Generalized Cauchy means

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Abstract

Given two means M and N, the operator $\mathcal{M}_{M,N}$ asigning to a given mean μ the mean

$$\mathcal{M}_{M,N}(\mu)(x,y) = M(\mu(x,N(x,y)),\mu(N(x,y),y))$$

has been defined in [7] in connection with Cauchy means: the Cauchy mean generated by the pair f, g of continuous and strictly monotonic functions is the unique solution μ to the fixed point equation

$$\mathcal{M}_{A_{(f)},A_{(g)}}(\mu) = \mu,$$

where $A_{(f)}$ and $A_{(g)}$ are the quasiarithmetic means respectively generated by f and g. In this article, the operator $\mathcal{M}_{M,N}$ is studied under less restrictive conditions and a general fixed point point theorem is derived from an explicit formula for the iterates $\mathcal{M}^n_{M,N}$. The concept of class of generalized Cauchy means associated to a given family of mixing pairs of means is introduced and some distinguished families of pairs are presented. The question of equality in these classes of means remains a challenging open problem.

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1 Introduction and preliminaries

Given a real interval I, a function $M : I^n \to I$ defined on I is a mean when it is *internal*; i.e., when it satisfies the property

$$\min\{x_1,\ldots,x_n\} \le M(x_1,\ldots,x_n) \le \max\{x_1,\ldots,x_n\},\ x_1,\ldots,x_n \in I. \tag{1}$$

The mean is said to be *strict* when the inequalities in (1) are strict provided that $x_i \neq x_j$ for a pair $i \neq j$ (*strict internality*). As a consequence of (1), the points in the diagonal $\Delta(I^n) = \{(x,x,\ldots,x) :\ x \in I\}$ play a special role: on one hand, the equality

$$M(x,\ldots,x) = x,\ x \in I, \tag{2}$$

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holds for every mean M, so that means are *reflexive* functions; on the other, a mean M turns out to be continuous on every point of $\Delta(I^n)$. A mean M is said to be *symmetric* when

$$M(x_{\sigma_1}, \ldots, x_{\sigma_n}) = M(x_1, \ldots, x_n), \tag{3}$$

for every permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ of the set of indexes $S_n = \{1, \ldots, n\}$. The linear means $L_\alpha(x_1, \ldots, x_n) = \sum_{i=1}^n \alpha_i x_i$ ($\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$) as well as the linear symmetric two-variables mean $M_\alpha(x, y) = (1-\alpha)\min\{x, y\} + \alpha \max\{x, y\}$ allow making useful explicit computations.

The product order in I^n is defined by

$$(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n) \text{ if and only if } x_i \leq y_i,\ i = 1, 2, \ldots, n;$$

and it will be written $(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n)$ when $x_i < y_i,\ i = 1, 2, \ldots, n$. A mean M is said to be *isotone* when preserves the product order in I^n; i.e., when $M(x_1, \ldots, x_n) \leq M(y_1, \ldots, y_n)$ provided that $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$. M is said to be *strictly isotone* when $M(x_1, \ldots, x_n) < M(y_1, \ldots, y_n)$ provided that $(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n)$.

If M is a continuous mean and $f : I \to \mathbb{R}$ is a strictly monotonic and continuous function (i.e., a homeomorphism from I onto $f(I)$), the *f-conjugated* M_f of M is the (continuous) mean defined on $f(I)$ by

$$M_f = f \circ M \circ (\overbrace{f^{-1} \times \cdots \times f^{-1}}^{n \text{ times}});$$

i.e.,

$$M_f(y_1, \ldots, y_n) = f(M(f^{-1}(y_1), \ldots, f^{-1}(y_n))),\ y_1, \ldots, y_n \in f(I).$$

When M is a given mean and f varies on the set of homeomorphism from I onto $f(I)$, then M_f runs along the entire *class of conjugation* of M. For example, the class of conjugation of the arithmetic mean in n variables $A(x_1, \ldots, x_n) = (x_1 + \cdots + x_n)/n$ is the family of *quasiarithmetic means* in n-variables $QA_n(I) = \{A_{(f)} : f : I \to \mathbb{R} \text{ homeomorphism}\}$, where

$$A_{(f)}(x_1, \ldots, x_n) = f^{-1}\left(\frac{f(x_1) + \cdots + f(x_n)}{n}\right) = A_{f^{-1}}(x_1, \ldots, x_n).$$

The means considered throughout this paper will be *continuous means*; i.e., means that are continuous functions. A mean M satisfying the inequality

$$|M(y_1, \ldots, y_n) - M(x_1, \ldots, x_n)| \leq \max_{i=1,2,\ldots,n} |y_i - x_i|; \tag{4}$$

for every pair $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in I^n$ is said to be a *nonexpansive mean*; while it is said *(C)-nonexpansive* when the class of conjugation of M contains a nonexpansive mean; i.e., when there exists a homeomorphism $f : I \to \mathbb{R}$

such that M_f is nonexpansive. In this paper, a crucial role is reserved for (C)-nonexpansive means.

After reminding these elementary notions, let us pay attention to the main subject of this paper. Given a pair of two variables means M and N on an interval I, the *mixing operator* $\mathcal{M}_{M,N}$ assigns to a mean μ another mean $\mathcal{M}_{M,N}(\mu)$ defined by

$$\mathcal{M}_{M,N}(\mu)(x,y) = M(\mu(x,N(x,y)),\mu(N(x,y),y)),\ x,y \in I; \tag{5}$$

the relevant question being that of solving the fixed point equation

$$\mathcal{M}_{M,N}(\mu) = \mu. \tag{6}$$

The mixing operator was considered for the first time in [7]. A mean μ solving equation (6) was named there a *mixing mean* of the pair (M,N) and, in order to show the existence of mixing means, the Knaster-Tarski Fixed Point Theorem was applied to $\mathcal{M}_{M,N}$ when defined on the family of pairs (M,N) composed by two *generalized symmetric means* M, N; i.e., reflexive, symmetric and isotone functions M, $N : I\times I \to I$. For a pair (M,N) belonging to this family of means there are, in general, more that one mixing mean. An extreme case of multiplicity is furnished by the pair (max, min), since the equation $\mathcal{M}_{\max,\min}(\mu) = \mu$ is satisfied by every generalized symmetric mean.

Even if the uniqueness of mixing means is, in the above context, a hopeless question, it turns out to be that a unique solution to equation (6) exists when (M,N) belong to certain families of pairs of means. A relevant family of pairs is identified in the following ([7], Theor. 2):

Theorem 1 *If $M = A_{(f)}$ and $N = A_{(g)}$; then the equation*

$$\mathcal{M}_{A_{(f)},A_{(g)}}(\mu) = \mu$$

has a unique solution μ given by

$$\mu(x,y) = \begin{cases} f^{-1}\left(\frac{1}{g(y)-g(x)}\int_x^y f(\xi)\,dg(\xi)\right), & x \neq y \\ x, & x = y \end{cases}. \tag{7}$$

The mean defined by (7) is known as *Cauchy mean generated by f and g* (cf. [7] and [8], pg. 405 and ff.) by its connections with the Cauchy Mean Value Theorem. Indeed, if g is differentiable and $F(x) = \int_{x_0}^x f(\xi)\,dg(\xi)$ for a certain $x_0 \in I$; then (7) can be rewritten as

$$\frac{F(y)-F(x)}{g(y)-g(x)} = f(\mu(x,y)) = \frac{F'(\mu(x,y))}{g'(\mu(x,y))}. \tag{8}$$

Cauchy means generalize Lagrangian means (which are related to the Lagrange Mean Value Theorem): the *Lagrangian mean generated by f* is the Cauchy mean generated by f and $g = id$. More precisely, the class of Cauchy means is the smallest closed under conjugacy class of means containing the class of Lagrangian means ([7]).

Now, let us consider a family of pairs of means $\mathcal{F}$ such that, for every $(M, N) \in \mathcal{F}$, there exists a *unique* solution to the fixed point equation (6). Throughout this paper, a family $\mathcal{F}$ with this property is named a *mixing family* of pairs, while the unique mean μ satisfying $\mathcal{M}_{M,N}(\mu) = \mu$ for a given $(M, N) \in \mathcal{F}$ is said to be the *generalized Cauchy mean corresponding to the pair* (M, N). In generalizing the notation $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ used in [7] to denote the Cauchy mean generated by f and g, the symbol $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ will be employed for the generalized Cauchy mean corresponding to the pair (M, N). The class $GC(\mathcal{F})$ of *generalized Cauchy means* associated to a mixing family of pairs $\mathcal{F}$ is defined by

$$GC(\mathcal{F}) = \left\{ \begin{bmatrix} M \\ N \end{bmatrix} : \ (M, N) \in \mathcal{F} \right\}.$$

The identification of non trivial mixing families of pairs constitutes, in this approach, a question of capital importance. A general response to this question is offered in the subsequent sections of this paper. In Section 2, Dyadic iteration and binary tree expansion, two iterative algorithms involving means, enable us to write a formula for the iterates of the mixing operator $\mathcal{M}_{M,N}$. Based on this formula, a class of pairs $\mathcal{F}_{\mathrm{G}}$ with unique mixing mean is presented in Section 3: the class of pairs $(A_{(f)}, A_{(g)})$ composed by quasiarithmetic means is far exceeding by $\mathcal{F}_{\mathrm{G}}$. Examples and commentaries are gather together in Section 4, while the final Section 5 is devoted to study the basic properties of generalized Cauchy means. At the end of this section, the challenging problem of representation of generalized Cauchy means is commented.

2 A closed form expression for $\mathcal{M}^n_{M,N}$

In order to derive a closed form expression for the iterations $\mathcal{M}^n_{M,N}$ of the operator $\mathcal{M}_{M,N}$ defined by (5), two general algorithms involving compositions of a two variables function $F : I \times I \to I$ are now presented. The first one, named dyadic iteration, inductively defines a family $\{F^d(x, y) : d \in D([0, 1])\}$ of *dyadic iterates* on $[x, y]$ of F as follows (cf. [5], [4]): the first step consists in setting

$$F^0(x, y) \equiv x, \ \ F^1(x, y) \equiv y; \tag{9}$$

then, assuming that $F^{\frac{j}{2^n}}(x, y)$ is known for $n \geq 0$ and for every $0 \leq j \leq 2^n$, the inductive step establishes that

$$F^{\frac{k}{2^{n+1}}}(x, y) = \begin{cases} F^{\frac{h}{2^n}}(x, y), & \text{if } k = 2h, \ 0 \leq h \leq 2^n \\ \\ F\left(F^{\frac{h}{2^n}}(x, y), F^{\frac{h+1}{2^n}}(x, y)\right), & \text{if } k = 2h + 1, \ 0 \leq h \leq 2^n - 1 \end{cases}. \tag{10}$$

Two dyadic fractions p, $q \in D[0,1]$ are said to be *consecutive dyadic fractions* when there exist $m \in \mathbb{N}_0$ and $1 \leq k \leq 2^m$, such that

$$p = \frac{k-1}{2^m} \text{ and } q = \frac{k}{2^m}. \tag{11}$$

A useful property of dyadic iterations is stated by the following:

Lemma 2 *If p, $q \in D[0,1]$ are consecutive dyadic fractions; then, the equality*

$$F^r(F^p(x,y), F^q(x,y)) = F^{(1-r)p+rq} \tag{12}$$

holds for every dyadic fraction $r \in D[0,1]$.

Proof. Assume that the fractions p, q are given by (11) and that $r = j/2^n$. Let us prove the lemma by induction on n. If $n = 0$ or $n = 1$, the equality (12) reduces to trivial identities. In fact, for $n = 0$, the equality

$$F^j(F^p, F^q) = F^{(1-j)p+jq}$$

is true by (9) whichever be $j = 0, 1$. Analogously, if $n = 1$; then,

$$F^{\frac{j}{2}}(F^p, F^q) = F^{\left(1-\frac{j}{2}\right)p+\frac{j}{2}q}$$

is a consequence of (9) for $j = 0, 2$ while, taking into account that p and q are consecutive dyadic fractions, it is immediately derived from (10) for $j = 1$.

Now, suppose that the lemma is true for $r = j/2^n$ with $n \geq 1$ and every $j = 0, 1, \ldots, 2^n$; let us prove that it is true also for $j/2^{n+1}$with $j = 0, 1, \ldots, 2^{n+1}$. Indeed, if j is even, that is if $j = 2i$, then $j/2^{n+1} = i/2^n$ and (12) is true by the inductive hypothesis. On the other hand, if $j = 2i - 1$ is odd, then, by (10) and the inductive hypothesis, it can be written

$$\begin{aligned} F^{\frac{2i-1}{2^{n+1}}}(F^p, F^q) &= F(F^{\frac{i-1}{2^n}}(F^p, F^q), F^{\frac{i}{2^n}}(F^p, F^q)) \\ &= F(F^{\left(1-\frac{i-1}{2^n}\right)p+\frac{i-1}{2^n}q}, F^{\left(1-\frac{i}{2^n}\right)p+\frac{i}{2^n}q}), \end{aligned} \tag{13}$$

and, in view of

$$\begin{aligned} \left(1-\frac{i-1}{2^n}\right)\frac{k}{2^m} + \frac{i-1}{2^n}\frac{k+1}{2^m} &= \frac{2^n k + i - 1}{2^{m+n}}, \\ \left(1-\frac{i}{2^n}\right)\frac{k-1}{2^m} + \frac{i}{2^n}\frac{k+1}{2^m} &= \frac{2^n k + i}{2^{m+n}}, \end{aligned}$$

are consecutive dyadic fractions,

$$F(F^{\left(1-\frac{i-1}{2^n}\right)p+\frac{i-1}{2^n}q}, F^{\left(1-\frac{i}{2^n}\right)p+\frac{i}{2^n}q}) = F^{\frac{\left(1-\frac{i-1}{2^n}\right)p+\frac{i-1}{2^n}q+\left(1-\frac{i}{2^n}\right)p+\frac{i}{2^n}q}{2}} = F^{\left(1-\frac{2i-1}{2^{n+1}}\right)p+\frac{2i-1}{2^{n+1}}q}. \tag{14}$$

From (13) and (14) it is obtained

$$F^{\frac{2i-1}{2^{n+1}}}(F^p, F^q) = F^{\left(1-\frac{2i-1}{2^{n+1}}\right)p+\frac{2i-1}{2^{n+1}}q},$$

which completes the inductive proof. ■

In general, dyadic iterations of a symmetric mean are not symmetric; rather, one have the following:

Lemma 3 *For every $d \in D([0,1])$,*

$$M^d(y,x) = M^{1-d}(x,y),\ \ x, y \in I.$$

Proof. The simple inductive proof of this lemma can be found in [4]. ■

It should be observed that the dyadic iterations M^d of a mean M are means. Furthermore, for a strict continuous mean M, the dyadic iterations M^d can be extended from $D([0,1])$ to the whole interval $[0,1]$ by taking limits: for a given $\delta \in (0,1)$, there exists an increasing sequence $\{d_n\}_{n=1}^{\infty} \subseteq D([0,1])$ such that $d_n \uparrow \delta$ when $n \uparrow +\infty$, and $M^\delta(x,y)$ is defined by

$$M^\delta(x,y) = \lim_{n\uparrow+\infty} M^{d_n}(x,y). \tag{15}$$

Namely, the following result, whose proof can be found in [4] (see also [5]), holds.

Theorem 4 *For a strictly internal and reflexive function M, the function $d \mapsto M^d(x,y)$ defined on $D([0,1])$ is monotonically extended by (15) to the interval $[0,1]$. The extension $\delta \mapsto M^\delta(x,y)$ is a continuous function provided that M is a continuous mean. $\delta \mapsto M^\delta(x,y)$ is a monotonic function; increasing when $x < y$ and decreasing when $x > y$. Furthermore, M^δ is a continuous mean when $0 < \delta < 1$ and $M^0(x,y) = x,\ M^1(x,y) = y$.*

The second algorithm also applies to a function $F : I \times I \to I$, but this time the outcome is a family $\{F^{(n)} : I^{2^n} \to I\}$ in an increasing number of variables. Concretely, the *binary tree extension* $F^{(n)}$ of F is inductively defined by

$$F^{(1)}(x_1, x_2) = F(x_1, x_2) \tag{16}$$

and

$$F^{(n)}(x_1,\ldots,x_{2^n}) = F(F^{(n-1)}(x_1,\ldots,x_{2^{n-1}}), F^{(n-1)}(x_{2^{n-1}+1},\ldots,x_{2^n})),\ n > 1. \tag{17}$$

The simple inductive proof of the following result will be omitted.

Lemma 5 *The equality*

$$F^{(n)}(x_1,\ldots,x_{2^n}) = F^{(n-k)}(F^{(k)}(w_1^k,\ldots,w_{n-k}^k)),$$

where

$$w_1^k = (x_1,\ldots,x_{2^k}),\ w_2^k = (x_{2^k+1},\ldots,x_{2^k+2^k}),\ldots,\ w_{n-k}^k = (x_{2^n-2^k+1},\ldots,x_{2^n}),$$

holds for every $1 \le k \le n-1$.

Particularly useful is the case $k = 1$:

$$F^{(n)}(x_1, \ldots, x_{2^n}) = F^{(n-1)}((F(x_{2j-1}, x_{2j}))_{j=1}^{2^{n-1}}).$$

Note that a repeated application of Lemma 5 gives

$$F^{(n)} = F^{(n_1)}(F^{(n_2)}(\ldots(F^{(n_k)}, \ldots, F^{(n_k)}),$$

provided that $n_1 + n_2 + \ldots + n_k = n$.

The algorithms defined in the precedent paragraphs have a common characteristic: when $F = A_{(f)}$ is a quasiarithmetic mean, $A^d_{(f)}$, $d \in D([0,1])$, as well as $A^{(n)}_{(f)}$, $n \in \mathbb{N}$, can be computed in a closed form. As an easy inductive reasoning shows, the dyadic iteration $A^d_{(f)}$ of the quasiarithmetic mean $A_{(f)}$ are given by

$$A^d_{(f)}(x, y) = f^{-1}\left((1-d)f(x) + df(y)\right), \tag{18}$$

thus coinciding with the *weighted quasiarithmetic mean* with weight d (and same generator f). In its turn, the binary tree extension $A^{(n)}_{(f)}$ takes the form

$$A^{(n)}_{(f)}(x_1, \ldots, x_{2^n}) = f^{-1}\left(\frac{1}{2^n}\sum_{j=1}^{2^n} f(x_j)\right), \tag{19}$$

so that $A^{(n)}_{(f)}(x_1, \ldots, x_{2^n})$ coincides with the quasiarithmetic mean $A_{(f)}(x_1, \ldots, x_{2^n})$ in 2^n variables.

Many properties of a mean M are preserved by dyadic iteration or binary tree extension. Some of them are collected in the following result.

Lemma 6 *Let M be a two variables mean; then M^d, $d \in D([0,1])$, and $M^{(n)}$, $n \in \mathbb{N}$, are strict, continuous, (strictly) isotone, homogeneous or (C)-nonexpansive means provided that M is strict, continuous, (strictly) isotone, homogeneous or (C)-nonexpansive, respectively.*

Proof. Let us prove only the preservation of (C)-nonexpansiveness. Clearly, dyadic iterations and binary tree extensions commutes with conjugations; i.e., $(M_f)^d = \left(M^d\right)_f$ and $(M_f)^{(n)} = \left(M^{(n)}\right)_f$ for every homeomorphism f and every $d \in D([0,1])$ and $n \in \mathbb{N}$. In this way, it will be enough to prove that M^d or $M^{(n)}$ are nonexpansive when M provided that M is nonexpansive, but these follow by an inductive reasoning based respectively on (10) and (17). For instance, assuming that $M^{(n-1)}$ is nonexpansive for a certain $n \geq 2$, from (17) it is obtained

$$
\begin{aligned}
& \left|M^{(n)}((y_i)_{i=1}^{2^n}) - M^{(n)}((x_i)_{i=1}^{2^n})\right| \\
= & \left|M(M^{(n-1)}((y_i)_{i=1}^{2^{n-1}}), M^{(n-1)}((y_i)_{i=2^{n-1}+1}^{2^n})) - M(M^{(n-1)}((x_i)_{i=1}^{2^{n-1}}), M^{(n-1)}((x_i)_{i=2^{n-1}+1}^{2^n}))\right| \\
\leq & \max\left\{\left|M^{(n-1)}((y_i)_{i=1}^{2^{n-1}}) - M^{(n-1)}((x_i)_{i=1}^{2^{n-1}})\right|, \left|M^{(n-1)}((y_i)_{i=2^{n-1}+1}^{2^n}) - M^{(n-1)}((x_i)_{i=2^{n-1}+1}^{2^n})\right|\right\} \\
\leq & \max\left\{\max_{i=1,\ldots,2^{n-1}} |y_i - x_i|, \max_{i=2^{n-1}+1,\ldots,2^n} |y_i - x_i|\right\} \\
= & \max_{i=1,\ldots,2^n} |y_i - x_i|.
\end{aligned}
$$

■

Symmetry of a mean M is a property generally lost by its binary tree extensions $M^{(n)}$. This fact is already manifested for $n = 2$, since $M^{(2)}(x_1, x_2, x_3, x_4)$ is a symmetric mean if and only, besides of the symmetry condition, the *bisymmetry equation*

$$
M(M(x_1, x_2), M(x_3, x_4)) = M(M(x_1, x_3), M(x_2, x_4)), \tag{20}
$$

is satisfied by M. Indeed, the following result holds.

Theorem 7 *Assume that M is a symmetric mean; then*

i) *$M^{(n)}$ is symmetric for every $n \in \mathbb{N}$ if and only if the equation (20) is satisfied by M; moreover,*

ii) *if M is continuous and strictly isotone, $M^{(n)}$ is symmetric for every $n \in \mathbb{N}$ if and only if M is quasiarithmetic.*

Proof. The proof of this theorem will be only sketched here. The necessity and sufficiency of (20) is immediate for $n = 2$ and the proof of **i)** is completed by induction. To prove **ii)**, the Aczél´s characterization of quasiarithmetic means as symmetric, continuous and strictly isotone solutions to equation (20) ([1], Sect. 6.4) is employed. ■

Now, the iterates of the mixing operator $\mathcal{M}_{M,N}$ are expressed in terms of dyadic iterations of N and binary tree extensions of M.

Theorem 8 *For every $n \in \mathbb{N}$, the iterate $\mathcal{M}_{M,N}^n$ of $\mathcal{M}_{M,N}$ is expressed by*

$$
\mathcal{M}_{M,N}^n(\mu) = M^{(n)}\left(\left(\mu\left(N^{\frac{j-1}{2^n}}(x,y), N^{\frac{j}{2^n}}(x,y)\right)\right)_{j=1}^{2^n}\right). \tag{21}
$$

Observe that, when $M = N$,

$$
\mathcal{M}_{M,M}^n(M) = M^{(n)}\left(\left(M\left(M^{\frac{j-1}{2^n}}(x,y), M^{\frac{j}{2^n}}(x,y)\right)\right)_{j=1}^{2^n}\right) = M^{(n)}\left(\left(M^{\frac{2j-1}{2^{n+1}}}(x,y)\right)_{j=1}^{2^n}\right)
$$

by (10).

Proof. For $n = 1$ formula (21) gives

$$\mathcal{M}^1_{M,N}(\mu) = M\left(\left(\mu\left(N^{\frac{j-1}{2}}(x,y), N^{\frac{j}{2}}(x,y)\right)\right)_{j=1}^{2}\right) = \mathcal{M}_{M,N}(\mu).$$

Assuming that (21) holds for $n \geq 1$, (5) and (17) yield

$$\begin{aligned}
\mathcal{M}^{n+1}_{M,N}(\mu) &= M(\mathcal{M}^n_{M,N}(\mu)(x, N(x,y)), \mathcal{M}^n_{M,N}(\mu)(N(x,y), y)) \\
&= M\left(M^{(n)}\left(\left(\mu\left(N^{\frac{j-1}{2^n}}(x, N(x,y)), N^{\frac{j}{2^n}}(x, N(x,y))\right)\right)_{j=1}^{2^n}\right),\right. \\
&\qquad \left. M^{(n)}\left(\left(\mu\left(N^{\frac{j-1}{2^n}}(N(x,y), y), N^{\frac{j}{2^n}}(N(x,y), y)\right)\right)_{j=1}^{2^n}\right)\right);
\end{aligned}$$

but, by Lemma 2 with $p = 0$, $q = 1/2$ and $p = 1/2$, $q = 1$, the equalities

$$N^{\frac{k}{2^n}}(x, N(x,y)) = N^{\left(1-\frac{k}{2^n}\right)0+\frac{k}{2^n}\frac{1}{2}}(x,y) = N^{\frac{k}{2^{n+1}}}(x,y)$$

and

$$N^{\frac{k}{2^n}}(N(x,y), y) = N^{\left(1-\frac{k}{2^n}\right)\frac{1}{2}+\frac{k}{2^n}}(x,y) = N^{\frac{1}{2}+\frac{k}{2^{n+1}}}(x,y)$$

hold for every $k = 0, 1, \ldots, 2^n$, and therefore

$$\begin{aligned}
\mathcal{M}^{n+1}_{M,N}(\mu) &= M\left(M^{(n)}\left(\left(\mu\left(N^{\frac{j-1}{2^{n+1}}}(x,y), N^{\frac{j}{2^{n+1}}}(x,y)\right)\right)_{j=1}^{2^n}\right),\right. \\
&\qquad \left. M^{(n)}\left(\left(\mu\left(N^{\frac{1}{2}+\frac{j-1}{2^{n+1}}}(x,y), N^{\frac{1}{2}+\frac{j}{2^{n+1}}}(x,y)\right)\right)_{j=1}^{2^n}\right)\right) \\
&= M^{(n+1)}\left(\left(\mu\left(N^{\frac{j-1}{2^{n+1}}}(x,y), N^{\frac{j}{2^{n+1}}}(x,y)\right)\right)_{j=1}^{2^n}, \left(\mu\left(N^{\frac{1}{2}+\frac{j-1}{2^{n+1}}}(x,y), N^{\frac{1}{2}+\frac{j}{2^{n+1}}}(x,y)\right)\right)_{j=1}^{2^n}\right) \\
&= M^{(n+1)}\left(\left(\mu\left(N^{\frac{j-1}{2^{n+1}}}(x,y), N^{\frac{j}{2^{n+1}}}(x,y)\right)\right)_{j=1}^{2^{n+1}}\right),
\end{aligned}$$

which completes the inductive reasoning. ■

3 Generalized Cauchy means

In this section, the expression (21) for $\mathcal{M}^n_{M,N}$ given by Theor. 8 will be employed to study the fixed points of the mixing operator $\mathcal{M}_{M,N}$ in a context which is, in some sense, intermediate: on one hand, it is not so general as to require the application of fixed points theorems like that of Knaster-Tarski but, on the other, a class of means much more larger than the class of quasiarithmetic means is covered by the corresponding theory. The main tools in this approach are order theoretic and the assumption that M is an isotone mean will be essential since, if so, then the operator $\mathcal{M}_{M,N}$ turns out to be isotone; i.e., if μ, ν are two means and $\mu \leq \nu$, then $\mathcal{M}_{M,N}(\mu) \leq \mathcal{M}_{M,N}(\nu)$ (the isotonicity of $\mathcal{M}_{M,N}$ is strict provided that M is strictly isotone).

Let us begin by defining two sequences $\{L_n(x,y)\}$ and $\{U_n(x,y)\}$ of functions as follows: for every $n \in \mathbb{N}$,

$$L_n(x,y) = \begin{cases} M^{(n)}\left(\left(N^{\frac{j-1}{2^n}}(x,y)\right)_{j=1}^{2^n}\right), & x \le y \\ M^{(n)}\left(\left(N^{\frac{j}{2^n}}(x,y)\right)_{j=1}^{2^n}\right), & x \ge y \end{cases} \tag{22}$$

and

$$U_n(x,y) = \begin{cases} M^{(n)}\left(\left(N^{\frac{j}{2^n}}(x,y)\right)_{j=1}^{2^n}\right), & x \le y \\ M^{(n)}\left(\left(N^{\frac{j-1}{2^n}}(x,y)\right)_{j=1}^{2^n}\right), & x \ge y \end{cases}. \tag{23}$$

Since the second members of (22) and (23) are both compositions of means, L_n and U_n are means.

Theorem 9 *Let M, N two continuous means such that M is isotone and N is strict. Then, the means L_n and U_n enjoy the following properties:*

i) *L_n and U_n are continuous means satisfying the inequality*

$$L_n(x,y) \le U_n(x,y), \;\; x,y \in I; \tag{24}$$

ii) *there exist two means L_∞ and U_∞ such that, when $n \uparrow +\infty$, $L_n \nearrow L_\infty$ and $U_n \searrow U_\infty$, $x,y \in I$. L_∞ is l.s.c., while U_∞ is u.s.c. in I^2. L_∞ and U_∞ are comparable one each other:*

$$L_\infty(x,y) \le U_\infty(x,y), \;\; x,y \in I; \tag{25}$$

iii) *the equation*

$$K_{n+1}(x,y) = M(K_n(x, N(x,y)), K_n(N(x,y), y)) \tag{26}$$

is satisfied by $K_n = L_n$ and also by $K_n = U_n$, $n \in \mathbb{N}$.

Proof. The continuity of L_n and U_n is a consequence of Lemma 6. By Theor. 4, when $x \le y$,

$$N^{\frac{j-1}{2^n}}(x,y) \le N^{\frac{j}{2^n}}(x,y) \tag{27}$$

for every $j = 1, 2, \dots, 2^n$ and then, the inequality (24) in the case $x \le y$ follows from the isotonicity of M. Clearly, the inequality opposite to (27) holds when $x \ge y$, so that (24) also holds in this case. Now, by (23) and the case $k = 1$ of Lemma 5, when $x \le y$ it can be written

$$\begin{aligned} U_{n+1}(x,y) &= M^{(n+1)}\left(\left(N^{\frac{j}{2^{n+1}}}(x,y)\right)_{j=1}^{2^{n+1}}\right) \\ &= M^{(n)}\left(\left(M(N^{\frac{2j-1}{2^{n+1}}}(x,y), N^{\frac{2j}{2^{n+1}}}(x,y))\right)_{j=1}^{2^n}\right), \end{aligned}$$

and taking into account that $N^{\frac{2j-1}{2^{n+1}}}(x,y) \leq N^{\frac{2j}{2^{n+1}}}(x,y)$ by Theor. 4, the isotonicity of M implies

$$
\begin{aligned}
M^{(n)}\left(\left(M(N^{\frac{2j-1}{2^{n+1}}}(x,y),N^{\frac{2j}{2^{n+1}}}(x,y))\right)_{j=1}^{2^n}\right) &\leq M^{(n)}\left(\left(N^{\frac{2j}{2^{n+1}}}(x,y)\right)_{j=1}^{2^n}\right)\\
&= U_n(x,y),
\end{aligned}
$$

whence $U_{n+1}(x,y) \leq U_n(x,y)$ when $x \leq y$. If $x \geq y$, it can be similarly written

$$
\begin{aligned}
U_{n+1}(x,y) &= M^{(n+1)}\left(\left(N^{\frac{j-1}{2^{n+1}}}(x,y)\right)_{j=1}^{2^{n+1}}\right)\\
&= M^{(n)}\left(\left(M(N^{\frac{2(j-1)}{2^{n+1}}}(x,y),N^{\frac{2(j-1)+1}{2^{n+1}}}(x,y))\right)_{j=1}^{2^n}\right)\\
&\leq M^{(n)}\left(\left(N^{\frac{2(j-1)}{2^{n+1}}}(x,y)\right)_{j=1}^{2^n}\right) = U_n(x,y).
\end{aligned}
$$

Since

$$U_n \geq \min\{x,y\}\,,\ n \in \mathbb{N},$$

there exists the limit $U_\infty(x,y) = \lim_{n\uparrow+\infty} U_n(x,y)$ and, being the limit of a decreasing sequence of continuous means, it turn out to be an *u.s.c.* mean. A similar argument works in the case of L_n. By taking limits when $n \uparrow +\infty$, the inequality (25) follows from (24). Finally, to prove the equality (26) for $K_n = U_n$, let us note that, when $x \leq y$, the definition of the binary tree extension $M^{(n)}$ of M and Lemma 2 yield

$$
\begin{aligned}
U_{n+1}(x,y) &= M^{(n+1)}\left(\left(N^{\frac{j}{2^{n+1}}}(x,y)\right)_{j=1}^{2^{n+1}}\right)\\
&= M\left(M^{(n)}\left(N^{\frac{j}{2^{n+1}}}(x,y)\right)_{j=1}^{2^n}, M^{(n)}\left(N^{\frac{j}{2^{n+1}}}(x,y)\right)_{j=2^n+1}^{2^{n+1}}\right)\\
&= M\left(M^{(n)}\left(N^{\frac{j}{2^n}}(x,N(x,y))\right)_{j=1}^{2^n}, M^{(n)}\left(N^{\frac{j}{2^n}}(N(x,y),y)\right)_{j=1}^{2^n}\right)\\
&= M(U_n(x,N(x,y)),U_n(N(x,y),y)).
\end{aligned}
$$

The proof of (26) for $K_n = U_n$ and $x \geq y$ is analogous and the case $K_n = L_n$ can be similarly treated. ■

In what follows, the means L_∞ and U_∞ given by Theor. 9 are to be called *lower and upper means* corresponding to the mixing operator $\mathcal{M}_{M,N}$. The terminology is justified by the fact that the inequalities

$$U_n(x,y) \leq \mu(x,y) \leq L_n(y,x),\ x,y \in I, \tag{28}$$

are satisfied by every fixed point of the mixing operator $\mathcal{M}_{M,N}$ and therefore,

$$L_\infty(x,y) = \sup_{n\in\mathbb{N}} L_n(x,y) \leq \mu(x,y) \leq \inf_{n\in\mathbb{N}} U_n(y,x) = U_\infty(x,y),\ x,y \in I. \tag{29}$$

Furthermore, taking limits for $n \uparrow +\infty$ in the equality (26) it is seen that L_∞ and U_∞ are fixed points of $\mathcal{M}_{M,N}$. In other words, the set of mixing means of the pair (M, N) admit a minimum mean L_∞ and a maximum mean U_∞, after which the existence of a generalized Cauchy mean associated to $\mathcal{M}_{M,N}$ is guaranteed by the equality $L_\infty = U_\infty$. The converse is also true: if there exists a unique mean μ such that $\mathcal{M}_{M,N}(\mu) = \mu$; then $L_\infty = \mu = U_\infty$. In summary, the following theorem was established.

Theorem 10 *Let M, N be two continuous means such that M is isotone and N is strict. If L_∞ and U_∞ are the lower and upper means associated to the mixing operator $\mathcal{M}_{M,N}$ and μ is a mixing mean of the pair (M, N), then*

$$L_\infty(x, y) \leq \mu(x, y) \leq U_\infty(x, y), \;\; x, y \in I.$$

Furthermore, $L_\infty = U_\infty$ if and only if there exists the generalized Cauchy mean $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ corresponding to the pair (M, N).

Proof. See the previous discussion. ■

Note that, when there exists the generalized Cauchy mean $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ corresponding to the pair (M, N), it admits the representation

$$\begin{bmatrix} M \\ N \end{bmatrix}(x, y) = \lim_{n\uparrow+\infty} M^{(n)} \left(\left(N^{\frac{j}{2^n}}(x, y) \right)_{j=1}^{2^n} \right), \;\; x, y \in I. \tag{30}$$

A condition ensuring $L_\infty = U_\infty$ is furnished by the following result.

Theorem 11 *Assume that M, N fulfill the hypotheses made in Theor. 10 and, moreover, that M is a (C)-nonexpansive mean; then the equality $L_\infty(x, y) = U_\infty(x, y)$ holds for every $x, y \in I$.*

Note that $L_\infty(x, y) = U_\infty(x, y)$ is a continuous mean.

Proof. It will be sufficient to prove the theorem in the case in which M is nonexpansive. In fact, if M is (C)-nonexpansive on I; then, for any homeomorphism $f : I \to \mathbb{R}$, the f-conjugated $M_f = f \circ M \circ (f^{-1} \times f^{-1})$ is a nonexpansive mean on $f(I)$. Now, for $x, y \in I$, it can be written

$$\begin{aligned} \mathcal{M}_{M_f,N_f}(\mu_f)(f(x), f(y)) &= f\left(M(f^{-1}\left(\mu_f(f(x), f\left(N(x, y)\right))\right), f^{-1}\left(\mu_f(N(x, y), y)\right))\right) \\ &= f\left(M\left(\mu\left(x, N(x, y)\right), \mu(N(x, y), y)\right)\right) \\ &= f(\mathcal{M}_{M,N}(\mu))(x, y); \end{aligned}$$

whence μ is a fixed point of $\mathcal{M}_{M,N}$ if and only if

$$\mathcal{M}_{M_f,N_f}(\mu_f) = \mu_f;$$

i.e., if and only if μ_f is a fixed point of $\mathcal{M}_{M_f,N_f}$. In view of N_f is a continuous and strict mean on $f(I)$, this prove the assertion above. Now, after Lemma 6,

$M^{(n)}$ is nonexpansive for every $n \in \mathbb{N}$ provided that M is nonexpansive; thus, for every $x, y \in I$,

$$\begin{aligned} |U_n(x,y) - L_n(x,y)| &= \left| N^{(n)}\left(\left(N^{\frac{j}{2^n}}(x,y)\right)_{j=1}^{2^n}\right) - N^{(n)}\left(\left(N^{\frac{j-1}{2^n}}(x,y)\right)_{j=1}^{2^n}\right)\right| \\ &\le \left\| \left(N^{\frac{j}{2^n}}(x,y)\right)_{j=1}^{2^n} - \left(N^{\frac{j-1}{2^n}}(x,y)_{j=1}^{2^n}\right)_{j=1}^{2^n} \right\|_\infty . \end{aligned} \tag{31}$$

Since N is a strict continuous mean, $\delta \mapsto N^\delta(x,y)$ is continuous on $[0,1]$ by Theor. 4 and therefore, uniformly continuous there so that, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\left\| \left(N^{\frac{j}{2^n}}(x,y)\right)_{j=1}^{2^n} - \left(N^{\frac{j-1}{2^n}}(x,y)_{j=1}^{2^n}\right)_{j=1}^{2^n} \right\|_\infty < \varepsilon,\ n \ge n_0. \tag{32}$$

The equality $U_\infty = L_\infty$ follows from (31) and (32), which finishes the proof. ■

Another statement of Theor. 11 is the following: *the family of pairs* F_{G} *defined by*

$$\mathcal{F}_{\mathrm{G}} = \{(M,N) : M \text{ is isotone and (C)-nonexpansive, } N \text{ is strict and continuous}\}$$

constitutes a mixing family. It is clear that the class of pairs $(A_{(f)}, A_{(g)})$ composed by quasiarithmetic means is strictly contained in $\mathcal{F}_{\mathrm{G}}$.

4 Examples and remarks

Let M be a nonexpansive mean defined on I. Since the nonexpansiveness inequality (4) is a Lipschitz condition with an unitary Lipschitz constant, M turns out to be almost everywhere differentiable by the Rademacher´s Theorem. By virtue of the Lebesgue´s differentiation of monotonic functions Theorem, a homeomorphism $f : I \to \mathbb{R}$ is also almost everywhere differentiable. In this manner, a (C)-conjugated mean defined on I is almost everywhere differentiable on I^n. Now, useful criterions of nonexpansiveness and (C)-nonexpansiveness can be given for differentiable functions. Let us discuss them briefly in the context of two variables means (the case of n variables does not present appreciable differences).

A well-known criterion of nonexpansiveness of a differentiable function $F : I \times I \to \mathbb{R}$ is expressed by the inequality

$$\|\triangledown F(x,y)\|_1 = |F_x(x,y)| + |F_y(x,y)| \le 1,\ (x,y) \in I \times I. \tag{33}$$

For an isotone mean, the partial derivatives are non negative; hence, a differentiable isotone mean is nonexpansive if and only if the inequality (33) holds without the absolute-value bars. Now, assume that M is a differentiable mean such that, for a differentiable homeomorphism f, the f-conjugated

$M_f = f \circ M \circ (f^{-1} \times f^{-1})$ is nonexpansive. In this instance, the necessary and sufficient condition $\|\bigtriangledown M_f(x,y)\|_1 \leq 1,\ x,y \in f(I)$, takes the form

$$
\begin{aligned}
&\left|f'(M(f^{-1}(x), f^{-1}(y))\right| \left[\left|M_x(f^{-1}(x), f^{-1}(y)) \frac{1}{f'\left(f^{-1}(x)\right)}\right| + \right. \\
&\left. + \left|M_y(f^{-1}(x), f^{-1}(y)) \frac{1}{f'\left(f^{-1}(y)\right)}\right|\right] \\
\leq\ & 1,
\end{aligned}
$$

for every $x, y \in f(I)$ or, equivalently,

$$\left|M_x(x,y)\frac{1}{f'(x)}\right| + \left|M_y(x,y)\frac{1}{f'(y)}\right| \leq \frac{1}{|f'(M(x,y)|},\ x,y \in I. \tag{34}$$

In terms of $\phi(t) = 1/\left|f'(t)\right|,\ t \in I$, this inequality becomes

$$|M_x(x,y)|\,\phi(x) + |M_y(x,y)|\,\phi(y) \leq \phi\left(M(x,y)\right),\ x,y \in I. \tag{35}$$

Taking into account that $|f'(t)| > 0,\ t \in I$, one can state the following result.

Lemma 12 *A mean $M \in \mathcal{C}^1(I \times I)$ is $\mathcal{C}^1$-conjugated of a nonexpansive mean if and only if the inequality (35) is satisfied by a positive and continuous function ϕ defined on I.*

Observe that the inequality (35) with $\phi = const. > 0$ corresponds to the case of a nonexpansive mean M.

Proof. After the preceding discussion it remains to prove only the sufficiency. To this end, choose a point $a \in I$ and observe that the function defined by

$$f(x) = \int_a^x \frac{d\xi}{\phi(\xi)},\ x \in I,$$

is $\mathcal{C}^1$ and strictly increasing in I and therefore, the inverse $f^{-1} : \Phi(I) \to I$ exists and is a $\mathcal{C}^1$ function on $\Phi(I)$. Since $f'(x) = 1/\phi(x) > 0,\ x \in I$, the inequality (35) can be rewritten in the form (34) which, as seen in the discussion above, turns out to be equivalent to $\|\bigtriangledown M_f(x,y)\|_1 \leq 1$. ■

Let M be a differentiable and homogeneous mean on $\mathbb{R}^+$; then M satisfies the Euler´s equation

$$M_x(x,y)x + M_y(x,y)y = M(x,y),\ x,y > 0;$$

if M is, besides, isotone, then the inequality (35) holds with $\phi(x) \equiv x$ and therefore, the following consequence to Lemma 12 can be stated.

Corollary 13 *Every differentiable, isotone and homogeneous mean M on $\mathbb{R}^+$ is (C)-nonexpansive.*

In this way, most of the usual means are (C)-nonexpansive and therefore, this hypothesis is not so stringent as might appear at first sight. Under the hypotheses of the corollary, it is clear that $\ln\left(M(e^x, e^y)\right)$ turns out to be a nonexpansive mean.

Example 14 *The* Heronian mean $\mathfrak{H}_{\mathfrak{E}}$ *(cf. [8], pg. 399) is given by*

$$\mathfrak{H}_{\mathfrak{E}}(x,y) = \frac{x+y+\sqrt{xy}}{3}, \ x,y>0.$$

In view of

$$\begin{aligned}(\mathfrak{H}_{\mathfrak{E}})_x + (\mathfrak{H}_{\mathfrak{E}})_y &= \frac{1}{3}\left(2+\frac{1}{2}\left(\frac{x+y}{\sqrt{xy}}\right)\right) \\ &= \frac{1}{3}\left(2+\frac{A(x,y)}{G(x,y)}\right) \geq \frac{1}{3}(2+1) = 1,\end{aligned}$$

with equality if and only $x=y$, it turns out to be that $\mathfrak{H}_{\mathfrak{E}}$ is not a nonexpansive mean. However, $\mathfrak{H}_{\mathfrak{E}} \in \mathcal{C}^{\infty}(\mathbb{R}^+ \times \mathbb{R}^+)$ is (strictly) isotone and homogeneous, and then $\mathfrak{H}_{\mathfrak{E}}$ is (C)-nonexpansive by Cor. 13: $\ln \mathfrak{H}_{\mathfrak{E}}(e^x, e^y) = \ln\left(\left(e^x + e^y + e^{(x+y)/2}\right)/3\right)$ is a nonexpansive mean. On the other side, the generalized logarithmic mean of order 2 *is defined (cf. [8], pg. 385) by*

$$\mathcal{L}^{[2]}(x,y) = F(x,y) = \sqrt{\frac{x^2+xy+y^2}{3}}, \ x,y>0,$$

and, as a simple computation shows, it is the Lagrangian mean generated by the function $f(x)=x^2$; thus, it is (strictly) isotone. Adding the partial derivatives of $\mathcal{L}^{[2]}$ yields

$$\mathcal{L}_x^{[2]} + \mathcal{L}_y^{[2]} = \left(\sqrt{\frac{x^2+xy+y^2}{3}}\right)^{-1}\left(\frac{x+y}{2}\right) = \frac{A(x,y)}{\mathcal{L}^{[2]}(x,y)} \leq 1, \ x,y>0.$$

The last inequality is derived from the fact that $\mathcal{L}^{[2]}$ is a superarithmetic mean:

$$\mathcal{L}^{[2]}(x,y) \geq A(x,y), \ x,y>0.$$

In this way, $\mathcal{L}^{[2]}$ turns out to be a symmetric, isotone, strict and nonexpansive mean. Now, the mean conjugated of $\mathcal{L}^{[2]}$ by $f(x)=x^2$ is

$$\left(\mathcal{L}^{[2]}(\sqrt{x}, \sqrt{y})\right)^2 = \frac{x+y+\sqrt{xy}}{3}, \ x,y>0;$$

i.e., the Heronian mean $\mathfrak{H}_{\mathfrak{E}}$. *This example shows that, for a given (C)-nonexpansive mean M, there are in general more than one homeomorphism f such that M_f is nonexpansive.*

Theor. 2 in [7] is easily derived from Theor. 10. In fact, after (18) and (19) it can be written

$$U_n(x,y) = \begin{cases} f^{-1}\left(\frac{1}{2^n}\sum_{j=1}^{2^n} f\left(g^{-1}\left(g(x)+\frac{j}{2^n}\left(g(y)-g(x)\right)\right)\right)\right), & x \leq y \\ f^{-1}\left(\frac{1}{2^n}\sum_{j=1}^{2^n} f\left(g^{-1}\left(g(x)+\frac{j-1}{2^n}\left(g(y)-g(x)\right)\right)\right)\right), & x \geq y \end{cases},$$

and it is easy to see that

$$U_n(x,y) \to f^{-1}\left(\int_0^1 f \circ g^{-1}\left(g(x)+t\left(g(y)-g(x)\right)\right) dt\right)$$

when $n \uparrow +\infty$. Now, for $x \neq y$,

$$\begin{aligned} f^{-1}\left(\int_0^1 f \circ g^{-1}\left(g(x)+t\left(g(y)-g(x)\right)\right) dt\right) &= f^{-1}\left(\frac{1}{g(y)-g(x)}\int_{g(x)}^{g(y)} f \circ g^{-1}(\eta)\, d\eta\right) \\ &= f^{-1}\left(\frac{1}{g(y)-g(x)}\int_x^y f(\xi)\, dg(\xi)\right), \end{aligned}$$

which proves that the generalized Cauchy mean corresponding to the pair $(A_{(f)}, A_{(g)})$ is the Cauchy mean generated by f and g or, in symbols,

$$\begin{bmatrix} A_{(f)} \\ A_{(g)} \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{36}$$

In view of the fact that $\left[\begin{smallmatrix} f \\ f \end{smallmatrix}\right] = A_{(f)}$, it turns out to be

$$\begin{bmatrix} A_{(f)} \\ A_{(f)} \end{bmatrix} = A_{(f)}; \tag{37}$$

i.e., the mean generated by the pair $(A_{(f)}, A_{(f)})$ is $A_{(f)}$.

Partially closed expressions can be written for the generalized Cauchy mean corresponding to the pair (M, N) if only one component of the pair is quasi-arithmetic. If $M = A_{(f)}$ and N is a continuous strict mean; then,

$$\lim_{n\uparrow+\infty} U_n(x,y) = \lim_{n\uparrow+\infty} \begin{cases} f^{-1}\left(\frac{1}{2^n}\sum_{j=1}^{2^n} f\left(N^{\frac{j}{2^n}}(x,y)\right)\right), & x \leq y \\ f^{-1}\left(\frac{1}{2^n}\sum_{j=1}^{2^n} f\left(N^{\frac{j-1}{2^n}}(x,y)\right)\right), & x \geq y \end{cases} = f^{-1}\left(\int_0^1 f\left(N^{\delta}(x,y)\right) d\delta\right). \tag{38}$$

Now, the map $\delta \mapsto N^{\delta}(x,y)$ is continuous and strictly monotonic by Theor. 4, so that denoting by $\phi(x,y;\cdot)$ its (continuous and strictly monotonic) inverse, the integral in the last member of (38) can be written in the form

$$f^{-1}\left(\int_0^1 f\left(N^{\delta}(x,y)\right) d\delta\right) = f^{-1}\left(\int_x^y f(\xi)\, d\phi(x,y;\xi)\right),$$

where $\left\{d\phi(x,y;\xi) : (x,y) \in I^2\right\}$ is a family of Borel probability measures on $[0,1]$ (which are absolutely continuous with respect the Lebesgue measure). Basic results on this type of means can be found in [2].

On the other side, if M is an isotone and (C)-nonexpansive mean and $N = A_{(g)}$; then

$$\lim_{n\uparrow+\infty} U_n(x,y) = \lim_{n\uparrow+\infty} \begin{cases} M^{(n)}\left(\left(g^{-1}\left((1-\frac{j}{2^n})g(x)+\frac{j}{2^n}g(y)\right)\right)_{j=1}^{2^n}\right), & x\le y \\ M^{(n)}\left(\left(g^{-1}\left((1-\frac{j}{2^n})g(x)+\frac{j}{2^n}g(y)\right)\right)_{j=1}^{2^n}\right), & x\ge y \end{cases}$$
$$= \lim_{n\uparrow+\infty} g^{-1}\left(M_g^{(n)}\left(\left((1-\frac{j}{2^n})g(x)+\frac{j}{2^n}g(y)\right)_{j=1}^{2^n}\right)\right).$$

The next example shows an explicit computation of L_n and U_n in the case of linear means $M = L_\alpha,\ N = L_\beta$.

Example 15 *Let us assume that $0<\alpha,\ \beta<1$ and define $M(x,y) = L_\alpha(x,y) = (1-\alpha)\,x+\alpha y$ and $N(x,y) = L_\beta(x,y) = (1-\beta)\,x+\beta y$; then, the equalities (26) give*

$$K_{n+1}(x,y) = L_\alpha(K_n(x, L_\beta(x,y)), K_n(L_\beta(x,y),y)),$$

or, setting

$$K_n(x,y) = (1-\alpha_n)x+\alpha_n y,$$

$$\begin{aligned}(1-\alpha_{n+1})x+\alpha_{n+1}y &= (1-\alpha)\left[(1-\alpha_n)x+\alpha_n\left((1-\beta)\,x+\beta y\right)\right]+\alpha\left[(1-\alpha_n)\left((1-\beta)\,x+\beta y\right)+\alpha_n y\right] \\ &= \left[1-\alpha\beta-(\alpha+\beta-2\alpha\beta)\,\alpha_n\right]x+\left[(\alpha+\beta-2\alpha\beta)\,\alpha_n+\alpha\beta\right]y.\end{aligned}$$

Hence, the first order difference equation

$$\alpha_{n+1} = A\alpha_n+\alpha\beta \tag{39}$$

with $A = \alpha+\beta-2\alpha\beta$ is satisfied by α_n. Note that $0 < A \le 1/2$ when $0 < \alpha,\ \beta < 1$. Once the substitution $\alpha_n = A^n\beta_n$ is made in (39), it is obtained

$$\beta_{n+1} = \beta_n+\frac{\alpha\beta}{A^{n+1}}.$$

an equation for β_n which is easily solved in the form

$$\beta_n = \sum_{k=1}^{n-1}\frac{\alpha\beta}{A^{k+1}}+\beta_1 = -\frac{1}{A^{n+1}}\alpha\beta\frac{A-A^n}{A-1}+\beta_1.$$

Thence,

$$\alpha_n = A^n\beta_n = \frac{\alpha\beta}{A}\frac{A-A^n}{1-A}+\beta_1 A^n,$$

so that, in view of $0 < A < 1/2$, $\alpha_n \to \alpha\beta\,(1-A)^{-1}$ when $n\uparrow+\infty$, independently from the initial value of the sequence; thus, it turns out to be $L_\infty = M_\gamma = U_\infty$ with

$$\gamma = \frac{\alpha\beta}{1-A} = \frac{\alpha\beta}{1-(\alpha+\beta-2\alpha\beta)}. \tag{40}$$

5 Properties of the generalized Cauchy means

As said in the Introduction, Cauchy means constitute a closed under conjugacy class of means, being this property a clear indicative of the huge size of such class. A family of pairs $\mathcal{F}$ is said to be closed under conjugacy when $(M_f, N_f) \in \mathcal{F}$ for every homeomorphism $f : I \to I$ provided that $(M, N) \in \mathcal{F}$. For generalized Cauchy means, the following result holds.

Theorem 16 *A class of generalized Cauchy means $GC(\mathcal{F})$ associated to a mixing family $\mathcal{F}$ is closed under conjugacy provided that $\mathcal{F}$ is closed under conjugacy.*

Since $\mathcal{F}_{\mathrm{G}}$ is clearly closed under conjugacy, the class $GC(\mathcal{F}_{\mathrm{G}})$ is closed as well.

Proof. In the proof of Theor. 11, it was established that μ is a fixed point of $\mathcal{M}_{M,N}$ if and only if μ_f is a fixed point of $\mathcal{M}_{M_f,N_f}$. The theorem is a straightforward consequence of this fact. ■

Now, a basic result on comparison of generalized Cauchy means is established.

Theorem 17 *Let $\mathcal{F}$ be a mixing family of pairs (M, N) such that the first components M are isotone means. If $(M_i, N_i) \in \mathcal{F}$, $i = 1, 2$; then*

$$\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} \leq \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}$$

provided that $M_1 \leq M_2$ and $N_1 \leq N_2$.

Proof. A proof of this theorem can be given along the lines traced in [7], Lemma 3. Here, a proof based on the representation formula (30) is offered. Clearly, for a pair of comparable means M_1, M_2, $M_1^{(n)} \leq M_2^{(n)}$, $n \in \mathbb{N}$, and $M_1^d \leq M_2^d$, $d \in D([0,1])$, provided that $N_1 \leq N_2$. This fact together the isotonicity of M_2 yields

$$\begin{aligned} \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} &= \lim_{n\uparrow+\infty} M_1^{(n)} \left(\left(N_1^{\frac{j}{2^n}} \right)_{j=1}^{2^n} \right) \\ &\leq \lim_{n\uparrow+\infty} M_2^{(n)} \left(\left(N_1^{\frac{j}{2^n}} \right)_{j=1}^{2^n} \right) \\ &\leq \lim_{n\uparrow+\infty} M_2^{(n)} \left(\left(N_2^{\frac{j}{2^n}} \right)_{j=1}^{2^n} \right) = \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}. \end{aligned}$$

This finishes the proof. ■

Take, for instance, the mixing family $\mathcal{F}_{\mathrm{G}}$; then, the inequalities

$$G \leq \begin{bmatrix} M \\ N \end{bmatrix} \leq A$$

are satisfied by a mean $\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right] \in GC(\mathcal{F}_{\mathrm{G}})$ provided that

$$G \leq M,\, N \leq A.$$

Indeed, the previous theorem yields $\left[\begin{smallmatrix}G\\G\end{smallmatrix}\right] \leq \left[\begin{smallmatrix}M\\N\end{smallmatrix}\right] \leq \left[\begin{smallmatrix}A\\A\end{smallmatrix}\right]$ and $G = \left[\begin{smallmatrix}G\\G\end{smallmatrix}\right],\ A = \left[\begin{smallmatrix}A\\A\end{smallmatrix}\right]$ by (37).

As stated by the next result, other properties of the pair (M, N) are inherited by the generalized Cauchy mean $\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right]$.

Theorem 18 *Let $\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right]$ be the generalized Cauchy mean corresponding to the pair (M, N); then, the following assertions hold:*

i) $\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right]$ *is a strict mean provided that M and N are both strict;*

ii) $\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right]$ *is an isotone mean provided that M and N are both isotone;*

iii) $\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right]$ *is a homogeneous mean provided that M and N are both homogeneous;*

iv) $\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right]$ *is a continuous mean provided that M and N are both continuous.*

Proof. **i)** is a consequence of the case $n = 1$ of inequalities (25). In fact, if M and N are strict means; then L_1 and U_1 turn out to be strict and therefore,

$$\min\{x, y\} < L_1(x, y) \leq \begin{bmatrix}M\\N\end{bmatrix}(x, y) \leq U_1(x, y) < \max\{x, y\},\ x \neq y.$$

Taking into account Lemma 6, the assertions **ii)** and **iii)** are easily derived from the representation formula (30). To prove **iv)**, let us simply observe that the equality $L_\infty = \left[\begin{smallmatrix}M\\N\end{smallmatrix}\right] = U_\infty$ holds by Theor. 10 and L_∞ is *l.s.c.*, while U_∞ is *u.s.c. in* I^2 by Theor. 9-**ii)**. ■

Note that the symmetry of $\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right]$ does not figure in the list of properties inherited from the pair (M, N) given by Theor. 18. Indeed, mixing non symmetric means may well result in a symmetric mean. For instance, the weight γ given by (40) satisfies $\gamma = 1/2$ if and only if $\alpha + \beta = 1$, so that $\left[\begin{smallmatrix}L_\alpha\\L_{1-\alpha}\end{smallmatrix}\right] = L_{1/2} = A$ for every $0 < \alpha < 1$. Sufficient conditions for the symmetry of $\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right]$ are given by the following:

Theorem 19 *Assume that M is a quasiarithmetic mean and that N is a (strict, continuous) symmetric mean; then, the means L_n and U_n are symmetric for every $n \in \mathbb{N}$, as well as their common limit $\left[\begin{smallmatrix}M\\N\end{smallmatrix}\right]$.*

Proof. In view of $N^{\frac{j}{2^n}}(y,x) = N^{1-\frac{j}{2^n}}(y,x)$ by Lemma 3 and of $M^{(n)}$ turns out to be symmetric for every $n \in \mathbb{N}$ by Theor. 7, it can be written

$$\begin{aligned} U_n(y,x) &= \begin{cases} M^{(n)}\left(\left(N^{\frac{j}{2^n}}(y,x)\right)_{j=1}^{2^n}\right), & x \le y \\ M^{(n)}\left(\left(N^{\frac{j-1}{2^n}}(y,x)\right)_{j=1}^{2^n}\right), & x \ge y \end{cases} \\ &= \begin{cases} M^{(n)}\left(\left(N^{1-\frac{j}{2^n}}(x,y)\right)_{j=1}^{2^n}\right), & x \le y \\ M^{(n)}\left(\left(N^{1-\frac{j-1}{2^n}}(x,y)\right)_{j=1}^{2^n}\right), & x \ge y \end{cases} \\ &= \begin{cases} M^{(n)}\left(\left(N^{\frac{j}{2^n}}(x,y)\right)_{j=1}^{2^n}\right), & x \le y \\ M^{(n)}\left(\left(N^{\frac{j-1}{2^n}}(x,y)\right)_{j=1}^{2^n}\right), & x \ge y \end{cases} \\ &= U_n(x,y). \end{aligned}$$

A similar argument shows the symmetry of L_n. The symmetry of $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ follows by taking limits for n tending to $+\infty$ in the above equality. ■

To end this paper, let us recall that once defined a certain class $\mathfrak{M}(I)$ of means on an interval I, a basic question is the *problem of representation* (sometimes referred as *equality problem*) of the means belonging to $\mathfrak{M}(I)$: how many equivalent expressions of a mean $M \in \mathfrak{M}(I)$ are there? Probably, the first problem of representation was considered by G. Hardy, J. E. Littlewood and G. Pólya, who find in [9] all pairs f, g such that $A_{(f)} = A_{(g)}$. A suitable response to the problem is also known for several classes of mean besides of quasiarithmetic ones, majorly for such classes admitting a finite number of generators like Lagrangian or anti-Lagrangian means ([6], [7]), Bajraktarević means ([12]), generalized weighted means ([11]) and many others. In regards to (two variables) Cauchy means, L. Losonczi has solved in [14] the problem of representation in the case of sufficiently regular (seven times differentiable) generators (see also [13] and [3]). J. Matkowski has shown in [10] that the regularity hypothesis on the generators can be really omitted. Now well, given a mixing family of pairs $\mathcal{F}$, the problem of representation in the class $GC(\mathcal{F})$ consists of determining the pairs $(M_i, N_i) \in GC(\mathcal{F})$, $i = 1, 2$, such that

$$\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} = \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}, \tag{41}$$

or, equivalently, of finding the solutions μ to the simultaneous functional equations

$$\begin{cases} M_1(\mu(x, N_1(x,y)), \mu(N_1(x,y), y)) = \mu(x,y) \\ M_2(\mu(x, N_2(x,y)), \mu(N_2(x,y), y)) = \mu(x,y) \end{cases}, \ x, y \in I.$$

When M is isotone and N is strict, the representation formula (30) enable us to write the equality (41) in the form

$$\lim_{n\uparrow+\infty} M_1^{(n)}\left(\left(N_1^{\frac{j}{2^n}}(x,y)\right)_{j=1}^{2^n}\right) = \lim_{n\uparrow+\infty} M_2^{(n)}\left(\left(N_2^{\frac{j}{2^n}}(x,y)\right)_{j=1}^{2^n}\right), \ x, y \in I.$$

It is apparent that the difficulty of the problem of representation in the class $GC(\mathcal{F})$ increases with the size of the mixing family $\mathcal{F}$.

References

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York and London, (1966).

[2] L. R. Berrone, *Decreasing sequences of means appearing from non-decreasing functions*, Publ. Math. Debrecen **55,**1-2, (1999), 53-72.

[3] L. Berrone, *Invariance of the Cauchy mean value expression with an application to the problem of equality of Cauchy means*, Internat. J. Math. Math. Sci. **2005**, 18, (2005), 2895-2912.

[4] L. R. Berrone, *A dynamical characterization of quasilinear means*, Aequationes Math. **84**, 1, (2012), 51-70.

[5] L. R. Berrone, A. L. Lombardi, *A note on equivalence of means*, Publ. Math. Debrecen **58**, Fasc. **1-2**, (2001), 49-56.

[6] L. R. Berrone, J. Moro, *Lagrangian means*, Aequationes Math. **55**, (1998), 217-226.

[7] L. R. Berrone, J. Moro, *Cauchy means*, Aequationes Math. **60**, (2000), 1-14.

[8] P. S. Bullen, *Handbook of Means and Their Inequalities*, Series Mathematics and Its Applications, Kluwer Academic Publisher, 2nd. Ed., 2003.

[9] G. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1st. ed., 1934.

[10] J. Matkowski, *Solution of a regularity problem in equality of Cauchy means*, Publ. Math. Debrecen **64**, 3-4, (2004), 391-400.

[11] J. Matkowski, *Generalized weighted quasi-arithmetic means*, Aequationes Math. **79**, (2010), 203-212.

[12] L. Losonczi, *Equality of two variable weighted means: reduction to differential equations*, Aequationes Math. **58**, 3, (1999), 223–241.

[13] L. Losonczi, *Equality of Cauchy means values*, Publ. Math. Debrecen **57,** 1-2, (2000), 217–230.

[14] L. Losonczi, *Equality of two variable Cauchy mean values*, Aequationes Math. **65**, 1-2, (2003), 61-81.