SINGLE AGENTS AND THE SET OF

MANY-TO-ONE STABLE MATCHINGS*

by

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ABSTRACT: Some properties of the set of many-to-one stable matchings when firms have responsive preferences and quotas *are not* necessarily true when firms preferences are substitutable. In particular, we exhibit examples in which firms have substitutable preferences but firms and workers may be "single" in one stable matching but matched in another one. We identify a set of axioms on firms preferences guaranteeing that the set of unmatched agents is the same under every stable matching. We also propose a weaker condition than responsiveness, called separability with quotas or $q-$separability, that together with substitutability implies this set of axioms. *Journal of Economic Literature* Classification Number: J41.

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1 Introduction

Two-sided, many-to-one models have been used to study assignment problems where agents can be divided, from the very beginning, into two disjoint subsets. One contains institutions like *firms*, hospitals, colleges, sororities, orchestras, schools, clubs, etc. The other includes individuals like *workers*, medical interns, students, musicians, children, sportmen, etc. The fundamental question of these assignment problems consists of matching each firm, on one side, with a group of workers, on the other side.[1] Stability has been considered the main property to be satisfied by any sensible matching. A matching is called stable if all agents have acceptable partners and there is no unmatched worker-firm pair who both would prefer to be matched to each other rather than staying with their current partners. To give all blocking power to only individual agents and worker-firm pairs seems a weak requirement. Moreover, in many cases it may be the right solution concept since, to destroy an individually rational unstable matching, only a telephone call (or a couple of e-mails) is required.

The "college admissions model with substitutable preferences" is the name given by Roth and Sotomayor [9] to the most general many-to-one model with ordinal preferences. Firms are restricted to having substitutable preferences over subsets of workers, while workers may have all possible preference orderings over the set of firms. A preference of a firm is said to be substitutable if that firm continues to want to employ a worker even if other workers become unavailable.[2] Under this hypothesis Roth and Sotomayor [9] showed that the deferred-acceptance algorithms produce either the firm-optimal stable matching or the worker-optimal stable matching, depending on whether the firms or the workers make the offers. The firm (worker)-optimal stable matching is unanimously considered by all firms (respectively, workers) to be the best among all stable matchings.

A more specific many-to-one model, called the "college admissions problem" by Gale and Shapley [1], supposes that firms (colleges) have a maximum

[1] We will follow the convention of generically referring to institutions as firms and to individuals as workers. See Roth and Sotomayor [9] for an illuminating and comprehensive survey of this literature as well as an exhaustive bibliography.

[2] Kelso and Crawford [4] were the first to use this property (under the name of "gross substitutability condition") in a more general model with money. They proved the existence of a stable matching and of a firm-optimal stable matching (all firms agree it is the best stable matching).

number of positions to be filled (their quota), and that each firm (college), given its ranking of individual workers (students), orders subsets of workers in a responsive manner; namely, for any two subsets that differ in only one student a college prefers the subset containing the more preferred student. In this model the set of stable matchings satisfies the following additional properties: (1) There is a polarization of interests between the two sides of the market along the set of stable matchings. (2) The set of unmatched agents is the same under every stable matching. (3) The number of workers assigned to a firm through stable matchings is the same. (4) If a firm does not complete its quota under some stable matching then it gets the same set of workers at any stable matching, proved by Roth (1986).[3]

The purpose of this paper is twofold. On the one hand, its negative side: we exhibit examples with firms having substitutable preferences in which properties (1) to (4) are violated.[4]

On the other hand, its positive side: we propose a set of axioms on firms preferences (opposite optimality, acceptability, and desirability) under which properties (2), (3), and (4) are always true. Moreover, we identify a weaker condition than $q-$responsiveness, called separability with quota, or $q-$separability, that together with substitutability imply the set of axioms. We also show that those restricted preferences do not guarantee that property (1) holds (see Example 2). A firm is said to have separable preferences over all subsets of workers if its partition between acceptable and unacceptable workers has the property that only adding acceptable workers makes any given subset of workers a better one. However, in many applications as the entry-level professional labor markets, separability alone does not seem very reasonable because firms usually have much smaller number of openings (their quota) than that there are "good" workers looking for a job. In those cases it seems reasonable to restrict firm preferences in such a way that the separability condition operates only up to their quota, considering unacceptable all subsets with higher cardinality. Moreover, while responsiveness seems the relevant property for extending an ordered list of individual students to preferences on all subsets of students, it is too restrictive, though, to capture some degree of complementarity among workers, which can be very natural in other settings. The $q-$separability condition permits greater flex-

[3]Property (1) is a consequence of the decomposition lemma proved by Gale and Sotomayor [2] and [3]. Properties (2) and (3) were proved independently by gale and Sotomayor [2] and [3] and Roth [5]. Property (4) was proved by Roth [7].

[4]Property (1) is even violated under stronger restrictions (see a comment below).

ibility in going from orders on individuals to orders on subsets. For instance, candidates for a job can be grouped together by areas of specialization. A firm with quota two may consider as the best subset of workers not the set consisting of the first two candidates on the individual ranking (which may have both the same specialization) but rather the subset composed of the first and fourth candidates in the individual ranking (i.e.; the first in each area of specialization).

In the next section we present the notation and definitions. Section 3 contains the results and the main examples.

2 Notation and definitions

There are two disjoint sets of agents, the set of n *firms* $\mathcal{F} = \{F_1, ..., F_n\}$ and the set of m *workers* $\mathcal{W} = \{w_1, ..., w_m\}$. Each firm $F \in \mathcal{F}$ has a strict, transitive, and complete preference relation $P(F)$ over the set of all subsets of $\mathcal{W}$, and each worker $w \in \mathcal{W}$ has a strict, transitive, and complete preference relation $P(w)$ over $\mathcal{F} \cup \{\emptyset\}$. *Preferences profiles* are $(n+m)$-tuples of preference relations and they are represented by $P = (P(F_1), ..., P(F_n); P(w_1), ..., P(w_m))$. Given a preference relation of a firm $P(F)$ the subsets of workers preferred to the empty set by F are called acceptable. Similarly, given a preference relation of a worker $P(w)$ the firms preferred by w to the empty set are called acceptable. To express preference relations in a concise manner, and since only acceptable partners will matter, we will represent preference relations as lists of acceptable partners. We will denote by $\mathcal{P}$ a generic subset of preferences profiles.

The assignment problem consists of matching workers with firms keeping the bilateral nature of their relationship and allowing for the possibility that both, firms and workers, may remain unmatched. Formally,

Definition 1 *A* ***matching*** μ *is a mapping from the set* $\mathcal{F} \cup \mathcal{W}$ *into the set of all subsets of* $\mathcal{F} \cup \mathcal{W}$ *such that for all* $w \in \mathcal{W}$ *and* $F \in \mathcal{F}$*:*

1. *Either* $|\mu(w)| = 1$ *and* $\mu(w) \subseteq \mathcal{F}$ *or else* $\mu(w) = \emptyset$.

2. $\mu(F) \in 2^{\mathcal{W}}$.

3. $\mu(w) = F$ *if and only if* $w \in \mu(F)$.

We say that w and F are *single* in a matching μ if $\mu(w) = \emptyset$ and $\mu(F) = \emptyset$. Otherwise, they are matched. A matching μ is said to be *one-to-one* if firms can hire at most one worker; namely, condition *2* is replaced by: Either $|\mu(F)| = 1$ and $\mu(F) \subseteq \mathcal{W}$ or else $\mu(F) = \emptyset$. The model in which all matchings are one-to-one is also known in the literature as the *marriage model*. We will follow the widespread notation where

$$\mu = \begin{pmatrix} F_1 & F_2 & F_3 & \emptyset \\ \{w_3, w_4\} & \{w_1\} & \emptyset & \{w_2\} \end{pmatrix}$$

represents the matching where firm F_1 is matched to workers w_3 and w_4, firm F_2 is matched to worker w_1, and firm F_3 and worker w_2 are single.

Let P be a preference profile. Given a set of workers $S \subseteq \mathcal{W}$, let $Ch(S, P(F))$ denote firm F's *most-preferred* subset of S according to its preference ordering $P(F)$. A matching μ is *blocked by a worker* w if $\emptyset P(w)\,\mu(w)$. Similarly, μ is *blocked by a firm* F if $\mu(F) \neq Ch(\mu(F), P(F))$. We say that a matching is *individually rational* if it is not blocked by any individual agent. A matching μ is *blocked by a worker-firm pair* (w, F) if $w \notin \mu(F)$, $w \in Ch(\mu(F) \cup \{w\}, P(F))$, and $FP(w)\,\mu(w)$.

Definition 2 *A matching μ is* ***stable*** *if it is not blocked by any individual agent or any firm-worker pair.*

Given a preference profile P, denote the set of stable matchings by $S(P)$. It is easy to construct examples of preference profiles with the property that the set of stable matchings is empty. The literature has concentrated not only on subsets of preferences where the set of stable matchings is nonempty but also on subsets where firms (workers) unanimously agree that a matching $\mu_{\mathcal{F}}$ ($\mu_{\mathcal{W}}$) is the best stable matching. That is why $\mu_{\mathcal{F}}$ and $\mu_{\mathcal{W}}$ are called, respectively, the firms-optimal stable matching and the workers-optimal stable matching.[5] Moreover, there is an opposition of interests on these two optimal stable matchings. We state both properties as an axiom on sets of preferences profiles $\mathcal{P}$.

(OO) The set $\mathcal{P}$ satisfies OPPOSITE OPTIMALITY if for all $P \in \mathcal{P}$ there exist $\mu_{\mathcal{F}}, \mu_{\mathcal{W}} \in S(P)$ such that for all $\mu \in S(P)$ and all F and w: $\mu_{\mathcal{F}} R(F)\,\mu R(F)\,\mu_{\mathcal{W}}$ and $\mu_{\mathcal{W}} R(w)\,\mu R(w)\,\mu_{\mathcal{F}}$.

[5] We are following the convention of extending preferences from the original sets ($2^{\mathcal{W}}$ and $\mathcal{F} \cup \{\emptyset\}$) to the set of matchings. However, we now have to consider weak orderings since the matchings μ and μ' may associate the same partner with an individual. These orderings will be denoted by $R(F)$ and $R(w)$.

The literature has also focused on the restriction where workers are regarded as substitutes.

Definition 3 *A firm F's preference ordering $P(F)$ satisfies* ***substitutability*** *if for any set S containing workers w and $\bar{w}$ $(w \neq \bar{w})$, if $w \in Ch(S, P(F))$ then $w \in Ch(S \backslash \{\bar{w}\}, P(F))$.*

A preference profile P is *substitutable* if for each firm F, the preference ordering $P(F)$ satisfies substitutability.

Remark 1 *The set of substitutable preferences satisfies opposite optimality.*

We will also concentrate on subsets of preferences satisfying the axiom that, in a stable matching, firms only hire individually acceptable workers. Formally,

(A) The set $\mathcal{P}$ satisfies ACCEPTABILITY if for all $P \in \mathcal{P}$ and all $\mu \in S(P)$: $w \in \mu(F)$ implies $wP(F)\emptyset$.

We will assume that each firm F has a maximum number of positions to be filled: its quota q_F. This limitation may arise from, for example, technological, legal, or budgetary reasons. We will denote by $q = (q_F)_{F \in \mathcal{F}}$ the list of quotas and we will focus on the axiom saying that if a firm does not fill its quota it is willing to hire an acceptable worker. Formally,

(D) The set $\mathcal{P}$ satisfies $q-$DESIRABILITY if for all $P \in \mathcal{P}$, all $\mu \in S(P)$, and all $F \in \mathcal{F}$: $|\mu(F)| < q_F$ and $wP(F)\emptyset$ imply $w \in Ch(\mu(F) \cup \{w\}, P(F))$.

We now define the set of $q-$separable and $q-$responsive preferences. A firm F has $q-$separable preferences if the division between *good* workers $(wP(F)\emptyset)$ and *bad* workers $(\emptyset P(F) w)$ guides, up to his quota, the ordering of subsets in the sense that adding a good worker leads to a *better* set, while adding a bad worker leads to a *worse* set. Formally,

Definition 4 *A firm F's preference ordering $P(F)$ over sets of workers is q_F-**separable** if: (a) for all $S \subsetneq \mathcal{W}$ such that $|S| < q_F$ and $w \notin S$ we have that$(S \cup \{w\})\, P(F) S$ if and only if $wP(F)\emptyset$, and (b) $\emptyset P(F) S$ for all S such that $|S| > q_F$.*[6]

[6] For the purpose of studying the set of stable matchings, condition (b) in this definition could be replaced by the following condition: $|Ch(S, P(F))| \leq q_F$ for all S such that $|S| > q_F$. We choose condition (b) since it is simpler. Sönmez [10] used an alternative approach that consists of deleting condition (b) in the definition but then requiring in the definition of a matching that $|\mu(F)| \leq q_F$ for all $F \in \mathcal{F}$.

Remark 2 *The set of $q-$separable preferences satisfies acceptability and $q-$desirability.*

Following Roth and Sotomayor [9], a firm F's preference ordering $P(F)$ (over all *subsets* of workers) is said to be q_F-*responsive (*to its ordering over individual workers) if it is q_F-separable and for any two sets of workers that differ in only one worker, F prefers the subset containing the most-preferred worker. Formally,

Definition 5 *A firm F's preference ordering $P(F)$ over sets of workers is q_F-**responsive** if it is q_F-separable and for all S, $w' \in S$, and $w \notin S$ we have that $(S \backslash w' \cup \{w\}) P(F) S$ if and only if $w P(F) w'$.*

By definition, all q_F-responsive preference orderings are q_F-separable. We will say that a preference profile P is $q-$*separable* if each $P(F)$ is q_F-separable. Similarly, a preference profile P is $q-$*responsive* if each $P(F)$ is q_F-responsive. In principle we may have firms with different quotas. The case where all firms have 1-separable preferences is equivalent, from the point of view of the set of stable matchings, to the one-to-one model. Hence, our set-up includes the marriage model as a particular case.

The following ordering over $2^{\mathcal{W}}$, where $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$,

$$\begin{aligned} P(F) &= \{w_1, w_2\}, \{w_3, w_4\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_2, w_4\}, \\ &\quad \{w_1\}, \{w_2\}, \{w_3\}, \{w_4\} \end{aligned}$$

illustrates the fact that q-separability does not imply substitutability. To see this, notice that $P(F)$ is 2-separable but it is not substitutable since $w_1 \in Ch(\{w_1, w_2, w_3, w_4\}, P(F)) = \{w_1, w_2\}$, but $w_1 \notin Ch(\{w_1, w_3, w_4\}, P(F)) = \{w_3, w_4\}$. However, it is easy to see that all $(m-1)-$separable as well as all q_F-responsive preferences are substitutable. As a consequence of this later inclusion we have that the set of $q-$responsive preferences satisfies existence. The ordering

$$P(F') = \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \{w_3\}$$

illustrates the fact that the set of $q-$responsive preferences is a proper subset of the set of $q-$separable and substitutable preferences.

The following example shows that even if all firms have q–separable preferences the set of stable matchings may be empty.

Example 1 Let $\mathcal{F} = \{F_1, F_2\}$ and $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with a profile of preferences P defined by

$$\begin{aligned}
P(F_1) = & \ \{w_3, w_4\}, \{w_2, w_4\}, \{w_1, w_2\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_4\}, \{w_1\}, \\
& \ \{w_2\}, \{w_3\}, \{w_4\}, \\
P(F_2) = & \ \{w_3\}, \{w_4\}, \\
P(w_1) = & \ F_1, \\
P(w_2) = & \ F_1, \\
P(w_3) = & \ F_1, F_2, \text{ and} \\
P(w_4) = & \ F_2, F_1.
\end{aligned}$$

Notice that P is $(2,1)-$separable. However, it is a matter of verification to see that $S(P) = \emptyset$.

3 Results and examples

In the marriage model the decomposition lemma says that both sides of the market are in conflict on the set of stable matchings in the sense that the partners (through any stable matching μ_1) of the subset of agents of one side of the market that consider μ_1 to be at least as good as the stable matching μ_2, have to consider μ_2 to be at least as good as μ_1. In particular (and we refered to it in the Introduction as the existence of a polarization of interests (Property (1)), if all agents of one side of the market consider the stable matching μ_1 to be at least as good as the stable matching μ_2 then all agents of the other side have to consider μ_2 to beat least as good as μ_1. The decomposition lemma also holds for the college admissions model, with $q-$responsive preferences, as a consequence of the following result of Roth and Sotomayor [9]: For any given pair of stable matchings μ_1 and μ_2, if firm F prefers $\mu_1(F)$ to $\mu_2(F)$ then it prefers every worker in $\mu_1(F)$ to any worker in $\mu_2(F) \setminus \mu_1(F)$. This result also implies that the set of stable matchings depends only on how firms order individual workers and not on their specific responsive extensions. Roth [6] gives an example where the decomposition lemma is not true in a many-to-one model with money and substitutable preferences. Example 2 below shows that the decomposition lemma does not hold in our more restricted framework of ordinal, $q-$separable, and substitutable preferences.

Example 2 Let $\mathcal{F} = \{F_1, F_2\}$ and $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with a profile of preferences P defined by

$$\begin{array}{rl} P(F_1) = & \{w_1, w_2\}, \{w_1, w_3\}, \{w_2, w_4\}, \{w_3, w_4\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \\ & \{w_3\}, \{w_4\}, \\ P(F_2) = & \{w_3, w_4\}, \{w_2, w_4\}, \{w_1, w_3\}, \{w_1, w_2\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \\ & \{w_3\}, \{w_4\}, \\ P(w_1) = & F_2, F_1, \\ P(w_2) = & F_2, F_1, \\ P(w_3) = & F_1, F_2, \text{ and} \\ P(w_4) = & F_1, F_2. \end{array}$$

It is easy to see that P is $(2,2)-$separable and substitutable and that the set of stable matchings consists of the following four matchings:

$$\mu_{\mathcal{F}} = \begin{pmatrix} F_1 & F_2 \\ \{w_1, w_2\} & \{w_3, w_4\} \end{pmatrix},$$

$$\mu_1 = \begin{pmatrix} F_1 & F_2 \\ \{w_1, w_3\} & \{w_2, w_4\} \end{pmatrix},$$

$$\mu_2 = \begin{pmatrix} F_1 & F_2 \\ \{w_2, w_4\} & \{w_1, w_3\} \end{pmatrix}, \text{ and}$$

$$\mu_{\mathcal{W}} = \begin{pmatrix} F_1 & F_2 \\ \{w_3, w_4\} & \{w_1, w_2\} \end{pmatrix}.$$

Notice that $\mu_{\mathcal{F}} P(F) \mu_1 P(F) \mu_2 P(F) \mu_{\mathcal{W}}$ for all F. However, the preferences of the workers are:

$$\mu_{\mathcal{W}} R(w_1) \mu_2 P(w_1) \mu_1 R(w_1) \mu_{\mathcal{F}},$$

$$\mu_{\mathcal{W}} R(w_2) \mu_1 P(w_2) \mu_2 R(w_2) \mu_{\mathcal{F}},$$

$$\mu_{\mathcal{W}} R(w_3) \mu_1 P(w_3) \mu_2 R(w_3) \mu_{\mathcal{F}}, \text{ and}$$

$$\mu_{\mathcal{W}} R(w_4) \mu_2 P(w_4) \mu_1 R(w_4) \mu_{\mathcal{F}}.$$

Actually, μ_1 is strictly preferred to μ_2 by F_1, F_2, and w_2.

We turn now to establish the fact that under axioms (OO), (A), and (D) the set of unmatched agents is the same under every stable matching, and therefore this property also holds whenever firms have $q-$separable and substitutable preferences. This is an important property since otherwise a “single” agent would be able to argue that he was badly treated by a particular stable matching. Remember that the result also holds in both the marriage and the college admission models. This is in spite of the fact that its proof in these models, according to Roth and Sotomayor [9], “is a simple consequence of the decomposition lemma” which does not hold in our setting. Moreover, we would also like to note that when firms' preferences are $q-$responsive the structure of the set of stable matchings coincides in the marriage and the college admission models. This is because one can identify each firm F with q_F identical firms and any many-to-one matching μ with the one-to-one matching $\bar{\mu}$ where each $w \in \mu(F)$ is matched through $\bar{\mu}$ with one of the q_F replica of F. However, as soon as preferences are not $q-$responsive, the properties of the set of stable matchings have to be proven directly without relying on the properties of the marriage model.

Proposition 1 below states the result for the workers.

Proposition 1 *Assume $\mathcal{P}$ satisfies axioms* (OO), (A), *and* (D). *Then, for all $P \in \mathcal{P}$, if w is single in $\mu \in S(P)$, then w is single in any $\mu' \in S(P)$.*

P roof. Suppose the contrary; that is, there exist $\bar{w}$, $\widehat{F}$, and $\mu, \mu' \in S(P)$ such that $\mu(\bar{w}) = \emptyset$ and $\mu'(\bar{w}) = \widehat{F}$. By (OO), there exist matchings $\mu_{\mathcal{F}}$ and $\mu_{\mathcal{W}}$ that are the worst and the best stable matchings, respectively, for all workers. Therefore, we can also find a firm $\bar{F}$ such that $\bar{w} \in \mu_{\mathcal{W}}(\bar{F})$ and $\bar{w} \notin \bigcup_F \mu_{\mathcal{F}}(F)$. We will distinguish between two cases:

Case 1: $\bigcup_F \mu_{\mathcal{F}}(F) \subseteq \bigcup_F \mu_{\mathcal{W}}(F)$. In this case

$$\sum\nolimits_F |\mu_{\mathcal{F}}(F)| < \sum\nolimits_F |\mu_{\mathcal{W}}(F)| \leq \sum\nolimits_F q_F, \tag{1}$$

where the strict inequality follows because $\bar{w} \notin \bigcup_F \mu_{\mathcal{F}}(F)$. Then, $\left|\mu_{\mathcal{F}}(\tilde{F})\right| < q_{\tilde{F}}$ for at least one $\tilde{F}$. Denote by $\widetilde{\mathcal{F}}$ the set of all such firms. We claim that there exists $F_0 \in \widetilde{\mathcal{F}}$ such that $\mu_{\mathcal{W}}(F_0) \backslash \mu_{\mathcal{F}}(F_0) \neq \emptyset$, because otherwise, $\mu_{\mathcal{W}}(F) \subseteq \mu_{\mathcal{F}}(F)$ for all $F \in \widetilde{\mathcal{F}}$ would imply

$$\begin{aligned}
\sum\nolimits_F |\mu_{\mathcal{W}}(F)| &= \left|\bigcup\nolimits_F \mu_{\mathcal{W}}(F)\right| \\
&= \left|\bigcup\nolimits_{F \in \widetilde{\mathcal{F}}} \mu_{\mathcal{W}}(F)\right| + \left|\bigcup\nolimits_{F \notin \widetilde{\mathcal{F}}} \mu_{\mathcal{W}}(F)\right| \\
&\leq \left|\bigcup\nolimits_{F \in \widetilde{\mathcal{F}}} \mu_{\mathcal{F}}(F)\right| + \left|\bigcup\nolimits_{F \notin \widetilde{\mathcal{F}}} \mu_{\mathcal{W}}(F)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\bigcup\nolimits_{F\in\widetilde{\mathcal{F}}}\mu_{\mathcal{F}}(F)\right|+\left|\bigcup\nolimits_{F\notin\widetilde{\mathcal{F}}}\mu_{\mathcal{F}}(F)\right|\\
&= \left|\bigcup\nolimits_{F}\mu_{\mathcal{F}}(F)\right|\\
&= \sum\nolimits_{F}\left|\mu_{\mathcal{F}}(F)\right|
\end{aligned}
$$

which contradicts (1). Let $w_0 \in \mu_{\mathcal{W}}(F_0)\backslash\mu_{\mathcal{F}}(F_0)$. Then, the pair (w_0, F_0) blocks $\mu_{\mathcal{F}}$, since we have that $w_0 \notin \mu_{\mathcal{F}}(F_0)$, $F_0 = \mu_{\mathcal{W}}(w_0)P(w_0)\mu_{\mathcal{F}}(w_0)$, and

$$
w_0 \in Ch\left(\mu_{\mathcal{F}}(F_0)\cup\{w_0\}, P(F_0)\right). \tag{2}
$$

Condition (2) holds because $|\mu_{\mathcal{F}}(F_0)| < q_{F_0}$ and $w_0 \in \mu_{\mathcal{W}}(F_0)$ imply, by (C) and (D), that $w_0P(F_0)\emptyset$. Therefore, Case 1 is false.

Case 2: $\bigcup_F \mu_{\mathcal{F}}(F) \nsubseteq \bigcup_F \mu_{\mathcal{W}}(F)$. In this case, there exists a worker $\tilde{w} \in \bigcup_F \mu_{\mathcal{F}}(F)\backslash\bigcup_F \mu_{\mathcal{W}}(F)$. Hence, we can find $\tilde{F}$ such that $\tilde{w} \in \mu_{\mathcal{F}}(\tilde{F})$ while $\tilde{w} \notin \bigcup_F \mu_{\mathcal{W}}(F)$. But, this says that by (OO)

$$
\mu_{\mathcal{W}}(\tilde{w}) = \emptyset P(\tilde{w})\mu_{\mathcal{F}}(\tilde{w}) = \tilde{F}
$$

which contradicts that $\mu_{\mathcal{F}}$ is individually rational for $\tilde{w}$. ■

Corollary 2 *Assume firms have $q-$separable and substitutable preferences. If w is single in a stable matching μ, then w is single in any stable matching $\widehat{\mu}$.*

Examples 3 and 4 below show that the statement of Corollary 2 is false without either q-separability or substitutability

Example 3 Let $\mathcal{F} = \{F_1, F_2\}$ and $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with a substitutable profile of preferences P defined by

$$
\begin{aligned}
P(F_1) =&\ \{w_1,w_2\},\{w_1,w_3\},\{w_2,w_4\},\{w_3,w_4\},\{w_1\},\{w_2\},\{w_3\},\{w_4\},\\
P(F_2) =&\ \{w_3\},\{w_1,w_2\},\{w_1\},\{w_2\},\{w_4\},\\
P(w_1) =&\ F_2, F_1,\\
P(w_2) =&\ F_2, F_1,\\
P(w_3) =&\ F_1, F_2, \text{ and}\\
P(w_4) =&\ F_1, F_2.
\end{aligned}
$$

The substitutable preference ordering $P(F_2)$ is not q_{F_2}-separable for any $q_{F_2} \geq 1$. Axiom (D) is violated. The two optimal stable matchings

$$
\mu_{\mathcal{F}} = \begin{pmatrix} F_1 & F_2 & \emptyset \\ \{w_1,w_2\} & \{w_3\} & \{w_4\} \end{pmatrix} \text{ and}
$$

$$\mu_{\mathcal{W}} = \begin{pmatrix} F_1 & F_2 \\ \{w_3, w_4\} & \{w_1, w_2\} \end{pmatrix}$$

have the property that $\mu_{\mathcal{F}}(w_4) = \emptyset$ and $\mu_{\mathcal{W}}(w_4) = F_1$.

Example 4 Let $\mathcal{F} = \{F_1, F_2\}$ and $\mathcal{W} = \{w_1, w_2, w_3, w_4, w_5\}$ be the two sets of agents with a profile of preferences P defined by

$$\begin{array}{ll} P(F_1) = & \{w_1,w_2\},\{w_3,w_4\},\{w_1,w_3\},\{w_1,w_4\},\{w_2,w_3\},\{w_2,w_4\},\{w_1\},\{w_2\}, \\ & \{w_3\},\{w_4\}, \\ P(F_2) = & \{w_3,w_5\},\{w_1,w_2\},\{w_1,w_3\},\{w_1,w_5\},\{w_2,w_3\},\{w_2,w_5\},\{w_1\},\{w_2\}, \\ & \{w_3\},\{w_5\}, \\ P(w_1) = & F_2, F_1, \\ P(w_2) = & F_2, F_1, \\ P(w_3) = & F_1, F_2, \\ P(w_4) = & F_1, F_2, \text{ and} \\ P(w_5) = & F_2. \end{array}$$

Notice that while P is $(2,2)-$separable $P(F_1)$ is not substitutable since $w_1 \in Ch(\{w_1, w_2, w_3, w_4\}, P(F_1))$ but $w_1 \notin Ch(\{w_1, w_3, w_4\}, P(F_1))$. The following two stable matchings

$$\mu_1 = \begin{pmatrix} F_1 & F_2 & \emptyset \\ \{w_1, w_2\} & \{w_3, w_5\} & \{w_4\} \end{pmatrix} \text{ and}$$

$$\mu_2 = \begin{pmatrix} F_1 & F_2 & \emptyset \\ \{w_3, w_4\} & \{w_1, w_2\} & \{w_5\} \end{pmatrix}$$

have different single workers. Axiom (OO) is violated since $\mu_{\mathcal{W}}$ does not exit and $\mu_1 = \mu_{\mathcal{F}}$ is not the worst stable matching for w_5.

Proposition 3 below states that properties (1) and (2) also hold under our set of axioms (remember that they hold for the college admissions model).

Proposition 3 *Assume $\mathcal{P}$ satisfies axioms* (OO), (A), *and* (D). *Then, for all $P \in \mathcal{P}$, all pairs $\mu, \mu' \in S(P)$, and all $F \in \mathcal{F}$:*
(a) $|\mu(F)| = |\mu'(F)|$.
(b) If $|\mu(F)| < q_F$ *then* $\mu(F) = \mu'(F)$.

P roof. To prove (a) we will show that if $\mu \in S(P)$ then $|\mu(F)| = |\mu_{\mathcal{W}}(F)|$ for all $F \in \mathcal{F}$. Assume the contrary; that is, suppose there exist

$\mu \in S(P)$ and $F \in \mathcal{F}$ such that $|\mu(F)| \neq |\mu_{\mathcal{W}}(F)|$.
Case 1: Assume $|\mu_{\mathcal{W}}(F)| > |\mu(F)|$ holds. It means that there exist $w \in \mu_{\mathcal{W}}(F) \backslash \mu(F)$. Therefore,

$$w \notin \mu(F). \tag{3}$$

By (OO) we have that

$$F = \mu_{\mathcal{W}}(w)\, P(w)\, \mu(w). \tag{4}$$

By (A) and $w \in \mu_{\mathcal{W}}(F)$ we have that $wP(F)\,\emptyset$, which implies, by (D), that

$$w \in Ch(\mu(F) \cup \{w\}, P(F)) \tag{5}$$

since $|\mu(F)| < q_F$. Therefore, conditions (3), (4), and (5) imply that μ is not stable since (w, F) blocks it.
Case 2: Assume $|\mu_{\mathcal{W}}(F)| < |\mu(F)|$ holds. We claim that we can find $\hat{F} \in \mathcal{F}$ such that $\left|\mu(\hat{F})\right| < \left|\mu_{\mathcal{W}}(\hat{F})\right|$, otherwise the number of workers matched at μ would be greater than the number of workers matched at $\mu_{\mathcal{W}}$, contradicting Proposition 1. Applying Case 1 to $\hat{F}$ we conclude that μ is not stable.
To prove (b) suppose that $\mu \in S(P)$ and $|\mu(F)| < q_F$ for some $F \in \mathcal{F}$. By (a) we have that $|\mu_{\mathcal{W}}(F)| = |\mu(F)| < q_F$ holds. It is sufficient to show that $\mu(F) = \mu_{\mathcal{W}}(F)$. To get a contradiction let $w \in \mu_{\mathcal{W}}(F) \backslash \mu(F)$ and, by Proposition **??**, let $F' \neq F$ be such that

$$w \in \mu(F'). \tag{6}$$

Using (A) and Proposition 1 we obtain that $wP(F)\,\emptyset$ and (OO) implies that

$$\mu_{\mathcal{W}}(w) = FP(w)\, F' = \mu(w). \tag{7}$$

By assumption and (a) the condition $|\mu(F)| = |\mu_{\mathcal{W}}(F)| < q_F$ holds. Therefore, by (D)

$$w \in Ch(\mu(F) \cup \{w\}, P(F)). \tag{8}$$

Conditions (6), (7), and (8) imply that (w, F) blocks μ. ■

Since Proposition 3 implies that the set of unmatched firms is the same in all stable matchings, we can state the following Corollary.

Corollary 4 *Assume firms have $q-$separable and substitutable preferences. If F is single in a stable matching μ then F is single in any stable matching $\widehat{\mu}$.*

Examples 5 and 6 below show that the statement of Corollary 4 is false without either q-separability or substitutability.

Example 5 Let $\mathcal{F} = \{F_1, F_2, F_3\}$ and $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with a substitutable profile of preferences P defined by

$$\begin{aligned}
P(F_1) = &\ \{w_1, w_2\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_2, w_4\}, \{w_3, w_4\}, \{w_1\}, \{w_2\}, \\
&\ \{w_3\}, \{w_4\}, \\
P(F_2) = &\ \{w_3\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2\}, \{w_1\}, \{w_2\}, \\
P(F_3) = &\ \{w_4\}, \\
P(w_1) = &\ F_2, F_1, \\
P(w_2) = &\ F_2, F_1, \\
P(w_3) = &\ F_1, F_2, \text{ and} \\
P(w_4) = &\ F_1, F_3.
\end{aligned}$$

The preference ordering $P(F_2)$ is not q_{F_2}-separable for any $q_{F_2} \geq 1$ and violates axiom (D). The two optimal stable matchings

$$\mu_{\mathcal{F}} = \begin{pmatrix} F_1 & F_2 & F_3 \\ \{w_1, w_2\} & \{w_3\} & \{w_4\} \end{pmatrix} \text{ and}$$

$$\mu_{\mathcal{W}} = \begin{pmatrix} F_1 & F_2 & F_3 \\ \{w_3, w_4\} & \{w_1, w_2\} & \emptyset \end{pmatrix}$$

have the property that $\mu_{\mathcal{F}}(F_3) \neq \emptyset$ and $\mu_{\mathcal{W}}(F_3) = \emptyset$.

Example 6 Let $\mathcal{F} = \{F_1, F_2, F_3\}$ and $\mathcal{W} = \{w_1, w_2, w_3, w_4, w_5\}$ be the two sets of agents with a $(2, 2, 2)$ −separable profile of preferences P defined by

$$\begin{aligned}
P(F_1) = &\ \{w_1, w_2\}, \{w_3, w_5\}, \{w_2, w_5\}, \{w_1, w_5\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_1\}, \\
&\ \{w_2\}, \{w_3\}, \{w_5\}, \\
P(F_2) = &\ \{w_3, w_4\}, \{w_1, w_2\}, \{w_2, w_4\}, \{w_1, w_4\}, \{w_1, w_3\}, \{w_2, w_3\}, \{w_3\}, \\
&\ \{w_1, w_2\}, \{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}, \\
P(F_3) = &\ \{w_5\}, \\
P(w_1) = &\ F_2, F_1, \\
P(w_2) = &\ F_2, F_1, \\
P(w_3) = &\ F_1, F_2, \\
P(w_4) = &\ F_2, \text{ and} \\
P(w_5) = &\ F_1, F_3.
\end{aligned}$$

Notice that $P(F_2)$ is not substitutable since $w_3 \in Ch(\{w_1, w_2, w_3, w_4\}, P(F_2))$ but $w_3 \notin Ch(\{w_1, w_2, w_3\}, P(F_2))$. The two stable matchings

$$\mu_1 = \begin{pmatrix} F_1 & F_2 & F_3 \\ \{w_1, w_2\} & \{w_3, w_4\} & \{w_5\} \end{pmatrix} \text{ and}$$

$$\mu_2 = \begin{pmatrix} F_1 & F_2 & F_3 & \emptyset \\ \{w_3, w_5\} & \{w_1, w_2\} & \emptyset & \{w_4\} \end{pmatrix}$$

have the property that F_3 is single in μ_2 while it is matched with w_5 in μ_1. Axiom (OO) is violated since $\mu_{\mathcal{W}}$ does not exist and $\mu_1 = \mu_{\mathcal{F}}$ is not the worst stable matching for w_4.

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