Contemporary Mathematics
Volume **585**, 2013
http://dx.doi.org/10.1090/conm/585/11616

On twisted conjugacy classes of type D in sporadic simple groups

F. Fantino and L. Vendramin

Abstract. We determine twisted conjugacy classes of type D associated with the sporadic simple groups. This is an important step in the program of the classification of finite-dimensional pointed Hopf algebras with non-abelian coradical. As a by-product we prove that every complex finite-dimensional pointed Hopf algebra over the group of automorphisms of M_{12}, J_2, Suz, He, HN, T is the group algebra. In the appendix we improve the study of conjugacy classes of type D of sporadic simple groups.

1. Introduction

A fundamental step in the classification of finite-dimensional complex pointed Hopf algebras, in the context of the Lifting method [**AS1**], is the determination of all finite-dimensional Nichols algebras of braided vector spaces arising from Yetter-Drinfled modules over groups. This problem can be reformulated in other terms: to study finite-dimensional Nichols algebras of braided vector spaces arising from pairs (X, q), where X is a rack and q is a 2-cocycle of X.

A useful strategy to deal with this problem is to discard those pairs (X, q) whose associated Nichols algebra is infinite dimensional. A powerful tool to discard such pairs is the notion of rack of type D [**AFGV1**]. This notion is based on the theory of Weyl groupoids developed in [**AHS**] and [**HS**]. The importance of racks of type D lies in the following fact: if X is a rack of type D then the Nichols algebra associated to (X, q) is infinite-dimensional for all 2-cocycle q. The property of being of type D is well-behaved with respect to monomorphisms and epimorphisms, see Remark 2.2. On the other hand, it is well-known that a finite rack can be decomposed as an union of indecomposable subracks. Further, every indecomposable rack X admits a surjection $X \to Y$, where Y is a simple rack, and the classification of finite simple racks is known, see [**AG**] and [**J**]. These facts and the ubiquity of racks of type D suggests a powerful approach for the classification problem of finite-dimensional pointed Hopf algebras over non-abelian groups: to classify finite simple racks of type D. This program was described in [**AFGaV**, §2] and successfully applied to the classification of finite-dimensional pointed Hopf algebras over the alternating

2010 *Mathematics Subject Classification.* Primary 16T05, 17B37.

This work was partially supported by CONICET, ANPCyT-Foncyt, Secyt-UNC, Embajada de Francia en Argentina.

©2013 American Mathematical Society

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

simple groups [**AFGV1**] and over many of the sporadic simple groups [**AFGV3**]. This paper is a contribution to this program.

Towards the classification of simple racks of type D, we study an important family of simple racks: the twisted conjugacy classes of a sporadic simple group L. Our aim is to classify which of these racks are of type D. For that purpose, we use the fact that these racks can be realized as conjugacy classes of the group of automorphisms of L. The main result of our work is the following theorem.

THEOREM 1.1. *Let L be one of the simple groups*

$$M_{12},\ M_{22},\ J_2,\ J_3,\ Suz,\ HS,\ McL,\ He,\ Fi_{22},\ ON,\ Fi_{24}',\ HN,\ T.$$

Let $\mathcal{O}$ be a conjugacy class of $\mathrm{Aut}(L)$ *not contained in L which is not listed in Table* 1*. Then $\mathcal{O}$ is of type D.*

TABLE 1. Classes not of type D

$\mathrm{Aut}(M_{22})$	2B
$\mathrm{Aut}(HS)$	2C
$\mathrm{Aut}(Fi_{22})$	2D
$\mathrm{Aut}(J_3)$	34A, 34B
$\mathrm{Aut}(ON)$	38A, 38B, 38C
$\mathrm{Aut}(McL)$	22A, 22B
$\mathrm{Aut}(Fi_{24}')$	2C

Notice that the groups in Theorem 1.1 are the only sporadic simple groups with non-trivial outer automorphism group. Theorem 1.1 with [**AFGV3**] and the lifting method [**AS1**] imply the following classification result.

COROLLARY 1.2. *Let L be one of the simple groups*

$$M_{12},\ J_2,\ Suz,\ He,\ HN,\ T.$$

Then $\mathrm{Aut}(L)$ *does not have non-trivial finite-dimensional complex pointed Hopf algebras.* □

The study of twisted conjugacy classes of sporadic groups is suitable for being attacked case-by-case with the help of computer calculations. The strategy for proving Theorem 1.1 is the same as in [**AFGV2, AFGV3**]. We use the computer algebra system GAP to perform the computations [**GAP**] [**B**] [**WPN+**] [**WWT+**]. The main scripts and log files of this work can be found in: `http://www.famaf.unc.edu.ar/~fantino/fv.tar.gz` or `http://mate.dm.uba.ar/~lvendram/fv.tar.gz`.

The paper is organized as follows. In Section 2 we deal with the basic definitions and the basic techniques for studying Nichols algebras over simple racks. The proof of the main result is given in Section 3. In the appendix we improve the classification of racks of type D given in [**AFGV3**, Table 2]. With the exception of the Monster group, conjugacy classes of type D in sporadic simple groups are classified, see Remark 4.13.

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

2. Preliminaries

We refer to [**AS2**] for generalities about Nichols algebras and to [**AG**] for generalities about racks and their cohomologies in the context of Nichols algebras. We follow [**CCNPW**] for the notations concerning the sporadic simple groups.

A *rack* is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \to X$ is a map (considered as a binary operation on X) such that the map $\varphi_x : X \to X$, $\varphi_x(y) = x \triangleright y$, is bijective for all $x \in X$, and $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$. A *subrack* of a rack X is a non-empty subset $Y \subseteq X$ such that $(Y, \triangleright)$ is also a rack. A rack X is said to be a *quandle* if $x \triangleright x = x$ for all $x \in X$. All the racks considered in this work are indeed quandles.

A rack $(X, \triangleright)$ is said to be of *type D* if it contains a decomposable subrack $Y = R \sqcup S$ such that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$ for some $r \in R$, $s \in S$.

Remark 2.1. *Let G be a group. A conjugacy class $\mathcal{O}$ of G is of type D if and only if there exist $r, s \in \mathcal{O}$ such that $(rs)^2 \neq (sr)^2$ and r and s are not conjugate in the group generated by r and s, see* [**AFGV3**, Subsection 2.2].

Remark 2.2. *Racks of type D have the following properties:*

(i) If $Y \subseteq X$ is a subrack of type D, then X is of type D.
(ii) If Z is a finite rack and $p : Z \to X$ is an epimorphism, then X of type D implies Z of type D.

The following result is the reason why it is important to study racks of type D. This theorem is based on [**AHS**] and [**HS**].

Theorem 2.3. [**AFGV1**, Thm. 3.6] *Let X be a finite rack of type D. Then the Nichols algebra associated with the pair (X, q) is infinite-dimensional for all 2-cocycles q.* □

A rack is *simple* if it has no quotients except itself and the one-element rack. We recall the classification of finite simple racks given in [**AG**, Theorems 3.9 and 3.12], see also [**J**]. A finite simple rack belongs to one of the following classes:

(a) simple affine racks;
(b) non-trivial conjugacy classes of non-abelian finite simple groups;
(c) non-trivial twisted conjugacy classes of non-abelian finite simple groups;
(d) simple twisted homogeneous racks.

In this paper we study non-trivial twisted conjugacy classes of type D of sporadic simple groups. These racks belong to the class (c) mentioned above.

2.1. Twisted conjugacy classes. Let G be a finite group and $u \in \mathrm{Aut}(G)$. The group G acts on itself by $y \rightharpoonup_u x = y\, x\, u(y^{-1})$ for all $x, y \in G$. The orbit of x under this action will be called the *u-twisted conjugacy class* of x and it will be denoted by $\mathcal{O}_x^{G,u}$. It is easy to prove that the orbit $\mathcal{O}_x^{G,u}$ is a rack with

$$y \triangleright_u z = y\, u(z\, y^{-1})$$

for all $y, z \in \mathcal{O}_x^{G,u}$. Notice that $\mathcal{O}_x^{G,\mathrm{id}}$ is a conjugacy class in G.

We write $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ for the group of outer automorphisms of G and $\pi : \mathrm{Aut}(G) \to \mathrm{Out}(G)$ for the canonical surjection.

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

Assume that $\mathrm{Out}(G) \neq 1$. Let $u \in \mathrm{Aut}(G)$ such that $\pi(u) \neq 1$. Every u-twisted conjugacy class in G is isomorphic (as a rack) to a conjugacy class in the semidirect product $G \rtimes \langle u \rangle$. Indeed,

$$\mathcal{O}_{(x,u)}^{G\rtimes\langle u\rangle,\mathrm{id}} = \mathcal{O}_x^{G,u} \times \{u\}$$

for all $x \in G$. Therefore the problem of determining u-twisted conjugacy classes of type D in G can be reduced to study conjugacy classes of type D in $G \rtimes \langle u \rangle$ and contained in $G \times \{u\}$.

2.2. Conjugacy classes to study. Let L be one of the simple groups

$$M_{12},\ M_{22},\ J_2,\ J_3,\ Suz,\ HS,\ McL,\ He,\ Fi_{22},\ ON,\ Fi_{24}',\ HN,\ T.$$

It is well-known that $\mathrm{Aut}(L) \simeq L \rtimes \mathbb{Z}_2$ [**CCNPW**]. Hence, since L is a normal subgroup of $L \rtimes \mathbb{Z}_2$, it is possible to compute the list of conjugacy classes of $\mathrm{Aut}(L)$ not contained in L from the character table of $\mathrm{Aut}(L)$, see for example [**I**]. For that purpose, we use the GAP function `ClassPositionsOfDerivedSubgroup`. See the file `classes.log` for the information concerning the conjugacy classes of $\mathrm{Aut}(L)$ not contained in L.

2.3. Strategy. Our aim is to classify twisted conjugacy classes of sporadic simple groups of type D. By Subsection 2.1, we need to consider the conjugacy classes in $\mathrm{Aut}(L) \setminus L$, where L is a sporadic simple group with $\mathrm{Out}(L) \neq 1$. The strategy for studying these conjugacy classes is essentially based on studying conjugacy classes of type D in maximal subgroups of $\mathrm{Aut}(L)$. See [**AFGV2**, Subsection 1.1] for an exposition about the algorithms used.

2.4. Useful lemmata. Let G be a non-abelian group and $g \in G$. We write g^G for the conjugacy class of g in G. Let

$$\mathfrak{M}_g = \{M : M \text{ is a maximal subgroup of } G \text{ and } g^G \cap M \neq \emptyset\}.$$

LEMMA 2.4 (Breuer). *Assume that for all $M \in \mathfrak{M}_g$ there exists $m \in M$ such that*

$$g^G \cap M \subseteq m^M \subseteq g^G.$$

If g^G is of type D, then there exist $N \in \mathfrak{M}_g$ and $n \in N$ such that n^N is of type D.

PROOF. Since g^G is of type D, there exist $r, s \in g^G$ such that $(rs)^2 \neq (sr)^2$ and $r^H \cap s^H = \emptyset$ for $H = \langle r, s\rangle$. Since $r^H \cap s^H = \emptyset$ and $r^H \cup s^H \subseteq g^G$, the group H is contained in some maximal subgroup $N \in \mathfrak{M}_g$. Hence $r, s \in g^G \cap N \subseteq n^N$ for some $n \in N$ and the claim follows. □

LEMMA 2.5. *Let $\mathcal{O}$ be a conjugacy class of G and let H be a subgroup of G such that $\mathcal{O}$ contains two conjugacy classes $\mathcal{O}_1$, $\mathcal{O}_2$ of H. Assume that there exist $r \in \mathcal{O}_1$ and $s \in \mathcal{O}_2$ such that $(rs)^2$ does not belong to the centralizer of r in G. Then $\mathcal{O}$ is of type D.*

PROOF. Notice that $(rs)^2 = (sr)^2$ if and only if $(rs)^2$ commutes with r. Then the claim follows. □

LEMMA 2.6. *Let $\mathcal{O}$ be a conjugacy class of involutions of G. Then $\mathcal{O}$ is of type D if and only if there exist r, $s \in \mathcal{O}$ such that the order of rs is even and greater or equal to 6.*

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

PROOF. Assume that $|rs| = n$. Then $\langle r, s\rangle \simeq \mathbb{D}_n$ and the claim follows; see [**AFGV3**, §1.8]. □

3. Proof of Theorem 1.1

The claim concerning the automorphism groups of M_{12} and J_2 follows from the application of [**AFGV3**, Algorithm I]. The claim for the automorphism groups of M_{22}, Suz, HS, He, Fi_{22} and T follows from the application of [**AFGV3**, Algorithm III]. There is one log file for each of these groups, see Table 2. The automorphism groups of J_3, ON, McL, HN and Fi'_{24} are studied in Subsections 3.1, 3.2 and 3.3, respectively.

TABLE 2. Log files

L	log file	L	log file
M_{12}	`M12.2.log`	HS	`HS.2.log`
M_{22}	`M22.2.log`	He	`He.2.log`
J_2	`J2.2.log`	Fi_{22}	`Fi22.2.log`
Suz	`Suz.2.log`	T	`T.2.log`

3.1. The groups $\mathrm{Aut}(J_3)$, $\mathrm{Aut}(ON)$ and $\mathrm{Aut}(McL)$.

LEMMA 3.1.

(1) A conjugacy class $\mathcal{O}$ of $\mathrm{Aut}(J_3) \setminus J_3$ *is of type D if and only if* $\mathcal{O} \notin \{34\mathrm{A}, 34\mathrm{B}\}$.

(2) A conjugacy class $\mathcal{O}$ of $\mathrm{Aut}(ON) \setminus ON$ *is of type D if and only if* $\mathcal{O} \notin \{38\mathrm{A}, 38\mathrm{B}, 38\mathrm{C}\}$.

(3) A conjugacy class $\mathcal{O}$ of $\mathrm{Aut}(McL) \setminus McL$ *is of type D if and only if* $\mathcal{O} \notin \{22\mathrm{A}, 22\mathrm{B}\}$.

PROOF. We first prove (1). We claim that the classes 34A, 34B are not of type D. Let $G = \mathrm{Aut}(J_3)$ and let g be a representative of the conjugacy class 34A (the proof for the class 34B is analogous). By [**CCNPW**], the only maximal subgroup containing elements of order 34 is $\mathcal{M}_4 \simeq \mathbf{PSL}(2,17) \times \mathbb{Z}_2$. Further, it is easy to see that $\mathcal{M}_4 \in \mathfrak{M}_g$ satisfies $g^G \cap \mathcal{M}_4 \subseteq m^{\mathcal{M}_4} \subseteq g^G$ for some $m \in \mathcal{M}_4$ and the class $m^{\mathcal{M}_4}$ is not of type D. Hence Lemma 2.4 applies. See the file `J3.2/34AB.log` for more information. To prove that the remaining conjugacy classes are of type D, apply [**AFGV3**, Algorithm III]. See the file `J3.2/J3.2.log` for more information.

Now we prove (2). We claim that the classes 38A, 38B, 38C of $\mathrm{Aut}(ON)$ are not of type D. The only maximal subgroup (up to conjugation) of $\mathrm{Aut}(ON)$ containing elements of order 38 is the second maximal subgroup $\mathcal{M}_2$. By Lemma 2.4, it suffices to prove that the classes 38a, 38b, 38c of $\mathcal{M}_2$ are not of type D. This follows from a direct GAP computation. See the file `ON.2/38ABC.log` for more information. To prove that the remaining conjugacy classes are of type D we apply [**AFGV3**, Algorithm III]. See the file `ON.2/ON.2.log` for more information.

Now we prove (3). We claim that the classes 22A, 22B of $\mathrm{Aut}(McL)$ are not of type D. The only maximal subgroup (up to conjugation) of $\mathrm{Aut}(McL)$ containing elements of order 22 is the 8th maximal subgroup $\mathcal{M}_8$. By Lemma 2.4, it suffices to prove that the classes 22a, 22b of $\mathcal{M}_8$ are not of type D. This follows from a direct

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

GAP computation. See the file `McL.2/22AB.log` for more information. To prove that the remaining conjugacy classes are of type D, apply [**AFGV3**, Algorithm III]. See the file `McL.2/McL.2.log` for more information. ☐

3.2. The group $\mathrm{Aut}(HN)$.

LEMMA 3.2. *All the conjugacy classes in* $\mathrm{Aut}(HN) \setminus HN$ *are of type D.*

PROOF. With GAP it is possible to obtain the information related to the fusion of the conjugacy classes from the maximal subgroups of $\mathrm{Aut}(HN)$ into $\mathrm{Aut}(HN)$. The following table shows the maximal subgroup (and the log file) used and the conjugacy classes of $\mathrm{Aut}(HN)$ of type D.

File	Classes
`M2`	4D,4E,4F,6D,6E,6F,8C,8D,10G,10H,12D 12E,14B,18A,20F,24A,28A,30C,42A,60A
`M13`	8F,24B,24C
`M9`	8E
`M7`	20E,20G,20H,20I,40B,40C,40D
`M3`	44A,44B

It remains to prove that the class 2C is of type D. By [**AFGV1**, Thm. 4.1], the class of transpositions in $\mathbb{S}_{12}$ is the unique class of involutions which is not of type D. But there are three different conjugacy classes of involutions of the maximal subgroup $\mathcal{M}_2 \simeq \mathbb{S}_{12}$ contained in the class 2C of $\mathrm{Aut}(HN)$ and hence the latter is of type D. ☐

3.3. The group $\mathrm{Aut}(Fi_{24}')$.

LEMMA 3.3. *Let $\mathcal{O}$ be a conjugacy class of* $\mathrm{Aut}(Fi_{24}') \setminus Fi_{24}'$. *Then $\mathcal{O}$ is of type D if and only if* $\mathcal{O} \notin \{2\mathrm{C}\}$.

PROOF. As before, we study conjugacy classes in certain maximal subgroups. See the following table for more information:

File	Classes
`M5`	4G, 12N, 12U, 12V, 12Y, 24H, 40A
`M17`	30G
`M18`	18O
`M9`	66A, 66B
`M12`	36D, 36G
`M20`	12A1, 28C, 28D

To prove that the classes 6V, 42D, 84A are of type D we use the maximal subgroup $\mathcal{M}_{19} \simeq (\mathbb{Z}_7 \rtimes \mathbb{Z}_6) \times \mathbb{S}_7$. Notice that six conjugacy classes of $\mathcal{M}_{19}$ are contained in the class 6V. Further, three of them are of type D. On the other hand, the conjugacy classes of elements of order 42 and 84 in $\mathcal{M}_{19}$ are of type D.

The class 2D of $\mathrm{Aut}(Fi_{24}')$ contains the classes $\mathcal{O}_1 = 2\mathrm{d}$ and $\mathcal{O}_2 = 2\mathrm{g}$ of the maximal subgroup $\mathcal{M}_4 \simeq \mathbb{S}_3 \times O_8^+(3).\mathbb{S}_3$. With GAP we show that there exist $r \in \mathcal{O}_1$ and $s \in \mathcal{O}_2$ such that $(rs)^2$ has order 13 which does not divide order of the centralizer of the conjugacy class 2A of $\mathrm{Aut}(Fi_{24}')$. Hence $(rs)^2$ does not commute with r and therefore Lemma 2.5 applies and hence 2D is of type D.

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

TABLE 3. Classes in sporadic simple groups (different from the Monster M) not of type D

Group	Classes
T	2A
M_{11}	8A, 8B, 11A, 11B
M_{12}	11A, 11B
M_{22}	11A, 11B
M_{23}	23A, 23B
M_{24}	23A, 23B
Ru	29A, 29B
Suz	3A
HS	11A, 11B
McL	11A, 11B
Co_1	3A
Co_2	2A, 23A, 23B
Co_3	23A, 23B
J_1	15A, 15B, 19A, 19B, 19C
J_2	2A, 3A
J_3	5A, 5B, 19A, 19B
J_4	29A, 43A, 43B, 43C
Ly	37A, 37B, 67A, 67B, 67C
$O'N$	31A, 31B
Fi_{23}	2A
Fi_{22}	2A, 22A, 22B
Fi'_{24}	29A, 29B
B	2A, 46A, 46B, 47A, 47B

The class 2C is not of type D. Indeed, for all $r, s \in$ 2C, the order of rs is 1, 2 or 3; for this we use the GAP function `ClassMultiplicationCoefficient`. By Lemma 2.6, the claim holds.

For studying the remaining conjugacy classes we use the maximal subgroup $\mathcal{M}_2 \simeq \mathbb{Z}_2 \times Fi_{23}$. By [**AFGV3**, Thm. II], every conjugacy class of $\mathcal{M}_2$ with representative of order distinct from 2 is of type D, see Proposition 4.3 in the appendix. Hence the claim follows. □

4. Appendix: the sporadic simple groups

In this appendix we improve some of the results obtained in [**AFGV3**]. We remark that it is important to know if a rack X is of type D. By Theorem 2.3, if X is of type D, then $\dim \mathfrak{B}(X, q) = \infty$ for any 2-cocycle q of X, and hence the calculation of the 2-cocycles of X is not needed. Further, as a corollary we obtain that the Nichols algebras associated with any rack Y containing X and for any rack Z having X as a quotient are also infinite-dimensional; see [**AFGaV**] or [**AFGV3**]. Indeed, since any finite rack has a projection onto a simple rack, this shows the intrinsic importance of a conjugacy class of being of type D, not just for the specific group where it lives but to the whole classification program of finite-dimensional Nichols algebras associated to racks.

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

Table 3 contains the list of conjugacy classes of sporadic simple groups which are not of type D. The open cases are listed in Remark 4.13.

4.1. The groups T and Suz.

PROPOSITION 4.1.

(1) A conjugacy class $\mathcal{O}$ of T is of type D if and only if $\mathcal{O} \neq$ 2A.
(2) A conjugacy class $\mathcal{O}$ of Suz is of type D if and only if $\mathcal{O} \neq$ 3A.

PROOF. It follows from [**AFGV3**, Table 2] and a direct computer calculation. See the log files for details. □

4.2. The groups ON, McL, Co_3, Ru, HS and J_3.

PROPOSITION 4.2.

(1) A conjugacy class of ON is of type D if and only if it is different from 31A *and* 31B.
(2) A conjugacy class of McL is of type D if and only if it is different from 11A *and* 11B.
(3) A conjugacy class of Co_3 is of type D if and only if it is different from 23A *and* 23B.
(4) A conjugacy class of Ru is of type D if and only if it is different from 29A *and* 29B.
(5) A conjugacy class of HS is of type D if and only if it is different from 11A *and* 11B.
(6) A conjugacy class of J_3 is of type D if and only if it is different from 5A, 5B, 19A *and* 19B.

PROOF. We prove (1). By [**AFGV3**, Table 2], it remains to prove that the classes 31A, 31B are not of type D. Let g be a representative for the conjugacy class 31A of $G = ON$ (the proof for the class 31B is analogous). By [**CCNPW**], the only maximal subgroups (up to conjugacy) containing elements of order 31 are $\mathcal{M}_7$ and $\mathcal{M}_8$. Further, $\mathcal{M}_7 \simeq \mathcal{M}_8 \simeq \mathbf{PSL}(2,7)$ and it is easy to see that if $M = \mathcal{M}_7$ (or $\mathcal{M}_8$) then $g^G \cap M \subseteq m^M \subseteq g^G$ for some $m \in M$. Since the conjugacy class m^M is not of type D for all $m \in M$ of order 31, Lemma 2.4 applies.

To prove (2) the maximal subgroups to use are the Mathieu groups M_{11} and M_{22}. Then the claim follows from Lemma 2.4 and [**AFGV3**, Table 2]. To prove (3) the maximal subgroups to use are the Mathieu groups M_{23}. The proofs for (4)–(6) are similar. □

4.3. The group Fi_{23}.

PROPOSITION 4.3. *Let $\mathcal{O}$ be a conjugacy class of Fi_{23}. Then $\mathcal{O}$ is of type D if and only if $\mathcal{O}$ is not* 2A.

PROOF. [1] By [**AFGV3**, Table 2], it remains to prove that the classes 23A, 23B are of type D. Let N denote the normal subgroup of order 2^{11} in the maximal subgroup $\mathcal{M}_6 \simeq 2^{11}.M_{23}$ of Fi_{23}, and let x be an element of order 23 in the factor group $\mathcal{M}_6/N$. All preimages of x under the natural epimorphism from $\mathcal{M}_6$ to $\mathcal{M}_6/N$ have order 23, they are conjugate in $\mathcal{M}_6$, and their squares are also conjugate in $\mathcal{M}_6$. Take a preimage r of x under the natural epimorphism, choose

[1]This proposition is due to the referee.

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

a nonidentity element $n \in N$, and set $s = r^2 n$. Then r and s are conjugate in $\mathcal{M}_6$. Moreover, $(rs)^2$ and $(sr)^2 = r^{-1}(rs)^2 r$ are different. Indeed, $(rs)^2 = (r^3 n)^2 = r^6 n'$, with $n' := (r^{-3} n r^3) n$, whereas $(sr)^2 = r^6 (r^{-1} n' r)$.

The group U generated by r and s is also generated by r and n, and since r acts irreducibly on N, we get that U is a semidirect product of N and $\langle r \rangle$. In particular, r and s are not conjugate in U. Hence, the class of r in $\mathcal{M}_6$ is of type D. □

4.4. The group Fi_{22}.

PROPOSITION 4.4 (Breuer). *Let $\mathcal{O}$ be a conjugacy class of Fi_{22}. Then $\mathcal{O}$ is of type D if and only if $\mathcal{O} \notin \{$2A, 22A, 22B$\}$.*

PROOF. By [**AFGV3**, Table 2], it remains to prove that the classes 22A, 22B are not of type D. Assume that the class 22A of Fi_{22} is of type D (the proof for the class 22B is analogous). Let r and s be elements of the class 22A such that r and s are not conjugate in the group $H = \langle r, s \rangle$. By the fusion of conjugacy classes, H is a proper subgroup of some maximal subgroup M isomorphic to $2.U_6(2)$. Notice that the center of M is $Z(M) = \langle z \rangle \simeq \mathbb{Z}_2$. Using the GAP function `PowerMap` we get $r^{11} = s^{11} = z$ and hence $Z(M) \subseteq H$. We claim that the elements $rZ(M)$ and $sZ(M)$ are not conjugate in the quotient $H/Z(M)$. Let $p : H \to H/Z(M)$ be the canonical projection, and let $x \in H$ such that $p(x)p(r)p(x)^{-1} = p(s)$. Then $xrx^{-1} \in \{s, sz\}$ and hence $xrx^{-1} = s$ since sz has order 11. Now the claim follows from the following lemma. □

LEMMA 4.5. *Let $Q = U_6(2)$ and $x, y \in Q$ be two elements of order* 11 *such that $x^Q = y^Q$. Assume that x and y are not conjugate in the subgroup $\langle x, y \rangle$. Then $\langle x, y \rangle \simeq \mathbb{Z}_{11}$.*

PROOF. Let $U = \langle x, y \rangle$. Since $x^Q = y^Q$ and $x^U \neq y^U$, U is a proper subgroup of a maximal subgroup M and $M \simeq U_5(2)$ or $M \simeq M_{22}$. The only maximal subgroup of M which contains elements of order 11 is isomorphic to $L_2(11)$ and hence we may assume that U is a proper subgroup of $L_2(11)$ because $L_2(11)$ has exactly two conjugacy classes of elements of order 11. The only maximal subgroups of $L_2(11)$ that contain elements of order 11 are isomorphic to $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$. These groups have exactly two conjugacy classes of elements of order 11 and hence U must be a proper subgroup of $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$. From this the claim follows. □

4.5. The group Co_2.

PROPOSITION 4.6. *Let $\mathcal{O}$ be a conjugacy class of Co_2. Then $\mathcal{O}$ is of type D if and only if $\mathcal{O} \notin \{$2A, 23A, 23B$\}$.*

PROOF. By [**AFGV3**, Table 2], it remains to prove that the classes 23A, 23B are not of type D. This follows from the following lemma. □

LEMMA 4.7. *Let $Q = Co_2$ and $x, y \in Q$ be two elements of order* 23 *such that $x^Q = y^Q$. Assume that x and y are not conjugate in the subgroup $\langle x, y \rangle$. Then $\langle x, y \rangle \simeq \mathbb{Z}_{23}$.*

PROOF. Let $U = \langle x, y \rangle$. Since $x^Q = y^Q$ and $x^U \neq y^U$, U is a proper subgroup of a maximal subgroup M and $M \simeq M_{23}$. The only maximal subgroup of M which contains elements of order 23 is isomorphic to $\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}$. These groups have

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

exactly two conjugacy classes of elements of order 23 and hence U must be a proper subgroup of $\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}$. From this the claim follows. □

4.6. The group J_4.

PROPOSITION 4.8. *Let $\mathcal{O}$ be a conjugacy class of J_4. Then $\mathcal{O}$ is of type D if and only if* $\mathcal{O} \notin \{29A, 43A, 43B, 43C\}$.

PROOF. By [**AFGV3**, Table 2] it remains to study the classes 29A, 37A, 37B, 37C, 43A, 43B, 43C. We split the proof into two steps.

STEP 1. *The classes* 29A, 43A, 43B, 43C *of J_4 are not of type D.*

This is similar to the proof of Proposition 4.2. See the files in the folder `J4` for more information.

STEP 2. *The classes* 37A, 37B, 37C *of J_4 are of type D.*

The class 37A of J_4 contains the classes $\mathcal{O}_1 = 37a$ and $\mathcal{O}_2 = 37d$ of the maximal subgroup $\mathcal{M}_5 \simeq U_3(11).2$. With GAP we show that there exist $r \in \mathcal{O}_1$ and $s \in \mathcal{O}_2$ such that $(rs)^2$ has order 5. The centralizer associated with the conjugacy class 37A of J_4 is isomorphic to $\mathbb{Z}_{37}$ and therefore $(rs)^2$ does not commute with r. Hence Lemma 2.5 applies and the claim follows. The proof for the classes 37B, 37C of J_4 is analogous, see the file `J4/37ABC.log` for more information. □

4.7. The group Ly.

PROPOSITION 4.9. *Let $\mathcal{O}$ be a conjugacy class of Ly. Then $\mathcal{O}$ is of type D if and only if* $\mathcal{O} \notin \{37A, 37B, 67A, 67B, 67C\}$.

PROOF. By [**AFGV3**, Table 2], it remains to study the classes 33A, 33B, 37A, 37B, 37C, 37C, 67A, 67B, 67C. We split the proof into two steps.

STEP 1. *The classes* 37A, 37B, 67A, 67B, 67C *of Ly are not of type D.*

It is similar to the proof of Proposition 4.2. See the files `Ly/37AB.log` and `Ly/67ABC.log` for more information.

STEP 2. *The classes* 33A, 33B *of Ly are of type D.*

It suffices to prove that the classes 33A and 33B of the maximal subgroup $3.McL.2$ are of type D. This follows from Lemma 2.5 with the subgroup $3.McL$. See the file `Ly/33AB.log` for more information. □

4.8. The group Fi'_{24}.

PROPOSITION 4.10. *A conjugacy class $\mathcal{O}$ of Fi'_{24} is of type D if and only if* $\mathcal{O} \notin \{29A, 29B\}$.

PROOF. By [**AFGV3**, Table 2], it remains to prove that the classes 23A, 23B, 27B, 27C, 33A, 33B, 39C, 39D of Fi'_{24} are of type D and that the classes 29A, 29B are not. For the classes 23A and 23B the result follows from Proposition 4.3. The following six classes can be treated by Lemma 2.5. The table below contains the information concerning the maximal subgroups to use:

Classes	Maximal subgroup	Log file
27B,27C	$\mathcal{M}_5$	`F3+/27BC.log`
33A,33B	$\mathcal{M}_4$	`F3+/33AB.log`
39C,39D	$\mathcal{M}_3$	`F3+/39CD.log`

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

Now we prove that the classes 29A, 29B of Fi'_{24} are not of type D. The unique maximal subgroup (up to conjugacy) that contains elements of order 29 is $\mathcal{M}_{25} \simeq \mathbb{Z}_{29} \rtimes \mathbb{Z}_{14}$. This group has two classes of elements of order 29 and these classes are not of type D. Therefore Lemma 2.4 applies. □

4.9. The group Co_1.

Proposition 4.11. *Let $\mathcal{O}$ be a conjugacy class of Co_1. Then $\mathcal{O}$ is of type D if and only if $\mathcal{O} \notin \{3\text{A}\}$.*

Proof. By [**AFGV3**, Table 2], it remains to prove that the classes 23A, 23B are of type D and that the class 3A is not of type D. The proof for the classes 23A and 23B is analogous to the proof of Proposition 4.3 using the maximal subgroup $\mathcal{M}_3 \simeq 2^{11} : M_{24}$. The claim for the class 3A follows from a straightforward computer calculation. □

4.10. The group B.

Proposition 4.12. *Let $\mathcal{O}$ be a non-trivial conjugacy class of B. Then $\mathcal{O}$ is of type D if and only if $\mathcal{O} \notin \{2\text{A}, 46\text{A}, 46\text{B}, 47\text{A}, 47\text{B}\}$.*

Proof. By [**AFGV3**], it remains to study the conjugacy classes 2A, 16C, 16D, 32A, 32B, 32C, 32D, 34A, 46A, 46B, 47A, 47B. We split the proof into several steps.

Step 1. *The conjugacy class* 2A *is not of type D.*

With the GAP function `ClassMultiplicationCoefficient` we see that for all $r, s \in 2\text{A}$, $|rs|$ is 1, 2, 3 or 4. Then the claim follows from Lemma 2.6.

Step 2. *The conjugacy classes* 46A, 46B *of B are not of type D.*

Assume that the class 46A of B is of type D (the proof for the class 46B is analogous). Let r and s be elements of the class 46A such that r and s are not conjugate in the group $H = \langle r, s\rangle$. By the fusion of conjugacy classes, H is a proper subgroup of some maximal subgroup M isomorphic to $2^{1+22}.Co_2$. Notice that the center of M is $Z(M) = \langle z\rangle \simeq \mathbb{Z}_2$. With the GAP function `PowerMap` we get $r^{23} = s^{23} = z$ and hence $Z(M) \subseteq H$. We claim that the elements $rZ(M)$ and $sZ(M)$ are not conjugate in the quotient $H/Z(M)$. Let $p : H \to H/Z(M)$ be the canonical projection, and let $x \in H$ such that $p(x)p(r)p(x)^{-1} = p(s)$. Then $xrx^{-1} \in \{s, sz\}$ and hence $xrx^{-1} = s$ since sz has order 23. Now the claim follows from Lemma 4.7.

Step 3. *The conjugacy classes* 47A, 47B *of B are not of type D.*

It is easy to check that the only maximal subgroup of B (up to conjugacy) which contains elements of order 47 is $\mathcal{M}_{30} \simeq \mathbb{Z}_{47} \rtimes \mathbb{Z}_{23}$. (This is the only non-abelian group of order 1081.) This group has two conjugacy classes of elements of order 47 and these classes are not of type D. Then the claim follows from Lemma 2.4.

Step 4. *The class* 34A *of B is of type D.*

The conjugacy classes $\mathcal{O}_1 = 34\text{d}$ and $\mathcal{O}_2 = 34\text{f}$ of the first maximal subgroup of B are contained in the class 34A of B. With the GAP functions `ClassMultiplicationCoefficient` and `PowerMap` we see that there exist $r \in \mathcal{O}_1$ and $s \in \mathcal{O}_2$ such that $|(rs)^2| = 5$. Since the centralizer corresponding to the class 34A has order 68, the claim follows from Lemma 2.5.

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

STEP 5. *The classes* 16C, 16D *of* B *are of type D.*

Let $\mathcal{O}$ be the conjugacy class 16C (resp. 16D) of B. The conjugacy classes $\mathcal{O}_1 =$ 16g (resp. 16a) and $\mathcal{O}_2 =$ 16n (resp. 16f) of the first maximal subgroup of B are contained in the class $\mathcal{O}$. With the GAP functions `ClassMultiplicationCoefficient` and `PowerMap` it is easy to see that there exist $r \in \mathcal{O}_1$ and $s \in \mathcal{O}_2$ such that $(rs)^2$ has order 5. Since the centralizer corresponding to the class $\mathcal{O}$ has order 2^{11}, the claim follows from Lemma 2.5.

STEP 6. *The classes* 32A, 32B, 32C, 32D *of* B *are of type D.*

Let $\mathcal{O}$ be the conjugacy class 32A of B. We use the GAP function `PossibleClassFusions` to obtain a list with all the possible fusions from the maximal subgroup $\mathcal{M}_6$ into B. As in the previous step, with the GAP functions `ClassMultiplicationCoefficient` and `PowerMap` it is easy to show that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are two different conjugacy classes of $\mathcal{M}_6$ contained in the class $\mathcal{O}$ of B, then there exist $r \in \mathcal{O}_1$ and $s \in \mathcal{O}_2$ such that $(rs)^2$ has order 5. Since the centralizer related to the class $\mathcal{O}$ has size 2^7, the claim follows from Lemma 2.5. The proof for the claim concerning the classes 32B, 32C, 32D is analogous. □

REMARK 4.13. *The following conjugacy classes of the Monster group* M *are not known to be of type D:* 32A, 32B, 41A, 46A, 46B, 47A, 47B, 59A, 59B, 69A, 69B, 71A, 71B, 87A, 87B, 92A, 92B, 94A, 94B. *These are the only open cases related to the problem of classifying conjugacy classes of type D in sporadic simple groups.*

REMARK 4.14. *It is still unknown whether the Nichols algebras associated with the classes* 22A, 22B *of* Fi_{22}*, the classes* 46A, 46B *of* B*, and the classes* 32A, 32B, 46A, 46B, 92A, 92B, 94A, 94B *of* M *are finite-dimensional.*

Acknowledgement. First, we would like to thank the referee for his/her comments, which allowed us to improve the original version of the paper, and for kindly communicating us the proof of Proposition 4.3. We thank N. Andruskiewitsch for interesting conversations and T. Breuer for Lemma 2.4 and Proposition 4.4. We also thank Facultad de Matemática, Astronomía y Física (Universidad Nacional de Córdoba) for their computers shiva and ganesh where we have performed the computations. Part of the work of the first author was done during a postdoctoral position in Université Paris Diderot – Paris 7; he is very grateful to Prof. Marc Rosso for his kind hospitality.

References

[AFGaV] N. Andruskiewitsch, F. Fantino, G. A. Garcia, and L. Vendramin, *On Nichols algebras associated to simple racks*, Contemp. Math. **537** (2011) 31–56. MR2799090 (2012g:16065)

[AFGV1] N. Andruskiewitsch, F. Fantino, M. Graña, and L. Vendramin, *Finite-dimensional pointed Hopf algebras with alternating groups are trivial*, Ann. Mat. Pura Appl. (4) **190** (2011), no. 2, 225–245. MR2786171 (2012c:16095)

[AFGV2] ———, *The logbook of Pointed Hopf algebras over the sporadic simple groups*, J. Algebra **325** (2011), 282–304. MR2745541 (2012d:16092)

[AFGV3] ———, *Pointed Hopf algebras over the sporadic simple groups*, J. Algebra **325** (2011), 305–320. MR2745542 (2012d:16093)

[AG] N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math. **178** (2003), no. 2, 177–243. MR1994219 (2004i:16046)

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.

[AHS] N. Andruskiewitsch, I. Heckenberger, and H.-J. Schneider, *The Nichols algebra of a semisimple Yetter-Drinfeld module*, Amer. J. Math. **132** (2010), no. 6, 1493–1547. MR2766176 (2012a:16057)

[AS1] N. Andruskiewitsch and H.-J. Schneider, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3*, J. Algebra **209** (1998), no. 2, 658–691. MR1659895 (99k:16075)

[AS2] ———, *Pointed Hopf algebras*, New directions in Hopf algebras, Math. Sci. Res. Inst. Publ., vol. 43, Cambridge Univ. Press, Cambridge, 2002, pp. 1–68.

[B] T. Breuer, *The GAP Character Table Library, Version 1.2*; `http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib/`

[CCNPW] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray. MR827219 (88g:20025)

[GAP] *The GAP Group*, 2008, GAP – Groups, Algorithms, and Programming, Version 4.4.12. Available at `http://www.gap-system.org`.

[HS] I. Heckenberger and H.-J. Schneider, *Root systems and Weyl groupoids for Nichols algebras*, Proc. Lond. Math. Soc. (3) **101** (2010), no. 3, 623–654. MR2734956 (2011j:16067)

[I] M. Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006. MR2270898

[J] D. Joyce, *Simple quandles*, J. Algebra **79** (1982), no. 2, 307–318. MR682881 (84d:20078)

[WPN+] R. A. Wilson, R. A. Parker, S. Nickerson, J. N. Bray and T. Breuer, *AtlasRep, A GAP Interface to the ATLAS of Group Representations, Version 1.4*, 2007, Refereed GAP package, `http://www.math.rwth-aachen.de/~Thomas.Breuer/atlasrep`.

[WWT+] R. A. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray and R. Abbott, *A world-wide-web Atlas of finite group representations*, `http://brauer.maths.qmul.ac.uk/Atlas/v3/`.

Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba. CIEM – CONICET. Medina Allende s/n (5000) Ciudad Universitaria, Córdoba, Argentina

UFR de Mathématiques, Université Paris Diderot - Paris 7, 175 Rue du Chevaleret, 75013, Paris, France

E-mail address: `fantino@famaf.unc.edu.ar`

URL: `http://www.mate.uncor.edu/~fantino/`

Departamento de Matemática, FCEN, Universidad de Buenos Aires, Pab. I, Ciudad Universitaria (1428), Buenos Aires, Argentina

E-mail address: `lvendramin@dm.uba.ar`

URL: `http://mate.dm.uba.ar/~lvendram/`

This is a free offprint provided to the author by the publisher. Copyright restrictions may apply.