

ON A FORMULA OF S. RAMANUJAN

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ABSTRACT. Certain finite families of rational functions have the property that the coefficients of the expansions in power series of their elements are related by simple algebraic expressions. We prove an identity in the spirit of S. Ramanujan using a result of T. N. Sinha.

A remarkable formula of S. Ramanujan states that if

$$\frac{1 + 53x + 9x^2}{1 - 82x - 82x^2 + x^3} = \sum_{i \geq 0} \alpha_i x^i,$$

$$\frac{2 - 26x - 12x^2}{1 - 82x - 82x^2 + x^3} = \sum_{i \geq 0} \beta_i x^i$$

and

$$\frac{2 + 8x - 10x^2}{1 - 82x - 82x^2 + x^3} = \sum_{i \geq 0} \gamma_i x^i,$$

then

$$\alpha_i^3 + \beta_i^3 = \gamma_i^3 + (-1)^i.$$

This was proved by M. Hirschhorn in [2]. Two more proofs were given in [4] and [3], the last one is a *proof by example* which means that one verifies an identity by showing that it holds for a finite number of values (see [6], pp. 9). Recently J. McLaughlin proved a very nice and much more complicated result in [5]. This result is also discussed in [1].

In this note we prove the following theorem using a modified version of a result of T. N. Sinha.

Theorem. *If*

$$\sum_{i=0}^{\infty} a_i(1)x^i = \frac{15 - 61x + 21x^2}{1 - 2x - 2x^2 + x^3}, \quad \sum_{i=0}^{\infty} a_i(2)x^i = \frac{13 - 77x - 7x^2}{1 - 2x - 2x^2 + x^3},$$

$$\sum_{i=0}^{\infty} a_i(3)x^i = \frac{27 - 135x + 13x^2}{1 - 2x - 2x^2 + x^3}, \quad \sum_{i=0}^{\infty} a_i(4)x^i = \frac{-11 - 49x + 33x^2}{1 - 2x - 2x^2 + x^3},$$

$$\sum_{i=0}^{\infty} a_i(5)x^i = \frac{17 - 75x - 71x^2}{1 - 2x - 2x^2 + x^3}, \quad \sum_{i=0}^{\infty} a_i(6)x^i = \frac{3 - 17x - 91x^2}{1 - 2x - 2x^2 + x^3},$$

$$\sum_{i=0}^{\infty} a_i(7)x^i = \frac{29 - 79x - 51x^2}{1 - 2x - 2x^2 + x^3}, \quad \sum_{i=0}^{\infty} a_i(8)x^i = \frac{-1 + 31x - 79x^2}{1 - 2x - 2x^2 + x^3},$$

$$\begin{aligned}
\sum_{i=0}^{\infty} a_i(9)x^i &= \frac{-13 + 45x - 59x^2}{1 - 2x - 2x^2 + x^3}, & \sum_{i=0}^{\infty} b_i(1)x^i &= \frac{-1 - 9x + 13x^2}{1 - 2x - 2x^2 + x^3}, \\
\sum_{i=0}^{\infty} b_i(2)x^i &= \frac{13 - 67x + 33x^2}{1 - 2x - 2x^2 + x^3}, & \sum_{i=0}^{\infty} b_i(3)x^i &= \frac{-13 - 5x - 7x^2}{1 - 2x - 2x^2 + x^3}, \\
\sum_{i=0}^{\infty} b_i(4)x^i &= \frac{17 - 115x + 21x^2}{1 - 2x - 2x^2 + x^3}, & \sum_{i=0}^{\infty} b_i(5)x^i &= \frac{29 - 129x + x^2}{1 - 2x - 2x^2 + x^3}, \\
\sum_{i=0}^{\infty} b_i(6)x^i &= \frac{15 - 81x - 59x^2}{1 - 2x - 2x^2 + x^3}, & \sum_{i=0}^{\infty} b_i(7)x^i &= \frac{1 - 23x - 79x^2}{1 - 2x - 2x^2 + x^3}, \\
\sum_{i=0}^{\infty} b_i(8)x^i &= \frac{3 - 7x - 51x^2}{1 - 2x - 2x^2 + x^3}, & \sum_{i=0}^{\infty} b_i(9)x^i &= \frac{-11 + 51x - 71x^2}{1 - 2x - 2x^2 + x^3}, \\
\sum_{i=0}^{\infty} b_i(10)x^i &= \frac{27 - 35x - 91x^2}{1 - 2x - 2x^2 + x^3},
\end{aligned}$$

then

$$\sum_{j=1}^{10} b_i(j)^k - \sum_{j=1}^9 a_i(j)^k = (-1)^{ik},$$

for $k = 1, \dots, 8$ and all $i = 0, 1, 2, 3, \dots$

The proof will follow from two lemmas. In the next lemma we write for short $A_i = A_i(m, n)$.

Lemma 1. *If*

$$\begin{aligned}
A_1 &= m^2 - mn - n^2, & A_2 &= 15m^2 - 25mn - 21n^2, & A_3 &= 13m^2 - 71mn + 7n^2, \\
A_4 &= 27m^2 - 95mn - 13n^2, & A_5 &= -11m^2 - 27mn - 33n^2, & A_6 &= 17m^2 - 129mn + 71n^2, \\
A_7 &= 3m^2 - 105mn + 91n^2, & A_8 &= 29m^2 - 101mn + 51n^2, & A_9 &= -m^2 - 49mn + 79n^2, \\
A_{10} &= -13m^2 - 27mn + 59n^2,
\end{aligned}$$

and

$$\begin{aligned}
B_1 &= -m^2 + 3mn - 13n^2, & B_2 &= 13m^2 - 21mn - 33n^2, & B_3 &= -13m^2 - 25mn + 7n^2, \\
B_4 &= 17m^2 - 77mn - 21n^2, & B_5 &= 29m^2 - 99mn - n^2, & B_6 &= 15m^2 - 125mn + 59n^2, \\
B_7 &= m^2 - 101mn + 79n^2, & B_8 &= 3m^2 - 55mn + 51n^2, & B_9 &= -11m^2 - 31mn + 71n^2, \\
B_{10} &= 27m^2 - 99mn + 91n^2,
\end{aligned}$$

then the following identity

$$\sum_{i=1}^{10} A_i^k = \sum_{i=1}^{10} B_i^k, \tag{1}$$

is valid for $k = 1, \dots, 8$ where m, n are arbitrary.

The proof is just a check. We remark that these equalities are the heart of the proof and they are similar to a result of T.N. Sinha, see [7].

Next, recall that the Fibonacci sequence is defined by

$$F_0 = 0, F_1 = 1, F_{i+2} = F_{i+1} + F_i,$$

and one has the formula

$$F_i = \frac{1}{\sqrt{5}} \left\{ \alpha^i - \beta^i \right\},$$

with $\alpha = \frac{\sqrt{5}+1}{2}$; $\beta = \frac{-\sqrt{5}+1}{2}$. Hint: observe α, β are the roots of $x^2 - x - 1 = 0$. Therefore α^i, β^i both satisfy the Fibonacci recurrence for $\alpha^{i+2} - \alpha^{i+1} - \alpha^i = \alpha^i(\alpha^2 - \alpha - 1) = 0$.

Lemma 2. *If $i = 0, 1, 2, \dots$, then*

$$F_{i+1}^2 - F_i(F_{i+1} + F_i) = F_{i+1}^2 - F_i F_{i+2} = (-1)^i.$$

Also

$$\begin{aligned} \sum_{i=0}^{\infty} F_i^2 x^i &= \frac{x(1-x)}{1-2x-2x^2+x^3}, \\ \sum_{i=0}^{\infty} F_{i+1}^2 x^i &= \frac{(1-x)}{1-2x-2x^2+x^3} \quad \text{and} \\ \sum_{i=0}^{\infty} F_{i+1} F_i x^i &= \frac{x}{1-2x-2x^2+x^3}. \end{aligned}$$

Proof: One has

$$\begin{aligned} F_{i+1}^2 - F_i F_{i+2} &= \frac{(\alpha^{i+1} - \beta^{i+1})^2}{5} - \frac{(\alpha^i - \beta^i)(\alpha^{i+2} - \beta^{i+2})}{5} = \\ &= \frac{-2(\alpha\beta)^{i+1}}{5} + \frac{(\alpha\beta)^i(\alpha^2 + \beta^2)}{5}. \end{aligned}$$

The first identity of the lemma follows from this noticing that $\alpha\beta = -1$, $\alpha^2 + \beta^2 = 3$.

The first series is

$$\begin{aligned} \sum_{i=0}^{\infty} F_i^2 x^i &= \sum_{i=0}^{\infty} \frac{(\alpha^i - \beta^i)^2}{5} x^i = \frac{1}{5} \sum_{i=0}^{\infty} (\alpha^{2i} + \beta^{2i} - 2(-1)^i) x^i = \\ &= \frac{1}{5} \left\{ \frac{1}{1-\alpha^2 x} + \frac{1}{1-\beta^2 x} - \frac{2}{1+x} \right\}, \end{aligned}$$

and the result follows.

The other series are proved in a similar way. ■

Proof of theorem: In the identity of Lemma 1 set

$$m = F_{i+1}, \quad n = F_i.$$

Then

$$A_1 = m^2 - mn - n^2 = F_{i+1}^2 - F_i(F_{i+1} + F_i) = \\ F_{i+1}^2 - F_i F_{i+2} = (-1)^i,$$

where the last equality follows from Lemma 2.

The rational functions stated in the theorem are obtained letting $a_i(1), \dots, a_i(9)$ correspond to A_2, \dots, A_{10} and $b_i(1), \dots, b_i(10)$ correspond to B_1, \dots, B_{10} with the above definition of m and n . In what follows we calculate the first rational function stated, i.e. that corresponding to $a_i(1)$, but we omit the lengthy calculations for the rest.

One has

$$a_i(1) = A_2 = 15F_{i+1}^2 - 25F_{i+1}F_i - 21F_i^2.$$

Therefore using Lemma 2

$$\sum_{i=0}^{\infty} a_i(1)x^i = \frac{15 - 61x + 21x^2}{1 - 2x - 2x^2 + x^3},$$

as stated.

Now the theorem follows from identity (1). ■

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