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# On the a.s. convergence of certain random series to a fractional random field in $\mathscr{D}'(\mathbb{R}^d)$ ☆

Juan Miguel Medina[a,*], Bruno Cernuschi Frías[b]

[a]Dpto. de Matemática and LIPSIRN, Universidad de Buenos Aires, FI-UBA, Libertad 1274, 11-A (1012), Buenos Aires, Argentina
[b]Universidad de Buenos Aires, (LIPSIRN) and CONICET, Buenos Aires, Argentina

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**Abstract**

We prove the almost sure convergence in the sense of Schwartz distributions of certain random series. This result is useful to construct some type of fractional random fields. These series resemble the Karhunen–Loéve expansions.


*Keywords:* Almost sure convergence; Karhunen–Loéve expansions; Schwartz distributions; Long-range dependence; Fractional random fields; Fractional integrals

## 1. Introduction

Stochastic fields with $1/f$ spectral behavior, first introduced by Kolmogorov in the context of turbulent flows, have numerous applications in engineering, general science and whenever strong long-range (long memory) dependence (LRD) phenomena appear.

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*Corresponding author. Tel.: +54 11 4812 7023.
*E-mail addresses:* jmedina@fi.uba.ar (J.M. Medina), bcf@ieee.org (B.C. Frías).

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A long memory process or field $X(x)$ with spectral density $\Phi_X(\omega)$ (Section 3.1) verifies the following spectral condition (see Beran, 1994, Reed et al., 1995): there exist $\beta$ and $c_f$ such that

$$\lim_{\omega\to 0}\frac{\Phi_X(\omega)}{c_f|\omega|^{-\beta}}=1. \tag{1}$$

As some authors have pointed out (Medina and Cernuschi Frías, 2002; Bojdecki and Gorostiza, 1999) this suggests looking for a relation between these processes and certain fractional integration differencing operators (see Eqs. (19), (20)). Considering these processes not as point processes but as random elements in a space of distributions (in the Schwartz sense), we provide a method to construct a series which converges a.s. to a generalized fractional random field, that is, in the weak-* topology of the dual space of an appropriate linear vector space. The natural space is $\mathscr{D}(\mathbb{R}^d)$. In particular, it is useful to obtain random fields which show LRD or more generally with spectral density of the form

$$\Phi_X(\omega)=(1+|\omega|^2)^{-\gamma}|\omega|^{-\alpha}\quad \gamma\in\mathbb{R}_{>0},\ \ 0<\alpha<d/2. \tag{2}$$

Processes of this type are sometimes considered as solutions of the $d$-dimensional fractional order differential equation:

$$(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2}X=\eta, \tag{3}$$

where $\eta$ is white noise and $\Delta$ denotes the Laplacian operator. Here we construct series such that given $\{\xi_n\}_n$ a set of independent, identically distributed random variables, then if $\{g_k\}_k$ is a set of appropriate functions, then

$$\sum_n^{\infty}\xi_n g_n=X,$$

where $X$ verifies condition (2) and the convergence is a.s. in the sense of Schwartz distributions. The general discussion follows some of the ideas developed in Angulo and Ruiz-Medina (1997), Medina and Cernuschi Frías (2002) and Yves Meyer et al. (1999).

This work is organized as follows: first we give some definitions and auxiliary results (Section 2), and finally in Section 3 we describe a general method to construct a random series which converges (a.s.) to a generalized random field with a prescribed covariance structure (Theorem 3.1). This result which is interesting on its own, combined with some results on fractional integration, is in particular very useful for constructing a series which converges to a random field with spectral density as (2) (Theorem 3.3).

## 2. Some definitions and auxiliary results

**Remark.** In the following, if $x\in\mathbb{C}^d$ $(d\geqslant 1)$ we will denote its usual norm by $|x|$ and $\mathrm{Supp}(f)=\mathrm{Cl}\{x:f(x)\neq 0\}$.

The Schwartz class of functions $\mathscr{S}(\mathbb{R}^d)$ is defined as the linear space of smooth functions rapidly decreasing at infinity, together with its derivatives. This means that $\phi\in\mathscr{S}(\mathbb{R}^d)$ whenever

$\phi \in C^\infty(\mathbb{R}^d)$ and

$$\sup_{(x_1,\ldots,x_d)\in\mathbb{R}^d} \prod_{i=1}^d |x_i|^{\alpha_i} \left| \frac{\partial}{\partial x_1^{\beta_1}} \cdots \frac{\partial}{\partial x_d^{\beta_d}} \phi(x_1,\ldots,x_d) \right| < \infty \quad \forall \alpha_j, \beta_j \in \mathbb{N}$$

endowed with its usual topology. We will denote by $\mathscr{D}(\mathbb{R}^d)$ the space of functions which are in $C^\infty(\mathbb{R}^d)$ and have compact support. Both spaces are topological vector spaces (Stein and Weiss, 1970), and their duals are denoted as $\mathscr{S}'(\mathbb{R}^d)$ (*tempered distributions*) and $\mathscr{D}'(\mathbb{R}^d)$ (*distributions*), respectively. Clearly, $\mathscr{D}(\mathbb{R}^d) \subset \mathscr{S}(\mathbb{R}^d)$ and then $\mathscr{S}'(\mathbb{R}^d) \subset \mathscr{D}'(\mathbb{R}^d)$.

### 2.1. Fourier transforms

The *Fourier transform* $\widehat{f}$ of $f \in \mathscr{S}(\mathbb{R}^d)$ is defined as

$$\mathscr{F}(f)(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}^d} f(x)\,\mathrm{e}^{-2\pi i\omega.x}\,\mathrm{d}x;$$

from this $\mathscr{F}$ can be defined, as usual as a linear map $\mathscr{F} : L^1(\mathbb{R}^d) \mapsto C_\downarrow(\mathbb{R}^d)$, or as an isometry on $L^2(\mathbb{R}^d)$ and by duality over the class of tempered distributions, that is, $\mathscr{F} : \mathscr{S}'(\mathbb{R}^d) \mapsto \mathscr{S}'(\mathbb{R}^d)$.

**Definition 1.** The Sobolev spaces $H^s$ (Calderón, 1976) are defined as

$$H^s(\mathbb{R}^d) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 (1+|\omega|^2)^s\,\mathrm{d}\omega < \infty \right\}. \tag{4}$$

#### 2.1.1. Remark

Let $s \in \mathbb{R}$, then $H^s(\mathbb{R}^d)$ is a Hilbert space with the product $(.,.)_{H^s} : H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \mapsto \mathbb{C}$

$$(h,g)_{H^s} = \int_{\mathbb{R}^d} \widehat{h}(\omega)\overline{\widehat{g}}(\omega)(1+|\omega|^2)^s\,\mathrm{d}\omega. \tag{5}$$

**Definition 2.** For $f, g \in \mathscr{D}(\mathbb{R}^d)$ we define the pairing $\langle .,. \rangle : \mathscr{D}(\mathbb{R}^d) \times \mathscr{D}(\mathbb{R}^d) \longrightarrow \mathbb{R}$ as

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f(x) g(x)\,\mathrm{d}x,$$

This can be extended by a density argument over $L^p \times L^q$, $1/p + 1/q = 1$ (when $p = 2$ this is the usual inner product) or $H^s \times H^{-s}$.

**Definition 3.** Let $\mathbb{V}$ and $\mathbb{W}$ be two $\mathbb{R}(\mathbb{C})$ vectorial spaces. Then the vectorial space of all bounded linear mappings with domain in $\mathbb{V}$ and range in $\mathbb{W}$ is denoted by $\mathscr{L}(\mathbb{V}, \mathbb{W})$.

### 2.2. Generalized stochastic processes

In the following $(\Omega, \mathscr{F}, \mathbf{P})$ will denote a probability space. A *generalized stochastic process* is a random element in $\mathscr{D}'(\mathbb{R}^d)$ (or in $\mathscr{S}'(\mathbb{R}^d)$). This means that if $\varpi \in \Omega$ and $\varphi \in \mathscr{S}(\mathbb{R}^d)$ then a generalized stochastic process $X(x)$ is defined by the random variable $X(\varphi) : \Omega \mapsto \mathbb{R}$

(Rozanov, 1969):

$$X(\varphi) = \langle X, \varphi \rangle = \int_{\mathbb{R}^d} X(x)\varphi(x)\,\mathrm{d}x,$$

where the last equality may be only formal. The *covariance functional* is defined by the bilinear form $\Gamma : \mathscr{D}(\mathbb{R}^d) \times \mathscr{D}(\mathbb{R}^d) \mapsto \mathbb{R}$

$$\Gamma_X(u, v) = \mathbf{E}X(u)X(v).$$

If $\Gamma_X(u, v)$ can be written as $\Gamma_X(u, v) = \langle u, R * v \rangle$ where $R(x)$ may be a generalized function, sometimes, it informally as $\mathbf{E}[X(x)X(x')] = R(x - x')$. For example, if $X(x)$ is the white noise $R(x) = \delta(x)$ in the sense of $\langle \delta, u \rangle = u(0)$, then $\Gamma(u, v) = \int_{\mathbb{R}^d} u(x)v(x)\,\mathrm{d}x$ for all $u$ and $v$ in $\mathscr{D}(\mathbb{R}^d)$. If $R \in \mathscr{S}'(\mathbb{R}^d)$, it is also possible to define the *spectral density* of the process as $\Phi_X = \mathscr{F}R = \widehat{R}$.

The following result will be useful.

**Theorem 2.1** (*Variant of the Shannon–Kotoélnikov theorem*). *If* $f \in L^2(\mathbb{R}^d)$ *is such that* $\mathrm{Supp}(f) \subset [-\lambda_0, \lambda_0]^d$ *with* $\lambda_0 < 1/2$, *then there exists* $\theta \in \mathscr{S}(\mathbb{R}^d)$ *such that*

$$\widehat{f}(\omega) = \sum_{k \in \mathbb{Z}^d} \widehat{f}(k)\theta(\omega - k). \tag{6}$$

**Proof.** Let $\widetilde{f}(x) = \sum_{k \in \mathbb{Z}^d} f(x + k)$ be the periodization of $f$. The identification $\widetilde{f}$ with the torus verifies $\widetilde{f} \in L^2(\mathbb{T}^d) \subset L^1(\mathbb{T}^d)$, and, if $\widetilde{f} \sim \sum_{k \in \mathbb{Z}^d} a_k \mathrm{e}^{-2\pi i x.k}$, then $\lim_{\lambda \to \infty} \sum_{k \in D_\lambda} a_k \mathrm{e}^{-2\pi i x.k} \overset{L^2(\mathbb{T}^d)}{=} \widetilde{f}$ and in $L^1(\mathbb{T}^d)$ for a suitable domain $D_\lambda \subset \mathbb{R}^d$. Now, we can take $\theta(x) \in \mathscr{S}(\mathbb{R}^d)$ such that

$$\widehat{\theta}(\omega) = \begin{cases} 1, & |\omega_i| < \lambda_0 \\ 0, & |\omega_i| \geqslant 1 - \lambda_0, \end{cases}$$

and define $S_\lambda(x) = \widehat{\theta}(x) \sum_{k \in D_\lambda} a_k \mathrm{e}^{-2\pi i x.k}$. On the other hand $f = \widetilde{f}\widehat{\theta}$; then, it is easy to show that $\lim_{\lambda \to \infty} \|S_\lambda - f\|_{L^1(\mathbb{R}^d)} = 0$. This implies $\lim_{\lambda \to \infty} \mathrm{Sup}_{\omega \in \mathbb{R}^d} |\widehat{S_\lambda}(\omega) - \widehat{f}(\omega)| = 0$, but (see Stein and Weiss, 1970) $a_k = \widehat{f}(k)$. Then

$$\widehat{S_\lambda}(\omega) = \sum_{k \in D_\lambda} \widehat{f}(k)\theta(\omega - k).$$

Then (6) follows immediately from this. □

Now, we can prove the following proposition, which is an easily extended $d$-dimensional version of a result of Yves Meyer et al. (1999).

**Proposition 2.1.** *Let $f \in L^2(\mathbb{R}^d)$ be with the same hypotheses of the previous theorem. Then*

$$\|f\|_{H^s} \leqslant K(s)\left(\sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2(1+|k|^2)^s\right)^{1/2}.$$

**Proof.** Recall Peetre's inequality $(1+(a+b)^2)^s \leqslant 2^{|s|}(1+a^2)^{|s|}(1+b^2)^s$, and by Theorem 2.1 we have

$$\int_{\mathbb{R}^d} |\hat{f}(\omega)|^2(1+|\omega|^2)^s\,\mathrm{d}\omega \leqslant \int_{\mathbb{R}^d}\left(\sum_{k\in\mathbb{Z}^d} |\hat{f}(k)\theta(\omega-k)|(1+|\omega|^2)^{s/2}\right)^2 \mathrm{d}\omega$$
$$\leqslant \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} u_k^2(\omega) \sum_{k\in\mathbb{Z}^d} v_k^2(\omega)\,\mathrm{d}\omega,$$

where $v_k(\omega) = |\theta(\omega-k)|^{1/2}$ and

$$u_k(\omega) = |\hat{f}(k)|(1+|k|^2)^{s/2}2^{|s|/2}(1+||\omega|-|k||^2)^{|s|/2}|\theta(\omega-k)|^{1/2}.$$

Since $\theta(x) \in \mathscr{S}(\mathbb{R}^d)$ we have $C = \sum_{k\in\mathbb{Z}^d} v_k^2(\omega) = \sum_{k\in\mathbb{Z}^d} |\theta(\omega-k)| < \infty$ and

$$K(s)2^{-|s|} = \int_{\mathbb{R}^d}(1+||\omega|-|k||^2)^{|s|}|\theta(\omega-k)|\,\mathrm{d}\omega$$
$$\leqslant \int_{\mathbb{R}^d}(1+|\omega-k|^2)^{|s|}|\theta(\omega-k)|\,\mathrm{d}\omega < \infty.$$

Then,

$$\int_{\mathbb{R}^d} |\hat{f}(\omega)|^2(1+|\omega|^2)^s\,\mathrm{d}\omega \leqslant CK(s)\sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2(1+|k|^2)^s. \qquad \square$$

### *2.3. Some auxiliary results on a.s. convergence*

The following mimic a celebrated theorem of Kolmogorov about the convergence of sums of independent random variables, but here we need a version for random elements (for a definition of random element see Taylor, 1978) in a Hilbert space (Kahane, 1985).

**Theorem 2.2.** *Let $\{\xi_k\}$ be a sequence of independent random variables in $L^2(\Omega, \mathscr{F}, \mathbf{P})$ such that $\mathbf{E}\xi_k = 0$ and $\{f_k\}$ is a sequence in $H$ a Hilbert space. If*

$$\sum_{n=1}^{\infty} \mathbf{E}|\xi_n|^2\|f_n\|_H^2 < \infty, \tag{7}$$

*then $S_n$ converges in $H$ a.s., where $X_k = \xi_k f_k$ and $S_n = \sum_{k=1}^n X_k$ .*

$S_n$ are well-defined random elements in $H$ as a consequence of Lemma 2.1.5 or 2.1.1 by Taylor (1978).

*2.3.1. A basic result*

In finite measure spaces there is a basic relationship between almost everywhere (almost sure) convergence and convergence in norm (mean convergence) (Billingsley, 1968, 1994). For this purpose we need the following definition.

**Definition 4.** Let $\{\xi_n\}_n$ be a sequence of random variables. We say that $\{\xi_n\}_n$ is uniformly integrable if

$$\lim_{\alpha\to\infty} \sup_n \int_{\{|\xi_n|>\alpha\}} |\xi_n| \, \mathrm{d}\mathbf{P} = 0. \tag{8}$$

Then the following can be proved (Billingsley, 1968, 1994).

**Theorem 2.3.** *Let $p \geqslant 1$ and $\{\xi_n\}_n \subset L^p(\Omega, \mathscr{F}, \mathbf{P})$ be a sequence, such that $\xi_n \longrightarrow \xi$ a.s. as $n \longrightarrow \infty$. Then, $\mathbf{E}|\xi_n - \xi|^p \longrightarrow 0$ when $n \longrightarrow \infty \iff \{|\xi_n|^p\}_n$ is uniformly integrable.*

It is easy to prove that a sufficient condition for $\{\xi_n\}_n$ to be uniformly integrable is

$$\exists\, \varepsilon > 0, K > 0 \text{ such that } \mathbf{E}|\xi_n|^{1+\varepsilon} \leqslant K \ \forall n. \tag{9}$$

## 3. Main results

In this section we will prove in Theorem 3.1 that given $T \in \mathscr{L}(L^2(\mathbb{R}^d), L^p(\mathbb{R}^d))$ it is possible to construct a series which converges almost surely to a generalized random field, namely $X$, with covariance functional $\Gamma_X(\phi, \psi) = \langle \phi, T \circ T^* \psi \rangle$ with $\phi, \psi \in \mathscr{D}(\mathbb{R}^d)$. This result will be used in Section 3.1, Theorem 3.3 to construct a generalized fractional random field.

In the following $\mathscr{F}$ will denote any $\sigma$-algebra on $\Omega$ for which the family $\{\xi_n f_n\}_n$ is measurable considering the $\sigma$-algebra $B(\mathscr{D}'(\mathbb{R}^d))$. $\{f_n\}_n$ and $\{\xi_n\}_n$ are as above.

**Theorem 3.1.** *Let $\{\xi_n\}_{n\in\mathbb{N}} \subset L^4(\Omega, \mathscr{F}, \mathbf{P})$ be a sequence of independent identically distributed random variables such that $\mathbf{E}\xi_n = 0$ . If $\{f_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ and $T \in \mathscr{L}(L^2(\mathbb{R}^d), L^p(\mathbb{R}^d))$ with $p \geqslant 1$, then:*

(I) $$X = \sum_{n=0}^{\infty} \xi_n T f_n \tag{10}$$

*converges to a generalized process a.s.*

(II) *The covariance functional of $X$, $\Gamma_X : \mathscr{D}(\mathbb{R}^d) \times \mathscr{D}(\mathbb{R}^d) \longrightarrow \mathbb{R}$ is $\Gamma_X(\phi, \psi) = \langle \phi, T \circ T^* \psi \rangle$.*

(III) *Given $\varphi \in \mathscr{D}(\mathbb{R}^d)$, then*

$$X(\varphi) = \sum_{n=0}^{\infty} \xi_n \langle T f_n, \varphi \rangle \text{ in the } L^2(\Omega, \mathscr{F}, \mathbf{P}) \text{ sense.} \tag{11}$$

**Proof** (*Part I*). Let $\{Q_p\}_p$ be a denumerable family of disjoint cubes such that by some translation $\tau_p$ equals $(-1/2, 1/2]^d$ and $\mathbb{R}^d = \bigsqcup_p Q_p$. Then by Proposition 2.1,

$$\|(Tf_n)\mathbf{1}_{Q_p}\|_{H^s} \leqslant K(s) \left( \sum_{k\in\mathbb{Z}^d} |\widehat{(Tf_n)\mathbf{1}_{Q_p}}(k)|^2, (1+|k|^2)^s \right)^{1/2},$$

with

$$|\widehat{(Tf_n)\mathbf{1}_{Q_p}}(k)| = |\langle (Tf_n)\mathbf{1}_{Q_p}, e_k\rangle| \text{ and } e_k = \mathrm{e}^{\mathrm{i}2\pi k.x}\mathbf{1}_{\tau_p^{-1}[-1/2,1/2]^d}.$$

Then,

$$\sum_n \|(Tf_n)\mathbf{1}_{Q_p}\|^2_{H^s} \leqslant \sum_n K(s) \sum_{k\in\mathbb{Z}^d} |\widehat{(Tf_n)\mathbf{1}_{Q_p}}(k)|^2(1+|k|^2)^s.$$

Taking $s=-d$, and since $e_k \in L^{p'}(\mathbb{R}^d)$, $1/p+1/p'=1$ and $\mathrm{Supp}((Tf_n)\mathbf{1}_{Q_p}) = \mathrm{Supp}(e_k)$, the last term equals

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d} K(s)(1+|k|^2)^{-d} \sum_n |\langle (Tf_n)\mathbf{1}_{Q_p}, e_k\rangle|^2 \\
&\quad = \sum_{k\in\mathbb{Z}^d} K(s)(1+|k|^2)^{-d} \sum_n |\langle f_n, T^*e_k\rangle|^2 \leqslant \sum_{k\in\mathbb{Z}^d} K(s)(1+|k|^2)^{-d}\|T^*e_k\|^2_{L^2} \\
&\quad \leqslant \sum_{k\in\mathbb{Z}^d} K(s)(1+|k|^2)^{-d}K''\|e_k\|^2_{L^{p'}} \leqslant K''' \int_{\mathbb{R}^d}(1+|x|^2)^{-d}\,\mathrm{d}x|Q_p|^{2/p'} < \infty.
\end{aligned} \tag{12}$$

Since $\{\xi_n\}_{n\in\mathbb{N}}$ are independent random variables for which we can assume, without loss of generality, $\mathbf{E}|\xi_n|^2 = 1$, then

$$\sum_n \mathbf{E}|\xi_n|^2\|(Tf_n)\mathbf{1}_{Q_p}\|^2_{H^{-d}} = \sum_n \|(Tf_n)\mathbf{1}_{Q_p}\|^2_{H^{-d}} < \infty.$$

By Theorem 2.2 and Remark 2.1.1 we have $\|\sum_n \xi_n (Tf_n)\mathbf{1}_{Q_p}\|_{H^{-d}} < \infty$ a.s. But convergence in $H^{-d} \cong (H^d)^*$ implies convergence in $\mathscr{S}'(\mathbb{R}^d) \subset \mathscr{D}'(\mathbb{R}^d)$.

(*Part II*). If $X:\Omega \longrightarrow \mathscr{D}'(\mathbb{R}^d)$ is the limit field, then its covariance is $\Gamma_X(\phi,\psi) := \mathbf{E}X(\phi)X(\psi)$. This is well defined since $X(\varphi)$ is a random variable as a consequence of Lemma 2.2.1 in Taylor (1978), since $\varphi$ is Borel-measurable and $X$ is $\mathscr{F}$-measurable as a consequence of Lemma 2.1.3 in Taylor (1978, p. 22), since $X_m = \sum_{k=0}^m \xi_n Tf_n$ is $\mathscr{F}$-measurable (Taylor, 1978, Lemma 2.1.5, p. 24).

In order to prove that $\mathbf{E}X_m(\phi)X_m(\psi) \longrightarrow \mathbf{E}X(\phi)X(\psi) = \langle \phi, T\circ T^*\psi\rangle$ when $m\longrightarrow\infty$, with $\phi,\psi \in \mathscr{D}(\mathbb{R}^d)$, first we prove the uniform integrability of the sequence $\{X_m(\phi)X_m(\psi)\}_m$. This result will follow if we find $\varepsilon>0$, $K>0$ such that

$$\mathbf{E}|X_m(\phi)X_m(\psi)|^{1+\varepsilon} \leqslant K \ \forall m. \tag{13}$$

Given $\phi \in \mathscr{D}(\mathbb{R}^d)$ let us call $c_m := \langle Tf_m, \phi\rangle = \langle f_m, T^*\phi\rangle$, $c := (c_m)_m \in \mathbb{R}^{\mathbb{N}}$; then

$$\mathbf{E}|X_m(\phi)|^4 = \mathbf{E}\left(\sum_{ijkl=0}^m c_ic_jc_kc_l\xi_i\xi_j\xi_k\xi_l\right),$$

but, since the $\xi_m$ are independent, we have the following factorization: $d_{ijkl} := \mathbf{E}(\xi_i\xi_j\xi_k\xi_l) = \mathbf{E}(\xi_i)\mathbf{E}(\xi_j\xi_k\xi_l) = 0$ whenever $i \neq j,k,l$. From this fact and since the $\xi_m$ are identically distributed, we get

$$d_{ijkl} = \begin{cases} (\mathbf{E}|\xi_1|^2)^2 & \text{whenever two pairs of indexes are equal,} \\ \mathbf{E}|\xi_1|^4 & \text{if } i = j = k = l, \\ 0 & \text{whenever only one index differs from the others.} \end{cases}$$

From this,

$$\begin{aligned} \mathbf{E}|X_m(\phi)|^4 &= \sum_{i=0}^{m} c_i^4 \mathbf{E}|\xi_1|^4 + 3 \sum_{i,j=0\ i\neq j}^{m} c_i^2 c_j^2 (\mathbf{E}|\xi_1|^2)^2 \\ &\leqslant \mathbf{E}|\xi_1|^4 \|c^2\|_{l^\infty} \|T^*\phi\|^2_{L^2(\mathbb{R}^d)} + 3\|T^*\phi\|^2_{L^2(\mathbb{R}^d)} \end{aligned} \tag{14}$$

$$\leqslant (\mathbf{E}|\xi_1|^4 \|c^2\|_{l^\infty} + 3)\|T^*\|\|\phi\|^2_{L^{p'}(\mathbb{R}^d)} < \infty. \tag{15}$$

Now, since $\mathbf{E}|X_m(\phi)X_m(\psi)|^2 \leqslant (\mathbf{E}|X_m(\phi)|^4)^{1/2}(\mathbf{E}|X_m(\psi)|^4)^{1/2}$ and from (14) condition (13) is verified for $\varepsilon = 1 \Longrightarrow \mathbf{E}X_m(\phi)X_m(\psi) \longrightarrow \mathbf{E}X(\phi)X(\psi) = \Gamma_X(\phi, \psi)$ when $m \longrightarrow \infty$.

Let us prove that $\Gamma_X(\phi, \psi) = \langle \phi, T \circ T^*\psi \rangle$. Given $m$, let us define the bilinear form $\Gamma_m : \mathscr{D}(\mathbb{R}^d) \times \mathscr{D}(\mathbb{R}^d) \mapsto \mathbb{R}$ as follows:

Let $k_m(x, y) := \sum_{jk=0}^{m} \mathbf{E}\xi_j\xi_k Tf_k(x)Tf_j(y)$, and for $\phi, \psi \in \mathscr{D}(\mathbb{R}^d)$, define

$$\Gamma_m(\phi, \psi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k_m(x, y)\phi(y)\psi(x)\,\mathrm{d}x\,\mathrm{d}y.$$

Since $\{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of independent random variables with $\mathrm{Var}(\xi_n) = 1$ and $\mathbf{E}[\xi_n] = 0$, then $\mathbf{E}\xi_n\xi_m = \delta_{nm}$. From this it follows that $k_m(x, y) = \sum_{k=0}^{m} Tf_k(x)Tf_k(y)$. Hence,

$$\begin{aligned} \Gamma_m(\phi, \psi) &= \int_{\mathbb{R}^d} \left( \sum_{k=0}^{m} \int_{\mathbb{R}^d} Tf_k(x)\psi(x)\,\mathrm{d}x Tf_k(y) \right) \phi(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} T\left( \sum_{k=0}^{m} \int_{\mathbb{R}^d} f_k(x)T^*\psi(x)\,\mathrm{d}x f_k(y) \right) \phi(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \left( \sum_{k=0}^{m} \int_{\mathbb{R}^d} f_k(x)T^*\psi(x)\,\mathrm{d}x f_k(y) \right) T^*\phi(y)\,\mathrm{d}y. \end{aligned} \tag{16}$$

Then, if $P_m \in \mathscr{L}(L^2(\mathbb{R}^d))$ is the orthogonal projection over $\mathrm{Span}\{f_0, \ldots, f_m\}$, (16) equals $\langle P_m \circ T^*\psi, T^*\phi \rangle_{L^2(\mathbb{R}^d)}$, and since the $\{f_n\}_n$ is complete, given $\varepsilon > 0$, there exists $M(\varepsilon) \in \mathbb{N}$ such that $\|P_m \circ T^*\phi - T^*\phi\|_{L^2} < \varepsilon/\|T^*\psi\|_{L^2}$ if $m \geqslant M$. On the other hand, $\langle \phi, T \circ T^*\psi \rangle = \langle T^*\phi, T^*\psi \rangle$, and from these facts, taking for example $m \geqslant M(\varepsilon)$, it follows that

$$\begin{aligned} |\langle \phi, T \circ T^*\psi \rangle - \Gamma_m(\phi, \psi)| &= |\langle T^*\phi, T^*\psi \rangle - \langle P_m \circ T^*\phi, T^*\psi \rangle| \\ &= |\langle P_m \circ T^*\phi - T^*\phi, T^*\psi \rangle| \\ &\leqslant \|T^*\psi\|_{L^2} \|P_m \circ T^*\phi - T^*\phi\|_{L^2} < \varepsilon. \end{aligned} \tag{17}$$

(*Part III*). From Eqs. (14), (15), given $\varphi \in \mathscr{D}(\mathbb{R}^d)$ we have that $\{|X_n(\varphi)|^2\}_n$ is uniformly integrable, since condition (9) is verified for $\varepsilon = 2$. Since $X_n(\varphi) \longrightarrow X(\varphi)$ a.s. from Part I, then from

Theorem 2.3 we have

$$\lim_{n\to\infty} \mathbf{E}|X_n(\varphi) - X(\varphi)|^2 = 0. \qquad \square$$

### *3.1. Some consequences and applications: Construction of a fractional random field*

We will need the following well-known result:

**Theorem 3.2.** *Let $T \in \mathscr{L}(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))$; if $T$ conmute with translations then there exists a unique tempered distribution $\mu$ such that for every $f \in \mathscr{S}(\mathbb{R}^d)$: $Tf = \mu * f$.*

Then, from the above theorem and the definition of Fourier transform of a distribution, we have the following immediate and intuitive result on the covariance functional of the limit process $X$ of Proposition 3.1: if $T$ is translation invariant (conmute with translations) and $\mu \in \mathscr{S}'(\mathbb{R}^d)$ is the distribution of Theorem 3.2, then

$$\Gamma_X(\phi, \psi) = \int_{\mathbb{R}^d} \hat{\phi}(\omega)\, |\hat{\mu}(\omega)|^2 \overline{\hat{\psi}}(\omega)\, \mathrm{d}\omega. \tag{18}$$

Moreover, $\Phi_X(\omega) = |\hat{\mu}(\omega)|^2$. The previous results are useful for constructing random fields with spectral behaviour given by Eq. (2). For this purpose we need some results on fractional potentials.

Let us consider the usual Laplacian of $f: \Delta f = \sum_{j=1}^d (\partial^2 f/\partial x_j^2)$. Then, at least formally, $\widehat{\Delta f}(\omega) = -(2\pi)^2|\omega|^2\widehat{f}(\omega)$. From this we could define the operators $(-\Delta)^{-\alpha/2}$ as

$$(-\Delta)^{-\alpha/2} f = \mathscr{F}^{-1}(2\pi)^{-\alpha}|.|^{-\alpha}\mathscr{F}f. \tag{19}$$

The formal manipulations have a precise meaning (Stein, 1970).

**Definition 5.** Let $0<\alpha<d$. For $f \in \mathscr{S}(\mathbb{R}^d)$ we can define its *Riesz potential*:

$$((-\Delta)^{-\alpha/2} f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, \mathrm{d}y, \tag{20}$$

where $\gamma(\alpha) = \pi^{d/2}2^{\alpha}\Gamma(\alpha/2)/\Gamma(d/2-\alpha/2)$.

This linear operator has the following properties (Stein, 1970).

**Proposition 3.1.** *Let $0<\alpha<d$. Then*: (a) *The Fourier transform of $|x|^{-d+\alpha}$ is $\gamma(\alpha)(2\pi)^{-\alpha}|\omega|^{-\alpha}$ in the sense*

$$\int_{\mathbb{R}^d} |x|^{-d+\alpha}\varphi(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} \gamma(\alpha)(2\pi)^{-\alpha}|\omega|^{-\alpha}\widehat{\varphi}(\omega)\, \mathrm{d}\omega$$

*for all $\varphi \in \mathscr{S}(\mathbb{R}^d)$.*

(b) *The Fourier transform of $((-\Delta)^{-\alpha/2} f)(x)$ is $(2\pi)^{-\alpha}|\omega|^{-\alpha}\widehat{f}(\omega)$ in the sense*

$$\int_{\mathbb{R}^d} ((-\Delta)^{-\alpha/2} f)(x) g(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} \widehat{f}(\omega)(2\pi)^{-\alpha}|\omega|^{-\alpha}\widehat{g}(\omega)\, \mathrm{d}\omega$$

*for all $f, g \in \mathscr{S}(\mathbb{R}^d)$.*

It is easy to check that $\forall f \in \mathscr{S}(\mathbb{R}^d)$: if $\alpha+\beta<d$ then $(-\Delta)^{-\alpha/2}((-\Delta)^{-\beta/2}f)=(-\Delta)^{-(\alpha+\beta)/2}(f)$; and $\Delta((-\Delta)^{-\alpha/2}f)=(-\Delta)^{1-(\alpha/2)}(f)$.

We recall the following bound for these operators acting in $L^p(\mathbb{R}^d)$ (Calderón, 1960, Stein, 1970).

**Proposition 3.2** (*Hardy, Littlewood and Sobolev*). *Let* $0<\alpha<d$, $1\leqslant p<q<\infty$ *and* $1/q=1/p-\alpha/d$. *Then*:

(a) $\forall f \in L^p(\mathbb{R}^d)$, *the integral that defines* $(-\Delta)^{-\alpha/2}f$ *converges a.e.*
(b) *If* $p>1$ *then*

$$\|(-\Delta)^{-\alpha/2}f\|_{L^q}\leqslant C_{pq}\|f\|_{L^p}. \tag{21}$$

**Remark.** These operators are the inverses of the (positive) fractional powers of the Laplacian operator. For the class $\mathscr{S}(\mathbb{R}^d)$, $(-\Delta)^{\alpha/2}$ is given by

$$-(-\Delta)^{\alpha/2}f(x)=c\int_{\mathbb{R}^d}\left[f(y)-f(x)-\frac{\nabla f(x)(y-x)}{1+|y-x|^2}\right]\frac{\mathrm{d}y}{|y-x|^{d+\alpha}}.$$

This expression follows from Stein (1970, Section 6.10), and from this formula a short proof of the existence of the fractional Brownian field with exponent $\alpha/2$ can be given (Bojdecki and Gorostiza, 1999).

Now, introduce another fractional integration operator defined formally as

$$(I-\Delta)^{s/2}f=\mathscr{F}^{-1}(1+|.|^2)^{s/2}\mathscr{F}f. \tag{22}$$

This operator is continuous (Stein, 1970).

**Proposition 3.3.** *If* $s<0$ *and* $p\geqslant 1$, $(I-\Delta)^{s/2}:L^p(\mathbb{R}^d)\longrightarrow L^p(\mathbb{R}^d)$ *defines a continuous linear operator, i.e. there exists* $C_p>0$ *such that*

$$\|(I-\Delta)^{s/2}f\|_{L^p}\leqslant C_p\|f\|_{L^p}.$$

With all this, now we can claim the following assertion on fractional random fields.

**Theorem 3.3.** *If* $T=(-\Delta)^{-\alpha/2}(I-\Delta)^{-\gamma/2}$ *with* $0<\alpha<d/2$, $\gamma>0$, *then the series defined by* (10) *converges to a generalized stochastic field with spectral density as* (2).

**Proof.** The operator $T$ is a well-defined bounded linear operator as a consequence of Theorems 3.2 and 3.3; moreover it maps $L^2(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ for some $p>1$ which depends on $\alpha$. Then the desired result follows from Theorems 3.1 and 3.2, and Eq. (18). $\square$

It is straightforward to see from the proof of Proposition 3.1 that this assertion may fail if $\alpha\geqslant d/2$. In Anh et al. (1999), by means of operators (22) and (20) is proved the existence of a process with spectral density as (2) with $\alpha\in(0,d)$. This result is based on the following assertion: if $D\subset\mathbb{R}^d$ is a measurable bounded domain, there exists $C>0$ such that for every $f\in L^2(\mathbb{R}^d)$ and $\mathrm{Supp}(f)\subset D$

$$\int_{\mathbb{R}^d}|\widehat{(-\Delta)^{-\alpha/2}f}(\omega)|^2\mathrm{d}\omega\leqslant C\int_{\mathbb{R}^d}|\hat{f}(\omega)|^2\,\mathrm{d}\omega. \tag{23}$$

But this is false for $\alpha \geqslant d/2$: take $D = B(0,1)$ the ball of radius 1 and $f = \mathbf{1}_D$. We prove that for such $f$, $(-\Delta)^{-\alpha/2} f$ does not belong to $L^2(\mathbb{R}^d)$. From (20) we have

$$(-\Delta)^{-\alpha/2} f(x) = \frac{1}{\gamma(\alpha)} \int_{B(0,1)} \frac{\mathrm{d}y}{|x-y|^{d-\alpha}},$$

but $|x-y| \leqslant |x| + |y| \leqslant |x| + 1$, then $(|x|+1)^{-d+\alpha} \leqslant |x-y|^{-d+\alpha}$ if $|y| \leqslant 1$, so

$$|(-\Delta)^{-\alpha/2} f(x)| \geqslant K \frac{|B(0,1)|}{(|x|+1)^{-d+\alpha}}$$

for all $x \in \mathbb{R}^d$. Then we have the following bound:

$$\begin{aligned}\|(-\Delta)^{-\alpha/2} f\|^2 &\geqslant \int_{\mathbb{R}^d} (|x|+1)^{-2d+2\alpha}\, \mathrm{d}x\, |B(0,1)|^2 \\ &= K \int_0^\infty (r+1)^{-2d+d\alpha} r^{d-1} \mathrm{d}r = k\beta(d, d-2\alpha),\end{aligned}$$

but this expression for Euler's beta function converges if and only if $d > 0$ and $d - 2\alpha > 0$.

## References


Angulo, J.M., Ruiz-Medina, M.D., 1997. On the orthogonal representation of generalized random fields. Statist. Probab. Lett. 31, 145–153.

Anh, V.V., Angulo, J.M., Ruiz-Medina, M.D., 1999. Possible long range dependance in fractional random fields. J. Statist. Plann. Inference 80, 95–110.

Beran, J., 1994. Statistics for Long-Memory Processes. Chapman & Hall, London, CRC, Boca Raton.

Billingsley, P., 1968. Convergence of Probability Measures, first ed. Wiley (Wiley series in probability and Mathematical Statics).

Billingsley, P., 1994. Probability and Measure, third ed. Wiley (Wiley series in probability and Mathematical Statics).

Bojdecki, T., Gorostiza, L.G., 1999. Fractional Brownian motion via fractional Laplacian. Statist. Probab. Lett. 44 (1), 107–108.

Calderón, A.P., 1960. Integrales Singulares y sus Aplicaciones a Ecuaciones Diferenciales Hiperbolicas, Cursos y seminarios de Matemática, Universidad de Buenos Aires, Fasc.3.

Calderón, A.P., 1976. Lecture Notes on Pseudo-Differential Operators and Elliptic Boundary Value Problems. Instituto Argentino de Matemática, CONICET, Buenos Aires.

Kahane, 1985. Some Random Series of Functions. Cambridge University Press, Cambridge.

Medina, J.M., Cernuschi Frías, B., 2002, A relationship between fractional integration and $1/|\omega|^\beta$ processes. Proceedings of the 2002 IEEE Inf. Theory Workshop, p. 205.

Meyer, Y.F., Sellan, M.S., Taqqu, 1999. Wavelets, genaralized white noise and fractional integration: The synthesis of fractional Brownian motion. J. Fourier Anal. Appl. 5 (5).

Reed, I.S., Lee, P.C., Truong, T.K., 1995. Spectral reperesentation of fractional Brownian motion in n dimensions and its properties. IEEE Trans. Inform. Theory 41 (5).

Rozanov, Y., 1969. Probability Theory. Springer, Berlin.

Stein, E.M., 1970. Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton.

Stein, E.M., Weiss, G., 1970. Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton.

Taylor, R.L., 1978. Stochastic convergence of weighted sums of random elements in linear spaces. Lecture Notes in Mathematics, vol. 672, Springer, Berlin.