal{H}_{01}) = \{0\},\ Q(\mathcal{H}_{01}) \subset \mathcal{H}_{01},\ P(\mathcal{H}_{10}) = \{0\},$$
and thus $\mathcal{H}_{10} \oplus \mathcal{H}_{01}$ is invariant for P and Q. Therefore $\mathcal{H}_0$ is also invariant fot P and Q (it is called the part of $\mathcal{H}$ where P and Q are in generic position [12]). The following results are well known.

Remark 2.2. [12]

1. The reduced projections $P_0 = P|_{\mathcal{H}_0}$ and $Q_0 = Q|_{\mathcal{H}_0}$ are unitarily equivalent in $\mathcal{H}_0$. Indeed, representing both operators as 2×2 matrices in terms of P_0, they are

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q_0 = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where C and S are commuting positive contractions, such that $N(C) = N(S) = \{0\}$ and $C^2 + S^2 = 1$. Note that

$$U_0 = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}$$

is a unitary operator satisfying $U_0 P_0 U_0^* = Q_0$.

2. Since P_0 and Q_0 are in generic position, the space $\mathcal{H}_0$ is isomorphic to a product $\mathcal{K} \times \mathcal{K}$, where $R(P_0)$ is mapped to $\mathcal{K} \times 0$ and $N(P_0)$ is mapped to $0 \times \mathcal{K}$. The above properties of C and S, and an elementary use of the spectral theorem implies that there exists a selfadjoint operator $X \in \mathcal{B}(\mathcal{K})$ with $\sigma(X) \subset [0, \pi/2]$, such that (via the unitary isomorphism)

$$C = \cos(X) \text{ and } S = \sin(X).$$

Thus picking the (self-adjoint) matrix operator

$$\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix},$$

acting in $\mathcal{K} \times \mathcal{K}$, and pulling it back with the unitary isomorphism, one obtains a selfadjoint operator $Z_0 \in \mathcal{B}(\mathcal{H}_0)$, with $\|Z\| \le \pi/2$ such that

$$e^{iZ_0} = U_0.$$

3. Note that because of the form of the above unitary isomorphism, the operator Z_0 above is co-diagonal with respect to P_0. Therefore the curve

$$\delta_0(t) = e^{itZ_0} P_0 e^{-itZ_0}$$

is a geodesic of $\mathcal{P}(\mathcal{H}_0)$ with $\|Z_0\| \le \pi/2$, it is minimal (though not necessarily unique) joining P_0 and Q_0.

4. It follows that if $\dim(\mathcal{H}_{10}) = \dim(\mathcal{H}_{01})$, then P and Q are unitarilly equivalent. Indeed, if this is the case, let $V : \mathcal{H}_{10} \to \mathcal{H}_{01}$ be an isometric isomorphism, and consider $U' = V + V^*$. Then U' is a unitary operator in $\mathcal{H}_{10} \oplus \mathcal{H}_{01}$. It is the exponential of a co-diagonal operator in this decomposition. Indeed, pick

$$Z' = i\frac{\pi}{2}(V - V^*).$$

This operator is selfadjoint, verifies $\|Z'\| = \pi/2$, is co-diagonal with respect to the decomposition $\mathcal{H}_{10} \oplus \mathcal{H}_{01}$ (which is the decomposition $R(P) \oplus N(P)$ reduced to this part of $\mathcal{H}$), and verifies $e^{iZ'} = U'$.

Note that the condition $\dim(\mathcal{H}_{10}) = \dim(\mathcal{H}_{01})$ is not necessary for P and Q to be unitarilly equivalent. Indeed, one can find for instance projections $P \geq Q$, both with infinite rank and corank (and therefore unitarilly equivalent) such that $\dim(\mathcal{H}_{10}) \neq \dim(\mathcal{H}_{01})$.

Finally, we collect the basic facts on the index of pairs in [5].

Remark 2.3. [5]

Let (P, Q) be a Fredholm pair.

1. P and Q are unitarilly equivalent if and only if $i(P, Q) = 0$. This fact is Thereom 3.3 in [5]. It can also be obtained with the above idea. Namely, the operator

$$QP|_{R(P)} : R(P) \to R(Q)$$

is Fredholm, apparently its kernel and co-rank are, respectively, $N(Q) \cap R(P) = \mathcal{H}_{10}$ and $N(P) \cap R(Q) = \mathcal{H}_{01}$. In particular, $\mathcal{H}_{01}$ and $\mathcal{H}_{10}$ are finite dimensional. Also $i(P, Q) = 0$ means precisely $\dim(\mathcal{H}_{01}) = \dim(\mathcal{H}_{10})$.

2. (Q, P) is also a Fredholm pair, with

$$i(Q, P) = -i(P, Q).$$

3. If U is a unitary operator, (UPU^*, UQU^*) is also a Fredholm pair, and

$$i(UPU^*, UQU^*) = i(P, Q).$$

4. If (Q, R) is another Fredhlom pair, and either $Q - R$ or $P - Q$ is compact, then (P, R) is a Fredholm pair and

$$i(P, R) = i(P, Q) + i(Q, R).$$

3 Geodesics between projections

Using the notations of the previous section, we have the following result:

Theorem 3.1. *Let P and Q be orthogonal projections in $\mathcal{H}$.*

The following are equivalent:

1. *There exists a geodesic $\delta(t)$ in $\mathcal{P}$, which joins P and Q, and has minimal length among all rectifiable curves in $\mathcal{P}$ joining the same endpoints,*

2. *There exists a geodesic $\delta(t)$ in $\mathcal{P}$, which joins P and Q.*

3. $\dim(\mathcal{H}_{10}) = \dim(\mathcal{H}_{01})$.

Proof. 1) $\Rightarrow$ 2) is apparent. 3)$\Rightarrow$ 1) is the construction outlined in the previous section. Namely, in the decomposition

$$\mathcal{H} = \mathcal{H}_{11} \oplus \mathcal{H}_{00} \oplus (\mathcal{H}_{10} \oplus \mathcal{H}_{01}) \oplus \mathcal{H}_0,$$

with the notations of Remark 2.2 , consider the operator

$$Z = 0 \oplus 0 \oplus Z' \oplus Z_0.$$

Clearly it is a P-codiagonal selfadjoint operator, with $\|Z\| \leq \pi/2$, and it verifies $e^{iZ}Pe^{-iZ} = Q$. Moreover, $\delta(t) = e^{iZ}Pe^{-iZ}$ is a minimal geodesic joining P and Q, by the observation following Remark 2.1.

2) $\Rightarrow$ 3): Suppose that there exists a P-co-diagonal selfadjoint operator Z such that

$$e^{iZ}Pe^{-iZ} = Q. \tag{1}$$

Pick $\xi \in \mathcal{H}_{10}$, i.e. $P\xi = \xi$ and $Q\xi = 0$. Apparently, (1) implies that $e^{iZ}(R(P)) = R(Q)$, so that $e^{iZ}\xi \in R(Q)$. Let us prove that $e^{iZ}\xi \in N(P)$, which would mean that $e^{iZ}(\mathcal{H}_{10}) \subset \mathcal{H}_{01}$. Indeed, note that (1) also implies that $Pe^{-iZ} = e^{-iZ}Q$, and since Z anti-commutes with $2P-1$, one has

$$(2P-1)e^{iZ}\xi = e^{-iZ}(2P-1)\xi = e^{-iZ}\xi,$$

and thus

$$Pe^{iZ}\xi = P(2P-1)e^{iZ}\xi = Pe^{-iZ}\xi = e^{-iZ}Q\xi = 0.$$

Reasoning with $P^\perp = 1-P$ and $Q^\perp = 1-Q$ in the place of P and Q, using that 1 holds for $P^\perp$ and $Q^\perp$, and that Z is $P^\perp$-co-diagonal, one obtains also that

$$e^{iZ}(\mathcal{H}_{01}) \subset \mathcal{H}_{10}.$$

Apparently, (1) also implies that $e^{iZ}(\mathcal{H}_{ii}) = \mathcal{H}_{ii}$ $(i = 0, 1)$. It follows that $e^{iZ}(\mathcal{H}_0) \subset \mathcal{H}_0$. Since these subspaces decompose $\mathcal{H}$ and e^{iZ} is a unitary operator, equality holds in the above inclusions. In particular

$$e^{iZ}(\mathcal{H}_{10}) = \mathcal{H}_{01},$$

and thus $\dim(\mathcal{H}_{10}) = \dim(\mathcal{H}_{01})$. $\square$

Remark 3.2. With the above notations, suppose that there exists a geodesic $\delta(t) = e^{itZ}Pe^{-itZ}$ joining P and Q, then

$$\mathcal{H}_{00} \oplus \mathcal{H}_{11} = \mathcal{S}_{+1} := \{\xi \in \mathcal{H} : e^{2iZ}\xi = \xi\}$$

and

$$\mathcal{H}_{01} \oplus \mathcal{H}_{10} = \mathcal{S}_{-1} := \{\xi \in \mathcal{H} : e^{2iZ}\xi = -\xi\}.$$

Let us prove the second assertion, the first is similar. If $\xi \in \mathcal{H}_{01} = N(P) \cap R(Q)$, then $(2P-1)\xi = -\xi$ and $(2Q-1)\xi = \xi$. Thus

$$e^{-2iZ}\xi = e^{-2iZ}(2Q-1)\xi = (2P-1)\xi = -\xi.$$

Similarly for $\xi \in \mathcal{H}_{10}$. Thus $\mathcal{H}_{01} \oplus \mathcal{H}_{10} \subset \mathcal{S}_{-1}$. Conversely, suppose $\xi \in \mathcal{S}_{-1}$. An analogous argument as above shows that $\mathcal{H}_{00} \oplus \mathcal{H}_{11} \subset \mathcal{S}_{+1}$, and then $\xi \perp \mathcal{H}_{01} \oplus \mathcal{H}_{10}$. Let $\xi' = P_{\mathcal{H}_{01}\oplus\mathcal{H}_{10}}\xi$. Then $\xi_0 = \xi - \xi_0$ lies in $\mathcal{S}_{-1}$, and is orthogonal to $\mathcal{H}_{01} \oplus \mathcal{H}_{10}$ and to $\mathcal{H}_{00} \oplus \mathcal{H}_{11}$. Thus $\xi_0 \in \mathcal{H}_0$. It suffices to show that $\xi_0 = 0$.

Note that e^{2iZ} acts in $\mathcal{H}_0$. Thus we are reduced to the case when $\xi_0 \in \mathcal{H}_0$, e^{2iZ} acts in $\mathcal{H}_0$, and $e^{2iZ}\xi_0 = -\xi_0$. From the results by Halmos [12] cited collected in Remark 2.2, changing variables with a suitable unitary transformation, we may suppose $\mathcal{H}_0 = \mathcal{K} \times \mathcal{K}$, P_0, Q_0 and

$$U_0 = \begin{pmatrix} C & -S \\ S & C \end{pmatrix} = \exp\begin{pmatrix} 0 & iX \\ -iX & 0 \end{pmatrix}$$

as in item 1. of 2.2. Then

$$e^{2iZ}|_{\mathcal{H}_0}(2P_0-1)=2Q_0-1=U_0(2P_0-1)U_0^*=U_0^2(2P_0-1),$$

and thus

$$e^{2iZ}|_{\mathcal{H}_0}=U_0^2=\begin{pmatrix} C^2-S^2 & -2CS \\ 2CS & C^2-S^2 \end{pmatrix}.$$

Recall that C and S are positive commuting contractions and verify $C^2+S^2=1_{\mathcal{K}}$.

Next note that $\xi_0':=P_0\xi_0$ and $\xi_0'':=(1-P_0)\xi_0$ also belong to $\mathcal{S}_{-1}$. Indeed, since also $e^{-2iZ}\xi_0=-\xi_0$,

$$e^{2iZ}(2P_0-1)\xi_0=(2P_0-1)e^{-2iZ}\xi_0=-(2P_0-1)\xi_0,$$

i.e. $2P_0-1$ leaves $\mathcal{S}_{-1}$ invariant, or equivalently P_0 commutes with $P_{\mathcal{S}_{-1}}$.

Summing up these facts

$$-\begin{pmatrix} \xi_0' \\ 0 \end{pmatrix}=e^{2iZ}P_0\xi_0=\begin{pmatrix} C^2-S^2 & -2CS \\ 2CS & C^2-S^2 \end{pmatrix}\begin{pmatrix} \xi_0' \\ 0 \end{pmatrix}=\begin{pmatrix} C^2\xi_0'-S^2\xi_0' \\ 2CS\xi_0' \end{pmatrix}.$$

In particular $C^2\xi_0'-S^2\xi_0'=-\xi_0'$. Combining this with $C^2+S^2=I_{\mathcal{K}}$, it follows that $C^2\xi_0'=0$, and thus $C\xi_0'=0$ because C is selfadjoint. Thus $CS\xi_0'=SC\xi_0'=0$. It follows that

$$Q_0\xi_0'=\begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}\begin{pmatrix} \xi_0' \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

That is, $\xi_0'\in R(P)\cap N(Q)$ which is orthogonal to $\mathcal{H}_0$, and thus $\xi_0'=0$. A similar argument shows that $\xi_0''=0$. It follows that $\xi_0=0$, and therefore $\mathcal{H}_{01}\oplus\mathcal{H}_{10}=\mathcal{S}_{-1}$.

Summarizing, the unitary operator e^{iZ} maps $\mathcal{H}_{10}$ onto $\mathcal{H}_{01}$, $\mathcal{H}_{01}$ onto $\mathcal{H}_{10}$, and its square e^{2iZ} acts as the identity in $\mathcal{H}_{00}\oplus\mathcal{H}_{11}=\mathcal{S}_{+1}$, as minus the identity in $\mathcal{H}_{01}\oplus\mathcal{H}_{10}=\mathcal{S}_{-1}$, and equals

$$U_0^2=\begin{pmatrix} C^2-S^2 & -2CS \\ 2CS & C^2-S^2 \end{pmatrix}$$

in $\mathcal{H}_0$.

4 Local structure of $\mathcal{F}$

An elementary argument shows that $\mathcal{F}$ is open in $\mathcal{P}\times\mathcal{P}$.

Proposition 4.1. *The set $\mathcal{F}$ is open in $\mathcal{P}\times\mathcal{P}$*

Proof. Let (P_0,Q_0) be a Fredholm pair. Suppose that $(P,Q)\in\mathcal{P}\times\mathcal{P}$ is such that

$$\|P-P_0\|<1,\quad \|Q-Q_0\|<1.$$

There exists unitary cross sections for the action of the unitary group on $\mathcal{P}$, i.e. there exist unitary operators U_P and V_Q, which are continuous functions in P and Q, respectively, which can be chosen so that $U_{P_0}=1$ and $V_{Q_0}=1$, such that

$$U_PP_0U_P^*=P,\quad V_QP_0V_Q^*=Q.$$

Note that $U_P(R(P_0)) = R(P)$ and $V_Q(R(Q_0)) = R(Q)$. Thus

$$QP = V_Q Q_0 V_Q^* U_P P_0 U_P^* : R(P) \to R(Q)$$

is Fredholm if and only if

$$Q_0 V_Q^* U_P P_0 : R(P_0) \to R(Q_0)$$

is Fredholm. If (P, Q) is sufficiently close to (P_0, Q_0), then both unitaries U_P, V_Q are close to 1, and therefore $Q_0 V_Q^* U_Q P_0$ is close to $Q_0 P_0$ as operators in $\mathcal{B}(R(P_0), R(Q_0))$. The conclusion follows, recalling that the set of Fredholm operators between two fixed spaces is open. □

In particular, this implies that $\mathcal{F}$ is a submanifold of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$. The same argument shows that pairs that are close enough have the same index. It follows that also the sets

$$\mathcal{F}_n = \{(P, Q) \in \mathcal{F} : i(P, Q) = n\}$$

are open in $\mathcal{P} \times \mathcal{P}$ (and submanifolds of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$).

If $\|P - Q\| < 1$, then the pair $(P, Q) \in \mathcal{F}_0$ [5]. The following result shows that in this case the Fredolm pairs (P, P) and (P, Q) can be joined in $\mathcal{F}_0$, with a curve that is a minimal geodesic in the product metric of $\mathcal{P} \times \mathcal{P}$

Theorem 4.2. *Let $P, Q \in \mathcal{P}$ with $\|P - Q\| < 1$. Let $\delta(t) = e^{itX} P e^{-itX}$ be the minimal geodesic of $\mathcal{P}$ such that $\delta(1) = Q$. Then the curve $(P, \delta(t))$, $t \in [0, 1]$ joins (P, P) and (P, Q) inside $\mathcal{F}_0$, with minimal length (for the Finsler metric given by $\|(A, B)\| = (\|A\|^2 + \|B\|^2)^{1/2}$ at every tangent space of $\mathcal{P} \times \mathcal{P}$).*

Proof. It is apparent that $(P, \delta(t))$ joins (P, P) and (P, Q). Let us prove that $(P, \delta(t))$ remains inside $\mathcal{F}$. We must check that for every $t \in [0, 1]$, the operator

$$\delta(t)P : R(P) \to R(\delta(t))$$

is Fredholm. Equivalently, that $Pe^{itX}P$ is Fredholm in $\mathcal{B}(R(P))$. It is known (see for instance [2]), that if $\|P - Q\| < 1$, and X as above, then

$$\|X\| = \arcsin(\|P - Q\|),$$

and that, written as a 2×2 matrix in terms of P,

$$e^{itX} = \begin{pmatrix} \cos(t(X_0X_0^*)^{1/2}) & itX_0 \operatorname{sinc}(t(X_0^*X_0)^{1/2}) \\ -itX_0^* \operatorname{sinc}(t(X_0X_0^*)^{1/2}) & \cos(t(X_0^*X_0)^{1/2}) \end{pmatrix},$$

where

$$X = \begin{pmatrix} 0 & X_0 \\ X_0^* & 0 \end{pmatrix},$$

and $\operatorname{sinc}(t) = \frac{\sin(t)}{t}$. Thus, since $\|X\| < \pi/2$, it follows that $\cos(t(X_0X_0^*)^{1/2})$ is invertible in $\mathcal{B}(R(P))$ for $t \in [0, 1]$. In particular, $Pe^{itX}P$ is a Fredholm operator of index zero in $\mathcal{B}(R(P))$.

Note that, since the first coordinate is fixed, the length of the curve (P, δ) equals the length of δ. On the other hand, if (γ_1, γ_2) is a curve in $\mathcal{F}$ joining (P, P) and (P, Q), apparently

$$length(\gamma_2) \le length(\gamma_1, \gamma_2).$$

Since δ is a minimal geodesic of $\mathcal{P}$, and γ_2 joins P and Q, $length(\delta) \le length(\gamma_2)$. The result follows. □

Remark 4.3. If $i(P,Q) = 0$, and δ is a minimal geodesic joining P and Q, then the curve $(P, \delta(t))$ remains inside $\mathcal{F}_0$ for all $t \in [0,1]$. This is apparent, for $t \in [0,1)$ it is proved in the above Theorem. For $t = 1$, $(P, \delta(1)) = (P,Q) \in \mathcal{F}_0$ by hypothesis.

We may adapt Theorem 3.1 to Fredholm pairs.

Corollary 4.4. *Let $(P,Q) \in \mathcal{F}$. Then the following are equivalent:*

1. *There exist a geodesic $\delta(t) = e^{itX}Pe^{-itX}$ with $\delta(1) = Q$, such that $(P, \delta(t)) \in \mathcal{F}$.*
2. *There exist a minimal geodesic $\delta(t) = e^{itX}Pe^{-itX}$ with $\delta(1) = Q$, such that $(P, \delta(t)) \in \mathcal{F}$.*
3. *$i(P,Q) = 0$*

Proof. Clearly $i(P,Q) = 0$ means $\dim(\mathcal{H}_{10}) = \dim(\mathcal{H}_{01})$. The assertion that the pairs $(P, \delta(t))$ remain inside $\mathcal{F}$ is clear if δ is minimal (i.e. $\|Z\| \leq \pi/2$) by the above remark. If δ is an arbitrary geodesic joining P and Q, Theorem 3.1 guarantees the existence of a minimal geodesic joining P and Q, and the proof follows. □

Remark 4.5. One can also prove that the existence of a geodesic implies the existence of a minimal geodesic by a direct argument, cutting down the spectrum of X in the following fashion. Let $\delta(t) = e^{itX}Pe^{-itX}$ such that $\delta(1) = Q$. Note that X is P-co-diagonal, i.e. X anti-commutes with $2P - 1$. Let us show that there exists Y selfadjoint, which is also P-co-diagonal, such that $e^{2iX} = e^{2iY}$ and $\|Y\| \leq \pi/2$. Then $\epsilon(t) = e^{itY}Pe^{itY}$ would be another geodesic, joining P and Q:

$$2Q - 1 = e^{iX}(2P - 1)e^{-iX} = e^{2iX}(2P - 1) = e^{2iY}(2P - 1) = e^{iY}(2P - 1)e^{-iY},$$

and with minimal length in the interval $[0,1]$, because $\|Y\| \leq \pi/2$. consider the bounded Borel function g given by

$$g(t) = \begin{cases} t \text{ if } t \in [-\pi, \pi] \\ t - 2k\pi \text{ if } t \in (-\pi + 2k\pi, \pi + 2k\pi] \text{ with } k > 0 \\ t + 2k\pi \text{ if } t \in [-\pi + 2k\pi, \pi + 2k\pi) \text{ with } k < -1 \end{cases}$$

Clearly $|g(t)| \leq \pi$, $g(-t) = -g(t)$ and $e^{it} = e^{ig(t)}$. Put $Y = \frac{1}{2}g(2X)$. Apparently $Y^* = Y$ and $\|Y\| \leq \pi/2$. Also it is clear that $e^{2iY} = e^{2iX}$. Finally let us show that Y anti-commutes with $2P - 1$. Since X is selfadjoint and anti-commutes with $2P - 1$, if $p(t)$ is a polynomial,

$$p(X)(2P - 1) = (2P - 1)p(-X),$$

i.e., if $\xi, \eta \in \mathcal{H}$, then (if we denote by $\mu_{\xi,\eta}$ the scalar spectral measure of X associated to the pair of vectors ξ, η)

$$\int_{\mathbb{R}} p(t) d\mu_{(2P-1)\xi,\eta}(t) =< p(X)(2P - 1)\xi, \eta >=< p(-X)\xi, (2P - 1)\eta >= \int_{\mathbb{R}} p(-t) d\mu_{\xi,(2P-1)\eta}(t).$$

Let $p_n(t)$ be polynomials which converge pointwise to $g(t)$. Then by the Lebesgue Theorem of bounded convergence,

$$\begin{aligned} < g(2X)(2P - 1)\xi, \eta >= \int_{\mathbb{R}} g(2t) d\mu_{(2P-1)\xi,\eta}(t) = -\int_{\mathbb{R}} g(-2t) d\mu_{\xi,(2P-1)\eta}(t) \\ = - < g(-2X)\xi, (2P - 1)\eta >, \end{aligned}$$

that is

$$g(2X)(2P - 1) = (2P - 1)g(-2X) = -(2P - 1)g(2X),$$

i.e. Y is P-co-diagonal.

5 Connected components of $\mathcal{F}$

We show that (P,Q) and (P',Q') lie in the same component of $\mathcal{F}$ if and only if they lie in the same component of $\mathcal{P}\times\mathcal{P}$ and have the same index.

Proposition 5.1. *Let (P,Q) be a Fredholm pair with $i(P,Q)=k$. If $k\le 0$, there exists a sub-projection $Q_0\le P$ such that the pairs (P,Q) and (P,Q_0) lie in the same connected component of $\mathcal{F}$. If $k>0$, then there exists a sub-projection $P_0\le Q$ such that (P,Q) and (P_0,Q) lie in the same connected component of $\mathcal{F}$. If $k=0$, the curve joining them can be chosen to be a minimal geodesic.*

Proof. Suppose $k\le 0$. Consider the subspaces $\mathcal{H}_{ij}$, $i,j=0,1$ and $\mathcal{H}_0$ defined in Remark 2.2. In the generic part $\mathcal{H}_0$, it has been remarked that both projections $P|_{\mathcal{H}_0}$ and $Q|_{\mathcal{H}_0}$ can be joined by a (non necesarilly unique) minimal geodesic $\delta_0(t)=e^{itX_0}P|_{\mathcal{H}_0}e^{-itX_0}$, $t\in[0,1]$, where X_0 is a P-co-diagonal selfadjoint operator acting in $\mathcal{H}_0$, with $\|X_0\|\le\pi/2$. In this part of $\mathcal{H}$, we pick $Q_0|_{\mathcal{H}_0}=P|_{\mathcal{H}_0}$. In $\mathcal{H}_{ii}$ both projections coincide. So we must consider

$$\mathcal{H}'=\mathcal{H}_{10}\oplus\mathcal{H}_{01}=[R(P)\cap N(Q)]\oplus[R(Q)\cap N(P)].$$

Note that $\dim(\mathcal{H}_{10})=\dim(\mathcal{H}_{01})+k$. Let $\mathcal{S}\subset\mathcal{H}_{10}$ such that $\dim(\mathcal{S})=\dim(\mathcal{H}_{01})$. Let $\epsilon:\mathcal{H}_{01}\to\mathcal{S}$ be an isometric isomorphism, and denote $\mathcal{S}'=\mathcal{H}_{10}\ominus\mathcal{S}$. In the decomposition

$$\mathcal{H}'=\mathcal{H}_{01}\oplus\mathcal{S}\oplus\mathcal{S}'$$

consider the operator

$$V(\xi,\eta_1,\eta_2)=(\epsilon^*\eta_1,\epsilon\xi,\eta_2).$$

Clearly V is a unitary operator in $\mathcal{H}'$, and a straightforward computation shows that

$$VQ|_{\mathcal{H}'}V^*(\xi,\eta_1,\eta_2)=(0,\eta_1,0).$$

Note that P acts in this decomposition as $P(\xi,\eta_1,\eta_2)=(0,\eta_1,\eta_2)$, and thus we pick

$$Q_0|_{\mathcal{H}'}=VQ|_{\mathcal{H}'}V^*\le P|_{\mathcal{H}'}.$$

By the second part of Remark 4.3, the curve $\delta_0(t)=e^{itX_0}P|_{\mathcal{H}_0}e^{-itX_0}$, joining the generic parts $P|_{\mathcal{H}_0}$ and $Q|_{\mathcal{H}_0}$, satisfies that $(P|_{\mathcal{H}_0},\delta(t))$ are Fredholm pairs (of index zero) in the Hilbert space $\mathcal{H}_0$. The unitary V of $\mathcal{H}'$ constructed above, is of the form $V=e^{iX'}$, for some $X'^*=X'$ acting in the finite dimensional space $\mathcal{H}'$. Put

$$\delta(t)=\delta_0\oplus 1_{\mathcal{H}_{00}}\oplus 1_{\mathcal{H}_{11}}\oplus e^{itX'}$$

acting in the decomposition $\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_{00}\oplus\mathcal{H}_{11}\oplus\mathcal{H}'$. Then the curve $(P,\delta(t))$ remains inside $\mathcal{F}$, because $\mathcal{H}'$ is finite dimensional. Since it is apparently continuous, the index is constant along it (this argument is developed in detail in the necessity part of the next result). Thus $(P,\delta(t))$ joins (P,Q) and (P,P_0) inside $\mathcal{F}_k$.

The case $k<0$ can be reduced to this case reversing the pair, since $i(Q,P)=-i(P,Q)$.

If $k=0$, P and Q are unitarilly equivalent, and the assertion is contained in Theorem 3.1. $\square$

Theorem 5.2. *Let (P,Q) and (P',Q') be two Fredholm pairs in the same connected component of $\mathcal{P}\times\mathcal{P}$. Then these pairs lie in the same connected component of $\mathcal{F}$ if and only if*

$$i(P,Q)=i(P',Q').$$

Proof. Suppose that (P,Q) and (P',Q') lie in the same component of $\mathcal{F}$. The assertion follows essentially from the local continuity of the index. Let $(P(t),Q(t))$, $t\in[0,1]$ be a continuous path in $\mathcal{F}$ joining (P,Q) and (P',Q'). Since the map $U\mapsto UPU^*$, from $\mathcal{U}(\mathcal{H})$ to the connected component of P in $\mathcal{P}$ is a submersion [8], there exist a continuous path $U(t)$ of unitary operators, $t\in[0,1]$, such that $U(t)PU(t)^*=P(t)$. Similarly for $Q(t)$: $Q(t)=V(t)QV(t)^*$, for a continuous path $V(t)$ of unitaries. By hypothesis,

$$Q(t)P(t)|_{R(P(t))}=V(t)QV(t)^*U(t)PU(t)^*|_{U(t)(R(P))}:U(t)(R(P))\to V(t)(R(Q))$$

are Fredholm operators. Thus

$$QV(t)^*U(t)P|_{R(P)}:R(P)\to R(Q)$$

is a continuous path of Fredholm operators in $\mathcal{B}(R(P),R(Q))$. It follows that the index is constant along this path. At $t=0$ one has $i(P,Q)$, at $t=1$, the index of the operator $QV(1)^*U(1)P$ equals the index of

$$V(1)QV(1)^*U(1)PU(1)^*=Q'P'$$

as operators in $\mathcal{B}(R(P'),R(Q'))$, i.e. $i(P,Q)=i(P',Q')$.

Conversely, suppose that $i(P,Q)=i(P',Q')=k$. Suppose, without loss of generality, that $k\geq 0$. By the above result, there exist sub-projections $P_0\leq P$ and $P_0'\leq P'$ such that the pair (P,Q) lies in the same component as (P,P_0), and (P',P_0') lies in the same component as (P',Q'). Note that the equality of the indexes, means that $\dim(R(P_0))=\dim(R(P_0'))$ and $\dim(R(P-P_0))=\dim(R(P'-P_0'))$. P and P' are unitarilly equivalent, $P'=UPU^*$. Thus it suffices to show that (P,P_0) and $(P,U^*P_0'U)=U(P',P_0')U^*$ lie in the same component of $\mathcal{F}$. Note that one can regard this question inside the space $R(P)$, and there it is equivalent to the question of wether P_0 and $U^*P_0'U$ are unitarilly equivalent (as projections in $R(P)$). This assertion is clearly true, by the equality of the dimensions and co-dimensions of P_0 and P_0'. □

6 Compact pairs

Inside $\mathcal{F}$ we may distinguish the following set of pairs of projections:

$$\mathcal{C}=\{(P,Q)\in\mathcal{P}\times\mathcal{P}:P-Q\in\mathcal{K}(\mathcal{H})\},$$

where $\mathcal{K}(\mathcal{H})$ denotes the space of compact operators. Note that indeed $\mathcal{C}\subset\mathcal{F}$. To prove this, recall that $A\in\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ is a Fredholm operator if and only if AA^* and A^*A are Fredholm operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. If $P-Q$ is compact, then

$$P-PQP=P(P-Q)P \text{ and } QPQ-Q=Q(P-Q)Q$$

are compact operators in $R(P)$ and $R(Q)$, respectively. Then $(QP)^*QP=PQP$ is a compact perturbation of the identity P in $R(P)$, thus it is a Fredholm operator in $R(P)$. Analogously, $QP(QP)^*$ is a Fredholm operator in $R(Q)$. Therefore QP is a Fredholm operator in $\mathcal{B}(R(P),R(Q))$, i.e. $(P,Q)\in\mathcal{F}$.

Let

$$\mathcal{U}_c(\mathcal{H}) = \{U \in \mathcal{U}(\mathcal{H}) : U - 1 \in \mathcal{K}(\mathcal{H})\}$$

be the unitary Fredholm group. It is one of the so called [10] classical Banach-Lie group. It is a differentiable manifold modelled in $\mathcal{K}(\mathcal{H})$. To assert the local structure of $\mathcal{C}$, the following known result will be useful (for instance, it shows how the exponential map provides the local charts for $\mathcal{U}_c(\mathcal{H})$). We include a proof.

Lemma 6.1. *The exponential map*

$$exp : \{X \in \mathcal{K}(\mathcal{H}) : X^* = X, \|X\| < \pi\} \to \{U \in \mathcal{U}_c(\mathcal{H}) : \|U - 1\| < 2\} \ , \ exp(X) = e^{iX}$$

is a real analytic diffeomorphism.

Proof. It is a straightforward consequence of the spectral Theorem that exp is a real analytic diffeomorphism from the open ball of radius π in $\mathcal{B}(\mathcal{H})$ onto the open ball of radius 2 centered at 1 in $\mathcal{U}(\mathcal{H})$. Aparently, if $X \in \mathcal{K}(\mathcal{H})$, then $exp(X) - 1 \in \mathcal{K}(\mathcal{H})$. Therefore it suffices to show that if $U \in \mathcal{U}_c(\mathcal{H})$ with $\|U - 1\| < 2$, then its unique $log(U)$ in the ball of radius π is in fact a compact operator. Since $\|U - 1\| < 2$, the spectrum of U is contained in the open set $\{z \in \mathbb{C} : |z - 1| < 2\}$. Therefore the map $f(t) = log(e^{it} - 1)$ can be approximated by polynomials $p_n(t)$ satisfying $p_n(1) = 0$, uniformly in $\sigma(U)$. Then $p_n(U - 1) \in \mathcal{K}(\mathcal{H})$, and thus $log(U) = f(U - 1) = \lim_n p_n(U) \in \mathcal{K}(\mathcal{H})$. □

One can characterize the pairs $(P, Q) \in \mathcal{C}$ such that $\|P - Q\| < 1$. By the results in Remark 2.1, if $\|P - Q\| < 1$, then there exist a unique $Z^* = Z$ with $\|Z\| < \pi/2$, such that Z is P-co-diagonal (or equivalently, anti-commutes with $2P-1$), such that $e^{iZ}Pe^{-iZ} = Q$ (or equivalently, $\epsilon_Q = e^{2iZ}\epsilon_P = \epsilon_P e^{-2iZ}$).

Proposition 6.2. *Let (P, Q) be a pair of projections such that $\|P - Q\| < 1$. Then $(P, Q) \in \mathcal{C}$ if and only if $Z \in \mathcal{K}(\mathcal{H})$.*

Proof. Suppose first that $Z \in \mathcal{K}(\mathcal{H})$. Then $e^{-2iZ} - 1 \in \mathcal{K}(\mathcal{H})$. Then

$$P - Q = \frac{1}{2}(\epsilon_Q - \epsilon_P) = \frac{1}{2}\epsilon_P(e^{-2iZ} - 1) \in \mathcal{K}(\mathcal{H}).$$

Conversely, suppose that $P - Q \in \mathcal{K}(\mathcal{H})$. With the same computation as above, it follows that $e^{-2iZ} - 1 \in \mathcal{K}(\mathcal{H})$. Since $\|Z\| < \pi/2$, by the above lemma, this implies that $Z \in \mathcal{K}(\mathcal{H})$. □

This proposition enables one to show that the inclusion $\mathcal{C} \subset \mathcal{F}$ is strict. Indeed, given a projection P, let $Z^* = Z$ be a non compact P-co-diagonal operator with $\|Z\| < \pi$ (it is an elementary fact that such operators can be constructed). Then the pair $(P, e^{iZ}Pe^{-iZ}) \notin \mathcal{C}$. On the other hand, since $\|P - e^{iZ}Pe^{-iZ}\| < 1$, the pair $(P, e^{iZ}Pe^{-iZ}) \in \mathcal{F}$.

Also this proposition enables one to examine the local structure of $\mathcal{C}$. Denote by

$$\mathcal{C}_n = \{(P, Q) : i(P, Q) = n\}.$$

Apparently, elements in the same connected component of $\mathcal{C}$ share the same index.

First, we shall need the following result on the structure of the orbit of a projection under the group $\mathcal{U}_c(\mathcal{H})$. A related result can be found in [15], for the ideal of Hilbert-Schmidt operators in the place of $\mathcal{K}(\mathcal{H})$.

Theorem 6.3. *Let $P \in \mathcal{P}$. Then*

$$\{UPU^* : U \in \mathcal{U}_c(\mathcal{H})\} = \{Q \in \mathcal{P} : Q - P \in \mathcal{K}(\mathcal{H}) \text{ and } i(P, Q) = 0\}.$$

This set is a real analytic manifold. Any pair Q, Q' of elements in this orbit can be joined by a minimal geodesic $\delta(t) = e^{itX}Qe^{-itX}$, where $X^* = X$ is compact and Q-co-diagonal.

Proof. In [3] it was proved that the orbit of a projection under the action of $\mathcal{U}_c(\mathcal{H})$ is an analytic manifold which satisfies the assertion on the minimal geodesics. Thus it only remains to be proved that the orbit coincides with the right hand set. The inclusion

$$\{UPU^* : U \in \mathcal{U}_c(\mathcal{H})\} \subset \{Q \in \mathcal{P} : Q - P \in \mathcal{K}(\mathcal{H})\}$$

is apparent: if $U - 1 \in \mathcal{K}(\mathcal{H})$, then

$$UPU^* - P = UPU^* - UP + UP - P = UP(U^* - 1) + (U - 1)P \in \mathcal{K}(\mathcal{H}).$$

Moreover, by the result in [3], any element $Q = UPU^*$ is of the form $Q = e^{iX}Pe^{-iX}$ for some compact $X^* = X$ with $\|X\| \leq \pi/2$. Then $(P, e^{itX}Pe^{-itX})$ is a continuous curve of projections whose differences are compact (by the computation above), and therefore connects (P, P) and (P, Q) inside $\mathcal{F}$. It follows that $i(P, Q) = 0$.

The next argument is taken partly from the proof of Theorem 3.3 in [5]. Pick $Q \in \mathcal{P}$ such that $Q - P \in \mathcal{K}(\mathcal{H})$ and $i(P, Q) = 0$. Let P_+ be the orthogonal projection onto $N(P - Q - 1) = \mathcal{H}_{10}$ and P_- the orthogonal projection onto $N(P - Q + 1) = \mathcal{H}_{01}$. Note that these spaces are finite dimensional. The zero index of $QP \in \mathcal{B}(R(P), R(Q))$ implies that $\mathcal{H}_{10}$ and $\mathcal{H}_{01}$ have the same dimension. Put

$$U_0 : \mathcal{H}_{10} \to \mathcal{H}_{01}$$

a linear isometry. Denote $\mathcal{H}' = \mathcal{H}_{10} \oplus \mathcal{H}_{01}$. A unitary operator V_0 can be defined,

$$V_0 : \mathcal{H}' \to \mathcal{H}', \quad V(\xi, \eta) = (U_0^*\eta, U_0\xi).$$

Recall that $\mathcal{H}' = \mathcal{H}_{10} \oplus \mathcal{H}_{01}$ reduces P and Q. Consider $B = 1 - P - Q$ and $S = QP - (1 - P)(1 - Q)$. Clearly $SP = QS$ and $S = (1 - 2Q)B = B(1 - 2P)$. Also both B and S act in $(\mathcal{H}')^\perp$. Note that

$$B^2 = 1 - (P - Q)^2.$$

This implies that B^2 in invertible in $(\mathcal{H}')^\perp$ (the eigenspaces of 1 and -1 for $P - Q$ lie inside $\mathcal{H}'$). Then B, being selfadjoint, is invertible in $(\mathcal{H}')^\perp$, and thus $S = B(1 - 2P)$ is also invertible there. Note also that

$$S = 1 + 2QP - P - Q = Q(Q - P) + (Q - P)P + 1 \in 1 + \mathcal{K}(\mathcal{H}).$$

Then its restriction $S_1 = S|_{(\mathcal{H}')^\perp}$ belongs to $1 + \mathcal{K}((\mathcal{H}')^\perp)$. A simple spectral argument shows that the unitary part V_1 in the polar decomposition $S_1 = V_1|S_1|$ is a unitary operator in $\mathcal{U}_c((\mathcal{H}')^\perp)$. It is a standard elementary fact that if an invertible operator (namely S_1) intertwines two selfadjoint operators, then the unitary part also intertwines them, in this case:

$$V_1 P|_{(\mathcal{H}')^\perp} = Q|_{(\mathcal{H}')^\perp} V_1.$$

Put $V = V_0 \oplus V_1$ in $\mathcal{H}' \oplus (\mathcal{H}')^\perp = \mathcal{H}$. Since $\mathcal{H}'$ is finite dimensional and $V_1 \in \mathcal{U}_c((\mathcal{H}')^\perp)$, it follows that $V \in \mathcal{U}_c(\mathcal{H})$. Clearly $VPV^* = Q$. □

As with Fredholm pairs, two compact pairs (P, Q) and (P', Q'), which lie in the same connected component of $\mathcal{P} \times \mathcal{P}$, lie in the same component of $\mathcal{C}$ if and only if $i(P, Q) = i(P', Q')$. The proof, which is omitted, is similar as in the former situation.

Proposition 6.4. *Let (P, Q) and (P', Q') be pairs in $\mathcal{C}$, with $i(P, Q) = k$ and $i(P', Q') = k'$.*

1. *If $k \leq 0$, there exists a sub-projection $Q_0 \leq P$ such that $\dim R(P - Q_0) = k$, and the pairs (P, Q) and (P, Q_0) lie in the same connected component of $\mathcal{C}$. If $k > 0$, there exists a sub-projection $P_0 \leq Q$ with $\dim(Q - P_0) = -k$, such that (P, Q) and (P_0, Q) lie in the same connected component of $\mathcal{C}$.*

2. *Suppose that (P, Q) and (P', Q') lie in the same component of $\mathcal{P} \times \mathcal{P}$. Then they lie in the same component of $\mathcal{C}$ if and only if $k = k'$.*

Finally, let us prove that $\mathcal{C}$, or rather, its open subsets $\mathcal{C}_n$, are C^∞ submanifolds of $\mathcal{P} \times \mathcal{P}$.

Theorem 6.5. *For any $k \in \mathbb{Z}$, the set $\mathcal{C}_k$ is a real analytic (non complemented) submanifold of $\mathcal{P} \times \mathcal{P}$.*

Proof. Suppose first that $k \leq 0$. Pick a pair $(P, P_0) \in \mathcal{C}_k$ such that $P_0 \leq P$. Note that $P - P_0$ is a projection of rank k. As noted above, a projection P' satisfies $\|P' - P\| < 1$ if and only if there exists a unique selfadjoint operator $X_{P'}$, which is co-diagonal with respect to P, such that $\|X_{P'}\| < \pi/2$ and $P' = e^{iX} P e^{-iX}$. Moreover, the map $P' \mapsto X_{P'}$ is real analytic, in particular it is continuous. Consider the open neighbourhood of (P, P_0) in $\mathcal{C}$:

$$\mathcal{B}_{P,P_0} = \{(P', Q') \in \mathcal{C} : \|P' - P\| < 1, \text{ and } \|e^{-iX} Q' e^{iX} - P_0\| < 1\}.$$

Note that

$$e^{-iX} Q' e^{iX} - P_0 = e^{-iX} Q' e^{iX} - e^{-iX} P' e^{iX} + e^{-iX} P' e^{iX} - P_0 =$$
$$e^{-iX}(Q' - P') e^{iX} + P - P_0 \in \mathcal{K}(\mathcal{H}).$$

By the above Theorem, if $(P', Q') \in \mathcal{B}_{P,P_0}$, then there exists a unique selfadjoint compact operator Y which is P_0-co-diagonal, with $\|Y\| < \pi/2$, such that $Q'' = e^{-iX} Q' e^{iX} = e^{iY} P_0 e^{-iY}$. Moreover, the map $Q'' \mapsto Y$ is an analytic diffeomorphism (being the local chart of the orbit of Q' under the action of the unitary Fredholm group). Therefore, the open set $\mathcal{B}_{P,P_0}$ is diffeomorphic to

$$\mathcal{B}' = \{(X, Y) \in \mathcal{B}_s(\mathcal{H})^2 : X \ P\text{-codiagonal}, Y \ P_0\text{-codiagonal}, \|X\|, \|Y\| < \pi/2, Y \in \mathcal{K}(\mathcal{H})\}.$$

Let us construct a local chart for an arbitrary pair $(P, Q) \in \mathcal{C}_k$. As in the proposition above, there exist $P_0 \leq P$ and a compact selfadjoint operator X, which is co-diagonal with respect to P_0, such that $Q = e^{iX} P_0 e^{-iX}$. Consider

$$\mathcal{B}_{P,Q} = \{(P', Q') \in \mathcal{C} : (P', e^{-iX} Q' e^{iX}) \in \mathcal{B}_{P,P_0}\}.$$

Clearly $(P, Q) \in \mathcal{B}_{P,Q}$. Moreover, since X is compact and selfadjoint, the map

$$(P', Q') \mapsto (P', e^{-iX} Q' e^{iX})$$

is a diffeomorphism which preserves $\mathcal{C}$: $P' - Q'$ is compact if and only $P' - e^{-iX} Q' e^{iX}$ is compact. This implies that $\mathcal{B}_{P,Q}$ is open, and this homeomorphism maps it into the local chart $\mathcal{B}_{P,P_0}$. $\square$

Proposition 6.6. *Let* (P, Q) *and* (P', Q') *be two pairs in the same connected component of* $\mathcal{C}$. *If* $P - P'$ *is compact with* $i(P, P') = 0$, *then there exists a curve* (δ_1, δ_2) *which satisfies the following conditions:*

1. (δ_1, δ_2) *joins* (P, Q) *and* (P', Q') *inside* $\mathcal{C}$.
2. δ_1, δ_2 *are geodesics of* $\mathcal{P}$.
3. (δ_1, δ_2) *has minimal length among curves in* $\mathcal{P} \times \mathcal{P}$ *joining these pairs. In particular, it is minimal in* $\mathcal{C}$. *As in Theorem 4.2, the Finsler metric considered in* $\mathcal{P} \times \mathcal{P}$ *is given by* $\|(A, B)\| = (\|A\|^2 + \|B\|^2)^{1/2}$.

Proof. The fact that $P - P'$ is compact implies that

$$Q - Q' = (Q - P) + (P - P') + (P' - Q')$$

is also compact. Therefore the result by [5] cited in Remark 2.3 applies:

$$i(Q, Q') = i(Q, P) + i(P, P') + i(P', Q') = -i(P, Q) + i(P', Q') = 0.$$

Thus, by Theorem 6.3, there exist minimal geodesics $\delta_1(t) = e^{itX_1} P e^{-itX_1}$ and $e^{itX_2} Q e^{-itX_2}$, with X_i compact and selfadjoint, $\|X_i\| \le \pi/2$ $(i = 1, 2)$, X_1 is P-co-diagonal, X_2 is Q-co-diagonal, and $\delta_1(1) = P'$, $\delta_2(1) = Q'$. Apparently, the curve (δ_1, δ_2) is minimal in $\mathcal{P} \times \mathcal{P}$ joining (P, Q) and (P', Q'). It only remains to prove that $(\delta_1(t), \delta_2(t)) \in \mathcal{C}$ for all t. Indeed, note that $e^{itX_i} = 1 + K_i$ and $e^{-itX_i} = 1 + K_i'$, with K_i, K_i' compact, for $i = 1, 2$. Then

$$\delta_1(t) - \delta_2(t) = P - Q + K_1 P + P K_1' + K_1 P K_1' + K_2 Q + Q K_2' + K_2 Q K_2',$$

which is clearly compact. □

References

[1] Amrein, W. O.; Sinha, K. B. On pairs of projections in a Hilbert space. Linear Algebra Appl. 208/209 (1994), 425–435.

[2] Andruchow, E.; Corach, G.; Stojanoff, D. Projective spaces of a C^*-algebra. Integral Equations Operator Theory 37 (2000), no. 2, 143–168.

[3] Andruchow E., G. Larotonda, G. The rectifiable distance in the unitary Fredholm group. Studia Math. 196 (2010), no. 2, 151–178.

[4] Atkin, C. J. The Finsler geometry of groups of isometries of Hilbert space. J. Austral. Math. Soc. Ser. A 42 (1987), no. 2, 196–222.

[5] Avron, J.; Seiler, R.; Simon, B. The index of a pair of projections. J. Funct. Anal. 120 (1994), no. 1, 220–237.

[6] Böttcher, A.; Spitkovsky, I. M. A gentle guide to the basics of two projections theory. Linear Algebra Appl. 432 (2010), no. 6, 1412–1459.

[7] Buckholtz, Don. Inverting the difference of Hilbert space projections. Amer. Math. Monthly 104 (1997), no. 1, 60–61.

[8] Corach, G.; Porta, H.; Recht, L. The geometry of spaces of projections in C^*-algebras. Adv. Math. 101 (1993), no. 1, 59–77.

[9] Davis, C. Separation of two linear subspaces. Acta Sci. Math. Szeged 19 1958 172–187.

[10] de la Harpe, P. Classical Banach-Lie algebras and Banach-Lie groups of operators in Hilbert space. Lecture Notes in Mathematics, Vol. 285. Springer-Verlag, Berlin-New York, 1972.

[11] Dixmier, J. Position relative de deux variétés linaires fermées dans un espace de Hilbert. (French) Revue Sci. 86, (1948). 387–399.

[12] Halmos, P. R. Two subspaces. Trans. Amer. Math. Soc. 144 1969 381–389.

[13] Kato, T. Perturbation theory for linear operators. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[14] Porta, H.; Recht, L. Minimality of geodesics in Grassmann manifolds. Proc. Amer. Math. Soc. 100 (1987), no. 3, 464–466.

[15] Beltita, D.; Ratiu, T. S.; Tumpach, A. B. The restricted Grassmannian, Banach Lie-Poisson spaces, and coadjoint orbits. J. Funct. Anal. 247 (2007), no. 1, 138–168.

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