SOLUTIONS OF THE BRAID EQUATION AND ORDERS

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ABSTRACT. We introduce the notion of non-degenerate solution of the braid equation on the incidence coalgebra of a locally finite order. Each one of these solutions induces by restriction a non-degenerate set-theoretic solution over the underlying set. So, it makes sense to ask if a non-degenerate set-theoretic solution on the underlying set of a locally finite order extends to a non-degenerate solution on its incidence coalgebra. In this paper we begin the study of this problem.

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Introduction

Let V be a vector space over a field K. One of the more basic equations of mathematical physics is the *quantum Yang-Baxter equation*

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12} \quad \text{in } \mathrm{Aut}(V \otimes_K V)$$

where $R\colon V \otimes_K V \longrightarrow V \otimes_K V$ is a bijective linear operator and R_{ij} denotes R acting on the i-th and j-th coordinates. Let $\tau \in \mathrm{Aut}(V \otimes_K V)$ be the flip. Then R satisfies the quantum Yang-Baxter equation if and only if $r := \tau \circ R$ satisfies the *braid equation*

$$r_{12} \circ r_{23} \circ r_{12} = r_{23} \circ r_{12} \circ r_{23}. \tag{0.1}$$

So, both equations are equivalent and working with one or the other is a matter of taste. In the present paper we consider the second one. Since the eighties many solutions of the braid equation have been found, many of them being deformations of the flip. It is interesting to obtain solutions that are not of this type, and in [4], Drinfeld proposed to study the most simple of them, namely, the set-theoretic ones, i.e. pairs (X, r), where X is a set and

$$r\colon X \times X \longrightarrow X \times X$$

is an invertible map satisfying (0.1). Each one of these solutions yields in an evident way a linear solution on the vector space with basis X. From a structural point

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of view this approach was considered first by Etingof, Schedler and Soloviev [14] and Gateva-Ivanova and Van den Bergh [7] for involutive solutions, and then by Lu, Yan and Zhu [10] and Soloviev [12] for non-degenerate not necessarily involutive solutions. In the last two decades the theory has developed rapidly, and now it is known that it has connections with bijective 1-cocycles, Bierbach groups and groups of I-type, involutive Yang-Baxter groups, Garside structures, biracks, cyclic sets, braces, Hopf algebras, matched pairs, left symmetric algebras, etcetera (see, for instance [1], [5], [2], [6], [3], [8], [11], [13]).

In [9], the authors began the study of set-type solutions of the braid equation in the context of symmetric tensor categories. The underlying idea is simple: replace the sets by cocommutative coalgebras. The central result of that paper was the existence of the universal solutions in this setting (this generalizes the main result of [10]), and the main technical tool was the generalization of the concept of a non-degenerate map. But this definition makes sense for non-cocommutative coalgebras in symmetric tensor categories, and in a forthcoming paper we will investigate the non-cocommutative versions of the theoretic results established in [14], [10] and [12]. In the present paper we are interested in another type of problems, involving an important particular case, and related with the search of non-degenerate solutions on the incidence coalgebra D of a locally finite poset $(X,\leq)$ (for the definitions see Subsection 1.2). Each non-degenerate coalgebra automorphism

$$r\colon D\otimes_K D \longrightarrow D\otimes_K D$$

induces by restriction a non-degenerate bijection

$$r_{|}\colon X\times X \longrightarrow X\times X.$$

Moreover, if r is a solution of the braid equation, then $r_{|}$ is a solution of the set-theoretic braid equation. So, it makes sense to study the following problems: given a linear automorphism

$$r\colon D\otimes_K D \longrightarrow D\otimes_K D$$

such that $r_{|}$ is a non-degenerate bijection:

i) Find necessary and sufficient conditions for r to be a non-degenerate coalgebra automorphism.
ii) Assuming that $r_{|}$ is a solution of the set-theoretic braid equation, find necessary and sufficient conditions for r to be a non-degenerate solution of the braid equation.

In Sections 2 and 3 we solve completely the first problem (Section 1 is devoted to the preliminaries). The main result is Theorem 3.4. In Sections 4, 5 and 6 we consider the second problem. In Proposition 4.3 we encode in a system of (nonlinear) equations the conditions for r to be a non-degenerate solution of the braid equation. Then, in Proposition 4.3, we analyze the meaning of these equations when the sum of the lengths of the involved intervals $[a,b]$, $[c,d]$ and $[e,f]$ is less than or equal to 1. This allows us to solve them in Proposition 4.5, under fairly general conditions. In Theorem 5.4, given $p\prec q$ in X, we determine all the solutions of the equations determined by subintervals of $[p,q]$, under the hypothesis that r induces the flip on $\{p,q\}\times\{p,q\}$. Using this, in Corollary 5.5 we find all the non-degenerate solutions of the braid equation associated with the poset $(\{p,q\},\leq)$, where $p<q$. Finally, in Section 6 we give the solution of the same problem for the configuration $o\prec q\succ p$, under the hypothesis that $r_{|}$ is the permutation of $\{o,q,p\}\times\{o,q,p\}$ given by $r_{|}(x,y)\coloneqq(\phi(y),\phi(x))$, where ϕ is the permutation of $\{o,p,q\}$ that interchanges o and p.

1 Preliminaries

In this paper we work in the category of vector spaces over a field K, all the maps between vector spaces are K-linear maps, and given vector spaces V and W, we let $V \otimes W$ denote the tensor product $V \otimes_K W$ and we set $V^2 := V \otimes V$.

1.1 Braided sets

Let C be a coalgebra. Let r be a coalgebra automorphism of C^2 and let

$$\sigma := (C \otimes \epsilon) \circ r \quad \text{and} \quad \tau := (\epsilon \otimes C) \circ r.$$

Then $r = (\sigma \otimes \tau) \circ \Delta_{C^2}$. Moreover σ and τ are the unique coalgebra morphisms with this property.

Definition 1.1. *A pair (C, r), where r is coalgebra automorphism of C^2, is called a* braided set *if r satisfies the braid equation*

$$r_{12} \circ r_{23} \circ r_{12} = r_{23} \circ r_{12} \circ r_{23}, \tag{1.1}$$

where $r_{12} := r \otimes C$ and $r_{23} := C \otimes r$, and it is called non-degenerate *if there exist maps $\overline{\sigma}\colon C^2 \to C$ and $\overline{\tau}\colon C^2 \to C$ such that*

$$\overline{\sigma} \circ (C \otimes \sigma) \circ (\Delta \otimes C) = \sigma \circ (C \otimes \overline{\sigma}) \circ (\Delta \otimes C) = \epsilon \otimes C \tag{1.2}$$

and

$$\overline{\tau} \circ (\tau \otimes C) \circ (C \otimes \Delta) = \tau \circ (\overline{\tau} \otimes C) \circ (C \otimes \Delta) = C \otimes \epsilon. \tag{1.3}$$

If (C, r) is a non-degenerate pair, then we say that r is non-degenerate.

A direct computation shows that r is non-degenerate if and only if the maps $(C \otimes \sigma) \circ (\Delta \otimes C)$ and $(\tau \otimes C) \circ (C \otimes \Delta)$ are isomorphisms. Moreover, their compositional inverses are the maps $(C \otimes \overline{\sigma}) \circ (\Delta \otimes C)$ and $(\overline{\tau} \otimes C) \circ (C \otimes \Delta)$, respectively. This implies that $\overline{\sigma}$ and $\overline{\tau}$ are coalgebra morphisms.

1.2 Posets

A *partially ordered set* or *poset* is a pair $(X, \leq)$ consisting of a set X endowed with a binary relation $\leq$, called *an order*, that is reflexive, antisymmetric and transitive. For the sake of brevity from now one we will say that X is a poset, without explicit mention of the order. As usual, for $a, b \in X$ we write $a < b$ to mean that $a \leq b$ and $a \neq b$. Two elements a, b of X are *comparable* if $a \leq b$ or $b \leq a$. Otherwise they are *incomparable*. A poset X is a *totally ordered set* if each pair of elements of X is comparable. A *connected component* of X is an equivalence class of the equivalence relation generated by the relation $x \sim y$ if x and y are comparable. Let X be a poset. Each subset Y of X becomes a poset simply by restricting the order relation of X to Y^2. A subset Y of X is a chain of X if it is a totally ordered set. The *height* of a finite chain $a_0 < \cdots < a_n$ is n. The *height* $h(X)$ of a finite poset X is the height of its largest chain. Let $a, b \in X$. The closed interval $[a, b]$ is the set of all the elements c of X such that $a \leq c \leq b$. We say that b covers a, and we write $a \prec b$ (or $b \succ a$), if $[a, b] = \{a, b\}$. A poset X is *locally finite* if $[a, b]$ is finite for all $a, b \in X$.

1.2.1 The incidence coalgebra of a locally finite poset

Let $(X, \leq)$ be a locally finite poset. Set $Y := \{(a, b) \in X \times X : a \leq b\}$. It is well known that $D := KY$ is a coalgebra, called the *incidence coalgebra of X*, via

$$\Delta(a, b) := \sum_{c \in [a,b]} (a, c) \otimes (c, b).$$

Consider KX endowed with the coalgebra structure determined by requiring that each $x \in X$ is a group like element. The K-linear map $\iota\colon KX \to D$ defined by $\iota(x) := (x,x)$ is an injective coalgebra morphism, whose image is the subcoalgebra of D spanned by its group like elements.

Let $r\colon D\otimes D \longrightarrow D\otimes D$ be a K-linear map and let

$$\left(\lambda_{a|b|c|d}^{e|f|g|h}\right)_{(a,b),(c,d),(e,f),(g,h)\in Y}$$

be the family of scalars defined by

$$r((a,b)\otimes(c,d)) = \sum_{\substack{e\le f\\ g\le h}} \lambda_{a|b|c|d}^{e|f|g|h}(e,f)\otimes(g,h). \tag{1.4}$$

From now on we assume that r is invertible.

Remark 1.2. Let $T := (a,b)\otimes(c,d)$. Since

$$(\epsilon\otimes\epsilon)(T) = \delta_{ab}\delta_{cd}, \qquad (\epsilon\otimes\epsilon)\circ r(T) = \sum_{e,g}\lambda_{a|b|c|d}^{e|e|g|g},$$

$$\Delta_{D^2}\circ r(T) = \sum_{\substack{e\le f\\ g\le h}}\sum_{\substack{y\in[e,f]\\ z\in[g,h]}} \lambda_{a|b|c|d}^{e|f|g|h}(e,y)\otimes(g,z)\otimes(y,f)\otimes(z,h)$$

and

$$(r\otimes r)\circ\Delta_{D^2}(T) = \sum_{\substack{p\in[a,b]\\ q\in[c,d]}}\sum_{\substack{e\le y\\ g\le z}}\sum_{\substack{y'\le f\\ z'\le h}} \lambda_{a|p|c|q}^{e|y|g|z}\lambda_{p|b|q|d}^{y'|f|z'|h}(e,y)\otimes(g,z)\otimes(y',f)\otimes(z',h),$$

the map r is a coalgebra automorphism if and only if the following facts hold:

- for $a\le b$ and $c\le d$,
$$\sum_{e,g}\lambda_{a|b|c|d}^{e|e|g|g} = \delta_{ab}\delta_{cd}, \tag{1.5}$$
- for each $a\le b$, $c\le d$, $e\le f$ and $g\le h$,
$$\sum_{(p,q)\in[a,b]\times[c,d]} \lambda_{a|p|c|q}^{e|y|g|z}\lambda_{p|b|q|d}^{y|f|z|h} = \lambda_{a|b|c|d}^{e|f|g|h} \tag{1.6}$$
for all y and z such that $e\le y\le f$ and $g\le z\le h$,
- For each $a\le b$, $c\le d$, $e\le f$ and $g\le h$,
$$\sum_{(p,q)\in[a,b]\times[c,d]} \lambda_{a|p|c|q}^{e|y|g|z}\lambda_{p|b|q|d}^{y'|f|z'|h} = 0, \tag{1.7}$$
for all $y'\le f$, $e\le y$, $z'\le h$ and $g\le z$ such that $(y,z)\ne(y',z')$.

Remark 1.3. By the very definitions of σ and τ, it is clear that

$$\sigma((a,b)\otimes(c,d)) = \sum_{\substack{e\le f\\ g}}\lambda_{a|b|c|d}^{e|f|g|g}(e,f) \quad\text{and}\quad \tau((a,b)\otimes(c,d)) = \sum_{\substack{e\\ g\le h}}\lambda_{a|b|c|d}^{e|e|g|h}(g,h).$$

2 Non-degenerate incidence coalgebra automorphisms

In this section we establish the main properties of the coefficients $\lambda_{a|b|c|d}^{e|f|g|h}$ and the maps ${}^{a}(-)$ and $(-)^{c}$ determined by a non-degenerate coalgebra automorphism of $D\otimes D$ (for the definition of these maps see Notation 2.2).

Remark 2.1. Let $r\colon D\otimes D \longrightarrow D\otimes D$ be a non-degenerate coalgebra automorphism. Since r maps group like elements to group like elements, for all $a,c\in X$ there exist $e,g\in X$ such that

$$\lambda_{a|a|c|c}^{e|e|g|g}=1 \quad\text{and}\quad \lambda_{a|a|c|c}^{e'|e''|g'|g''}=0 \quad \text{for all } (e',e'',g',g'')\neq(e,e,g,g).$$

So, r induces a map $r_|\colon X\times X\longrightarrow X\times X$. The same argument applied to r^{-1} shows that $r_|$ is a bijection. Moreover, since $\overline{\sigma}$ and $\overline{\tau}$ map group like elements to group like elements, $r_|$ is non-degenerate.

Notation 2.2. *In the sequel if* $r_|(a,c)=(e,g)$ *we write* ${}^ac:=e$ *and* $a^c:=g$.

For the rest of the section we fix a non-degenerate coalgebra automorphism r of $D\otimes D$ and we determine properties of the coefficients $\lambda_{a|b|c|d}^{e|f|g|h}$ and the maps ${}^a(-)$ and $(-)^c$. Note that $r_|$ being non-degenerate means that the maps ${}^a(-)$ and $(-)^c$ are bijections.

Proposition 2.3. *Let* $a,c,d\in X$. *If* d *covers* c, *then* $a^c=a^d$, ${}^ad$ *covers* ${}^ac$ *and there exists* $\alpha\in K^\times$ *and* $\beta\in K$ *such that*

$$r((a,a)\otimes(c,d))=\alpha({}^ac,{}^ad)\otimes(a^c,a^c)+\beta({}^ac,{}^ac)\otimes(a^c,a^c)-\beta({}^ad,{}^ad)\otimes(a^c,a^c).$$

Proof. Under the hypothesis, equality (1.6) says that for each $e\le y\le f$ and each $g\le z\le h$,

$$\lambda_{a|a|c|d}^{e|f|g|h}=\lambda_{a|a|c|c}^{e|y|g|z}\lambda_{a|a|c|d}^{y|f|z|h}+\lambda_{a|a|c|d}^{e|y|g|z}\lambda_{a|a|d|d}^{y|f|z|h}. \tag{2.1}$$

If $e<f$ and $g<h$, then taking $y=f$ and $z=g$, we obtain that $\lambda_{a|a|c|d}^{e|f|g|h}=0$. A similar argument proves that if $e=f$ and there exists z such that $g<z<h$, then also $\lambda_{a|a|c|d}^{e|f|g|h}=0$. Furthermore, by symmetry the same occurs if $g=h$ and there exists y such that $e<y<f$. So, if $\lambda_{a|a|c|d}^{e|f|g|h}\neq 0$, then we have the following possibilities:

a) $e=f$ and h covers g b) $g=h$ and f covers e c) $e=f$ and $h=g$

Next we consider each of these cases separately:

a) Taking $y=e$ and $z=g$ in equality (2.1), we obtain that
$$\lambda_{a|a|c|d}^{e|e|g|h}=\lambda_{a|a|c|c}^{e|e|g|g}\lambda_{a|a|c|d}^{e|e|g|h}+\lambda_{a|a|c|d}^{e|e|g|g}\lambda_{a|a|d|d}^{e|e|g|h}=\lambda_{a|a|c|c}^{e|e|g|g}\lambda_{a|a|c|d}^{e|e|g|h},$$
while taking $y=e$ and $z=h$ in equality (2.1), we obtain that
$$\lambda_{a|a|c|d}^{e|e|g|h}=\lambda_{a|a|c|c}^{e|e|g|h}\lambda_{a|a|c|d}^{e|e|h|h}+\lambda_{a|a|c|d}^{e|e|g|h}\lambda_{a|a|d|d}^{e|e|h|h}=\lambda_{a|a|c|d}^{e|e|g|h}\lambda_{a|a|d|d}^{e|e|h|h}.$$
Therefore, if $\lambda_{a|a|c|d}^{e|e|g|h}\neq 0$, then $e={}^ac={}^ad$, which is impossible, since ${}^a(-)$ is a bijection.

b) Taking $y=e$ and $z=g$ in equality (2.1), we obtain that
$$\lambda_{a|a|c|d}^{e|f|g|g}=\lambda_{a|a|c|c}^{e|e|g|g}\lambda_{a|a|c|d}^{e|f|g|g}+\lambda_{a|a|c|d}^{e|e|g|g}\lambda_{a|a|d|d}^{e|f|g|g}=\lambda_{a|a|c|c}^{e|e|g|g}\lambda_{a|a|c|d}^{e|f|g|g},$$
while taking $y=f$ and $z=g$ in equality (2.1), we obtain that
$$\lambda_{a|a|c|d}^{e|f|g|g}=\lambda_{a|a|c|c}^{e|f|g|g}\lambda_{a|a|c|d}^{f|f|g|g}+\lambda_{a|a|c|d}^{e|f|g|g}\lambda_{a|a|d|d}^{f|f|g|g}=\lambda_{a|a|c|d}^{e|f|g|g}\lambda_{a|a|d|d}^{f|f|g|g}.$$
Therefore, if $\lambda_{a|a|c|d}^{e|f|g|g}\neq 0$, then $(f,g)=({}^ad,a^d)$ and $(e,g)=({}^ac,a^c)$.

c) Taking $y=e$ and $z=g$ in equality (2.1), we obtain that
$$\lambda_{a|a|c|d}^{e|e|g|g}=\lambda_{a|a|c|c}^{e|e|g|g}\lambda_{a|a|c|d}^{e|e|g|g}+\lambda_{a|a|c|d}^{e|e|g|g}\lambda_{a|a|d|d}^{e|e|g|g}=(\lambda_{a|a|c|c}^{e|e|g|g}+\lambda_{a|a|d|d}^{e|e|g|g})\lambda_{a|a|c|d}^{e|e|g|g},$$
which implies that $(e,g)=({}^ac,a^c)$ or $(e,g)=({}^ad,a^d)$, when $\lambda_{a|a|c|d}^{e|e|g|g}\neq 0$.

Thus,

$$r((a,a)\otimes(c,d)) = \lambda_{a|a|c|d}^{{}^ac|{}^ad|a^c|a^c}({}^ac,{}^ad)\otimes(a^c,a^c) + \lambda_{a|a|c|d}^{{}^ac|{}^ac|a^c|a^c}({}^ac,{}^ac)\otimes(a^c,a^c)$$
$$+ \lambda_{a|a|c|d}^{{}^ad|{}^ad|a^d|a^d}({}^ad,{}^ad)\otimes(a^d,a^d).$$

Since r and $r_|$ are bijective, $\alpha := \lambda_{a|a|c|d}^{{}^ac|{}^ad|a^c|a^c} \neq 0$, and then $a^c = a^d$ and ${}^ad$ covers ${}^ac$. Also notice that

$$\beta := \lambda_{a|a|c|d}^{{}^ac|{}^ac|a^c|a^c} = -\lambda_{a|a|c|d}^{{}^ad|{}^ad|a^d|a^d},$$

because $(\epsilon\otimes\epsilon)\circ r((a,a)\otimes(c,d)) = 0$. □

Proposition 2.4. *Let $a,b,c \in X$. If b covers a, then b^c covers a^c, ${}^ac = {}^bc$ and there exists $\alpha \in K^\times$ and $\beta \in K$ such that*

$$r((a,b)\otimes(c,c)) = \alpha({}^ac,{}^ac)\otimes(a^c,b^c) + \beta({}^ac,{}^ac)\otimes(a^c,a^c) - \beta({}^ac,{}^ac)\otimes(b^c,b^c).$$

Proof. Apply Proposition 2.3 to $\tau\circ r\circ\tau$, where τ is the flip. □

Corollary 2.5. *For each $a\in X$ the maps ${}^a(-)$ and $(-)^a$ are automorphisms of orders. Moreover, if a and b are comparable or, more generally, if a and b belong to the same component of X, then ${}^a(-) = {}^b(-)$ and $(-)^a = (-)^b$.*

Notations 2.6. *We let ${}^{\bar a}(-)$ and $(-)^{\bar a}$ denote the inverse maps of ${}^a(-)$ and $(-)^a$, respectively. Note that here $\bar a$ is not an element of X.*

Lemma 2.7. *Let $a,c,d,e,f,g,h \in X$ such that $c\le d$, $e\le f$ and $g \le h$. If $g\neq a^c$ or $h \neq a^c$, then $\lambda_{a|a|c|d}^{e|f|g|h} = 0$.*

Proof. By Remark 2.1 and Proposition 2.3 we know that the statement is true when $\mathfrak{h}[c,d]\le 1$. Assume that it is true when $\mathfrak{h}[c,d]\le n$ and that $\mathfrak{h}[c,d] = n+1$. If $g\neq a^c$, then

$$\lambda_{a|a|c|d}^{e|f|g|h} = \sum_{q\in[c,d]} \lambda_{a|a|c|q}^{e|e|g|g}\lambda_{a|a|q|d}^{e|f|g|h} = 0,$$

because $\lambda_{a|a|d|d}^{e|f|g|h} = 0$, since $g\neq a^d = a^c$, while $\lambda_{a|a|c|q}^{e|e|g|g} = 0$ when $q<d$, by the inductive hypothesis; while if $h\neq a^c$, then

$$\lambda_{a|a|c|d}^{e|f|g|h} = \sum_{q\in[c,d]} \lambda_{a|a|c|q}^{e|e|g|h}\lambda_{a|a|q|d}^{e|f|h|h} = 0,$$

because $\lambda_{a|a|c|c}^{e|e|g|h} = 0$, since $h \neq a^c$, while $\lambda_{a|a|q|d}^{e|f|h|h} = 0$ when $q>c$, by the inductive hypothesis, since $a^q = a^c \neq h$. □

Proposition 2.8. *Let $a\le b$, $c\le d$, $e\le f$ and $g\le h$. If $\lambda_{a|b|c|d}^{e|f|g|h}\neq 0$, then it is true that $a^c\le g\le h\le b^c$ and ${}^ac \le e \le f \le {}^ad$.*

Proof. By symmetry it is sufficient to prove that if $a^c\le g\le h\le b^c$ is false, then $\lambda_{a|b|c|d}^{e|f|g|h} = 0$. By Lemma 2.7 this is true when $a=b$. Assume that it is true when $\mathfrak{h}[a,b] = m$ and that $\mathfrak{h}[a,b] = m+1$. On one hand, if $a^c \not\le g$, then

$$\begin{aligned}\lambda_{a|b|c|d}^{e|f|g|h} &= \sum_{(p,q)\in[a,b]\times[c,d]} \lambda_{a|p|c|q}^{e|e|g|g}\lambda_{p|b|q|d}^{e|f|g|h}\\ &= \sum_{(p,q)\in[a,b)\times[c,d]} \lambda_{a|p|c|q}^{e|e|g|g}\lambda_{p|b|q|d}^{e|f|g|h} + \sum_{q\in[c,d]} \lambda_{a|b|c|q}^{e|e|g|g}\lambda_{b|b|q|d}^{e|f|g|h}\\ &= 0\end{aligned}$$

because

- $\lambda_{b|b|q|d}^{e|f|g|h} = 0$ by Lemma 2.7, since by Corollary 2.5, we have $b^q = b^c \not\le g$,

- $\lambda^{e|e|g|g}_{a|p|c|q} = 0$ when $p < b$, by the inductive hypothesis.

On the other hand, if $h \nleq b^c$, then

$$\begin{aligned}\lambda^{e|f|g|h}_{a|b|c|d} &= \sum_{(p,q)\in[a,b]\times[c,d]} \lambda^{e|e|g|h}_{a|p|c|q}\lambda^{e|f|h|h}_{p|b|q|d}\\ &= \sum_{(p,q)\in(a,b]\times[c,d]} \lambda^{e|e|g|h}_{a|p|c|q}\lambda^{e|f|h|h}_{p|b|q|d} + \sum_{q\in[c,d]} \lambda^{e|e|g|h}_{a|a|c|q}\lambda^{e|f|h|h}_{a|b|q|d}\\ &= 0\end{aligned}$$

because

- $\lambda^{e|f|h|h}_{p|b|q|d} = 0$ when $p > a$, by the inductive hypothesis, since by Corollary 2.5, we have $h \nleq b^q = b^c$,
- $\lambda^{e|e|g|h}_{a|a|c|q} = 0$ by Lemma 2.7, since by Corollary 2.5, we have $h \nleq a^c$.

This finishes the proof. □

Corollary 2.9. *The following formula holds:*

$$r((a,b)\otimes(c,d)) = \sum_{\substack{\{(x,y):a\le x\le y\le b\}\\ \{(w,z):c\le w\le z\le d\}}} \lambda^{{}^a w|{}^a z|x^c|y^c}_{a|b|c|d}\,({}^a w, {}^a z)\otimes(x^c, y^c).$$

Proof. It follows immediately from Proposition 2.8 and Corollary 2.5. □

Proposition 2.10. *Let $a \le b$, $c \le d$ $e \le f$ and $g \le h$ such that $a^c \le g \le h \le b^c$ and ${}^a c \le e \le f \le {}^a d$. For each $y, z \in X$ such that $e \le y \le f$ and $g \le z \le h$, the following equality holds:*

$$\lambda^{e|f|g|h}_{a|b|c|d} = \lambda^{e|y|g|z}_{a|z^{\bar c}|c|{}^{\bar a}y}\lambda^{y|f|z|h}_{z^{\bar c}|b|{}^{\bar a}y|d}. \tag{2.2}$$

Proof. By Proposition 2.8 and Corollary 2.5, if $\lambda^{e|y|g|z}_{a|p|c|q}\lambda^{y|f|z|h}_{p|b|q|d} \neq 0$, then

$${}^a c \le e \le y \le {}^a q,\quad {}^a q \le y \le f \le {}^a d,\quad a^c \le g \le z \le p^c \quad\text{and}\quad p^c \le z \le h \le b^c,$$

So, $q = {}^{\bar a}y$, $p = z^{\bar c}$, and the result follows from equality (1.6). □

Examples 2.11. *Let $a \le b$ and $c \le d$. From the previous proposition it follows that:*

1) *for each $e, g \in X$ such that $a^c \le g \le b^c$ and ${}^a c \le e \le {}^a d$,*
$$\lambda^{e|e|g|g}_{a|b|c|d} = \lambda^{e|e|g|g}_{a|g^{\bar c}|c|{}^{\bar a}e}\lambda^{e|e|g|g}_{g^{\bar c}|b|{}^{\bar a}e|d},$$
2) *for each $e, f, g \in X$ such that $a^c \le g \le b^c$ and ${}^a c \le e \prec f \le {}^a d$,*
$$\lambda^{e|f|g|g}_{a|b|c|d} = \lambda^{e|e|g|g}_{a|g^{\bar c}|c|{}^{\bar a}e}\lambda^{e|f|g|g}_{g^{\bar c}|b|{}^{\bar a}e|d} = \lambda^{e|f|g|g}_{a|g^{\bar c}|c|{}^{\bar a}f}\lambda^{f|f|g|g}_{g^{\bar c}|b|{}^{\bar a}f|d},$$
3) *for each $e, g, h \in X$ such that ${}^a c \le e \le {}^a d$ and $a^c \le g \prec h \le b^c$,*
$$\lambda^{e|e|g|h}_{a|b|c|d} = \lambda^{e|e|g|g}_{a|g^{\bar c}|c|{}^{\bar a}e}\lambda^{e|e|g|h}_{g^{\bar c}|b|{}^{\bar a}e|d} = \lambda^{e|e|g|h}_{a|h^{\bar c}|c|{}^{\bar a}e}\lambda^{e|e|h|h}_{h^{\bar c}|b|{}^{\bar a}e|d}.$$

Notations 2.12. *For $p, q, m, n \in \mathbb{N}_0$, we let ${}_r\Lambda^{mn}_{pq}$ denote the restriction of the family*

$$\left(\lambda^{e|f|g|h}_{a|b|c|d}\right)_{(a,b),(c,d),(e,f),(g,h)\in Y}$$

to the set of indices $\{((a,b),(c,d),(e,f),(g,h))\}$ such that

$$\mathfrak{h}[a,b] = p,\ \mathfrak{h}[c,d] = q,\ \mathfrak{h}[e,f] = m,\ \mathfrak{h}[g,h] = n,\ [g,h] \subseteq [a^c, b^c] \text{ and } [e,f] \subseteq [{}^a c, {}^a d].$$

Moreover, we set

$${}_r\Lambda^{v}_{u} := \bigcup_{\substack{p+q=u\\ m+n=v}} {}_r\Lambda^{mn}_{pq},$$

and we denote by $\mathrm{Ex}({}_r\Lambda_n^0)$ *the restriction of* ${}_r\Lambda_n^0$ *to the set*

$$\big\{\big((a,b),(c,d),(e,e),(g,g)\big)\big) : (e,g) = ({}^a c, a^c) \text{ or } (e,g) = ({}^b d, b^d)\big\}.$$

Note that $\mathrm{Ex}({}_r\Lambda_0^0) = {}_r\Lambda_0^0$ *and* $\mathrm{Ex}({}_r\Lambda_1^0) = {}_r\Lambda_1^0$.

Proposition 2.13. *Let* $\tilde{r}\colon D\otimes D \longrightarrow D\otimes D$ *be a non-degenerate coalgebra automorphism. Then* $\tilde{r} = r$ *if and only if*

$$\mathrm{Ex}({}_{\tilde{r}}\Lambda_n^0) = \mathrm{Ex}({}_r\Lambda_n^0) \quad \text{for all } n \qquad \text{and} \qquad {}_{\tilde{r}}\Lambda_1^1 = {}_r\Lambda_1^1. \tag{2.3}$$

Proof. Clearly the conditions are necessary. So, we only need to prove that they are sufficient. For the sake of brevity we write

$$\widetilde{\lambda}_{a|b|c|d}^{e|f|g|h} := {}_{\tilde{r}}\Lambda\big((a,b),(c,d),(e,f),(g,h)\big).$$

For each $e,g\in X$ such that $a^c \le g \le b^c$ and ${}^a c \le e \le {}^a d$, we have

$$\lambda_{a|b|c|d}^{e|e|g|g} = \lambda_{a|g^{\bar c}|c|^{\bar a}e}^{e|e|g|g}\lambda_{g^{\bar c}|b|^{\bar a}e|d}^{e|e|g|g} = \widetilde{\lambda}_{a|g^{\bar c}|c|^{\bar a}e}^{e|e|g|g}\widetilde{\lambda}_{g^{\bar c}|b|^{\bar a}e|d}^{e|e|g|g} = \widetilde{\lambda}_{a|b|c|d}^{e|e|g|g},$$

where the first and the last equality hold by item 1) of Examples 2.11, and the second equality holds since

$$\lambda_{a|g^{\bar c}|c|^{\bar a}e}^{e|e|g|g}, \lambda_{g^{\bar c}|b|^{\bar a}e|d}^{e|e|g|g} \in \bigcup_{u\in\mathbb{N}_0} \mathrm{Ex}({}_r\Lambda_u^0).$$

So, ${}_r\Lambda_u^0 = {}_{\tilde{r}}\Lambda_u^0$ for all u.

Next we prove by induction on u that ${}_r\Lambda_u^1 = {}_{\tilde{r}}\Lambda_u^1$ for all u. For $u=0$ this is true, since ${}_r\Lambda_0^1 = {}_{\tilde{r}}\Lambda_0^1 = \emptyset$, and for $u=1$ this is true by hypothesis. Take $u>1$ and assume that ${}_r\Lambda_{u'}^1 = {}_{\tilde{r}}\Lambda_{u'}^1$ for all $u'<u$. Consider $\lambda_{a|b|c|d}^{e|f|g|h}$ with

$$\mathfrak{h}[a,b]+\mathfrak{h}[c,d]=u,\ \mathfrak{h}[e,f]+\mathfrak{h}[g,h]=1,\ a^c\le g\le h\le b^c \text{ and } {}^a c\le e\le f\le {}^a d.$$

Assume first that $g=h$. If $g^{\bar c} > a$ or ${}^{\bar a}e > c$, then taking $y=e$ in (2.2), we obtain

$$\lambda_{a|b|c|d}^{e|f|g|g} = \lambda_{a|g^{\bar c}|c|^{\bar a}e}^{e|e|g|g}\lambda_{g^{\bar c}|b|^{\bar a}e|d}^{e|f|g|g} = \widetilde{\lambda}_{a|g^{\bar c}|c|^{\bar a}e}^{e|e|g|g}\widetilde{\lambda}_{g^{\bar c}|b|^{\bar a}e|d}^{e|f|g|g} = \widetilde{\lambda}_{a|b|c|d}^{e|f|g|g},$$

since $\mathfrak{h}[g^{\bar c},b]+\mathfrak{h}[{}^{\bar a}e,d] < \mathfrak{h}[a,b]+\mathfrak{h}[c,d]=u$. Else, taking $y=f$, we obtain

$$\lambda_{a|b|c|d}^{e|f|g|g} = \lambda_{a|g^{\bar c}|c|^{\bar a}f}^{e|f|g|g}\lambda_{g^{\bar c}|b|^{\bar a}f|d}^{f|f|g|g} = \widetilde{\lambda}_{a|g^{\bar c}|c|^{\bar a}f}^{e|f|g|g}\widetilde{\lambda}_{g^{\bar c}|b|^{\bar a}f|d}^{f|f|g|g} = \widetilde{\lambda}_{a|b|c|d}^{e|f|g|g},$$

because $b>g^{\bar c}$ or $d>{}^{\bar a}f$, since otherwise $u=\mathfrak{h}[a,b]+\mathfrak{h}[c,d]=\mathfrak{h}[a,a]+\mathfrak{h}[e,f]=1$.

A similar argument yields $\lambda_{a|b|c|d}^{e|e|g|h} = \widetilde{\lambda}_{a|b|c|d}^{e|e|g|h}$ for $g\prec h$ and concludes the proof that ${}_r\Lambda_u^1 = {}_{\tilde{r}}\Lambda_u^1$.

Finally we prove using induction on v, that ${}_r\Lambda_u^v = {}_{\tilde{r}}\Lambda_u^v$ for all $v>1$ and all u. Take $v>1$ and assume that ${}_r\Lambda_u^{v'} = {}_{\tilde{r}}\Lambda_u^{v'}$ for all $v'<v$. Consider

$$\lambda_{a|b|c|d}^{e|f|g|h} \quad \text{with } \mathfrak{h}[e,f]+\mathfrak{h}[g,h]=v,\ a^c\le g\le h\le b^c \text{ and } {}^a c\le e\le f\le {}^a d.$$

If $e<f$ and $g<h$, then we take $y=f$ and $z=g$ in (2.2) and we obtain

$$\lambda_{a|b|c|d}^{e|f|g|h} = \lambda_{a|g^{\bar c}|c|^{\bar a}f}^{e|f|g|g}\lambda_{g^{\bar c}|b|^{\bar a}f|d}^{f|f|g|h} = \widetilde{\lambda}_{a|g^{\bar c}|c|^{\bar a}f}^{e|f|g|g}\widetilde{\lambda}_{g^{\bar c}|b|^{\bar a}f|d}^{f|f|g|h} = \widetilde{\lambda}_{a|b|c|d}^{e|f|g|h},$$

since $\mathfrak{h}[e,f]+\mathfrak{h}[g,g]<v$ and $\mathfrak{h}[f,f]+\mathfrak{h}[g,h]<v$.

Else $g=h$ or $e=f$. In the first case there exists y with $e<y<f$, and so, by (2.2) we obtain

$$\lambda_{a|b|c|d}^{e|f|g|g} = \lambda_{a|g^{\bar c}|c|^{\bar a}y}^{e|y|g|g}\lambda_{g^{\bar c}|b|^{\bar a}y|d}^{y|f|g|g} = \widetilde{\lambda}_{a|g^{\bar c}|c|^{\bar a}y}^{e|y|g|g}\widetilde{\lambda}_{g^{\bar c}|b|^{\bar a}y|d}^{y|f|g|g} = \widetilde{\lambda}_{a|b|c|d}^{e|f|g|g},$$

since $\mathfrak{h}[e,y]+\mathfrak{h}[g,g]<v$ and $\mathfrak{h}[y,f]+\mathfrak{h}[g,g]<v$. A similar argument proves the case $e=f$. By Proposition 2.8 this concludes the proof. □

Definition 2.14. *Let $e \le f$ and $g \le h$, and let*

$$e = y_0 < \cdots < y_j = f \quad \text{and} \quad g = z_0 < \cdots < z_k = h$$

be maximal chains. A configuration for the two given chains is a family $(a_i)_{i=0,\dots,j+k}$ with $a_i = (\alpha_i, \beta_i) \in \mathbb{N}_0^2$ such that $a_0 = (0,0)$, $a_{j+k} = (j,k)$, $\alpha_i \le \alpha_{i+1}$, $\beta_i \le \beta_{i+1}$ and $\alpha_{i+1} - \alpha_i + \beta_{i+1} - \beta_i = 1$ for $i = 0, \dots, j+k-1$.

Proposition 2.15. *Let $e \le f$, $g \le h$, $a \le b$ and $c \le d$, such that $a^c \le g \le h \le b^c$ and ${}^a c \le e \le f \le {}^a d$. Let $e = y_0 < \cdots < y_j = f$ and $g = z_0 < z_1 < \cdots < z_k = h$ be maximal chains and let $(a_i)_{i=0,\dots,j+k}$ be a configuration for the two given chains. Then*

$$\lambda_{a|b|c|d}^{e|f|g|h} = \lambda_{a|g^{\bar c}|c|^{\bar a}e}^{e|e|g|g} \lambda_{h^{\bar c}|b|^{\bar a}f|d}^{f|f|h|h} \prod_{i=1}^{k+j} \lambda_i, \qquad \text{where} \quad \lambda_i = \lambda_{z_{\beta_{i-1}}{}^{\bar c}|z_{\beta_i}{}^{\bar c}|^{\bar a}y_{\alpha_{i-1}}|^{\bar a}y_{\alpha_i}}^{y_{\alpha_{i-1}}|y_{\alpha_i}|z_{\beta_{i-1}}|z_{\beta_i}}.$$

Proof. We proceed by induction on $k+j$. If $k+j=0$, then, by Example 2.11(1),

$$\lambda_{a|b|c|d}^{e|e|g|g} = \lambda_{a|g^{\bar c}|c|^{\bar a}e}^{e|e|g|g} \lambda_{g^{\bar c}|b|^{\bar a}e|d}^{e|e|g|g}.$$

Assume $k+j>0$ and that the proposition holds for all pair of chains with the sum of the lengths smaller than $k+j$. Necessarily $z_{k+j-1} < h$ and $y_{k+j-1} = f$ or $z_{k+j-1} = h$ and $y_{k+j-1} < f$. In the first case $(a_i)_{i=0,\dots,k+j-1}$ is a configuration for $e = y_0 < \cdots < y_j = f$ and $g = z_0 < z_1 < \cdots < z_{k-1}$ and so by inductive hypothesis

$$\lambda_{a|z_{k-1}{}^{\bar c}|c|^{\bar a}f}^{e|f|g|z_{k-1}} = \lambda_{a|g^{\bar c}|c|^{\bar a}e}^{e|e|g|g} \lambda_{z_{k-1}{}^{\bar c}|z_{k-1}{}^{\bar c}|^{\bar a}f|^{\bar a}f}^{f|f|z_{k-1}|z_{k-1}} \prod_{i=1}^{k+j-1} \lambda_{z_{\beta_{i-1}}{}^{\bar c}|z_{\beta_i}{}^{\bar c}|^{\bar a}y_{\alpha_{i-1}}|^{\bar a}y_{\alpha_i}}^{y_{\alpha_{i-1}}|y_{\alpha_i}|z_{\beta_{i-1}}|z_{\beta_i}}.$$

Since $\lambda_{z_{k-1}{}^{\bar c}|z_{k-1}{}^{\bar c}|^{\bar a}f|^{\bar a}f}^{f|f|z_{k-1}|z_{k-1}} = 1$,

$$\lambda_{a|b|c|d}^{e|f|g|h} = \lambda_{a|z_{k-1}{}^{\bar c}|c|^{\bar a}f}^{e|f|g|z_{k-1}} \lambda_{z_{k-1}{}^{\bar c}|b|^{\bar a}f|d}^{f|f|z_{k-1}|h} \quad \text{and} \quad \lambda_{z_{k-1}{}^{\bar c}|b|^{\bar a}f|d}^{f|f|z_{k-1}|h} = \lambda_{z_{k-1}{}^{\bar c}|h^{\bar c}|^{\bar a}f|^{\bar a}f}^{f|f|z_{k-1}|h} \lambda_{h^{\bar c}|b|^{\bar a}f|d}^{f|f|h|h},$$

the result is true in this case. If $z_{k+j-1} = h$ and $y_{k+j-1} < f$, a similar argument proves the formula and concludes the proof. □

Corollary 2.16. *Let $e \le f$, $g \le h$, $a \le b$ and $c \le d$, such that $a^c \le g \le h \le b^c$ and ${}^a c \le e \le f \le {}^a d$. The product*

$$\lambda_{a|g^{\bar c}|c|^{\bar a}e}^{e|e|g|g} \lambda_{h^{\bar c}|b|^{\bar a}f|d}^{f|f|h|h} \prod_{i=1}^{k+j} \lambda_i$$

in Proposition 2.15 does not depend neither on the maximal chains nor on the chosen configuration.

3 Construction of non-degenerate automorphisms

In Section 2 we proved that each non-degenerate coalgebra automorphism r of $D \otimes D$ induces by restriction a non-degenerate bijection

$$r_| \colon X \times X \longrightarrow X \times X$$

and fulfills condition (1.5) and the statements of Corollary 2.5 and Propositions 2.8 and 2.10. In other words r satisfies (1.5) and, for all $a, b, c, d, e, f, g, h \in X$,

1) the maps ${}^a(-)$ and $(-)^b$, defined by $({}^a b, a^b) := r_|(a,b)$, are automorphisms of orders;
2) if a and b belong to the same component of X, then ${}^a(-) = {}^b(-)$ and $(-)^a = (-)^b$;

3) if $a \le b$, $c \le d$, $e \le f$, $g \le h$ and $\lambda_{a|b|c|d}^{e|f|g|h} \ne 0$, then $a^c \le g \le h \le b^c$ and ${}^a c \le e \le f \le {}^a d$;

4) if $a \le b$, $c \le d$ $e \le f$, $g \le h$, $a^c \le g \le h \le b^c$ and ${}^a c \le e \le f \le {}^a d$, then
$$\lambda_{a|b|c|d}^{e|f|g|h} = \lambda_{a|z^{\bar{c}}|c|{}^{\bar{a}}y}^{e|y|g|z} \lambda_{z^{\bar{c}}|b|{}^{\bar{a}}y|d}^{y|f|z|h}$$
for each $y, z \in X$ such that $e \le y \le f$ and $g \le z \le h$;

In this section we fix a linear automorphism r of $D \otimes D$, and we prove that, conversely, if r induces by restriction a non-degenerate bijection
$$r_| \colon X \times X \longrightarrow X \times X$$
and satisfies condition (1.5) and items 1) – 4), then r is a non-degenerate coalgebra automorphism.

Remark 3.1. Let $F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$ be the filtration of $D \otimes D$ defined setting F_i as the K-subspace of $D \otimes D$ generated by the tensors $(a,b) \otimes (c,d)$ with $\mathfrak{h}[a,b] + \mathfrak{h}[c,d] \le i$. It is clear that a linear map $r \colon D \otimes D \longrightarrow D \otimes D$ that satisfies item 3) preserves this filtration. Assume that r induces a non-degenerate permutation $r_|$ on $X \times X$. We claim that r is bijective if and only if $\lambda_{a|b|c|d}^{{}^a c|{}^a d|a^c|b^c} \in K^\times$ for all $(a,b),(c,d) \in Y$. In fact, in order to prove this it is sufficient to show that the last condition holds if and only if the graded morphism r_l induced by r is bijective. Now using again item 3) we obtain that
$$r_l((a,b) \otimes (c,d)) = \lambda_{a|b|c|d}^{{}^a c|{}^a d|a^c|b^c} ({}^a c, {}^a d) \otimes (a^c, b^c).$$
Consequently the condition is clearly necessary. The converse follows easily using that if $(e,g) = r_|(a,c) = ({}^a c, a^c)$, then $((a, h^{\bar{c}}), (c, {}^{\bar{a}} f))$, is the unique element of $Y \times Y$ such that $r_l((a, h^{\bar{c}}) \otimes (c, {}^{\bar{a}} f))$ is a multiple of $(e,f) \otimes (g,h)$ by a non-zero scalar.

Lemma 3.2. *Let $a, c, f, h \in X$. If $f \ge e := {}^a c$, $h \ge g := a^c$ and $\mathfrak{h}[e,f] + \mathfrak{h}[g,h] > 0$, then*
$$\sum_{(p,q) \in [g^{\bar{c}}, h^{\bar{c}}] \times [{}^{\bar{a}} e, {}^{\bar{a}} f]} \lambda_{g^{\bar{c}}|p|{}^{\bar{a}} e|q}^{e|e|g|g} \lambda_{p|h^{\bar{c}}|q|{}^{\bar{a}} f}^{f|f|h|h} = 0.$$

Proof. For the sake of brevity we let S denote the sum at the left hand of the above equality. We will proceed by induction on $N := \mathfrak{h}[e,f] + \mathfrak{h}[g,h]$. So assume that the assertion holds when $\mathfrak{h}[e,f] + \mathfrak{h}[g,h] < N$. By Remark 2.1
$$S = \lambda_{g^{\bar{c}}|h^{\bar{c}}|{}^{\bar{a}} e|{}^{\bar{a}} f}^{e|e|g|g} + \lambda_{g^{\bar{c}}|h^{\bar{c}}|{}^{\bar{a}} e|{}^{\bar{a}} f}^{f|f|h|h} + \sum_{\substack{(p,q) \in [g^{\bar{c}}, h^{\bar{c}}] \times [{}^{\bar{a}} e, {}^{\bar{a}} f] \\ (p,q) \notin \{(g^{\bar{c}}, {}^{\bar{a}} e), (h^{\bar{c}}, {}^{\bar{a}} f)\}}} \lambda_{g^{\bar{c}}|p|{}^{\bar{a}} e|q}^{e|e|g|g} \lambda_{p|h^{\bar{c}}|q|{}^{\bar{a}} f}^{f|f|h|h}.$$
But, by condition (1.5) and Statement 3), we have
$$\lambda_{g^{\bar{c}}|p|{}^{\bar{a}} e|q}^{e|e|g|g} = - \sum_{\substack{(i,j) \in [e, {}^a q] \times [g, p^c] \\ (i,j) \ne (e,g)}} \lambda_{g^{\bar{c}}|p|{}^{\bar{a}} e|q}^{i|i|j|j} \quad \text{and} \quad \lambda_{p|h^{\bar{c}}|q|{}^{\bar{a}} f}^{f|f|h|h} = - \sum_{\substack{(k,l) \in [{}^a q, f] \times [p^c, h] \\ (k,l) \ne (f,h)}} \lambda_{p|h^{\bar{c}}|q|{}^{\bar{a}} f}^{k|k|l|l},$$
and so
$$S = \lambda_{g^{\bar{c}}|h^{\bar{c}}|{}^{\bar{a}} e|{}^{\bar{a}} f}^{e|e|g|g} + \lambda_{g^{\bar{c}}|h^{\bar{c}}|{}^{\bar{a}} e|{}^{\bar{a}} f}^{f|f|h|h} + \sum_{\substack{i,j,p,q,k,l \in A \\ (i,j) \ne (e,g), (k,l) \ne (f,h)}} \lambda_{g^{\bar{c}}|p|{}^{\bar{a}} e|q}^{i|i|j|j} \lambda_{p|h^{\bar{c}}|q|{}^{\bar{a}} f}^{k|k|l|l}, \tag{3.1}$$
where $A := \{i,j,p,q,k,l : g \le j \le p^c \le l \le h \text{ and } e \le i \le {}^a q \le k \le f\}$. On the other hand
$$\lambda_{g^{\bar{c}}|h^{\bar{c}}|{}^{\bar{a}} e|{}^{\bar{a}} f}^{e|e|g|g} + \lambda_{g^{\bar{c}}|h^{\bar{c}}|{}^{\bar{a}} e|{}^{\bar{a}} f}^{f|f|h|h} + \sum_{\substack{(p,q) \in [g^{\bar{c}}, h^{\bar{c}}] \times [{}^{\bar{a}} e, {}^{\bar{a}} f] \\ (p,q) \notin \{(g^{\bar{c}}, {}^{\bar{a}} e), (h^{\bar{c}}, {}^{\bar{a}} f)\}}} \lambda_{g^{\bar{c}}|p|{}^{\bar{a}} e|q}^{{}^a q|{}^a q|p^c|p^c} \lambda_{p|h^{\bar{c}}|q|{}^{\bar{a}} f}^{{}^a q|{}^a q|p^c|p^c} = 0, \tag{3.2}$$

since by condition (1.5) and statements 3) and 4),

$$\lambda^{^aq|^aq|p^c|p^c}_{g^{\bar c}|p|^{\bar a}e|q}\lambda^{^aq|^aq|p^c|p^c}_{p|h^{\bar c}|q|^{\bar a}f}=\lambda^{^aq|^aq|p^c|p^c}_{g^{\bar c}|h^{\bar c}|^{\bar a}e|^{\bar a}f}\quad\text{and}\quad\sum_{\substack{\{p:g^{\bar c}\le p\le h^{\bar c}\}\\ \{q:^{\bar a}e\le q\le ^{\bar a}f\}}}\lambda^{^aq|^aq|p^c|p^c}_{g^{\bar c}|h^{\bar c}|^{\bar a}e|^{\bar a}f}=0.$$

Combining equalities (3.1) and (3.2), we obtain

$$\begin{aligned}S&=\sum_{\substack{i,j,p,q,k,l\in A\\ (i,j)\neq(e,g),(k,l)\neq(f,h)\\ \mathfrak{h}[i,k]+\mathfrak{h}[j,l]>0}}\lambda^{i|i|j|j}_{g^{\bar c}|p|^{\bar a}e|q}\lambda^{k|k|l|l}_{p|h^{\bar c}|q|^{\bar a}f}\\&=\sum_{\substack{\{j,l:g\le j\le l\le h\}\\ \{i,k:e\le i\le k\le f\}\\ (i,j)\neq(e,g),(k,l)\neq(f,h)\\ \mathfrak{h}[i,k]+\mathfrak{h}[j,l]>0}}\;\sum_{\substack{p\in[j^{\bar c},l^{\bar c}]\\ q\in[^{\bar a}i,^{\bar a}k]}}\lambda^{i|i|j|j}_{g^{\bar c}|p|^{\bar a}e|q}\lambda^{k|k|l|l}_{p|h^{\bar c}|q|^{\bar a}f}.\end{aligned}$$

But, since $(i,j)\neq(e,g)$ and $(k,l)\neq(f,h)$ imply

$$\mathfrak{h}[i,k]+\mathfrak{h}[j,l]<\mathfrak{h}[e,f]+\mathfrak{h}[g,h]=N,$$

by the inductive hypothesis $S=0$, as desired. □

Lemma 3.3. *The map r satisfies condition* (1.7).

Proof. By conditions 2) and 3) we know that if $\lambda^{e|y|g|z}_{a|p|c|q}\lambda^{y'|f|z'|g}_{p|b|q|d}\neq 0$, then

$$^ac\le e\le y\le {}^aq\le y'\le f\le {}^ad\quad\text{and}\quad a^c\le g\le z\le p^c\le z'\le g\le b^c.$$

So, we are reduced to prove that for each $a,b,c,d,e,f,g,h,y,y',z,z'\in X$ such that $^ac\le e\le y\le y'\le f\le {}^ad$, $a^c\le g\le z\le z'\le g\le b^c$ and $(y,z)\neq(y',z')$.

$$\sum_{(p,q)\in[z^{\bar c},z'^{\bar c}]\times[^{\bar a}y,^{\bar a}y']}\lambda^{e|y|g|z}_{a|p|c|q}\lambda^{y'|f|z'|h}_{p|b|q|d}=0.$$

But by condition 4), we have

$$\lambda^{e|y|g|z}_{a|p|c|q}=\lambda^{e|y|g|z}_{a|z^{\bar c}|c|^{\bar a}y}\lambda^{y|y|z|z}_{z^{\bar c}|p|^{\bar a}y|q}\qquad\text{and}\qquad\lambda^{y'|f|z'|h}_{p|b|q|d}=\lambda^{y'|y'|z'|z'}_{p|z'^{\bar c}|q|^{\bar a}y'}\lambda^{y'|f|z'|h}_{z'^{\bar c}|b|^{\bar a}y'|d},$$

and therefore it suffices to check that

$$\sum_{(p,q)\in[z^{\bar c},z'^{\bar c}]\times[^{\bar a}y,^{\bar a}y']}\lambda^{y|y|z|z}_{z^{\bar c}|p|^{\bar a}y|q}\lambda^{y'|y'|z'|z'}_{p|z'^{\bar c}|q|^{\bar a}y'}=0,$$

which is true by Lemma 3.2. □

Theorem 3.4. *Let $r\colon D\otimes D\longrightarrow D\otimes D$ be a linear map that induces by restriction a non-degenerate bijection $r_|\colon X\times X\longrightarrow X\times X$. Let*

$$\big(\lambda^{e|f|g|h}_{a|b|c|d}\big)_{(a,b),(c,d),(e,f),(g,h)\in Y}$$

be as in the discussion above Remark 1.2, and for each $a,c\in X$ let $^a(-)$ and $(-)^c$ be the maps introduced in Notation 2.2. If r satisfies items 1) – 4) at the beginning of the section, condition (1.5) *and $\lambda^{^ac|^ad|a^c|b^c}_{a|b|c|d}\in K^\times$ for all $(a,b),(c,d)\in Y$, then r is a non-degenerate coalgebra automorphism.*

Proof. By Remark 3.1 we know that the map r is bijective. By hypothesis r satisfies condition (1.5), and using items 2), 3) and 4), and arguing as in the proof of Proposition 2.10, we obtain that r satisfies condition (1.6). Moreover by Lemma 3.3 we know that r also satisfies condition (1.7). Hence r is a coalgebra automorphism and so it only remains to check that it is non-degenerate. Let G_l be the graded map induced by $(D\otimes\sigma)\circ(\Delta\otimes D)$. In order to prove that $(D\otimes\sigma)\circ(\Delta\otimes D)$ is invertible it suffices to show that so is G_l. Let $a,b,c,d \in x$ with $a \le b$ and $c \le d$. A direct computation (using item 3)) shows that

$$G_l((a,b)\otimes(c,d)) = \lambda_{b|b|c|d}^{{}^a c|{}^a d|b^c|b^c}(a,b)\otimes({}^a c, {}^a d).$$

So G_l is invertible, because $\lambda_{b|b|c|d}^{{}^a c|{}^a d|b^c|b^c} \ne 0$ by hypothesis. A similar computation shows that the map $(\tau\otimes D)\circ(D\otimes\Delta)$ is also bijective and finishes the proof. □

4 Non-degenerate solutions on incidence coalgebras

In this section we assume that $r\colon D\otimes D \longrightarrow D\otimes D$ is a non-degenerate coalgebra automorphism that induces a non-degenerate solution $r_|\colon X\times X \longrightarrow X\times X$ of the braid equation, and we determine necessary and sufficient conditions for r to be a solution of the braid equation. We also study these conditions when the sum of the lengths of the involved intervals is smaller than or equal to 1.

Notations 4.1. *For all $a,b,c\in X$, in this section we set $a^{bc} := (a^b)^c$, ${}^{ab}c := {}^{a}({}^{b}c)$, ${}^{a^b}c := {}^{(a^b)}c$, $a^{{}^b c} := a^{({}^b c)}$, ${}^{a}b^{{}^{a^b}c} := ({}^a b)^{({}^{(a^b)}c)}$ and ${}^{c^{{}^b a}}b^a := {}^{(c^{({}^b a)})}(b^a)$.*

Remark 4.2. It is well known and easy to check that a permutation $(a,b)\mapsto({}^a b, a^b)$ of $X\times X$ is a solution of the braid equation if and only if

$${}^{ab}c = {}^{{}^a b\, a^b}c,\quad a^{bc} = a^{{}^b c\, b^c}\quad \text{and}\quad {}^{a^{{}^b c}}b^c = {}^a b^{{}^{a^b}c}\quad \text{for all } a,b,c\in X.$$

Proposition 4.3. *The map r is a solution of the braid equation if and only if for each family of six closed intervals $[a,b]$, $[c,d]$, $[e,f]$, $[g,h]$, $[i,j]$ and $[k,l]$, with*

$$[g,h]\subseteq[a,b],\quad [i,j]\subseteq[c,d]\quad \textit{and}\quad [k,l]\subseteq[e,f],$$

the following equality holds:

$$\begin{aligned}&\sum_{\substack{x\in[a,g]\\ y\in[h,b]}}\sum_{\substack{w\in[c,i]\\ z\in[j,d]}}\sum_{\substack{u\in[e,k]\\ v\in[l,f]}} \lambda_{a|b|c|d}^{{}^a w|{}^a z|x^c|y^c}\lambda_{x^c|y^c|e|f}^{{}^{x^c}u|{}^{x^c}v|g^{ce}|h^{ce}}\lambda_{{}^a w|{}^a z|{}^{x^c}u|{}^{x^c}v}^{{}^{ac}k|{}^{ac}l|{}^a i^{{}^{a^i}e}|{}^a j^{{}^{a^j}e}}\\ &\qquad = \sum_{\substack{x\in[a,g]\\ y\in[h,b]}}\sum_{\substack{w\in[c,i]\\ z\in[j,d]}}\sum_{\substack{u\in[e,k]\\ v\in[l,f]}} \lambda_{c|d|e|f}^{{}^c u|{}^c v|w^e|z^e}\lambda_{a|b|{}^c u|{}^c v}^{{}^{ac}k|{}^{ac}l|x^{{}^c u}|y^{{}^c u}}\lambda_{x^{{}^c u}|y^{{}^c u}|w^e|z^e}^{{}^{a^{{}^i e}}i^e|{}^{a^{{}^j e}}j^e|g^{ce}|h^{ce}}.\end{aligned}\tag{4.1}$$

Proof. A direct computation using that r induces a solution $r_|$ of the set-theoretic braid equation on $X\times X$, Proposition 2.8 and Corollary 2.5, shows that

$$\begin{aligned}&(r\otimes D)\circ(D\otimes r)\circ(r\otimes D)((a,b)\otimes(c,d)\otimes(e,f))\\ &= \sum_{\substack{[x,y]\subseteq[a,b]\\ [w,z]\subseteq[c,d]}} \lambda_{a|b|c|d}^{{}^a w|{}^a z|x^c|y^c}(r\otimes D)\circ(D\otimes r)\big(({}^a w, {}^a z)\otimes(x^c,y^c)\otimes(e,f)\big)\\ &= \sum_{\substack{[x,y]\subseteq[a,b]\\ [w,z]\subseteq[c,d]}}\sum_{\substack{[g,h]\subseteq[x,y]\\ [u,v]\subseteq[e,f]}} \lambda_{a|b|c|d}^{{}^a w|{}^a z|x^c|y^c}\lambda_{x^c|y^c|e|f}^{{}^{x^c}u|{}^{x^c}v|g^{ce}|h^{ce}}(r\otimes D)\big(({}^a w, {}^a z)\otimes({}^{x^c}u, {}^{x^c}v)\otimes(g^{ce},h^{ce})\big)\\ &= \sum_{\substack{[x,y]\subseteq[a,b]\\ [w,z]\subseteq[c,d]}}\sum_{\substack{[g,h]\subseteq[x,y]\\ [u,v]\subseteq[e,f]}}\sum_{\substack{[i,j]\subseteq[w,z]\\ [k,l]\subseteq[u,v]}} \lambda_{a|b|c|d}^{{}^a w|{}^a z|x^c|y^c}\lambda_{x^c|y^c|e|f}^{{}^{x^c}u|{}^{x^c}v|g^{ce}|h^{ce}}\lambda_{{}^a w|{}^a z|{}^{x^c}u|{}^{x^c}v}^{{}^{{}^a w\, x^c}k|{}^{{}^a w\, x^c}l|{}^a i^{{}^{x^c}u}|{}^a j^{{}^{x^c}u}}\\ &\qquad\qquad \times\big({}^{{}^a w\, x^c}k, {}^{{}^a w\, x^c}l\big)\otimes\big({}^a i^{{}^{x^c}u}, {}^a j^{{}^{x^c}u}\big)\otimes(g^{ce},h^{ce})\end{aligned}$$

and

$$
\begin{aligned}
&(D\otimes r)\circ(r\otimes D)\circ(D\otimes r)((a,b)\otimes(c,d)\otimes(e,f))\\
&= \sum_{\substack{[w,z]\subseteq[c,d]\\ [u,v]\subseteq[e,f]}} \lambda_{c|d|e|f}^{{}^{c}u|{}^{c}v|w^{e}|z^{e}} (D\otimes r)\circ(r\otimes D)\big((a,b)\otimes({}^{c}u,{}^{c}v)\otimes(w^{e},z^{e})\big)\\
&= \sum_{\substack{[w,z]\subseteq[c,d]\\ [u,v]\subseteq[e,f]}} \sum_{\substack{[x,y]\subseteq[a,b]\\ [k,l]\subseteq[u,v]}} \lambda_{c|d|e|f}^{{}^{c}u|{}^{c}v|w^{e}|z^{e}} \lambda_{a|b|{}^{c}u|{}^{c}v}^{{}^{ac}k|{}^{ac}l|x^{{}^{c}u}|y^{{}^{c}u}} (D\otimes r)\big(({}^{ac}k,{}^{ac}l)\otimes(x^{{}^{c}u},y^{{}^{c}u})\otimes(w^{e},z^{e})\big)\\
&= \sum_{\substack{[w,z]\subseteq[c,d]\\ [u,v]\subseteq[e,f]}} \sum_{\substack{[x,y]\subseteq[a,b]\\ [k,l]\subseteq[u,v]}} \sum_{\substack{[g,h]\subseteq[x,y]\\ [i,j]\subseteq[w,z]}} \lambda_{c|d|e|f}^{{}^{c}u|{}^{c}v|w^{e}|z^{e}} \lambda_{a|b|{}^{c}u|{}^{c}v}^{{}^{ac}k|{}^{ac}l|x^{{}^{c}u}|y^{{}^{c}u}} \lambda_{x^{{}^{c}u}|y^{{}^{c}u}|w^{e}|z^{e}}^{{}^{x^{{}^{c}u}}i^{e}|{}^{x^{{}^{c}u}}j^{e}|g^{{}^{c}u\,w^{e}}|h^{{}^{c}u\,w^{e}}}\\
&\qquad\qquad \times ({}^{ac}k,{}^{ac}l)\otimes({}^{x^{{}^{c}u}}i^{e},{}^{x^{{}^{c}u}}j^{e})\otimes(g^{{}^{c}u\,w^{e}},h^{{}^{c}u\,w^{e}}).
\end{aligned}
$$

Since, by Corollary 2.5 and Remark 4.2

$$
\begin{aligned}
&{}^{{}^{a}w\,x^{c}}k = {}^{{}^{a}c\,a^{c}}k = {}^{ac}k, \quad {}^{a_{i}^{x^{c}}}u = {}^{a_{i}^{a^{i}}}e = {}^{a^{i_{e}}}i^{e} = {}^{x^{{}^{c}u}}i^{e}, \quad g^{ce} = g^{{}^{c}e\,c^{e}} = g^{{}^{c}u\,w^{e}},\\
&{}^{{}^{a}w\,x^{c}}l = {}^{{}^{a}c\,a^{c}}l = {}^{ac}l, \quad {}^{a_{j}^{x^{c}}}u = {}^{a_{j}^{a^{j}}}e = {}^{a^{j_{e}}}j^{e} = {}^{x^{{}^{c}u}}j^{e}, \quad h^{ce} = h^{{}^{c}e\,c^{e}} = h^{{}^{c}u\,w^{e}},
\end{aligned}
$$

the result follows immediately from the above equalities. □

4.1 Small intervals

Next we analyze exhaustively the meaning of equations (4.1) when the sum of the lengths of the intervals $[a,b]$, $[c,d]$ and $[e,f]$ is smaller than or equal to 1:

1) When $a=b$, $c=d$ and $e=f$ this equation reduces to

$$
\lambda_{a|a|c|c}^{{}^{a}c|{}^{a}c|a^{c}|a^{c}} \lambda_{a^{c}|a^{c}|e|e}^{{}^{a^{c}}e|{}^{a^{c}}e|a^{ce}|a^{ce}} \lambda_{{}^{a}c|{}^{a}c|{}^{a^{c}}e|{}^{a^{c}}e}^{{}^{ac}e|{}^{ac}e|{}^{a^{{}^{c}e}}c^{e}|{}^{a^{{}^{c}e}}c^{e}} = \lambda_{c|c|e|e}^{{}^{c}e|{}^{c}e|c^{e}|c^{e}} \lambda_{a|a|{}^{c}e|{}^{c}e}^{{}^{ac}e|{}^{ac}e|a^{{}^{c}e}|a^{{}^{c}e}} \lambda_{a^{{}^{c}e}|a^{{}^{c}e}|c^{e}|c^{e}}^{{}^{a^{{}^{c}e}}c^{e}|{}^{a^{{}^{c}e}}c^{e}|a^{ce}|a^{ce}}.
$$

This is true since the expressions at the both sides of the equal sign are 1.

2) When $a=g=h\prec b$, $c=d$ and $e=f$, it reduces to

$$
\lambda_{a|b|c|c}^{{}^{a}c|{}^{a}c|a^{c}|a^{c}} + \lambda_{a|b|c|c}^{{}^{a}c|{}^{a}c|a^{c}|b^{c}} \lambda_{a^{c}|b^{c}|e|e}^{{}^{a^{c}}e|{}^{a^{c}}e|a^{ce}|a^{ce}} = \lambda_{a|b|{}^{c}e|{}^{c}e}^{{}^{ac}e|{}^{ac}e|a^{{}^{c}e}|a^{{}^{c}e}} + \lambda_{a|b|{}^{c}e|{}^{c}e}^{{}^{ac}e|{}^{ac}e|a^{{}^{c}e}|b^{{}^{c}e}} \lambda_{a^{{}^{c}e}|b^{{}^{c}e}|c^{e}|c^{e}}^{{}^{a^{{}^{c}e}}c^{e}|{}^{a^{{}^{c}e}}c^{e}|a^{ce}|a^{ce}}.
$$

3) When $a=g\prec h=b$, $c=d$ and $e=f$, it reduces to

$$
\lambda_{a|b|c|c}^{{}^{a}c|{}^{a}c|a^{c}|b^{c}} \lambda_{a^{c}|b^{c}|e|e}^{{}^{a^{c}}e|{}^{a^{c}}e|a^{ce}|b^{ce}} = \lambda_{a|b|{}^{c}e|{}^{c}e}^{{}^{ac}e|{}^{ac}e|a^{{}^{c}e}|b^{{}^{c}e}} \lambda_{a^{{}^{c}e}|b^{{}^{c}e}|c^{e}|c^{e}}^{{}^{a^{{}^{c}e}}c^{e}|{}^{a^{{}^{c}e}}c^{e}|a^{ce}|b^{ce}}.
$$

4) When $a\prec g=h=b$, $c=d$ and $e=f$, it reduces to

$$
\lambda_{a|b|c|c}^{{}^{a}c|{}^{a}c|b^{c}|b^{c}} + \lambda_{a|b|c|c}^{{}^{a}c|{}^{a}c|a^{c}|b^{c}} \lambda_{a^{c}|b^{c}|e|e}^{{}^{a^{c}}e|{}^{a^{c}}e|b^{ce}|b^{ce}} = \lambda_{a|b|{}^{c}e|{}^{c}e}^{{}^{ac}e|{}^{ac}e|b^{{}^{c}e}|b^{{}^{c}e}} + \lambda_{a|b|{}^{c}e|{}^{c}e}^{{}^{ac}e|{}^{ac}e|a^{{}^{c}e}|b^{{}^{c}e}} \lambda_{a^{{}^{c}e}|b^{{}^{c}e}|c^{e}|c^{e}}^{{}^{a^{{}^{c}e}}c^{e}|{}^{a^{{}^{c}e}}c^{e}|b^{ce}|b^{ce}},
$$

which is equivalent to the condition obtained in item 2), because r satisfies condition (1.5) and the condition required in item 3) at the beginning of Section 3 is fulfilled.

5) When $a=b$, $c=i=j\prec d$ and $e=f$, it reduces to

$$\lambda_{a|a|c|d}^{{}^{a}c|{}^{a}c|a^{c}|a^{c}}+\lambda_{a|a|c|d}^{{}^{a}c|{}^{a}d|a^{c}|a^{c}}\lambda_{{}^{a}c|{}^{a}d|{}^{a^{c}}e|{}^{a^{c}}e}^{{}^{a^c}e|{}^{a^c}e|{}^{a}c^{{}^{a^{c}}e}|{}^{a}c^{{}^{a^{c}}e}}$$
$$=\lambda_{c|d|e|e}^{{}^{c}e|{}^{c}e|c^{e}|c^{e}}+\lambda_{c|d|e|e}^{{}^{c}e|{}^{c}e|c^{e}|d^{e}}\lambda_{a^{c^e}|a^{c^e}|c^{e}|d^{e}}^{{}^{a^{c^e}}c^{e}|{}^{a^{c^e}}c^{e}|a^{ce}|a^{ce}}.$$

6) When $a=b$, $c=i\prec j=d$ and $e=f$, it reduces to

$$\lambda_{a|a|c|d}^{{}^{a}c|{}^{a}d|a^{c}|a^{c}}\lambda_{{}^{a}c|{}^{a}d|{}^{a^{c}}e|{}^{a^{c}}e}^{{}^{a^c}e|{}^{a^c}e|{}^{a}c^{{}^{a^{c}}e}|{}^{a}d^{{}^{a^{d}}e}}=\lambda_{c|d|e|e}^{{}^{c}e|{}^{c}e|c^{e}|d^{e}}\lambda_{a^{c^e}|a^{c^e}|c^{e}|d^{e}}^{{}^{a^{c^e}}c^{e}|{}^{a^{d^e}}d^{e}|a^{ce}|a^{ce}}.$$

7) When $a=b$, $c\prec i=j=d$ and $e=f$, it reduces to

$$\lambda_{a|a|c|d}^{{}^{a}d|{}^{a}d|a^{c}|a^{c}}+\lambda_{a|a|c|d}^{{}^{a}c|{}^{a}d|a^{c}|a^{c}}\lambda_{{}^{a}c|{}^{a}d|{}^{a^{c}}e|{}^{a^{c}}e}^{{}^{a^c}e|{}^{a^c}e|{}^{a}d^{{}^{a^{d}}e}|{}^{a}d^{{}^{a^{d}}e}}$$
$$=\lambda_{c|d|e|e}^{{}^{c}e|{}^{c}e|d^{e}|d^{e}}+\lambda_{c|d|e|e}^{{}^{c}e|{}^{c}e|c^{e}|d^{e}}\lambda_{a^{c^e}|a^{c^e}|c^{e}|d^{e}}^{{}^{a^{d^e}}d^{e}|{}^{a^{d^e}}d^{e}|a^{ce}|a^{ce}},$$

which is equivalent to the condition obtained in item 5), by the same argument as in item 4).

8) When $a=b$, $c=d$ and $e=k=l\prec f$, it reduces to

$$\lambda_{a^{c}|a^{c}|e|f}^{{}^{a^c}e|{}^{a^c}e|a^{ce}|a^{ce}}+\lambda_{a^{c}|a^{c}|e|f}^{{}^{a^c}e|{}^{a^c}f|a^{ce}|a^{ce}}\lambda_{{}^{a}c|{}^{a}c|{}^{a^{c}}e|{}^{a^{c}}f}^{{}^{a^c}e|{}^{a^c}e|{}^{a^{c^e}}c^{e}|{}^{a^{c^e}}c^{e}}$$
$$=\lambda_{c|c|e|f}^{{}^{c}e|{}^{c}e|c^{e}|c^{e}}+\lambda_{c|c|e|f}^{{}^{c}e|{}^{c}f|c^{e}|c^{e}}\lambda_{a|a|{}^{c}e|{}^{c}f}^{{}^{a^c}e|{}^{a^c}e|a^{c^e}|a^{c^e}}.$$

9) When $a=b$, $c=d$ and $e=k\prec l=f$, it reduces to

$$\lambda_{a^{c}|a^{c}|e|f}^{{}^{a^c}e|{}^{a^c}f|a^{ce}|a^{ce}}\lambda_{{}^{a}c|{}^{a}c|{}^{a^{c}}e|{}^{a^{c}}f}^{{}^{a^c}e|{}^{a^c}f|{}^{a^{c^e}}c^{e}|{}^{a^{c^e}}c^{e}}=\lambda_{c|c|e|f}^{{}^{c}e|{}^{c}f|c^{e}|c^{e}}\lambda_{a|a|{}^{c}e|{}^{c}f}^{{}^{a^c}e|{}^{a^c}f|a^{c^e}|a^{c^e}}.$$

10) When $a=b$, $c=d$ and $e\prec k=l=f$, it reduces to

$$\lambda_{a^{c}|a^{c}|e|f}^{{}^{a^c}f|{}^{a^c}f|a^{ce}|a^{ce}}+\lambda_{a^{c}|a^{c}|e|f}^{{}^{a^c}e|{}^{a^c}f|a^{ce}|a^{ce}}\lambda_{{}^{a}c|{}^{a}c|{}^{a^{c}}e|{}^{a^{c}}f}^{{}^{a^c}f|{}^{a^c}f|{}^{a^{c^e}}c^{e}|{}^{a^{c^e}}c^{e}}$$
$$=\lambda_{c|c|e|f}^{{}^{c}f|{}^{c}f|c^{e}|c^{e}}+\lambda_{c|c|e|f}^{{}^{c}e|{}^{c}f|c^{e}|c^{e}}\lambda_{a|a|{}^{c}e|{}^{c}f}^{{}^{a^c}f|{}^{a^c}f|a^{c^e}|a^{c^e}},$$

which is equivalent to the condition obtained in item 8), by the same argument as in item 4).

In the sequel for $v=(v_1,v_2)$ and $w=(w_1,w_2)$ in K^2 we say that v and w are aligned and we write $v\sim w$ if $\det\left(\begin{smallmatrix} v_1 & v_2\\ w_1 & w_2\end{smallmatrix}\right)=0$.

Example 4.4. *Let $X_{a,h}\subseteq X$ be the set $\{a,b,c,d,e,f,g,h\}$. Assume that $r_|$ is the flip on $X_{a,h}\times X_{a,h}$. Then items 3), 6) and 9) are automatically fulfilled; whereas items 2), 5) and 8) say that when $a\prec b$, $c=d$ and $e=f$,*

$$\lambda_{a|b|c|c}^{c|c|a|a}+\lambda_{a|b|c|c}^{c|c|a|b}\lambda_{a|b|e|e}^{e|e|a|a}=\lambda_{a|b|e|e}^{e|e|a|a}+\lambda_{a|b|e|e}^{e|e|a|b}\lambda_{a|b|c|c}^{c|c|a|a}, \tag{4.2}$$

that when $a=b$, $c\prec d$ and $e=f$,

$$\lambda_{a|a|c|d}^{c|c|a|a}+\lambda_{a|a|c|d}^{c|d|a|a}\lambda_{c|d|e|e}^{e|e|c|c}=\lambda_{c|d|e|e}^{e|e|c|c}+\lambda_{c|d|e|e}^{e|e|c|d}\lambda_{a|a|c|d}^{c|c|a|a}, \tag{4.3}$$

and that when $a=b$, $c=d$ and $e\prec f$,

$$\lambda_{a|a|e|f}^{e|e|a|a}+\lambda_{a|a|e|f}^{e|f|a|a}\lambda_{c|c|e|f}^{e|e|c|c}=\lambda_{c|c|e|f}^{e|e|c|c}+\lambda_{c|c|e|f}^{e|f|c|c}\lambda_{a|a|e|f}^{e|e|a|a}. \tag{4.4}$$

Equality (4.2) *says that*

$$(\lambda_{a|b|e|e}^{e|e|a|a},\lambda_{a|b|e|e}^{e|e|a|b}-1)\sim(\lambda_{a|b|c|c}^{c|c|a|a},\lambda_{a|b|c|c}^{c|c|a|b}-1),$$

equality (4.3) *says that*

$$(\lambda_{c|d|e|e}^{e|e|c|c},\lambda_{c|d|e|e}^{e|e|c|d}-1)\sim(\lambda_{a|a|c|d}^{c|c|a|a},\lambda_{a|a|c|d}^{c|d|a|a}-1)$$

and equality (4.4) *says that*

$$(\lambda^{e|e|c|c}_{c|c|e|f}, \lambda^{e|f|c|c}_{c|c|e|f} - 1) \sim (\lambda^{e|e|a|a}_{a|a|e|f}, \lambda^{e|f|a|a}_{a|a|e|f} - 1).$$

Let $r\colon D\otimes D \longrightarrow D\otimes D$ be a non-degenerate coalgebra automorphism that induces a non-degenerate solution $r_{|}\colon X\times X \longrightarrow X\times X$ of the set-theoretic braid equation. Assume that there exist two commuting order automorphisms $\phi_r, \phi_l\colon X\to X$ such that $\phi_l = {}^{x}(-)$ and $\phi_r = (-)^{y}$ for all $x,y\in X$. For all $s,a,b\in X$ with $a\prec b$ and $i\in\mathbb{Z}$, we will write

$$s^{(i)} := \phi_r^i(s), \qquad {}^{(i)}s := \phi_l^i(s), \tag{4.5}$$

$$\alpha_r(s)(a,b) := \lambda^{{}^{(1)}s|{}^{(1)}s|a^{(1)}|b^{(1)}}_{a|b|s|s}, \qquad \beta_r(s)(a,b) := \lambda^{{}^{(1)}s|{}^{(1)}s|a^{(1)}|a^{(1)}}_{a|b|s|s}, \tag{4.6}$$

$$\alpha_l(s)(a,b) := \lambda^{{}^{(1)}a|{}^{(1)}b|s^{(1)}|s^{(1)}}_{s|s|a|b}, \qquad \beta_l(s)(a,b) := \lambda^{{}^{(1)}a|{}^{(1)}a|s^{(1)}|s^{(1)}}_{s|s|a|b}. \tag{4.7}$$

For the sake of brevity in the following result we write

$$\alpha_r^{(i)}(s) := \alpha_r(s^{(i)})(a^{(i)}, b^{(i)}) \quad \text{and} \quad \alpha_l^{(i)}(s) := \alpha_l({}^{(i)}s)({}^{(i)}a, {}^{(i)}b), \tag{4.8}$$

and we define $\beta_r^{(i)}(s)$ and $\beta_l^{(i)}(s)$ in a similar way. The following proposition generalizes the result obtained in the previous example.

Proposition 4.5. *Let* $n\in\mathbb{N}$. *Assume that* $\phi_r^n = \phi_l^n = \mathrm{id}$ *and that each element of* $K^\times$ *has* n *distinct* n*th roots, and fix a primitive* n*th root of unity* w. *The following facts hold:*

1) *Item 3) of Subsection 4.1 is satisfied if and only if for all* $a\prec b$ *in* X *there exists a constant* $C_r(a,b)\in K^\times$ *such that*

$$\frac{\alpha_r^{(1)}(s)}{\alpha_r(s)} = C_r(a,b) \quad \textit{and} \quad \alpha_r(s) = \alpha_r({}^{(1)}s^{(1)}) \textit{ for all } s\in X.$$

2) *Item 9) of Subsection 4.1 is satisfied if and only if for all* $a\prec b$ *in* X *there exists a constant* $C_l(a,b)\in K^\times$ *such that*

$$\frac{\alpha_l^{(1)}(s)}{\alpha_l(s)} = C_l(a,b) \quad \textit{and} \quad \alpha_l(s) = \alpha_l({}^{(1)}s^{(1)}) \textit{ for all } s\in X.$$

3) *Assume that the conditions in item 1) are fulfilled. Then item 2) of Subsection 4.1 is satisfied if and only if for all* $s\in X$ *and* $a\prec b$ *in* X

$$\beta_r(s) = \beta_r({}^{(1)}s^{(1)}),$$

and for all $a,b,s,t\in X$ *with* $a\prec b$ *and each* $0\le i<n$

$$\left(\gamma_r\alpha_r(s) - w^i, \sum_{j=0}^{n-1}\wp_j w^{ij}\beta_r^{(j)}(s)\right) \sim \left(\gamma_r\alpha_r(t) - w^i, \sum_{j=0}^{n-1}\wp_j w^{ij}\beta_r^{(j)}(t)\right),$$

where

- γ_r *is a fixed* n*th root of* $\displaystyle\prod_{u=0}^{n-2} C_r(a^{(u)}, b^{(u)})^{n-u-1}$,
- $\displaystyle\wp_j := \frac{1}{\gamma_r^{j+1}}\prod_{u=0}^{n-2} C_r(a^{(u)}, b^{(u)})\prod_{u=0}^{j-2} C_r(a^{(u)}, b^{(u)})^{j-u-1}$.

4) *Assume that the conditions in item 2) are fulfilled. Then item 8) of Subsection 4.1 is satisfied if and only if for all* $s\in X$ *and* $a\prec b$ *in* X

$$\beta_l(s) = \beta_l({}^{(1)}s^{(1)}),$$

and for all $a,b,s,t\in X$ *with* $a\prec b$ *and each* $0\le i<n$

$$\Bigl(\gamma_l\alpha_l(s)-w^i,\sum_{j=0}^{n-1}\ell_j w^{ij}\beta_l^{(j)}(s)\Bigr)\sim\Bigl(\gamma_l\alpha_l(t)-w^i,\sum_{j=0}^{n-1}\ell_j w^{ij}\beta_l^{(j)}(t)\Bigr),$$

where

- γ_l *is a fixed* n*th root of* $\prod_{u=0}^{n-2} C_l({}^{(u)}a,{}^{(u)}b)^{n-u-1}$,
- $\ell_j:=\dfrac{1}{\gamma_l^{j+1}}\prod_{u=0}^{n-2} C_l({}^{(u)}a,{}^{(u)}b)\prod_{u=0}^{j-2} C_l({}^{(u)}a,{}^{(u)}b)^{j-u-1}$.

Proof. Assume that the equality in item 3) of Subsection 4.1 holds. By Remark 3.1 all terms in that equality are non-zero. Replacing e by $s^{(1)}$ in it, we obtain

$$\frac{\alpha_r^{(1)}(c)}{\alpha_r(c)}=\frac{\alpha_r^{(1)}(s)}{\alpha_r({}^{(1)}s^{(1)})}=\frac{\alpha_r^{(1)}(s)}{\alpha_r(s)}\qquad\text{for all } c,s\in X,$$

where the last equality follows from the first one taking $c=s$. From this it follows immediately that there exists $C_r(a,b)\in K^\times$ such that the equalities in item 1) are true. The converse is straightforward. A similarly argument proves item 2).

Assume now that the conditions in item 1) are fulfilled and that the equality in item 2) of Subsection 4.1 holds. By Remark 3.1, setting $(a,b):=(a^{(-1)},b^{(-1)})$ and $c:={}^{(1)}e$ the equality yields $\beta_r(e)=\beta_r({}^{(1)}e^{(1)})$ for all $e\in X$. Using the same equality with $c:=s^{(i)}$, $(a,b):=(a^{(i)},b^{(i)})$ and $e:=t^{(i+1)}$, where $i\in\{0,\dots,n-1\}$, we obtain

$$\begin{aligned}\beta_r^{(0)}(s)+\alpha_r^{(0)}(s)\beta_r^{(1)}(t)&=\beta_r^{(0)}(t)+\alpha_r^{(0)}(t)\beta_r^{(1)}(s)\\ \beta_r^{(1)}(s)+\alpha_r^{(1)}(s)\beta_r^{(2)}(t)&=\beta_r^{(1)}(t)+\alpha_r^{(1)}(t)\beta_r^{(2)}(s)\\ &\vdots\\ \beta_r^{(n-1)}(s)+\alpha_r^{(n-1)}(s)\beta_r^{(0)}(t)&=\beta_r^{(n-1)}(t)+\alpha_r^{(n-1)}(t)\beta_r^{(0)}(s),\end{aligned}\tag{4.9}$$

where in the last equation we have used that $\beta_r^{(n)}=\beta_r^{(0)}$. For each $j\in\mathbb{N}_0$, let

$$C(j):=\prod_{u=0}^{j-1}C_r(a^{(u)},b^{(u)}).$$

Using that $\wp_jC(j)=\gamma_r\wp_{j+1}$ and $\wp_n=\wp_0$ we obtain that

$$\sum_{j=0}^{n-1}w^{ij}\wp_jC(j)\beta_r^{(j+1)}(x)=\sum_{j=0}^{n-1}w^{ij}\gamma_r\wp_{j+1}\beta_r^{(j+1)}(x)=\frac{\gamma_r}{w^i}\sum_{j=0}^{n-1}w^{ij}\wp_j\beta_r^{(j)}(x),\tag{4.10}$$

for all $x\in X$ and $i\in\{0,\dots,n-1\}$. Adding the first equality in (4.9) multiplied by $\wp_0$ to the second one multiplied by $w^i\wp_1$, and so on until we add the last equality multiplied by $w^{i(n-1)}\wp_{n-1}$, and using that $\alpha_r^{(j)}(s)=\alpha_r(s)C(j)$, we obtain

$$\begin{aligned}&\sum_{j=0}^{n-1}w^{ij}\wp_j\beta_r^{(j)}(s)+\alpha_r(s)\sum_{j=0}^{n-1}w^{ij}\wp_jC(j)\beta_r^{(j+1)}(t)\\ &\qquad=\sum_{j=0}^{n-1}w^{ij}\wp_j\beta_r^{(j)}(t)+\alpha_r(t)\sum_{j=0}^{n-1}w^{ij}\wp_jC(j)\beta_r^{(j+1)}(s),\end{aligned}\tag{4.11}$$

for $i\in\{0,\dots,n-1\}$. Hence, by (4.10),

$$S_r^{(i)}(s)+\alpha_r(s)\frac{\gamma_r}{w^i}S_r^{(i)}(t)=S_r^{(i)}(t)+\alpha_r(t)\frac{\gamma_r}{w^i}S_r^{(i)}(s),\tag{4.12}$$

where $S_r^{(i)}(x) := \sum_{j=0}^{n-1} w^{ij} \wp_j \beta_r^{(j)}(x)$ for $x \in X$, and so, for $0 \le i < n$, we have

$$(\gamma_r \alpha_r(s) - w^i, S_r^{(i)}(s)) \sim (\gamma_r \alpha_r(t) - w^i, S_r^{(i)}(t)) \quad \text{for all } s, t \in X, \tag{4.13}$$

as desired.

Conversely assume that $\beta_r(e) = \beta_r({}^{(1)}e^{(1)})$ for all $e \in X$, and that (4.13) holds, which means that (4.12) holds. By (4.10) the systems (4.12) and (4.11) are equivalent. We claim that the systems (4.11) and (4.9) are also equivalent. Indeed, this follows easily from the fact that all the $\wp_j$'s are non-zero and that the matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & w & \dots & w^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & \dots & w^{(n-1)^2} \end{pmatrix}$$

is invertible, because it is the Vandermonde matrix associated with the elements $1, w, w^2, \dots, w^{n-1}$, which are all different. Item 2) of Subsection 4.1 follows immediately from the first equality in (4.9) with s replaced by c and t replaced by ${}^{(1)}e$, using that $\beta_r({}^{(1)}e^{(1)})(a^{(1)}, b^{(1)}) = \beta_r(e)(a^{(1)}, b^{(1)})$. A similar argument proves item (4). □

Let $\phi_r, \phi_l, \alpha_r^{(i)}, \alpha_l^{(i)}, \beta_r^{(i)}$ and $\beta_l^{(i)}$ be as in the discussion above Proposition 4.5.

Proposition 4.6. *Let $n \in \mathbb{N}$. Assume that $\phi_r = \phi_l$, that $\phi_r^n = \mathrm{id}$ and that each element of $K^\times$ has n distinct n-roots and fix a primitive n-root of unity w. Then equality (4.1) is satisfied for all the intervals $[a,b]$, $[c,d]$, $[e,f]$, $[g,h]$, $[i,j]$ and $[k,l]$ such that*

$$[g,h] \subseteq [a,b], \quad [i,j] \subseteq [c,d] \quad \text{and} \quad [k,l] \subseteq [e,f],$$

and $\mathfrak{h}[a,b] + \mathfrak{h}[c,d] + \mathfrak{h}[e,f] = 1$, if and only if the following facts hold:

1) *For all $a \prec b$ in X there exists a constant $C(a,b) \in K^\times$, such that*
$$\frac{\alpha_l(s)}{\alpha_l^{(1)}(s)} = \frac{\alpha_r(s)}{\alpha_r^{(1)}(s)} = C(a,b), \quad \text{for all } s \in X.$$
2) *For all $a \prec b$ and s in X, it is true that*
$$\alpha_r(s) = \alpha_r(s^{(2)}), \quad \alpha_r(s) = \alpha_r(s^{(2)}), \quad \beta_r(s) = \beta_r(s^{(2)}), \quad \beta_l(s) = \beta_l(s^{(2)}).$$
3) *For all $a \prec b$ in X and each $0 \le i < n$, there exists a one dimensional vector subspace of $K \times K$, which contains all the vectors*
$$\left(\gamma\alpha_r(s) - w^i, \sum_{j=0}^{n-1} \wp_j w^{ij} \beta_r^{(j)}(s)\right) \quad \text{and} \quad \left(\gamma\alpha_l(s) - w^i, \sum_{j=0}^{n-1} \wp_j w^{ij} \beta_l^{(j)}(s)\right),$$
where $\gamma := \gamma_r = \gamma_l$ and $\wp_j = \ell_j$ are as in Proposition 4.5.

Proof. We know that if $\mathfrak{h}[a,b] + \mathfrak{h}[c,d] + \mathfrak{h}[e,f] = 1$, then equality (4.1) is equivalent to items 2), 3), 5), 6), 8) and 9) of Subsection 4.1. Moreover item 6) of Subsection 4.1 is satisfied if and only if for all $a \prec b$ in X there exists $C(a,b) \in K^\times$ such that

$$\frac{\alpha_l(s^{(1)})(a^{(1)}, b^{(1)})}{\alpha_l(s)(a,b)} = \frac{\alpha_r({}^{(1)}s)({}^{(1)}a, {}^{(1)}b)}{\alpha_r(s)(a,b)} = C(a,b) \quad \text{for all } s \in X.$$

On the other hand Proposition 4.5 gives necessary and sufficient conditions in order that items 2), 3), 8) and 9) of Subsection 4.1 are satisfied. Since $\phi_r = \phi_l$, we have $s^{(i)} = {}^{(i)}s$ for all $s \in X$ and all $i \in \mathbb{Z}$, and $C_r(a,b) = C(a,b) = C_l(a,b)$ for all $a \prec b$ in X. Consequently γ_r and γ_l are nth roots of the same element, and so we can choose $\gamma_r = \gamma_l$. It follows that $\wp_j = \ell_j$ for $0 \le j < n$ and that the conditions in

Proposition 4.5 are equivalent to items 1) and 2) together with the fact that there exist two one dimensional vector subspaces of $K\times K$ that contain all the vectors

$$\Big(\gamma\alpha_r(s)-w^i,\sum_{j=0}^{n-1}\wp_j w^{ij}\beta_r^{(j)}(s)\Big)\quad\text{and}\quad\Big(\gamma\alpha_l(s)-w^i,\sum_{j=0}^{n-1}\wp_j w^{ij}\beta_l^{(j)}(s)\Big),$$

respectively. Assume now that item 5) of Subsection 4.1 is satisfied. Since $\phi_r=\phi_l$, using the equality in that item with $c:=s^{(i)}$, $(a,b):=(a^{(i)},b^{(i)})$ and $e:=t^{(i)}$, where i runs on $\{0,\dots,n-1\}$, we obtain

$$\begin{aligned}\beta_l^{(0)}(s)+\alpha_l^{(0)}(s)\beta_r^{(1)}(t)&=\beta_r^{(0)}(t)+\alpha_r^{(0)}(t)\beta_l^{(1)}(s)\\ \beta_l^{(1)}(s)+\alpha_l^{(1)}(s)\beta_r^{(2)}(t)&=\beta_r^{(1)}(t)+\alpha_r^{(1)}(t)\beta_l^{(2)}(s)\\ &\vdots\\ \beta_l^{(n-1)}(s)+\alpha_l^{(n-1)}(s)\beta_r^{(0)}(t)&=\beta_r^{(n-1)}(t)+\alpha_r^{(n-1)}(t)\beta_l^{(0)}(s),\end{aligned}\tag{4.14}$$

where in the last equation we have used that $\beta_r^{(n)}=\beta_r^{(0)}$ and $\beta_l^{(n)}=\beta_l^{(0)}$. Mimicking the proof of item 3) of Proposition 4.5 we obtain that the equalities in (4.14) hold if and only if for $0\le i<n$

$$(\gamma\alpha_l(s)-w^i,S_l^{(i)}(s))\sim(\gamma\alpha_r(t)-w^i,S_r^{(i)}(t))\quad\text{for all } s,t\in X,\tag{4.15}$$

where $S_l^{(i)}(x)=\sum_{j=0}^{n-1}w^{ij}\wp_j\beta_l^{(j)}(x)$ and $S_r^{(i)}(x)$ is as in the proof of Proposition 4.5. So item 3) is true. We leave the proof of the converse to the reader. □

5 The configuration $p\prec q$ when $r_|$ is the flip

Let $(X,\le)$ and D be as in Section 2, let (D,r) be a non-degenerate braided set and let $p,q\in X$ be such that $p\prec q$. In this section we determine all the possibilities for the coefficients $\lambda_{a_1|b_1|a_2|b_2}^{a_3|b_3|a_4|b_4}$ with $a_i,b_i\in\{p,q\}$ and $a_i\le b_i$, under the assumption that

$$r_|(p,p)=(p,p),\quad r_|(p,q)=(q,p),\quad r_|(q,p)=(p,q)\quad\text{and}\quad r_|(q,q)=(q,q).$$

As a corollary we obtain all the non-degenerate braided sets (D,r) such that D is the incidence coalgebra of the linearly ordered set with two elements.

Let $f(p,p):=0$, $f(p,q):=1$ and $f(q,q):=2$. We can codify the 81 coefficients $\lambda_{a_1|b_1|a_2|b_2}^{a_3|b_3|a_4|b_4}$ in a 9×9 matrix M, setting

$$M_{i,j}=\lambda_{a_1|b_1|a_2|b_2}^{a_3|b_3|a_4|b_4}\qquad\begin{array}{l}\text{if } i:=3f(a_3,b_3)+f(a_4,b_4)+1\\ \text{and } j:=3f(a_1,b_1)+f(a_2,b_2)+1\end{array}.\tag{5.1}$$

Remark 5.1. By Proposition 2.8 and equality (1.5),

$$M=\begin{pmatrix}1&\beta_1&0&\beta_2&\Gamma_1&0&0&0&0\\0&0&0&\alpha_2&B_1&0&0&0&0\\0&0&0&-\beta_2&\Gamma_2&0&1&\beta_4&0\\0&\alpha_1&0&0&B_2&0&0&0&0\\0&0&0&0&A&0&0&0&0\\0&0&0&0&B_3&0&0&\alpha_4&0\\0&-\beta_1&1&0&\Gamma_3&\beta_3&0&0&0\\0&0&0&0&B_4&\alpha_3&0&0&0\\0&0&0&0&\Gamma_4&-\beta_3&0&-\beta_4&1\end{pmatrix},$$

where

$$\alpha_1 := \lambda^{p|q|p|p}_{p|p|p|q}, \quad \alpha_2 := \lambda^{p|p|p|q}_{p|q|p|p}, \quad \alpha_3 := \lambda^{q|q|p|q}_{p|q|q|q}, \quad \alpha_4 := \lambda^{p|q|q|q}_{q|q|p|q},$$

$$\beta_1 := \lambda^{p|p|p|p}_{p|p|p|q}, \quad \beta_2 := \lambda^{p|p|p|p}_{p|q|p|p}, \quad \beta_3 := \lambda^{q|q|p|p}_{p|q|q|q}, \quad \beta_4 := \lambda^{p|p|q|q}_{q|q|p|q},$$

$$A := \lambda^{p|q|p|q}_{p|q|p|q},$$

$$B_1 := \lambda^{p|p|p|q}_{p|q|p|q}, \quad B_2 := \lambda^{p|q|p|p}_{p|q|p|q}, \quad B_3 := \lambda^{p|q|q|q}_{p|q|p|q}, \quad B_4 := \lambda^{q|q|p|q}_{p|q|p|q},$$

$$\Gamma_1 := \lambda^{p|p|p|p}_{p|q|p|q}, \quad \Gamma_2 := \lambda^{p|p|q|q}_{p|q|p|q}, \quad \Gamma_3 := \lambda^{q|q|p|p}_{p|q|p|q}, \quad \Gamma_4 := \lambda^{q|q|q|q}_{p|q|p|q}.$$

Remark 5.2. By Proposition 2.10 and equality (1.5) we know that

$$A = \alpha_1\alpha_3 = \alpha_2\alpha_4, \tag{5.2}$$

$$B_1 = \alpha_2\beta_4, \quad B_2 = \alpha_1\beta_3, \quad B_3 = -\alpha_4\beta_2, \quad B_4 = -\alpha_3\beta_1, \tag{5.3}$$

$$\Gamma_2 = -\beta_2\beta_4 \quad \text{and} \quad \Gamma_3 = -\beta_1\beta_3, \tag{5.4}$$

and by equality (1.5) and Proposition 2.8 we know that

$$\Gamma_4 = -(\Gamma_1 + \Gamma_2 + \Gamma_3). \tag{5.5}$$

We will use these equalities (which in particular show that M depends exclusively on Γ_1, α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4), in order to determine M in all the cases. Moreover, by Remark 3.1, since r is an isomorphism, $\alpha_i \neq 0$ for all i.

Remark 5.3. If some $\beta_i \neq 0$, then by equalities (4.2), (4.3) and (4.4) there exists an element $C \in K$ such that

$$\alpha_i - 1 = C\beta_i, \quad \text{for } i = 1,2,3,4. \tag{5.6}$$

Theorem 5.4. *For the matrix M given by (5.1) necessarily one of the following eight cases occur:*

1) *If $\beta_i = 0$ for $i = 1,2,3,4$, and $\Gamma_1 = 0$, then M belongs to the family*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1\alpha_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\alpha_4 = \frac{\alpha_1\alpha_3}{\alpha_2}$, parameterized by $\alpha_1, \alpha_2, \alpha_3 \in K^\times$.

2) *If $\beta_i = 0$ for $i = 1,2,3,4$ and $\Gamma_1 \neq 0$, then M belongs to the family*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \Gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_1\alpha_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Gamma_1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

parameterized by $\alpha_1 = \pm 1$, $\alpha_3 = \pm 1$ and $\Gamma_1 \in K^\times$.

3) *If there exists i such that $\beta_i \neq 0$ and $C = 0$ (see equality (5.6)), then either*

$$\beta_3 = \beta_2 \quad \text{and} \quad \beta_4 = \beta_1, \qquad \text{or} \qquad \beta_4 = -\beta_3, \quad \beta_2 = -\beta_1 \quad \text{and} \quad \beta_3 \neq -\beta_1.$$

In the first case M belongs to the family

$$\begin{pmatrix} 1 & \beta_1 & 0 & \beta_2 & \Gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_2 & -\beta_1\beta_2 & 0 & 1 & \beta_1 & 0 \\ 0 & 1 & 0 & 0 & \beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_2 & 0 & 0 & 1 & 0 \\ 0 & -\beta_1 & 1 & 0 & -\beta_1\beta_2 & \beta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\beta_1\beta_2 - \Gamma_1 & -\beta_2 & 0 & -\beta_1 & 1 \end{pmatrix},$$

parameterized by $\Gamma_1, \beta_1, \beta_2 \in K$ with $(\beta_1, \beta_2) \neq (0,0)$; while, in the second case $\Gamma_1 = \beta_1\beta_3$ and M belongs to the family

$$\begin{pmatrix} 1 & \beta_1 & 0 & -\beta_1 & \beta_1\beta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\beta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_1 & -\beta_1\beta_3 & 0 & 1 & -\beta_3 & 0 \\ 0 & 1 & 0 & 0 & \beta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 1 & 0 \\ 0 & -\beta_1 & 1 & 0 & -\beta_1\beta_3 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1\beta_3 & -\beta_3 & 0 & \beta_3 & 1 \end{pmatrix},$$

parameterized by $\beta_1, \beta_3 \in K$ with $\beta_1 + \beta_3 \neq 0$.

4) *If $\beta_1 = 0$, some $\beta_i \neq 0$, $C \neq 0$ and $\Gamma_1 C = \alpha_2\beta_4$, then M belongs to the family*

$$\begin{pmatrix} 1 & 0 & 0 & \beta_2 & \frac{\alpha_2\beta_4}{C} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \alpha_2\beta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_2 & -\beta_2\beta_4 & 0 & 1 & \beta_4 & 0 \\ 0 & 1 & 0 & 0 & \beta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2\alpha_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_4\beta_2 & 0 & 0 & \alpha_4 & 0 \\ 0 & 0 & 1 & 0 & 0 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_2\beta_4 - \frac{\alpha_2\beta_4}{C} & -\beta_3 & 0 & -\beta_4 & 1 \end{pmatrix},$$

where $\alpha_i = 1 + C\beta_i$ and $\beta_3 = \beta_2 + \alpha_2\beta_4$, parameterized by $C \in K^\times$ and $\beta_2, \beta_4 \in K$ with $(\beta_2, \beta_4) \neq (0,0)$ such that $C\beta_i \neq -1$ for all i.

5) *If $\beta_1 = 0$, some $\beta_i \neq 0$, $C \neq 0$ and $\Gamma_1 C \neq \alpha_2\beta_4$, then $C\beta_4 = -2$, $\beta_2 = 0$, $\beta_3 = \beta_4$ and M belongs to the family*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \Gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \beta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \beta_4 & 0 \\ 0 & 1 & 0 & 0 & \beta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \beta_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Gamma_1 & -\beta_4 & 0 & -\beta_4 & 1 \end{pmatrix},$$

parameterized by $\Gamma_1 \in K \setminus \{-\beta_4^2/2\}$ *and* $\beta_4 \in K^{\times}$.

6) *If* $\beta_1 \neq 0$, $\alpha_1 = -1$, $C \neq 0$ *and* $\alpha_2\beta_4 C - C^2\Gamma_1 = -2$, *then* $C = -2/\beta_1$ *and* M *belongs to the family*

$$\begin{pmatrix} 1 & \beta_1 & 0 & \beta_2 & \frac{1}{2}(\beta_1^2 - \beta_1\beta_4 + 2\beta_2\beta_4) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \alpha_2\beta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_2 & -\beta_2\beta_4 & 0 & 1 & \beta_4 & 0 \\ 0 & -1 & 0 & 0 & -\beta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2\alpha_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_4\beta_2 & 0 & 0 & \alpha_4 & 0 \\ 0 & -\beta_1 & 1 & 0 & -\beta_1\beta_3 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\beta_3 - \beta_1 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\beta_1(2\beta_3 - \beta_1 + \beta_4) & -\beta_3 & 0 & -\beta_4 & 1 \end{pmatrix},$$

where $\beta_3 = \frac{2\beta_2\beta_4}{\beta_1} + \beta_1 - \beta_2 - \beta_4$ *and* $\alpha_i = 1 - \frac{2\beta_i}{\beta_1}$, *parameterized by* $\beta_1 \in K^{\times}$ *and* $\beta_2, \beta_4 \in K \setminus \{\beta_1/2\}$.

7) *If* $\beta_1 \neq 0$, $\alpha_1 = -1$, $C \neq 0$ *and* $\alpha_2\beta_4 C - C^2\Gamma_1 \neq -2$, *then* $\alpha_2 = -1$, $\beta_4 = \beta_3$, $C = -2/\beta_1$ *and either*

$$\beta_3 = \beta_1 \qquad or \qquad \beta_3 = 0.$$

In the first case $\alpha_i = -1$ *and* $\beta_i = \beta_1$ *for all* i *and* M *belongs to the family*

$$\begin{pmatrix} 1 & \beta_1 & 0 & \beta_1 & \Gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -\beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_1 & -\beta_1^2 & 0 & 1 & \beta_1 & 0 \\ 0 & -1 & 0 & 0 & -\beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & 0 & 0 & -1 & 0 \\ 0 & -\beta_1 & 1 & 0 & -\beta_1^2 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\beta_1^2 - \Gamma_1 & -\beta_1 & 0 & -\beta_1 & 1 \end{pmatrix},$$

parameterized by $\beta_1 \in K^{\times}$ *and* $\Gamma_1 \in K \setminus \{\beta_1^2\}$; *while in the second case* $\alpha_3 = \alpha_4 = 1$ *and* M *belongs to the family*

$$\begin{pmatrix} 1 & \beta_1 & 0 & \beta_1 & \Gamma_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_1 & 0 & 0 & 1 & 0 \\ 0 & -\beta_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Gamma_1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

parameterized by $\beta_1 \in K^{\times}$ *and* $\Gamma_1 \in K \setminus \{\beta_1^2/2\}$.

8) *If* $\beta_1 \neq 0$, $\alpha_1 \neq -1$ *and* $C \neq 0$, *then* M *belongs to the family*

$$\begin{pmatrix} 1 & \beta_1 & 0 & \beta_2 & \frac{\beta_3\alpha_1 - \beta_2}{C} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \alpha_2\beta_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_2 & -\beta_2\beta_4 & 0 & 1 & \beta_4 & 0 \\ 0 & \alpha_1 & 0 & 0 & \alpha_1\beta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2\alpha_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_4\beta_2 & 0 & 0 & \alpha_4 & 0 \\ 0 & -\beta_1 & 1 & 0 & -\beta_1\beta_3 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_3\beta_1 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\beta_4 C\beta_2 + \beta_2 - \beta_3}{C} & -\beta_3 & 0 & -\beta_4 & 1 \end{pmatrix},$$

where $\alpha_i = \beta_i C + 1$ *for all* i *and* $\beta_4 = (\beta_1 - \beta_2 + \beta_3 + \beta_1\beta_3 C)/(1 + \beta_2 C)$, *parameterized by* $C \in K^\times$, $\beta_1 \in K^\times \setminus \{-1/C\}$ *and* $\beta_2, \beta_3 \in K \setminus \{-1/C\}$.

Proof. 1) This follows immediately from Remark 5.2.

Before considering the other cases we derive equalities (5.7) to (5.12). Using equality (4.1) with $a = b = c = e = g = h = i = j = k = l = p$ and $d = f = q$, we obtain

$$\sum_{z,v\in[p,q]} \lambda^{p|z|p|p}_{p|p|p|q} \lambda^{p|v|p|p}_{p|p|p|q} \lambda^{p|p|p|p}_{p|z|p|v} = \sum_{z,v\in[p,q]} \lambda^{p|v|p|z}_{p|q|p|q} \lambda^{p|p|p|p}_{p|p|p|v} \lambda^{p|p|p|p}_{p|p|p|z}.$$

Combining this with the fact that $A = \alpha_1\alpha_3$, $B_1 = \alpha_2\beta_4$ and $B_2 = \alpha_1\beta_3$, we obtain

$$\beta_1\beta_1 + \beta_1\alpha_1\beta_1 + \alpha_1\beta_1\beta_2 + \alpha_1\alpha_1\Gamma_1 = \Gamma_1 + (\alpha_1\beta_3)\beta_1 + (\alpha_2\beta_4)\beta_1 + (\alpha_1\alpha_3)\beta_1\beta_1,$$

which we can write as

$$\Gamma_1(\alpha_1^2 - 1) = \beta_1(\alpha_1\beta_3 + \alpha_2\beta_4 + \alpha_1\alpha_3\beta_1 - \beta_1 - \alpha_1\beta_1 - \alpha_1\beta_2). \tag{5.7}$$

Equality (4.1) with $a = c = d = e = g = h = i = j = k = l = p$ and $b = f = q$, gives

$$\sum_{y,v\in[p,q]} \lambda^{p|p|p|y}_{p|q|p|p} \lambda^{p|v|p|p}_{p|y|p|q} \lambda^{p|p|p|p}_{p|p|p|v} = \sum_{y,v\in[p,q]} \lambda^{p|v|p|p}_{p|p|p|q} \lambda^{p|p|p|y}_{p|q|p|v} \lambda^{p|p|p|p}_{p|y|p|p}.$$

Combining this with the fact that $B_1 = \alpha_2\beta_4$ and $B_2 = \alpha_1\beta_3$, we obtain

$$\beta_2\beta_1 + \beta_2\alpha_1\beta_1 + \alpha_2\Gamma_1 + \alpha_2(\alpha_1\beta_3)\beta_1 = \beta_1\beta_2 + \alpha_1\Gamma_1 + \beta_1\alpha_2\beta_2 + \alpha_1(\alpha_2\beta_4)\beta_2,$$

which simplifies to

$$\alpha_1(\beta_1\beta_2 - \Gamma_1) = \alpha_2(\beta_1\beta_2 - \Gamma_1 + \alpha_1(\beta_2\beta_4 - \beta_1\beta_3)). \tag{5.8}$$

Similarly, using equality (4.1) with $a = b = d = f = g = h = i = j = k = l = q$ and $c = e = p$, we obtain

$$\Gamma_4(\alpha_4^2 - 1) = \beta_4(\alpha_3\beta_1 + \alpha_4\beta_2 + \alpha_1\alpha_3\beta_4 - \beta_4 - \alpha_4\beta_4 - \alpha_4\beta_3); \tag{5.9}$$

using equality (4.1) with $b = c = d = f = g = h = i = j = k = l = q$ and $a = e = p$, we obtain

$$\alpha_3(\beta_3\beta_4 - \Gamma_4) = \alpha_4(\beta_3\beta_4 - \Gamma_4 + \alpha_3(\beta_2\beta_4 - \beta_1\beta_3)); \tag{5.10}$$

using equality (4.1) with $a = c = e = f = g = h = i = j = k = l = p$ and $b = d = q$, we can see that

$$\Gamma_1(\alpha_2^2 - 1) = \beta_2(\alpha_2\beta_4 + \alpha_1\beta_3 + \alpha_1\alpha_3\beta_2 - \beta_2 - \alpha_2\beta_2 - \alpha_2\beta_1); \tag{5.11}$$

while using equality (4.1) with $b = d = e = f = g = h = i = j = k = l = q$ and $a = c = p$, we can see that

$$\Gamma_4(\alpha_3^2 - 1) = \beta_3(\alpha_3\beta_1 + \alpha_4\beta_2 + \alpha_1\alpha_3\beta_3 - \beta_3 - \alpha_3\beta_3 - \alpha_3\beta_4). \tag{5.12}$$

2) Since $\beta_1 = \beta_2 = 0$ and $\Gamma_1 \neq 0$, from equalities (5.4) and (5.5) we obtain that $\Gamma_2 = \Gamma_3 = 0$ and $\Gamma_4 = -\Gamma_1 \neq 0$, from equality (5.7) we obtain that $\alpha_1^2 = 1$, and from equality (5.8) we obtain that $\alpha_1 = \alpha_2$. Similarly, by equalities (5.9) and (5.10) we have $\alpha_4^2 = 1$ and $\alpha_3 = \alpha_4$. Using now equalities (5.2) and (5.3) we obtain that M belongs to the family in item 2).

3) Since $C = 0$, from equalities 5.6 it follows that $\alpha_j = 1$ for all j. Furthermore equalities (5.7), (5.8), (5.9), (5.11) and (5.12) yield

$$\beta_i(\beta_3 + \beta_4 - \beta_1 - \beta_2) = 0 \quad \text{for all } i, \qquad \text{and} \qquad \beta_2\beta_4 - \beta_1\beta_3 = 0.$$

Since at least one β_i is non-zero, from these facts it follows that

$$\beta_3 - \beta_1 = \beta_2 - \beta_4 \quad \text{and} \quad \beta_1\beta_3 = \beta_2\beta_4. \tag{5.13}$$

Hence

$$\beta_3 = \beta_2 \text{ and } \beta_4 = \beta_1, \qquad \text{or} \qquad \beta_2 = -\beta_1, \ \beta_4 = -\beta_3 \text{ and } \beta_3 \neq -\beta_1.$$

In the first case we obtain for M the first family in item 3). In the second case equality (4.1) with $a=c=e=g=h=i=j=k=l=p$ and $b=d=f=q$, gives

$$\sum_{v,y,z\in[p,q]} \lambda_{p|q|p|q}^{p|z|p|y}\lambda_{p|y|p|q}^{p|v|p|p}\lambda_{p|z|p|v}^{p|p|p|p} = \sum_{v,y,z\in[p,q]} \lambda_{p|q|p|q}^{p|v|p|z}\lambda_{p|q|p|v}^{p|p|p|y}\lambda_{p|y|p|z}^{p|p|p|p}.$$

Combining this with the fact that $A=\alpha_1\alpha_3$, $B_1=\alpha_2\beta_4$, $B_2=\alpha_1\beta_3$, $\alpha_i=1$ for all i, $\beta_2=-\beta_1$ and $\beta_4=-\beta_3$, we obtain

$$\Gamma_1\beta_1-\beta_3^2\beta_1-\beta_3\beta_1^2+\beta_3\Gamma_1=-\Gamma_1\beta_1+\beta_3^2\beta_1+\beta_3\beta_1^2-\beta_3\Gamma_1,$$

which can be written as

$$2(\beta_1+\beta_3)(\beta_1\beta_3-\Gamma_1)=0.$$

Since $\beta_3\neq-\beta_1$, we conclude that $\Gamma_1=\beta_1\beta_3$ and we obtain for M the second family in item 3).

Before considering the remaining cases we obtain some equalities which are valid under the assumption that $C\neq 0$. By equalities (5.2) and (5.6), we have

$$-\beta_2+\alpha_3\beta_1-\alpha_2\beta_4+\beta_3=-\frac{\alpha_2-1}{C}+\alpha_3\frac{\alpha_1-1}{C}-\alpha_2\frac{\alpha_4-1}{C}+\frac{\alpha_3-1}{C}=0. \quad (5.14)$$

So, by equality (5.9) and (5.2),

$$\begin{aligned}\Gamma_4C\beta_4(\alpha_4+1)&=\beta_4(\alpha_4\beta_2+\alpha_3\beta_1+\alpha_2\alpha_4\beta_4-\beta_4-\alpha_4\beta_4-\alpha_4\beta_3)\\&=\beta_4((\alpha_4+1)\beta_2+(\alpha_4+1)\alpha_2\beta_4-(\alpha_4+1)\beta_4-(\alpha_4+1)\beta_3)\\&=\beta_4(\alpha_4+1)(\beta_2+\alpha_2\beta_4-\beta_4-\beta_3)\\&=\beta_4(\alpha_4+1)(\alpha_3\beta_1-\beta_4),\end{aligned}$$

which implies

$$\beta_4(\alpha_4+1)(\alpha_3\beta_1-\beta_4-\Gamma_4C)=0. \quad (5.15)$$

Arguing in a similar way we can prove that

$$-\beta_3+\alpha_2\beta_4-\alpha_3\beta_1+\beta_2=0 \quad\text{and}\quad \beta_1(\alpha_1+1)(\alpha_2\beta_4-\beta_1-\Gamma_1C)=0. \quad (5.16)$$

Moreover we also have

$$\alpha_2\beta_4-\beta_1=\frac{\alpha_2\alpha_4-\alpha_2-\alpha_1+1}{C}=\frac{\alpha_1\alpha_3-\alpha_1-\alpha_2+1}{C}=\alpha_1\beta_3-\beta_2. \quad (5.17)$$

4) and 5) Since $\beta_1=0$, equality (5.6) implies $\alpha_1=1$. Thus equality (5.8) reduces to $-\Gamma_1=\alpha_2(-\Gamma_1+\beta_2\beta_4)$ and by equality (5.14) we have $\beta_3=\beta_2+\alpha_2\beta_4$. Using the first equality and again (5.6), we conclude that

$$\beta_2(C\Gamma_1-\alpha_2\beta_4)=\beta_2C\Gamma_1+\Gamma_1-\alpha_2\Gamma_1=0.$$

When $\Gamma_1C=\alpha_2\beta_4$, we obtain for M the family in item 4). Otherwise $\beta_2=0$, which by (5.2) and (5.6) implies $\alpha_2=1$, $\alpha_3=\alpha_4$ and $\beta_3=\beta_4$ (so $\beta_4\neq 0$ because at least one of the β_i's is non-zero). By equalities (5.4) and (5.5), we have $\Gamma_2=\Gamma_3=0$ and $\Gamma_4=-\Gamma_1$. Thus, from (5.15) we obtain

$$\beta_4(\alpha_4+1)(\Gamma_1C-\beta_4)=0.$$

Since $\Gamma_1C-\beta_4\neq 0$ and $\beta_4\neq 0$, we conclude that $\alpha_4=-1$. Thus, by (5.6) we have $\beta_4C=\alpha_4-1=-2$, and by (5.2) and (5.3) we obtain for M the family in item 5).

6) and 7) Since $\alpha_1=-1$, by equality (5.6) we have $\beta_1C=-2$ (consequently $C\neq 0$, $\beta_1\neq 0$ and $\beta_i\neq\beta_1/2$ for all i), and by equality (5.2) we have $\alpha_3=-\alpha_2\alpha_4$. Thus, again by equality (5.6),

$$\beta_2\beta_4-\beta_1\beta_3=\frac{1}{C^2}((\alpha_2-1)(\alpha_4-1)-2(\alpha_2\alpha_4+1))=-\frac{1}{C^2}(\alpha_2+1)(\alpha_4+1).$$

Therefore equality (5.8) reads

$$0 = (\alpha_2 + 1)\left(\beta_1\beta_2 - \Gamma_1 + \frac{1}{C^2}\alpha_2(\alpha_4 + 1)\right). \tag{5.18}$$

Moreover, again by equality (5.6),

$$C^2\beta_1\beta_2 = -2C\beta_2 = -2(\alpha_2 - 1) = -2\alpha_2 + 2 \quad \text{and} \quad \alpha_4 + 1 = C\beta_4 + 2,$$

and so

$$C^2\beta_1\beta_2 + \alpha_2(\alpha_4 + 1) = -2\alpha_2 + 2 + \alpha_2(C\beta_4 + 2) = \alpha_2 C\beta_4 + 2,$$

Hence, equality (5.18) implies that

$$0 = (\alpha_2 + 1)(2 + \alpha_2\beta_4 C - C^2\Gamma_1). \tag{5.19}$$

If $\alpha_2\beta_4 C - C^2\Gamma_1 = -2$, then using that $C = -2/\beta_1$ and equality (5.6) we obtain

$$\Gamma_1 = \frac{2}{C^2} + \frac{\alpha_2 - 1}{C}\beta_4 + \frac{\beta_4}{C} = \frac{1}{2}(\beta_1^2 - \beta_1\beta_4 + 2\beta_2\beta_4),$$

while using that $C = -2/\beta_1$ and equalities (5.16) and (5.6), we obtain

$$\beta_3 = \alpha_2\beta_4 - \alpha_3\beta_1 + \beta_2 = \left(1 - \frac{2}{\beta_1}\beta_2\right)\beta_4 - \left(1 - \frac{2}{\beta_1}\beta_3\right)\beta_1 + \beta_2,$$

which implies

$$\beta_3 = \frac{2\beta_2\beta_4}{\beta_1} + \beta_1 - \beta_2 - \beta_4.$$

So, we obtain for M the family in item 6). On the other hand, if $\alpha_2\beta_4 C - C^2\Gamma_1 \neq -2$, then (5.19) implies $\alpha_2 = \alpha_1 = -1$, and so, by equality (5.2) we have $\alpha_3 = \alpha_4$. Consequently, equality (5.6) implies that $\beta_2 = \beta_1$ and $\beta_4 = \beta_3$. Replacing these values in (5.9) yields

$$\Gamma_4(\alpha_3^2 - 1) = \beta_3(2\alpha_3\beta_1 - 3\alpha_3\beta_3 - \beta_3).$$

Combining this with the fact that, by equalities (5.6), (5.5) and (5.4),

$$\alpha_3 = 1 + C\beta_3 = 1 - 2\frac{\beta_3}{\beta_1} \quad \text{and} \quad \Gamma_4 = -\Gamma_1 + \beta_2\beta_4 + \beta_1\beta_3 = -\Gamma_1 + 2\beta_1\beta_3,$$

we obtain

$$-2\Gamma_4\frac{\beta_3}{\beta_1}\left(2 - 2\frac{\beta_3}{\beta_1}\right) = \beta_3\left(\left(1 - 2\frac{\beta_3}{\beta_1}\right)(2\beta_1 - 3\beta_3) - \beta_3\right),$$

which implies

$$(\beta_1 - \beta_3)\beta_3(\beta_1^2 + \beta_1\beta_3 - 2\Gamma_1) = 0.$$

Since

$$\beta_1^2 + \beta_1\beta_3 - 2\Gamma_1 = \frac{\beta_1^2}{2}(\alpha_2\beta_4 C - C^2\Gamma_1 + 2) \neq 0,$$

we conclude that either $\beta_3 = \beta_1$ or $\beta_3 = 0$. In the first case we obtain the first family in item 7) and, in the second case, we obtain the second family in item 7).

8) Since $\beta_1 \neq 0$ and $\alpha_1 + 1 \neq 0$, from equalities (5.16) and (5.17), it follows that $C\Gamma_1 = \alpha_1\beta_3 - \beta_2$. Moreover, by equality (5.6) we know that $\alpha_i = \beta_i C + 1$ for all i, and by equalities (5.16) and (5.6) we have

$$(1 + \beta_2 C)\beta_4 = \beta_3 - \beta_2 + (\alpha_3 - 1)\beta_1 + \beta_1 = \beta_3 - \beta_2 + C\beta_3\beta_1 + \beta_1.$$

So, we obtain for M the family in item 8). □

Corollary 5.5. *Let X be the poset $(\{p,q\},\leq)$, where $p<q$, let D be the incidence coalgebra of X and let $r\colon D\otimes D\longrightarrow D\otimes D$ be a map. If (D,r) is a non-degenerate braided set, then $r_|$ is the flip and the matrix M associated with r via* (5.1) *belongs to one of the families in the previous theorem. On the other hand each member M of the families yields a solution of the Yang-Baxter equation.*

Proof. The first assertion follows immediately from Corollary 2.5, the second one is a corollary of Theorem 5.4, and the third one follows by a direct computation, that can be done with the aid of a Computer Algebra System (set $M_1 := \mathrm{id}_D\otimes M$ and $M_2 := M\otimes \mathrm{id}_D$, and verify that $M_1M_2M_1 - M_2M_1M_2 = 0$). □

6 A case of the configuration $o\prec q\succ p$

Let $(X,\leq)$ and D be as in Section 1, let (D,r) be a non-degenerate braided set and let $o,p,q\in X$ be such that $o\prec q\succ p$. Let ϕ be the permutation of $\{o,p,q\}$ that interchanges o and p. In this section we determine all the possibilities for the coefficients $\lambda^{a_3|b_3|a_4|b_4}_{a_1|b_1|a_2|b_2}$, with $a_i,b_i\in\{o,p,q\}$ and $a_i\leq b_i$, under the assumptions that the characteristic of K is different than 2 and that ${}^{a}(c)=(c)^{b}=\phi(c)$ for all $a,b,c\in\{o,p,q\}$. Let $f(o,o):=0$, $f(o,q):=1$, $f(q,q):=2$, $f(p,q):=3$ and $f(p,p):=4$. We can codify the 625 coefficients $\lambda^{a_3|b_3|a_4|b_4}_{a_1|b_1|a_2|b_2}$ in a 25×25 matrix M, setting

$$M_{i,j} := \lambda^{a_3|b_3|a_4|b_4}_{a_1|b_1|a_2|b_2} \qquad \begin{array}{l}\text{if } i := 5f(a_3,b_3)+f(a_4,b_4)+1\\ \text{and } j := 5f(a_1,b_1)+f(a_2,b_2)+1.\end{array} \tag{6.1}$$

We begin by showing that M only depends on the entries

$$\begin{aligned}
&\alpha_1 := \lambda^{p|q|p|p}_{o|o|o|q}, \quad \alpha_4 := \lambda^{p|q|q|q}_{q|q|o|q}, \quad \alpha_6 := \lambda^{p|q|o|o}_{p|p|o|q},\\
&\beta_1 := \lambda^{p|p|p|p}_{o|o|o|q}, \quad \beta_2 := \lambda^{p|p|p|p}_{o|q|o|o}, \quad \beta_3 := \lambda^{q|q|p|p}_{o|q|q|q}, \quad \beta_4 := \lambda^{p|p|q|q}_{q|q|o|q},\\
&\beta_5 := \lambda^{o|o|p|p}_{o|q|p|p}, \quad \beta_6 := \lambda^{p|p|o|o}_{p|p|o|q}, \quad \beta_7 := \lambda^{o|p|p|p}_{o|o|p|q}, \quad \beta_8 := \lambda^{p|p|o|o}_{p|q|p|p},\\
&\beta_9 := \lambda^{o|o|q|q}_{q|q|p|q}, \quad \beta_{10} := \lambda^{q|q|o|o}_{p|q|q|q}, \quad \beta_{11} := \lambda^{o|o|o|o}_{p|q|p|p}, \quad \beta_{12} := \lambda^{o|o|o|o}_{p|p|p|q},\\
&\Gamma_1 := \lambda^{q|q|q|q}_{o|q|o|q}, \quad \Gamma_7 := \lambda^{q|q|q|q}_{o|q|p|q}, \quad \Gamma_{10} := \lambda^{q|q|q|q}_{p|q|o|q}, \quad \Gamma_{16} := \lambda^{q|q|q|q}_{p|q|p|q},
\end{aligned}$$

and two parameters $C,C_1\in K^\times$. For this we first note that by Proposition 2.8, the matrix M has the shape showed in Figure 6.1, where Γ_1, Γ_7, Γ_{10}, Γ_{16}, α_1, α_4, α_6, $\beta_1,\dots,\beta_{12}$ are as above, and

$$\begin{aligned}
&\alpha_2 := \lambda^{p|p|p|q}_{o|q|o|o}, \quad \alpha_3 := \lambda^{q|q|p|q}_{o|q|q|q}, \quad \alpha_5 := \lambda^{o|o|p|q}_{o|q|p|p},\\
&\alpha_7 := \lambda^{o|q|p|p}_{o|o|p|q}, \quad \alpha_8 := \lambda^{p|p|o|q}_{p|q|o|o}, \quad \alpha_9 := \lambda^{o|q|q|q}_{q|q|p|q},\\
&\alpha_{10} := \lambda^{q|q|o|q}_{p|q|q|q}, \quad \alpha_{11} := \lambda^{o|o|o|q}_{p|q|p|p}, \quad \alpha_{12} := \lambda^{o|q|o|o}_{p|p|p|q},\\
&A_1 := \lambda^{p|q|p|q}_{o|q|o|q}, \quad A_2 := \lambda^{o|q|p|q}_{o|q|p|q}, \quad A_3 := \lambda^{p|q|o|q}_{p|q|o|q}, \quad A_4 := \lambda^{o|q|o|q}_{p|q|p|q},\\
&B_1 := \lambda^{q|q|p|q}_{o|q|o|q}, \quad B_2 := \lambda^{p|q|q|q}_{o|q|o|q}, \quad B_3 := \lambda^{p|q|p|p}_{o|q|o|q}, \quad B_4 := \lambda^{p|p|p|q}_{o|q|o|q},\\
&B_5 := \lambda^{o|o|p|q}_{o|q|p|q}, \quad B_6 := \lambda^{o|q|q|q}_{o|q|p|q}, \quad B_7 := \lambda^{o|q|p|p}_{o|q|p|q}, \quad B_8 := \lambda^{q|q|p|q}_{o|q|p|q},\\
&B_9 := \lambda^{q|q|o|q}_{p|q|o|q}, \quad B_{10} := \lambda^{p|q|o|o}_{p|q|o|q}, \quad B_{11} := \lambda^{p|q|q|q}_{p|q|o|q}, \quad B_{12} := \lambda^{p|p|o|q}_{p|q|o|q},\\
&B_{13} := \lambda^{o|o|o|q}_{p|q|p|q}, \quad B_{14} := \lambda^{o|q|o|o}_{p|q|p|q}, \quad B_{15} := \lambda^{o|q|q|q}_{p|q|p|q}, \quad B_{16} := \lambda^{q|q|o|q}_{p|q|p|q},\\
&\Gamma_2 := \lambda^{q|q|p|p}_{o|q|o|q}, \quad \Gamma_3 := \lambda^{p|p|q|q}_{o|q|o|q}, \quad \Gamma_4 := \lambda^{p|p|p|p}_{o|q|o|q}, \quad \Gamma_5 := \lambda^{o|o|q|q}_{o|q|p|q},
\end{aligned}$$

$$\Gamma_6 := \lambda^{o|o|p|p}_{o|q|p|q}, \quad \Gamma_8 := \lambda^{q|q|p|p}_{o|q|p|q}, \quad \Gamma_9 := \lambda^{q|q|o|o}_{p|q|o|q}, \quad \Gamma_{11} := \lambda^{p|p|o|o}_{p|q|o|q},$$
$$\Gamma_{12} := \lambda^{p|p|q|q}_{p|q|o|q}, \quad \Gamma_{13} := \lambda^{o|o|o|o}_{p|q|p|q}, \quad \Gamma_{14} := \lambda^{o|o|q|q}_{p|q|p|q}, \quad \Gamma_{15} := \lambda^{q|q|o|o}_{p|q|p|q}.$$

A direct computation using item 1) of Proposition 4.6 proves that there exists $C \in K^\times$ such that

$$\alpha_k = \frac{\alpha_{13-k}}{C} \qquad \text{for } k = 7, \dots, 12. \tag{6.2}$$

For example, since $\alpha_{12} = \alpha_l(p)(p,q) = \alpha_l^{(1)}(o)(o,q)$, we have

$$\frac{\alpha_1}{\alpha_{12}} = \frac{\alpha_l(o)(o,q)}{\alpha_l(p)(p,q)} = \frac{\alpha_l(o)(o,q)}{\alpha_l^{(1)}(o)(o,q)} = C(o,q) =: C.$$

Moreover, by Proposition 2.10 and equality (1.5) we know that

$$\begin{aligned} &A_1 = \alpha_1\alpha_3 = \alpha_2\alpha_4, && A_2 = \alpha_3\alpha_7 = \alpha_5\alpha_9,\\ &A_3 = \alpha_4\alpha_8 = \alpha_6\alpha_{10}, && A_4 = \alpha_9\alpha_{11} = \alpha_{10}\alpha_{12}, \end{aligned} \tag{6.3}$$

$$\begin{aligned} &B_1 = -\alpha_3\beta_1, && B_2 = -\alpha_4\beta_2, && B_3 = \alpha_1\beta_3, && B_4 = \alpha_2\beta_4,\\ &B_5 = \alpha_5\beta_9, && B_6 = -\alpha_9\beta_5, && B_7 = \alpha_7\beta_3, && B_8 = -\alpha_3\beta_7,\\ &B_9 = -\alpha_{10}\beta_6, && B_{10} = \alpha_6\beta_{10}, && B_{11} = -\alpha_4\beta_8, && B_{12} = \alpha_8\beta_4,\\ &B_{13} = \alpha_{11}\beta_9, && B_{14} = \alpha_{12}\beta_{10}, && B_{15} = -\alpha_9\beta_{11}, && B_{16} = -\alpha_{10}\beta_{12} \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} &\Gamma_2 = -\beta_1\beta_3, && \Gamma_3 = -\beta_2\beta_4, && \Gamma_5 = -\beta_5\beta_9, && \Gamma_8 = -\beta_7\beta_3,\\ &\Gamma_9 = -\beta_6\beta_{10}, && \Gamma_{12} = -\beta_8\beta_4, && \Gamma_{14} = -\beta_{11}\beta_9, && \Gamma_{15} = -\beta_{10}\beta_{12}, \end{aligned} \tag{6.5}$$

and by equality (1.5) and Proposition 2.8, we know that

$$\begin{aligned} &\Gamma_4 = -(\Gamma_1 + \Gamma_2 + \Gamma_3), && \Gamma_6 = -(\Gamma_5 + \Gamma_7 + \Gamma_8),\\ &\Gamma_{11} = -(\Gamma_9 + \Gamma_{10} + \Gamma_{12}), && \Gamma_{13} = -(\Gamma_{14} + \Gamma_{15} + \Gamma_{16}). \end{aligned} \tag{6.6}$$

Equalities (6.2)–(6.6) imply that Γ_1, Γ_7, Γ_{10}, Γ_{16}, $\alpha_1,\dots,\alpha_6$, $\beta_1,\dots,\beta_{12}$ and C determine M. A direct computation using equalities (6.3) and (6.4) proves that there exists $C_1 \in K^\times$ such that

$$\alpha_2 = C_1\alpha_1, \quad \alpha_3 = C_1\alpha_4 \quad \text{and} \quad \alpha_5 = C_1\alpha_6. \tag{6.7}$$

So, M only depends of $\Gamma_1, \Gamma_7, \Gamma_{10}, \Gamma_{16}, \alpha_1, \alpha_4, \alpha_6, \beta_1, \dots, \beta_{12}$, C and C_1, as desired.

In the sequel we will provide without proofs analogous results to Theorem 5.4 and Corollary 5.5, for the configuration that we are considering. Similar arguments as in the proof of Theorem 5.4 show that M necessarily belongs to one of the families listed in the Table 6.1. In the fourth case we add a new parameter $C_2 \in K^\times$, which satisfies $C_2^2 = 1/C$ and comes from Proposition 4.6. We also add two new parameters C_3 and C_4 and expressions F_j, $j = 1,\dots,12$ and $G_1,\dots,G_4$ given by

$$\begin{aligned} &F_j := \alpha_j\frac{C_4 - C_3}{2} - \frac{C_3 + C_4}{2C_2} \quad \text{for } j = 1,\dots,6,\\ &F_j := \alpha_{13-j}C_2\frac{C_3 + C_4}{2} + \frac{C_3 - C_4}{2} \quad \text{for } j = 7,\dots,12, \end{aligned}$$

$$G_1 \coloneqq \frac{1}{4C_2^2}\Big(2(C_1+1)C_3C_4\left(1-\alpha_1\alpha_4C_1C_2^2\right)$$
$$-(C_1-1)C_3^2(\alpha_4C_2(\alpha_1C_1C_2+C_1+1)+1)$$
$$-(C_1-1)C_4^2(\alpha_4C_2(C_1(\alpha_1C_2-1)-1)+1)+4C_1\Gamma_{16}\Big)$$
$$G_2 \coloneqq \frac{1}{4C_2}\Big(C_3^2(-(\alpha_4C_2(\alpha_6C_1C_2+C_1+1)+1))+2\alpha_4(C_1-1)C_2C_3C_4$$
$$+C_4^2(\alpha_4C_2(\alpha_6C_1C_2-C_1-1)+1)\Big)$$
$$G_3 \coloneqq \frac{1}{4}\Big(2C_3C_4\left(\alpha_1\alpha_4C_1C_2^2-1\right)+C_3^2(\alpha_4C_2(\alpha_1C_1C_2+C_1+1)+1)$$
$$+C_4^2(\alpha_4C_2(\alpha_1C_1C_2-C_1-1)+1)\Big)$$
$$G_4 \coloneqq C_1\Gamma_{10}+\frac{1}{4\alpha_4C_2}(1-C_1)(\alpha_4+\alpha_6)\left(C_4^2-C_3^2\right)$$

and

$$G_5 \coloneqq \frac{(\alpha_1-\alpha_6)}{8\alpha_1\alpha_4}\Big(\alpha_1(C_3+C_4)^2+\alpha_4(C_3-C_4)(C_3(\alpha_4(C_1+1)C_2+1)$$
$$+C_4(\alpha_4(C_1+1)C_2-1))\Big).$$

We use the parameters C_3 and C_4 merely by convenience. For instance, if $\alpha_jC_1 \neq \pm 1$ for some j, then from the equalities $\beta_j = F_j$ and $\beta_{13-j} = F_{13-j}$, we obtain that

$$C_3 = \frac{\beta_jC_2-\beta_{13-j}}{\alpha_jC_2+1} \quad \text{and} \quad C_4 = \frac{\beta_jC_2+\beta_{13-j}}{-\alpha_jC_2+1},$$

which implies that C_3 and C_4 can be replaced by β_j and β_{13-j}.

Remark 6.1. Let X be the poset $(\{o,p,q\}, o \prec q \succ p)$, let D be the incidence coalgebra of X and let

$$r\colon D\otimes D \longrightarrow D\otimes D$$

be a map. By the same argument as in the proof of Corollary 5.5, if (D,r) is a non-degenerate braided set such that $r_|$ is not the flip, then the matrix M associated with r via (6.1) belongs to one of the families in Table 6.1. On the other hand each member M of the families yields a solution of the Yang-Baxter equation.

Table 6.1: Families for M

#	Dependent values in each family	Dependent values in subfamilies	Parameters
1.	$\beta_1=\cdots=\beta_{12}=0$ $\Gamma_1=\Gamma_7=\Gamma_{10}=\Gamma_{16}=0$		$\alpha_1,\alpha_4,\alpha_6\in K^\times$ $C, C_1\in K^\times$
2.	$\beta_1=\cdots=\beta_{12}=0$ $\Gamma_1=\alpha_1\alpha_6\Gamma_{16}$ $\Gamma_{10}=C_1\Gamma_7$	$\Gamma_{16}=0$	$C,\Gamma_7\in K^\times$ $C_1\in\{\pm1\}$ $\alpha_1,\alpha_4,\alpha_6\in K^\times$ such that $\alpha_1^2=\alpha_4^2=\alpha_6^2=C_1C$
		$\alpha_6=\alpha_1$	$C,\Gamma_{16}\in K^\times$ $\Gamma_7\in K$ $C_1\in\{\pm1\}$ $\alpha_1,\alpha_4\in K^\times$ such that

#	Dependent values in each family	Dependent values in subfamilies	Parameters
			$\alpha_1^2=\alpha_4^2=C_1C$
3.	$\alpha_4=\alpha_6=\alpha_1$ $C=\alpha_1^2$ $C_1=1$	$\beta_2=\dots=\beta_6=\beta_1$ $\beta_7=\dots=\beta_{11}=\beta_{12}$ $\Gamma_7=\Gamma_{10}$ $\Gamma_{16}=\Gamma_1\beta_{12}^2/\beta_1^2$	$\Gamma_1,\Gamma_{10}\in K$ $\beta_1,\beta_{12}\in K^\times$ $\alpha_1\in\{\pm\beta_1/\beta_{12}\}$
		$\alpha_1=\frac{\beta_2-\beta_1}{\beta_{11}-\beta_{12}}$ $\beta_3=\beta_5=\beta_2$ $\beta_4=\beta_6=\beta_1$ $\beta_7=\beta_9=\beta_{12}$ $\beta_8=\beta_{10}=\beta_{11}$ $\Gamma_7=\beta_{12}\beta_2-\beta_{11}\beta_1+\Gamma_{10}$ $\Gamma_{16}=\beta_{11}\beta_{12}+\frac{\Gamma_1-\beta_1\beta_2}{C}$	$\Gamma_1,\Gamma_{10}\in K$ $\beta_1,\beta_2\in K,\ \beta_1\neq\beta_2$ $\beta_{11},\beta_{12}\in K,\ \beta_{11}\neq\beta_{12}$
		$\alpha_1=-\frac{\beta_1}{\beta_{11}}$ $\beta_3=\beta_2$ $\beta_4=\beta_1$ $\beta_5=\beta_1+\beta_2-\beta_6$ $\beta_7=\frac{\beta_{11}}{\beta_1}\beta_5$ $\beta_{12}=\frac{\beta_{11}}{\beta_1}\beta_2$ $\beta_8=\beta_{11}+\beta_{12}-\beta_7$ $\beta_9=\beta_{12}$ $\beta_{10}=\beta_{11}$ $\Gamma_1=\beta_1\beta_2$ $\Gamma_7=\beta_5\beta_{12}$ $\Gamma_{10}=\beta_6\beta_{11}$ $\Gamma_{16}=\beta_{11}\beta_{12}$	$\beta_1,\beta_{11}\in K^\times$ $\beta_2,\beta_6\in K$ $\beta_6\neq\beta_1$
4.	For $j=1,\dots,12$ $\beta_j=F_j$	$\Gamma_7=G_2$ $\Gamma_{10}=\Gamma_7+\alpha_4C_3C_4(1-C_1)$ $\Gamma_{16}=G_3$	$\alpha_1,\alpha_4,\alpha_6\in K^\times$ $C_1,C_2\in K^\times$ $C_3,C_4\in K$
	$\Gamma_1=G_1$	$\Gamma_7=G_4$ $\Gamma_{16}=G_5$	$C_1\in\{\pm1\}$ $C_2\in K^\times$ $C_3,C_4,\Gamma_{10}\in K$ $\alpha_1,\alpha_4,\alpha_6\in K^\times$ such that $\alpha_1^2=\alpha_4^2=\alpha_6^2=C_1/C_2^2$

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\Gamma_{13}&\beta_{11}&0&0&0&\beta_{12}&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&B_{13}&\alpha_{11}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\Gamma_5&-\beta_5&0&0&0&\beta_9&1&0&0&0&\Gamma_{14}&-\beta_{11}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&B_5&\alpha_5&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\beta_7&1&0&0&0&\Gamma_6&\beta_5&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&B_{14}&0&0&0&0&\alpha_{12}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&A_4&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&B_6&0&0&0&0&\alpha_9&0&0&0&0&B_{15}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&A_2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\alpha_7&0&0&0&0&B_7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\Gamma_9&\beta_{10}&\Gamma_{15}&0&0&-\beta_6&1&-\beta_{12}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&B_9&\alpha_{10}&B_{16}&0&0&0&0&0&0\\
0&0&0&0&0&0&\Gamma_1&-\beta_3&\Gamma_7&0&0&-\beta_4&1&-\beta_9&0&0&\Gamma_{10}&-\beta_{10}&\Gamma_{16}&0&0&0&0&0&0\\
0&0&0&0&0&0&B_1&\alpha_3&B_8&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-\beta_1&1&-\beta_7&0&0&\Gamma_2&\beta_3&\Gamma_8&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&B_{10}&0&0&0&0&\alpha_6&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&A_3&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&B_2&0&0&0&0&\alpha_4&0&0&0&0&B_{11}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&A_1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\alpha_1&0&0&0&0&B_3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\beta_8&\Gamma_{11}&0&0&0&1&\beta_6&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\alpha_8&B_{12}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-\beta_2&\Gamma_3&0&0&0&1&\beta_4&0&0&0&-\beta_8&\Gamma_{12}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\alpha_2&B_4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&\beta_1&0&0&0&\beta_2&\Gamma_4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}
$$

FIGURE 6.1. The matrix M

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