# $\lambda$-Aluthge transforms and Schatten ideals.

Jorge Antezana, Pedro Massey and Demetrio Stojanoff *

Depto. de Matemática, FCE-UNLP and IAM-CONICET,
La Plata, Argentina. [1]

*Dedicated to the memory of Jorge Samur.*

**Abstract**

Let $T \in L(\mathcal{H})$, and let $T = U|T| = |T^*|U$ be the polar decomposition of $T$. Then, for every $\lambda \in [0,1]$ the $\lambda$-Aluthge transform is defined by $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$. We show that several properties which are known for the usual Aluthge transform (i.e. the case $\lambda = 1/2$) also hold for $\lambda$-Aluthge transforms with $\lambda \in (0,1)$. Moreover, we get several results which are new, even for the usual Aluthge transform.

**Keywords:** Aluthge transform, Schatten norms, Riesz calculus, polar decomposition.

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[1]E-mail address: antezana@mate.unlp.edu.ar, massey@mate.unlp.edu.ar and demetrio@mate.unlp.edu.ar

# 1 Introduction.

Let $\mathcal{H}$ be a complex Hilbert space, and let $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Given $T \in L(\mathcal{H})$, consider its (left) polar decomposition $T = U|T|$. In order to study the relationship among p-hyponormal operators, Aluthge introduced in [1] the transformation $\Delta_{1/2}(\cdot) : L(\mathcal{H}) \to L(\mathcal{H})$ defined by

$$\Delta_{1/2}(T) = |T|^{1/2}U|T|^{1/2}.$$

Later on, this transformation, now called Aluthge tranform, was also studied in other contexts by several authors, such as Jung, Ko and Pearcy [15] and [16], Foias, Jung, Ko and Pearcy [12], Ando [2], Ando and Yamazaki [3], Yamazaki [23], Okubo [17], Wang [21] and Wu [22] among others.
In this paper, given $\lambda \in [0, 1]$ and $T \in L(\mathcal{H})$, we study the so-called $\lambda$-Aluthge transform of $T$ defined by

$$\Delta_\lambda(T) = |T|^\lambda \, U \, |T|^{1-\lambda}.$$

This notion has already been considered by Okubo in [17]. For $\lambda = 0$, $|T|^\lambda$ will be considered as the orthogonal projection onto the closure of $R\big(|T|\big)$. For $\lambda = 1$, $\Delta_\lambda(T) = |T|U$, which is known as Duggal's transform of $T$([12]), or *hinge* of $T$ ([19]).
The main tool we use to study the $\lambda$-Aluthge transforms is Young's inequality (see, [4], [14] or Section 2). Some results of this paper are devoted to the generalization of well known properties of Aluthge transform to $\lambda$-Aluthge transforms. For $\lambda \in (0, 1)$, we prove that the map $T \mapsto \Delta_\lambda(T)$ is continuous at every closed range operator $T$ (see [15] for the case $\lambda = 1/2$). For every analytic function $f$ defined in an open neighborhood of $\sigma(T)$, we show that

$$\|f(\Delta_\lambda(T))\| \leqslant \|f(\Delta_1(T))\|^\lambda \, \|f(\Delta_0(T))\|^{1-\lambda} \leqslant \|f(T)\|,$$

(see [12] and [17]). When, $\dim \mathcal{H} = n < \infty$, we prove that the limit points of the sequence $\{\Delta_\lambda^m(T)\}$ are normal matrices, from which we deduce Yamazaki's spectral radius formula $\rho(T) = \lim_{n\to\infty} \|\Delta_\lambda^m(T)\|$ (only in the finite dimensional case), where $\rho(T)$ denotes the spectral radius of $T$.
On the other hand, we show several results which are new even for the usual Aluthge transform. Given $1 \leqslant p < \infty$, we prove that the Schatten $p$-norms of the $\lambda$-Aluthge transforms decrease with respect to the Schatten $p$-norms of the original operator. Moreover, if $\|\Delta_\lambda(T)\|_p = \|T\|_p < \infty$ (for any fixed $1 \leqslant p < \infty$), then $T$ must be normal. This was proved for $\lambda = 1/2$ and $p = 2$ in [12]. In this case, we show the following estimation: if $T$ is a Hilbert Schmidt operator, $\lambda \in (0, 1)$, and $\alpha = \min\{\lambda, 1-\lambda\}$, then

$$\alpha^2 \| \, |T| - |T^*| \, \|_2^2 \leqslant \|T\|_2^2 - \|\Delta_\lambda(T)\|_2^2 \, .$$

When $\dim \mathcal{H} = 2$, Ando and Yamazaki proved that the sequence of iterated Aluthge transforms $\{\Delta_{1/2}^m(T)\}$ converges (see [3]). Motivated by their ideas, we show that the sequence $\{\Delta_\lambda^m(T)\}$ converges for every $\lambda \in (0, 1)$ and every $2 \times 2$ matrix $T$. Moreover, if

$\Delta_\lambda^\infty(T) = \lim_{m\to\infty} \Delta_\lambda^m(T)$, we prove that the map $T \mapsto \Delta_\lambda^\infty(T)$ is jointly continuous in both parameters, $\lambda \in (0,1)$ and $T \in \mathcal{M}_2(\mathbb{C})$.
Finally, we study some properties of the Jordan structure of the iterated Aluthge transforms. Given $T \in \mathcal{M}_n(\mathbb{C})$ and $\mu \in \sigma(T)$, let $\mathcal{H}_{\mu,T}$ denote the spectral subspace of $T$ associated to the eigenvalue $\mu$ (see 4.18 for a precise definition). We prove that given two different eigenvalues of $T$, $\gamma$ and $\mu$, the angle between $\mathcal{H}_{\mu,\Delta_\lambda^m(T)}$ and $\mathcal{H}_{\gamma,\Delta_\lambda^m(T)}$ converges to $\pi/2$, for every $\lambda \in (0,1)$. In other words

$$P_{\mathcal{H}_{\mu,\Delta_\lambda^m(T)}} P_{\mathcal{H}_{\gamma,\Delta_\lambda^m(T)}} \xrightarrow[m\to\infty]{} 0 ,$$

where, for any subspace $\mathcal{S} \subseteq \mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$. Concerning the conjecture of the convergence of the sequence $\{\Delta_\lambda^m(T)\}$ for $T \in \mathcal{M}_n(\mathbb{C})$, we show a reduction to the invertible case.
The paper is organized as follows: Section 2 contains preliminary results on Riesz's functional calculus, Schatten ideals, and a list of known inequalities which we use in the paper. Section 3 deals with the properties of $\lambda$-Aluthge transform in the infinite dimensional setting. In section 4 we study the finite dimensional case.
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# 2 Preliminaries.

In this paper $\mathcal{H}$ denotes a complex Hilbert space, $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$, $GL(\mathcal{H})$ the group of all invertible elements of $L(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$ the group of unitary operators, $L(\mathcal{H})^+$ the cone of all positive operators and $L_0(\mathcal{H})$ the ideal of compact operators. When $\dim \mathcal{H} = n < \infty$ the elements of $L(\mathcal{H})$ are identified with $n \times n$ matrices, and we write $\mathcal{M}_n(\mathbb{C})$ instead of $L(\mathcal{H})$. Given $T \in L(\mathcal{H})$, $R(T)$ denotes the range or image of $T$, $N(T)$ the null space of $T$, $\sigma(T)$ the spectrum of $T$, $\rho(T)$ the spectral radius of $T$, $T^*$ the adjoint of $T$, and $\|T\|$ the usual norm of $T$ (also called spectral norm, we sometimes write $\|T\|_{sp}$); a norm $|||\cdot|||$ in $\mathcal{M}_n(\mathbb{C})$ (or defined in some adequate ideal of compact opeators) is called unitarily invariant if $|||UTV||| = |||T|||$ for unitary $U, V$. If $R(T)$ is closed, $T^\dagger$ denotes the Moore-Penrose pseudoinverse of $T$. Given a closed subspace $\mathcal{S} \subseteq \mathcal{H}$, $P_{\mathcal{S}} \in L(\mathcal{H})$ denotes the orthogonal projection onto $\mathcal{S}$.
Given $T \in L(\mathcal{H})$, $\mathrm{Hol}(\sigma(T))$ denotes the set of all complex analytic functions defined in an open neighborhood of $\sigma(T)$. In this set, we identify two functions if they agree in an open neighborhood of $\sigma(T)$. If $T \in L(\mathcal{H})$ and $f \in \mathrm{Hol}(\sigma(T))$, $f(T)$ indicates the evaluation of $f$ at $T$, by using the Riesz functional calculus. The reader is referred to Brown and Pearcy's book [8] (see also [9]) for general properties of this calculus, and a proof of the following statement.

**Proposition 2.1.** *Given $T_0 \in L(\mathcal{H})$ such that $\sigma(T_0)$ is contained in an open set $U \subseteq \mathbb{C}$, let $\{f_n\}$ be a sequence of locally analytic functions on $U$ converging to a limit $f_0$ uniformly on compact subsets of $U$, and likewise let $\{T_n\}$ be a sequence in $L(\mathcal{H})$, converging to $T_0$ (in norm). Then, $f_n(T_n)$ is defined for all sufficiently large $n$ and $f_n(T_n) \xrightarrow[n\to\infty]{\|\cdot\|} f_0(T_0)$.*

Given $A \in L_0(\mathcal{H})$, $s_k(A)$, $k \in \mathbb{N}$ denote the singular values of $A$, arranged in non-increasing order. If we denote by $tr$ the canonical semifinite trace in $L(\mathcal{H})$ then the Schatten $p$-ideals $(1 \leqslant p < \infty)$ are defined in the following way:

$$L^p(\mathcal{H}) = \{T \in L_0(\mathcal{H}) : \operatorname{tr}\left(|T|^p\right) < \infty\}.$$

Each $L^p(\mathcal{H})$, endowed with the norm

$$\|T\|_p = \Big(\operatorname{tr}\left(|T|^p\right)\Big)^{1/p} = \Big(\sum_{k\in\mathbb{N}} s_k(T)^p\Big)^{1/p} ,$$

is a Banach space. If $p > 1$, then $L^p(\mathcal{H})^* \cong L^q(\mathcal{H})$, where $1/p + 1/q = 1$.
In this rest of this section, we list some inequalities which will be useful in the sequel. We begin with the following two versions of Young's inequality.

**Proposition 2.2 (Argerami-Farenick [4]).** *Let $A \in L^p(\mathcal{H})$ and $B \in L^q(\mathcal{H})$ be positive operators and $1/p + 1/q = 1$. Then, $AB \in L^1(\mathcal{H})$ and*

$$\operatorname{tr}(|AB|) \leqslant \frac{\operatorname{tr}(A^p)}{p} + \frac{\operatorname{tr}(B^q)}{q} .$$

*Moreover, equality holds if and only if $A^p = B^q$.*

**Proposition 2.3 (Hirzallah-Kittaneh [14]).** *Let $A, B \in L(\mathcal{H})^+$, and let $p,\, q > 1$ with $1/p + 1/q = 1$. Suppose that $A^p, B^q \in L^2(\mathcal{H})$. Then $AB \in L^2(\mathcal{H})$, and*

$$\|AB\|_2^2 + \frac{1}{r^2}\, \|A^p - B^q\|_2^2 \leqslant \left\| \frac{A^p}{p} + \frac{B^q}{q} \right\|_2^2 ,$$

*where $r = \max\{p, q\}$.*

Now, we state a version of the well known Corde's inequality [10], for unitarily invariant norms. In the proof we use standard techniques and properties of the $k$th antisymmetric tensor powers $\bigwedge^k A$, $A \in L(\mathcal{H})$ and majorisation, which can be found in B. Simon's book [20] or Bhatia's book [6].

**Proposition 2.4.** *Let $A$ and $B$ be positive compact operators. If $p \geqslant 1$, then*

$$\sum_{i=1}^{k} s_i\left(|AB|^p\right) \leqslant \sum_{i=1}^{k} s_i\left(A^pB^p\right), \quad k \in \mathbb{N} . \tag{1}$$

*Sketch of proof.* Fix $k \in \mathbb{N}$. Since $\| \bigwedge^k A\| = \prod_{i=1}^{k} s_i(A)$, Cordes' inequality

$$\|CD\|^p \leqslant \|C^p D^p\| , \quad C, D \in L(\mathcal{H})^+ ,$$

implies that

$$\begin{aligned} \| \textstyle\bigwedge^k A^pB^p\| &= \left\|(\textstyle\bigwedge^k A)^p(\textstyle\bigwedge^k B)^p\right\| \geqslant \left\| \textstyle\bigwedge^k A \ \textstyle\bigwedge^k B\right\|^p \\ &= \left\| \textstyle\bigwedge^k AB\right\|^p = \left\| \textstyle\bigwedge^k |AB|^p \right\| . \end{aligned}$$

Then, $\prod_{i=1}^{k} s_i\left(|AB|^p\right) \leqslant \prod_{i=1}^{k} s_i\left(A^p B^p\right)$, $k \in \mathbb{N}$, which implies inequality (1) . $\square$

Finally, we include the next inequality, proved by Bhatia and Kittaneh [7]:

**Proposition 2.5.** *Let $A, B \in \mathcal{M}_n(\mathbb{C})^+$, and $r \in [0,1]$. Then*

$$|||A^r - B^r||| \leqslant |||I|||^{1-r}\, |||A-B|||^r$$

*for every unitarily invariant norm $|||\cdot|||$.*

# 3 $\lambda$-Aluthge Transforms.

**Definition 3.1.** Let $T \in L(\mathcal{H})$, and suppose that $T = U|T| = |T^*|U$ is the polar decomposition of $T$. Then, for every $\lambda \in [0,1]$ we define the $\lambda$-Aluthge transform of $T$ in the following way:

$$\Delta_\lambda(T) = |T|^\lambda\, U\, |T|^{1-\lambda}$$

When $\lambda = 0$, $|T|^\lambda$ will be considered as the orthogonal projection onto $\overline{R\big(|T|\big)}$.

**Remark 3.2.** Let $T \in L(\mathcal{H})$ and let $T = W|T|$ be an arbitrary polar decomposition of $T$. It was shown in [17] that $\Delta_\lambda(T) = |T|^\lambda W |T|^{1-\lambda}$ for every $\lambda \in [0,1)$ i.e., the $\lambda$-Aluthge transform does not depend on the partial isometry for $\lambda \in [0,1)$. We shall use this fact repeatedly in the sequel. On the other hand, for $\lambda = 1$, it is necessary to fix the unique partial isometry $U$ such that $T = U|T|$ and $N(U) = N(T)$. For example, if $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $U = T$ and $|T| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, but the unitary matrix $W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ also satisfies $T = W|T|$, while $\Delta_1(T) = |T|U = 0 \neq |T|W = T^*$. $\triangle$

In the next proposition, we describe some properties which follow easily from the definitions.

**Proposition 3.3.** *Let $T \in L(\mathcal{H})$ and $\lambda \in [0,1]$. Then:*

1. *$\Delta_\lambda(VTV^*) = V\Delta_\lambda(T)\,V^*$ for every $V \in \mathcal{U}(\mathcal{H})$.*
2. *$\|\Delta_\lambda(T)\| \leqslant \|T\|$.*
3. *$\sigma(\Delta_\lambda(T)) = \sigma(T)$.*
4. *If $\dim \mathcal{H} < \infty$, then $T$ and $\Delta_\lambda(T)$ have the same characteristic polynomial.*

**Proposition 3.4.** *Let $T \in L(\mathcal{H})$, $\lambda \in [0,1]$ and let $f$ be a function, which is locally analytic in a neighborhood of $\sigma(T)$. If $T = U|T|$ is the polar decomposition of $T$ then,*

1. *$f(T)U = Uf(\Delta_1(T))$.*
2. *$|T|^\lambda f(T) = f(\Delta_\lambda(T))|T|^\lambda$.*

*Sketch of proof.* A simple induction argument proves the statement for $f(t) = t^n$. This can be extended to every polynomial by linearity. This can be applied to show the statement for rational functions (with poles outside $\sigma(T)$ ). Finally, using Runge's theorem (see, for example, Conway's book [9]), the result generalizes to analytic functions. $\square$

In [15], Jung, Ko and Pearcy proved that the Aluthge transformation is continuous at every closed range operator, with respect to the norm topology, for $\lambda = 1/2$. In order to generalize this property for $\lambda \in (0,1)$, we need the following result. Recall that, if $B \in L(\mathcal{H})$ has closed range, there exists a unique pseudo-inverse $B^\dagger$ of $B$ such that $BB^\dagger$ and $B^\dagger B$ are selfadjoint projections. $B^\dagger$ is called the Moore-Penrose pseudo-inverse of $B$ (see, for example, [5]).

**Lemma 3.5.** *Let $B \in L(\mathcal{H})$, selfadjoint with closed range, and let $\{B_n\}$ be a sequence of closed range selfadjoint operators such that $B_n \xrightarrow[n\to\infty]{} B$ in norm. If $P_{R(B_n)} \xrightarrow[n\to\infty]{} P_{R(B)}$ in norm, then also $B_n^\dagger \xrightarrow[n\to\infty]{} B^\dagger$ in norm.*

*Proof.* Denote by $P_n = P_{R(B_n)}$ and $P = P_{R(B)}$. If $P_n \xrightarrow[n\to\infty]{} P$ then there exists a sequence $\{U_n\}$ of unitary operators such that $U_n \xrightarrow[n\to\infty]{} 1$ and $U_n^* P U_n = P_n$, $n \in \mathbb{N}$. Indeed, we can take $U_n$ as the unitary part in the polar decomposition of $PP_n + (1-P)(1-P_n)$, which is invertible for large $n$. Note that, if $S_n = U_n B_n U_n^*$, then $S_n \xrightarrow[n\to\infty]{} B$ in norm, $R(S_n) = R(B)$ and $S_n^\dagger = U_n B_n^\dagger U_n^*$, $n \in \mathbb{N}$. Hence, it suffices to prove that $S_n^\dagger \xrightarrow[n\to\infty]{} B^\dagger$. But this is clear by continuity of the map $A \mapsto A^{-1}$ (on the fixed subspace $R(B) = R(S_n)$, $n \in \mathbb{N}$). $\square$

**Theorem 3.6.** *Let $T$ be an operator with closed range. Then, for every $\lambda \in (0,1)$, the $\lambda$-Aluthge transform $\Delta_\lambda(\cdot)$ is continuous at $T$.*

*Proof.* Let $\{T_n\}$ be a sequence of operators such that $\|T_n - T\| \to 0$. For each $n \in \mathbb{N}$, let $T_n = U_n|T_n|$ be a polar decomposition of $T_n$. On the other hand, take $\varepsilon > 0$ such that $\sigma(|T|) \subseteq \{0\} \cup (2\varepsilon, +\infty)$ and suppose, without loss of generality, that $\sigma(|T_n|) \subseteq (-\varepsilon, \varepsilon) \cup (2\varepsilon, +\infty)$ for all $n$. Define, for $n \in \mathbb{N}$,

$$P_n = |T_n| E_{|T_n|}(-\varepsilon, \varepsilon) \quad \text{and} \quad A_n = U_n P_n \tag{2}$$

$$Q_n = |T_n| E_{|T_n|}(2\varepsilon, +\infty) \quad \text{and} \quad B_n = U_n Q_n \ , \tag{3}$$

where $E_{|T_n|}(I)$ denotes the spectral projection of $|T_n|$ corresponding to the interval $I \subseteq \mathbb{R}$. Note that $A_n + B_n = T_n$, and (2) and (3) are polar decompositions of $A_n$ and $B_n$, respectively. Therefore

$$\begin{aligned} \|\Delta_\lambda(T) - \Delta_\lambda(T_n)\| \ &\leqslant \|\Delta_\lambda(A_n)\| + \|P_n^\lambda U_n Q_n^{1-\lambda}\| + \\ &+ \|Q_n^\lambda U_n P_n^{1-\lambda}\| + \|\Delta_\lambda(T) - \Delta_\lambda(B_n)\| \ . \end{aligned}$$

By Proposition 2.1, $P_n = |T_n| E_{|T_n|}(-\varepsilon, \varepsilon) \xrightarrow[n\to\infty]{\|\cdot\|} |T| E_{|T|}(-\varepsilon, \varepsilon) = 0$. Then

$$\|\Delta_\lambda(A_n)\| + \|P_n^\lambda U_n Q_n^{1-\lambda}\| + \|Q_n^\lambda U_n P_n^{1-\lambda}\| \xrightarrow[n\to\infty]{} 0 \ .$$

On the other hand, $|B_n| = Q_n$ which have closed ranges. Since the maps $\chi_{(-\varepsilon,\varepsilon)}$ and $\chi_{(2\varepsilon,+\infty)}$ admit complex analytic extensions to the set $\{z \in \mathbb{C} : \mathrm{Re}(z) \in (-\varepsilon, \varepsilon) \cup (2\varepsilon, +\infty)\}$, we can apply Proposition 2.1, and obtain that

$$P_{R(Q_n)} = E_{|T_n|}(2\varepsilon, +\infty) \xrightarrow[n\to\infty]{\|\cdot\|} E_{|T|}(2\varepsilon, +\infty) = P_{R(|T|)}.$$

Hence, $|B_n| \xrightarrow[n\to\infty]{} |T|$ and $P_{R(|B_n|)} \xrightarrow[n\to\infty]{} P_{R(|T|)}$, both in the norm topology. By Lemma 3.5, we conclude that $|B_n|^\dagger \xrightarrow[n\to\infty]{} |T|^\dagger$ in norm. Therefore

$$\|\Delta_\lambda(T) - \Delta_\lambda(B_n)\| = \|\,|T|^\lambda T(|T|^\dagger)^\lambda - |B_n|^\lambda B_n(B_n^\dagger)^\lambda\| \xrightarrow[n\to\infty]{} 0\ ,$$

which completes the proof. □

**Remark 3.7.** Theorem 3.6 fails for $\lambda = 0$ and $\lambda = 1$, even in the finite dimensional case. Indeed, take $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T_n = \begin{pmatrix} 0 & 1 \\ 1/n & 0 \end{pmatrix}$, $n \in \mathbb{N}$. It is easy to check that $\Delta_0(T_n) = T_n$ and $\Delta_1(T_n) = T_n^*$, which do not converge to $0 = \Delta_0(T) = \Delta_1(T)$. Compare with Remark 3.2.

## Schatten norms and ideals

In this subsection we characterize those operators in $L^p(\mathcal{H})$ which satisfy $\|\Delta_\lambda(T)\|_p = \|T\|_p$. Naturally, the equality holds if $T$ is normal, because $T = \Delta_\lambda(T)$. It was proved in [16] that, for the Frobenius norm and for $\lambda = 1/2$, the equality holds if and only if $T$ is normal. In the following proposition we estimate from below the difference between the Frobenius norms of $T$ and $\Delta_\lambda(T)$.

**Proposition 3.8.** *Let $T \in L^2(\mathcal{H})$ and $\lambda \in (0,1)$. If $\alpha = \min\{\lambda, 1-\lambda\}$, then*

$$\alpha^2\|\,|T| - |T^*|\,\|_2^2 \leqslant \|T\|_2^2 - \|\Delta_\lambda(T)\|_2^2\ . \tag{4}$$

*Proof.* Note that, if $T = U|T|$ is the polar decomposition of $T$, then $|T^*|^r = U|T|^rU^*$, for every $r > 0$. Then

$$\begin{aligned}\|\Delta_\lambda(T)\|_2^2 &= \operatorname{tr}\left(\Delta_\lambda(T)\,\Delta_\lambda(T)^*\right) = \operatorname{tr}\left(|T|^\lambda U|T|^{2(1-\lambda)}U^*|T|^\lambda\right) \\ &= \operatorname{tr}\left(|T|^\lambda|T^*|^{2(1-\lambda)}|T|^\lambda\right) = \|\,|T|^\lambda|T^*|^{(1-\lambda)}\|_2^2\ .\end{aligned}$$

Using Hirzallah-Kittaneh's inequality (Proposition 2.3) with $A = |T|^\lambda$, $B = |T^*|^{1-\lambda}$, $p = \lambda^{-1}$, $q = (1-\lambda)^{-1}$ and $\alpha = \min\{\lambda, 1-\lambda\} = \max\{\lambda^{-1}, (1-\lambda)^{-1}\}^{-1}$, we get

$$\|\Delta_\lambda(T)\|_2^2 + \alpha^2\,\|\,|T| - |T^*|\,\|_2^2 \leqslant \|\lambda|T| + (1-\lambda)|T^*|\,\|_2^2 \leqslant \|T\|_2^2\ ,$$

where the last inequality follows from the triangle inequality. □

Now, we prove that equality in other Schatten norms also implies that $T$ is normal.

**Theorem 3.9.** *Let $\lambda \in (0,1)$, $1 \leq p < \infty$ and $T \in L^p(\mathcal{H})$. Then, $\Delta_\lambda(T) \in L^p(\mathcal{H})$ and*

$$\|\Delta_\lambda(T)\|_p \leq \|T\|_p.$$

*Moreover, the equality holds if and only if $T$ is normal.*

In order to prove this result, we need the following lemma.

**Lemma 3.10.** *Let $A, B \in L(\mathcal{H})$ and let $B = U|B|$ be the polar decomposition of $B$. Then, for every $p > 0$,*

$$|AB^*|^p = U \left| |A|\, |B| \right|^p U^*$$

*Proof.* Let $P = \left| |A|\, |B| \right|^2$. Then, for every continuous function $f$ defined on $[0, +\infty)$ such that $f(0) = 0$,

$$f(UPU^*) = Uf(P)U^*. \tag{5}$$

In fact, since $R(P) \subseteq R(|B|)$, and $U^*U$ is the orthogonal projection onto $\overline{R(|B|)}$, then $(UPU^*)^n = UP^nU^*$, for every $n \geq 1$. Therefore, by linearity, formula (5) holds for every polynomial $f$ such that $f(0) = 0$. On the other hand, given a continuous function $f$ defined in $[0, +\infty)$ such that $f(0) = 0$, there exists a sequence $\{p_n\}_{n \in \mathbb{N}}$ of polynomials such that $p_n(0) = 0$, $n \in \mathbb{N}$, and $p_n \xrightarrow[n\to\infty]{} f$ uniformly on $\sigma(P) \cup \{0\} = \sigma(UPU^*) \cup \{0\}$. So, standard limit arguments prove formula (5). Now, the result follows from the equality

$$|AB^*|^2 = BA^*AB^* = U|B||A|^2|B|U^* = U \left| |A||B| \right|^2 U^* ,$$

by applying the function $f(x) = x^{p/2}$ to both sides. □

**Proof of Theorem 3.9:** Let $T = U|T|$ be the polar decomposition of $T$. Fix $1 \leq p < \infty$. Then, using Lemma 3.10 with $A = |T|^\lambda$ and $B^* = U|T|^{1-\lambda}$, we get

$$\operatorname{tr} |\Delta_\lambda(T)|^p = \operatorname{tr} \left| |T|^\lambda\, |T^*|^{1-\lambda} \right|^p .$$

Using Proposition 2.4 with $A = |T|^\lambda$ and $B = |T^*|^{1-\lambda}$, we get

$$\operatorname{tr} \left| |T|^\lambda\, |T^*|^{1-\lambda} \right|^p \leqslant \operatorname{tr} \left|\, |T|^{p\lambda}\, |T^*|^{p(1-\lambda)} \right| .$$

Then, by Proposition 2.2, for the conjugate numbers $\lambda^{-1}$ and $(1-\lambda)^{-1}$,

$$\begin{aligned} \operatorname{tr} |\Delta_\lambda(T)|^p &\leqslant \operatorname{tr} \left|\, |T|^{p\lambda}\, |T^*|^{p(1-\lambda)} \right| \\ &\leqslant \lambda \operatorname{tr} |T|^p + (1-\lambda) \operatorname{tr} |T^*|^p = \operatorname{tr} |T|^p . \end{aligned}$$

Therefore, if $\|\Delta_\lambda(T)\|_p = \|T\|_p$, then equality holds in Young's inequality, and by Proposition 2.2, we conclude that $|T|^p = |T^*|^p$. Hence $T$ is normal. □

**Remark 3.11.** Theorem 3.9 fails for $\lambda = 1$. Take, for example, $T \in L^2(\mathcal{H})$ with polar decomposition $T = U|T|$, with $U \in \mathcal{U}(\mathcal{H})$. In this case, $\|\Delta_1(T)\|_2 = \|T\|_2$. The following example shows that Theorem 3.9 may be false for other unitarily invariant norms. In particular, for the spectral norm.

Let $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then, $\Delta_\lambda(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for every $\lambda \in (0, 1)$, and therefore

$$1 = \|\Delta_\lambda(T)\|_p < \|T\|_p = 2^{1/p} \qquad \text{but} \qquad \|\Delta_\lambda(T)\| = \|T\| = 1.$$

The reader interested in the equality for the spectral norm is referred to [24]. In that work, Yamazaki proves that $\|\Delta_\lambda(T)\| = \|T\|$ if an only if $T$ is normaloid, i.e., if $\rho(T) = \|T\|$. △

**Remark 3.12.** Using standard techniques of alternate tensor powers, it can be proved that given $T \in L_0(\mathcal{H})$ and $\lambda \in [0,1]$, then

$$\prod_{i=1}^{k} s_i\left(\Delta_\lambda\left(T\right)\right) \leqslant \prod_{i=1}^{k} s_i\left(T\right) \quad , \quad k \in \mathbb{N} \ .$$

This inequality says that the singular values of $\Delta_\lambda\left(T\right)$ are log-majorized by the singular values of $T$. Hence, we can deduce that for every unitarily invariant norm $|||\cdot|||$, we have that $|||\Delta_\lambda\left(T\right)||| \leq |||T|||$. △

## Riesz's functional calculus.

An interesting result proved by Foias, Jung, Ko and Pearcy [12] relates the Aluthge transform with completely contractive maps by using Riesz' functional calculus. Following similar ideas, in this subsection we study the relationship between Riesz's functional calculus and $\lambda$-Aluthge transforms. We begin with the following technical lemma.

**Lemma 3.13.** *Let $X \in L(\mathcal{H})$, $A \in GL(\mathcal{H})^+$ and $\lambda \in [0,1]$. Then, given $n \in \mathbb{N}$, and $f_{11}, \dots, f_{nn}$ analytic functions defined in a neighborhood of $\sigma\left(XA\right)$, we have*

$$\left\|\big(f_{ij}(A^\lambda X A^{1-\lambda})\big)_{ij}\ \right\| \leqslant \left\|\big(f_{ij}(AX)\big)_{ij}\ \right\|^{\lambda} \cdot \left\|\big(f_{ij}(XA)\big)_{ij}\ \right\|^{1-\lambda}$$

*Proof.* Let $\Omega_{0,1}$ denote the open subset of the complex plane defined by

$$\Omega_{0,1} = \big\{z \in \mathbb{C}: \quad \mathrm{Re}(z) \in (0,1)\big\}.$$

Given two unitary vectors $x = (x_1, \dots, x_n)$, and $y = (y_1, \dots, y_n)$ belonging to $\mathcal{H}^n$, define $\varphi_{x,y} : \overline{\Omega_{0,1}} \to \mathbb{C}$ in the following way

$$\varphi_{xy}(z) = \Big\langle \big(f_{ij}(A^z X A^{1-z})\big)_{ij}\ x,\ y\Big\rangle .$$

If $I_n$ denotes the identity operator on $\mathbb{C}^n$, then

$$\big(f_{ij}(A^z X A^{1-z})\big)_{ij} = \big(A^z f_{ij}(XA) A^{-z}\big)_{ij} = (A^z \otimes I_n)\big(f_{ij}(XA)\big)_{ij}(A^{-z} \otimes I_n) \ .$$

Hence, it is easy to see that $\varphi_{x,y}$ is analytic in $\Omega_{0,1}$ and continuous in $\overline{\Omega_{0,1}}$. On the other hand, since $A^{it}$ is unitary for every $t \in \mathbb{R}$,

$$\begin{aligned} |\varphi_{x,y}(it)| &= \Big|\Big\langle \big(f_{ij}(A^{it} X A^{1-it})\big)_{ij} x,\ y\Big\rangle\Big| \\ &= \Big|\Big\langle \Big((A^{it} \otimes I_n)\big(f_{ij}(XA)\big)_{ij}(A^{-it} \otimes I_n)\Big)\ x,\ y\Big\rangle\Big| \\ &\leqslant \Big\|\big(f_{ij}(XA)\big)_{ij}\Big\| \ . \end{aligned}$$

Analogously

$$\begin{aligned} |\varphi_{x,y}(1+it)| &= \Big|\Big\langle \big(f_{ij}(A^{1+it} X A^{-it})\big)_{ij} x,\ y\Big\rangle\Big| \\ &= \Big|\Big\langle \Big((A^{it} \otimes I_n)\big(f_{ij}(AX)\big)_{ij}(A^{-it} \otimes I_n)\Big)\ x,\ y\Big\rangle\Big| \\ &\leqslant \Big\|\big(f_{ij}(AX)\big)_{ij}\Big\| \ . \end{aligned}$$

Therefore, by the three lines theorem (see, for example, [18]), if $\lambda = \mathrm{Re}(z)$,

$$\left|\left\langle \big(f_{ij}(A^z X A^{1-z})\big)_{ij} x, y \right\rangle\right| \leqslant \left\| \big(f_{ij}(AX)\big)_{ij} \right\|^{\lambda} \cdot \left\| \big(f_{ij}(XA)\big)_{ij} \right\|^{1-\lambda} .$$

Taking supremum over all $x, y \in \mathcal{H}^n$, we get the desired inequality. □

Lemma 3.13 allows us to give an alternative proof of Jung Ko and Pearcy's result, which also generalizes it for $\lambda \in (0,1)$.

**Proposition 3.14.** *Let $T \in L(\mathcal{H})$, $\lambda \in (0,1)$ and $f \in Hol(\sigma(T))$. Then*

1. $\|f(\Delta_0(T))\| \leqslant \|f(T)\|$ *and* $\|f(\Delta_1(T))\| \leqslant \|f(T)\|$.
2. $\|f(\Delta_\lambda(T))\| \leqslant \|f(\Delta_1(T))\|^{\lambda} \, \|f(\Delta_0(T))\|^{1-\lambda} \leqslant \|f(T)\|$.

*Proof.* The inequality $\|f(\Delta_1(T))\| \leqslant \|f(T)\|$ was proved by Foias, Jung, Ko and Pearcy in [12], using Proposition 3.4. The inequality for $\Delta_0(T)$ can be proved by following exactly the same lines.
In order to prove the inequality of item 2, Let $T = U|T|$ be the polar decomposition of $T$ and $E$ the orthogonal projection onto $\overline{R(|T|)}$. Note that $\big(|T| + n^{-1}\big)^{\lambda} \xrightarrow[n\to\infty]{\|\cdot\|} |T|^{\lambda}$, because the sequence of functions $f_n(x) = \big(x + n^{-1}\big)^{\lambda}$ ($n \in \mathbb{N}$) converges uniformly to $f(x) = x^{\lambda}$ on compact subsets. So, given $f \in \mathrm{Hol}(\sigma(T))$, by Proposition 2.1 we have that

$$f\Big(\big(|T| + n^{-1}\big)^{\lambda} E \, U \big(|T| + n^{-1}\big)^{1-\lambda}\Big) ,$$

$f\Big(E\,U\big(|T|+n^{-1}\big)\Big)$, and $f\Big(\big(|T|+n^{-1}\big)E\,U\Big)$ are defined for all sufficiently large $n$. Moreover,

$$\begin{aligned} f\Big(U\big(|T| + n^{-1}\big)\Big) &\xrightarrow[n\to\infty]{\|\cdot\|} f\big(E\,U|T|\big), \\ f\Big(\big(|T| + n^{-1}\big)E\,U\Big) &\xrightarrow[n\to\infty]{\|\cdot\|} f\big(|T|E\,U\big) = f\big(|T|U\big), \text{ and} \\ f\Big(\big(|T| + n^{-1}\big)^{\lambda} E\,U\big(|T| + n^{-1}\big)^{1-\lambda}\Big) &\xrightarrow[n\to\infty]{\|\cdot\|} f\big(|T|^{\lambda} U |T|^{1-\lambda}\big). \end{aligned}$$

Using Lemma 3.13 and standard limit arguments, we get inequality 2. □

**Remark 3.15.** Using Lemma 3.13, it can be proved that given $n \in \mathbb{N}$, and $f_{11}, \dots, f_{nn} \in \mathrm{Hol}(\sigma(T))$,

$$\left\|\big(f_{ij}(\Delta_\lambda(T))\big)_{ij}\right\| \leqslant \left\|\big(f_{ij}(\Delta_1(T))\big)_{ij}\right\|^{\lambda} \left\|\big(f_{ij}(\Delta_0(T))\big)_{ij}\right\|^{1-\lambda} .$$

It should be mentioned that $\|\big(f_{ij}(\Delta_0(T))\big)_{ij}\| \leq \|\big(f_{ij}(T)\big)_{ij}\|$ . △

For $T \in L(\mathcal{H})$, we denote $W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$, its numerical range. As a corollary of Proposition 3.14, we obtain the next result about numerical ranges.

**Corollary 3.16.** *Let $T \in L(\mathcal{H})$ and $\lambda \in [0,1]$. Then, for every complex analytic function $f$ defined in a neighborhood of $\sigma(T)$,*

$$\overline{\mathrm{W}\left(f(\Delta_\lambda(T))\right)} \subseteq \overline{\mathrm{W}\left(f(T)\right)} \ .$$

*Proof.* Indeed, by Proposition 3.14 (item 1), for every $\mu \in \mathbb{C}$ it holds that $\|f(\Delta_\lambda(T)) - \mu I\| \leqslant \|f(T) - \mu I\|$. So, if $B(r,\zeta) = \{z \in \mathbb{C} : |z-\zeta| \leqslant r\}$, using the well known formula

$$\overline{\mathrm{W}(T)} = \bigcap_{\lambda \in \mathbb{C}} B\big(\|T - \lambda I\|, \lambda\big),$$

we have that

$$\begin{aligned} \overline{\mathrm{W}\left(f(\Delta_\lambda(T))\right)} &= \textstyle\bigcap_{\mu\in\mathbb{C}} B\big(\|f(\Delta_\lambda(T)) - \mu I\|, \lambda\big) \\ &\subseteq \textstyle\bigcap_{\mu\in\mathbb{C}} B\big(\|f(T) - \mu I\|, \lambda\big) = \overline{W\big(f(T)\big)} \ . \end{aligned}$$

□

**Remark 3.17.** The above Corollary, was proved in [12], for $\lambda = 1/2$, using that $\overline{W(T)}$ is the intersection of all half-planes $H$ containing $W(T)$, which are spectral sets for $T$. In [17], Okubo obtains the same result for a polynomial function $f$, for every $\lambda \in (0,1)$. △

# 4 The finite dimensional case.

In this section, we study the $\lambda$-Aluthge transformation in finite dimensional spaces. Given $T \in \mathcal{M}_n(\mathbb{C})$ and $\lambda \in (0,1)$, we denote by $\Delta_\lambda^n(T)$ the $n$-times iterated $\lambda$-Aluthge transform of $T$, i.e.,

$$\Delta_\lambda^0(T) = T \ , \qquad \text{and} \qquad \Delta_\lambda^n(T) = \Delta_\lambda\left(\Delta_\lambda^{n-1}(T)\right) \ , \quad n \in \mathbb{N}.$$

The following proposition was proved, for $\lambda = 1/2$, by Ando in [2], and by Jung, Ko and Pearcy in [16].

**Proposition 4.1.** *Let $T \in \mathcal{M}_n(\mathbb{C})$. Then, the limit points of the sequence $\{\Delta_\lambda^n(T)\}_{n\in\mathbb{N}}$ are normal. Moreover, if $L$ is a limit point, then $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity.*

*Proof.* Let $\{\Delta_\lambda^{n_k}(T)\}_{k\in\mathbb{N}}$ be a subsequence which converge in norm to a limit point $L$. By the continuity of Aluthge transforms, $\Delta_\lambda^{n_k+1}(T) \xrightarrow[k\to\infty]{} \Delta_\lambda(L)$. Then

$$\begin{aligned} \|\Delta_\lambda(L)\|_2 &= \lim_{k\to\infty} \|\Delta_\lambda^{n_k+1}(T)\|_2 = \lim_{n\to\infty} \|\Delta_\lambda^n(T)\|_2 \\ &= \lim_{k\to\infty} \|\Delta_\lambda^{n_k}(T)\|_2 = \|L\|_2 \ . \end{aligned}$$

Hence, by Theorem 3.9 $L$ is normal. It only remains to prove that $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity, or equivalently, that $\mathrm{tr}(T^m) = \mathrm{tr}(L^m)$ for every $m \in \mathbb{N}$. Indeed,

$$\mathrm{tr}\, L^m = \lim_{k\to\infty} \mathrm{tr}\ \Delta_\lambda^{n_k}(T)^m = \mathrm{tr}\, T^m, \quad m \in \mathbb{N} \ ,$$

because, for each $k \in \mathbb{N}$, $\sigma\left(\Delta_\lambda^{n_k}(T)\right) = \sigma(T)$ (with algebraic multiplicity), and therefore $\mathrm{tr}\ \Delta_\lambda^{n_k}(T)^m = \mathrm{tr}\, T^m$. □

As a consequence of this result, we obtain Yamazaki's spectral radius formula, for every $\lambda \in (0,1)$. It should be mentioned that Yamazaki's formula holds for operators in Hilbert spaces (with $\lambda = 1/2$), but we can only prove the general case ($\lambda \neq 1/2$) in the finite dimensional case.

**Corollary 4.2.** *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\lambda \in (0,1)$. Then,*

$$\rho(T) = \lim_{n\to\infty} \|\Delta_\lambda^n(T)\| \ .$$

*Proof.* Take a subsequence $\{\Delta_\lambda^{n_k}(T)\}$ that converges to a limit point $L$. Since $L$ is normal and $\sigma(L) = \sigma(T)$, it holds that $\|L\| = \rho(L) = \rho(T)$. Hence

$$\lim_{k\to\infty} \|\Delta_\lambda^{n_k}(T)\| = \|L\| = \rho(L) = \rho(T).$$

Finally, since the whole sequence $\{\|\Delta_\lambda^n(T)\|\}$ converges because it is non-increasing, we obtain the desired result. ☐

Analogously we can deduce the following result, proved by Ando in [2] for $\lambda = 1/2$. We use the notation co$(X)$ for the convex hull of the set $X$.

**Corollary 4.3.** *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\lambda \in (0,1)$. Then,*

$$co\Big(\sigma(T)\Big) = \bigcap_{n=1}^{\infty} W(\Delta_\lambda^n(T)) \ .$$

Now we state the following result, which is a direct consequence of Theorem 3.6 and the fact that the map $T \to |T|^r$ is norm-continuous in $\mathcal{M}_n(\mathbb{C})$.

**Proposition 4.4.** *The map $(\lambda, T) \to \Delta_\lambda(T)$ from $(0,1)\times\mathcal{M}_n(\mathbb{C})$ into $\mathcal{M}_n(\mathbb{C})$ is continuous when $\mathcal{M}_n(\mathbb{C})$ is endowed with the norm-topology and the interval $(0,1)$ with the usual one.*

*Proof.* It follows by a standard $\frac{\varepsilon}{2}$ -argument. ☐

## The iterated Aluthge transforms in $\mathcal{M}_2(\mathbb{C})$.

In this subsection we study the convergence of the sequence $\{\Delta_\lambda^n(T)\}$ when $T$ is a $2\times 2$ matrix. The convergence of this sequence for $n\times n$ matrices and $\lambda = 1/2$ was conjectured by Jung, Ko, and Pearcy in [15]. Although this conjecture is still open, there exists a result, due to T. Ando and T. Yamazaki [3], which answers the conjecture affirmatively for $2\times 2$ matrices and $\lambda = 1/2$. We generalize this result for arbitrary $\lambda \in (0,1)$ and we also prove that the map which assigns to each pair $(\lambda, T)$ the limit of the sequence $\{\Delta_\lambda^n(T)\}$ is continuous in both variables $T$ and $\lambda$.

**Lemma 4.5.** *Let $T \in \mathcal{M}_2(\mathbb{C})$ and $\lambda \in (0,1)$. Suppose that $\sigma(T) = \{\mu_1, \mu_2\}$ with $\mu_1 \neq \mu_2$. Then, there exists $\gamma(T,\lambda) \in (0,1)$ such that, for all $n \in \mathbb{N}$,*

$$\|\Delta_\lambda^n(T)^*\,\Delta_\lambda^n(T) - \Delta_\lambda^n(T)\,\Delta_\lambda^n(T)^*\|_2 \leqslant \gamma(T,\lambda)^n\ \|T^*T - TT^*\|_2.$$

*Moreover, if $\alpha = \min\{\lambda\ ,\ 1-\lambda\}$, then we can take*

$$\gamma(T,\lambda) = \left(1 - \frac{2\,\alpha^2\ |\ \mu_1 - \mu_2\ |^2}{2\,|\mu_1\mu_2| + \|T\|_2^2}\right)^{1/2} \ .$$

*Proof.* Denote $T_n = \Delta_\lambda^n(T)$, $n \in \mathbb{N}$. In some orthonormal basis, which may be different for each $n \in \mathbb{N}$, $T_n$ has the form

$$T_n = \begin{pmatrix} \mu_1 & a_n \\ 0 & \mu_2 \end{pmatrix} \ , \qquad \text{with } a_n = \left( \|T_n\|_2^2 - [\ |\mu_1|^2 + |\mu_2|^2] \right)^{1/2} \geqslant 0.$$

Hence $a_{n+1} \leqslant a_n$, $n \in \mathbb{N}$, by Theorem 3.9. Easy computations show that, if $M = |\ \mu_1 - \mu_2\ |^2$ then

$$\|T_n^* T_n - T_n T_n^*\|_2^2 = 2\ a_n^2\ \left(\ M + a_n^2\ \right)\ , \quad n \in \mathbb{N}\ . \tag{6}$$

Therefore, for all $n \in \mathbb{N}$,

$$\frac{\|T_{n+1}^* T_{n+1} - T_{n+1} T_{n+1}^*\|_2^2}{\|T_n^* T_n - T_n T_n^*\|_2^2} = \frac{a_{n+1}^2}{a_n^2}\ \frac{(M + a_{n+1}^2)}{(M + a_n^2)} \leqslant \frac{a_{n+1}^2}{a_n^2}\ . \tag{7}$$

Since $a_n^2 - a_{n+1}^2 = \|T_n\|_2^2 - \|T_{n+1}\|_2^2$, by Proposition 3.8 the following inequality holds for all $n \in \mathbb{N}$,

$$\frac{a_{n+1}^2}{a_n^2} = 1 - \frac{\|T_n\|_2^2 - \|T_{n+1}\|_2^2}{a_n^2} \leqslant 1 - \frac{\alpha^2\ \|\ |T_n| - |T_n^*|\ \|_2^2}{a_n^2}\ .$$

On the other hand, if $X \in \mathcal{M}_2(\mathbb{C})^+$ and $d = \det(X)^{1/2}$, then it is known that

$$X^{1/2} = \frac{X + d\,I}{\sqrt{2\,d + \operatorname{tr}(X)}}\ .$$

Hence, if we denote $d = \det(T_n^* T_n)^{1/2} = \det(T_n T_n^*)^{1/2} = |\det T| = |\mu_1 \mu_2|$, we have that

$$\|\ |T_n| - |T_n^*|\ \|_2^2 = \frac{\|T_n^* T_n - T_n T_n^*\|_2^2}{2\,d + \|T_n\|_2^2}\ , \ n \in \mathbb{N}.$$

Therefore, by equation (6), for all $n \in \mathbb{N}$,

$$\frac{a_{n+1}^2}{a_n^2} \leqslant 1 - \frac{\alpha^2\ \|T_n^* T_n - T_n T_n^*\|_2^2}{a_n^2 (2\,d + \|T_n\|_2^2)} = 1 - \frac{2\,\alpha^2\ (M + a_n^2)}{2\,d + \|T_n\|_2^2} \leqslant 1 - \frac{2\,\alpha^2\ M}{2\,d + \|T\|_2^2}\ . \tag{8}$$

Finally, taking $\gamma(T,\lambda) = \left(1 - \dfrac{2\,\alpha^2\ M}{2\,d + \|T\|_2^2}\right)^{1/2}$, by equations (7) and (8), we get

$$\|T_{n+1}^* T_{n+1} - T_{n+1} T_{n+1}^*\|_2 \leqslant \gamma(T,\lambda) \|T_n^* T_n - T_n T_n^*\|_2\ ,\ n \in \mathbb{N},$$

and the result is proved by iterating this inequality. Note that $0 < \alpha^2 \leqslant 1/4$ and

$$0 < M = |\ \mu_1 - \mu_2\ |^2 \leqslant 2\,|\mu_1 \mu_2| + |\mu_1|^2 + |\mu_2|^2 \leqslant 2\,d + \|T\|_2^2\ .$$

Then $0 < \gamma(T,\lambda) < 1$. □

**Theorem 4.6.** *Let $T \in \mathcal{M}_2(\mathbb{C})$ and $\lambda \in (0,1)$. Then, the sequence $\{\Delta_\lambda^n(T)\}$ converges.*

*Proof.* Suppose that $\sigma(T) = \{\mu_1, \mu_2\}$. Since we have proved (see Proposition 4.1) that the limit points of the sequence $\{\Delta_\lambda^n(T)\}$ are normal, if $\mu_1 = \mu_2 = c$, then $\Delta_\lambda^n(T) \xrightarrow[n\to\infty]{} cI$. Thus, from now on we only consider the case in which $\mu_1 \neq \mu_2$. As in the Lemma 4.5, we denote $T_n = \Delta_\lambda^n(T)$.
Fix $n \geqslant 0$. If $T_n = U_n|T_n|$ is the polar decomposition of $T_n$, then $|T_n^*|^s = U_n|T_n|^s U_n^*$, for every $s > 0$. Therefore we obtain

$$\begin{aligned}(T_{n+1} - T_n)U_n^* &= |T_n|^\lambda U_n|T_n|^{1-\lambda}U_n^* - U_n|T_n|U_n^* \\ &= |T_n|^\lambda|T_n^*|^{1-\lambda} - |T_n^*| = (|T_n|^\lambda - |T_n^*|^\lambda)|T_n^*|^{1-\lambda} .\end{aligned}$$

Since $\|AB\|_2 \leq \|A\|_2\|B\|$, we can deduce that

$$\|T_{n+1} - T_n\|_2 \leq \|\,|T_n|^\lambda - |T_n^*|^\lambda\|_2 \cdot \|\,|T_n^*|^{1-\lambda}\| \leq \|\,|T_n|^\lambda - |T_n^*|^\lambda\|_2 \cdot \|T\|^{1-\lambda} .$$

Using Proposition 2.5 with $A = T_n^*T_n$, $B = T_nT_n^*$ and $r = \lambda/2$, we get

$$\begin{aligned}\|T_{n+1} - T_n\|_2 \leqslant&\ \|\,|T_n|^\lambda - |T_n^*|^\lambda\|_2 \cdot \|\,T\,\|^{1-\lambda} \\ \leqslant&(2\,\|\,T\,\|^{1-\lambda})\ \|T_n^*T_n - T_nT_n^*\|_2^{\lambda/2} ,\end{aligned}$$

because $\|I_2\|_2^{1-\lambda/2} \leq 2$. Let $a = \gamma(T,\lambda)^{\lambda/2} < 1$, where $\gamma(T,\lambda) \in (0,1)$ is the constant of Lemma 4.5. Then

$$\begin{aligned}\|T_{n+1} - T_n\|_2 &\leqslant (2\,\|\,T\,\|^{1-\lambda})\ \|T_n^*T_n - T_nT_n^*\|_2^{\lambda/2} \\ &\leqslant\ a^n\ (2\,\|\,T\,\|^{1-\lambda}\ \|T^*T - TT^*\|_2^{\lambda/2}).\end{aligned}$$

Denote $N(T,\lambda) = 2\,\|\,T\,\|^{1-\lambda}\ \|T^*T - TT^*\|_2^{\lambda/2}$. Then, if $n, m \in \mathbb{N}$, with $n < m$,

$$\begin{aligned}\|T_m - T_n\|_2 \leqslant& \sum_{k=n}^{m-1} \|T_{k+1} - T_k\|_2 \\ \leqslant& N(T,\lambda)\ \sum_{k=n}^{m-1} a^k \xrightarrow[n,m\to\infty]{} 0 ,\end{aligned} \tag{9}$$

which shows that the $\lim\limits_{n\to\infty} T_n = \lim\limits_{n\to\infty} \Delta_\lambda^n(T)$ exists. □

In order to state precisely the next results, we need the following notations:

**Definition 4.7.**

1. Given $T \in \mathcal{M}_2(\mathbb{C})$ and $\lambda \in (0,1)$, denote $\Delta_\lambda^\infty(T) = \lim\limits_{n\to\infty} \Delta_\lambda^n(T)$.

2. Consider the map $\Gamma : (0,1) \times \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ defined by

$$\Gamma(\lambda, T) = \Delta_\lambda^\infty(T) \ , \quad (\lambda, T) \in (0,1) \times \mathcal{M}_2(\mathbb{C}) .$$

**Theorem 4.8.** *Let $\lambda \in (0,1)$ be fixed. Then the map $\Gamma(\lambda,\cdot) : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$, given by*

$$\mathcal{M}_2(\mathbb{C}) \ni T \mapsto \Delta_\lambda^\infty(T)$$

*is continuous. Therefore $\Delta_\lambda^\infty(\cdot)$ is a continuous retraction from $\mathcal{M}_2(\mathbb{C})$ onto the space of normal matrices in $\mathcal{M}_2(\mathbb{C})$.*

*Proof.* Take $T \in \mathcal{M}_2(\mathbb{C})$ and $\lambda \in (0,1)$. We shall consider two cases:
**Case 1.** Suppose that $\sigma(T) = \{\mu\}$. Let $S \in \mathcal{M}_2(\mathbb{C})$ with $\sigma(S) = \{\eta_1, \eta_2\}$. Since $\Delta_\lambda^\infty(T) = \mu I$ and $\Delta_\lambda^\infty(S)$ is a normal operator with the same spectrum as $S$, then

$$\|\Delta_\lambda^\infty(T) - \Delta_\lambda^\infty(S)\|_2^2 = |\mu - \eta_1|^2 + |\mu - \eta_2|^2 \ .$$

Clearly, this implies that $\Delta_\lambda^\infty(\cdot)$ is continuous at $T$.
**Case 2.** Suppose that $\sigma(T) = \{\mu_1, \mu_2\}$ with $\mu_1 \neq \mu_2$ and let $\varepsilon > 0$. Take $\delta_1 > 0$ such that for every matrix $S$ satisfying $\|T - S\|_2 \leqslant \delta_1$, the constant $\gamma(S,\lambda)$ of Lemma 4.5 applied to $S$ satisfies $\gamma(S,\lambda) \leqslant r$, for some $r < 1$. Indeed, note that the formula for $\gamma(S,\lambda)$ given in Lemma 4.5 depends continuously on $S$ (and its spectrum). Note that the constant $N(S,\lambda) = 4 \| S \|^{1-\lambda} \|S^*S - SS^*\|_2^{\lambda/2}$ is bounded on the set $\mathcal{U} = \{S \in \mathcal{M}_2(\mathbb{C}) : \|T - S\|_2 \leqslant \delta_1\}$. Then, by formula (9), we can deduce that there exists $n \in N$, such that

$$\|\Delta_\lambda^\infty(S) - \Delta_\lambda^n(S)\|_2 \ \leqslant \ N(S,\lambda) \sum_{k=n}^{\infty} r^{k\lambda/2} \ \leqslant \ \frac{\varepsilon}{3} \ ,$$

for every $S \in \mathcal{U}$. Finally, since the map $\Delta_\lambda^n(\cdot)$ is continuous on $\mathcal{M}_2(\mathbb{C})$, we can take $0 < \delta_2 < \delta_1$ such that, if $\|T - S\|_2 \leqslant \delta_2$, then

$$\|\Delta_\lambda^n(T) - \Delta_\lambda^n(S)\|_2 \leqslant \frac{\varepsilon}{3} \ .$$

So, if $\|T - S\|_2 \leqslant \delta_2$, then

$$\begin{aligned}\|\Delta_\lambda^\infty(T) - \Delta_\lambda^\infty(S)\|_2 \leqslant \ & \|\Delta_\lambda^\infty(T) - \Delta_\lambda^n(T)\|_2 + \|\Delta_\lambda^n(T) - \Delta_\lambda^n(S)\|_2 + \\ & + \|\Delta_\lambda^n(S) - \Delta_\lambda^\infty(S)\|_2 \leqslant \varepsilon \ ,\end{aligned}$$

which completes the proof. □

**Theorem 4.9.** *Let $T \in \mathcal{M}_2(\mathbb{C})$ be fixed. Then the map $\Gamma(\cdot, T) : (0,1) \to \mathcal{M}_2(\mathbb{C})$, given by*

$$(0,1) \ni \lambda \mapsto \Delta_\lambda^\infty(T)$$

*is continuous. Moreover, if $\sigma(T) = \{\mu_1, \mu_2\}$ with $|\mu_1| = |\mu_2|$, then the map is constant.*

*Proof.* The proof of the continuity is similar to the proof of the previous theorem (see also Remark 4.10 below). Note that the constants $\gamma(T,\lambda)$ and $N(T,\lambda)$ depend continuously on both variables, in particular on $\lambda$. Also, by Proposition 4.4, the map $\lambda \mapsto \Delta_\lambda^n(T)$ is continuous, for every $n \in \mathbb{N}$. Let $T \in \mathcal{M}_2(\mathbb{C})$ such that $|\mu_1| = |\mu_2|$. As Ando and Yamazaki pointed out in [3], without loss of generality we can assume that $T = \begin{pmatrix} a & b \\ -b & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$, with $b > 0$, and $\sigma(T) = \{u + iv, u - iv\}$ with $u^2 + v^2 = 1$ and $v > 0$. Then,

$$\Gamma(\lambda, T) = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \quad , \quad \lambda \in (0,1) \ .$$

Indeed, if $\Delta_\lambda^n(T) = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$, by Theorem 4.6 and some simple computations, we get $\Delta_\lambda^n(T)^* \, \Delta_\lambda^n(T) - \Delta_\lambda^n(T) \, \Delta_\lambda^n(T)^* =$

$$(b_n - c_n) \begin{pmatrix} -(b_n + c_n) & a_n - d_n \\ a_n - d_n & b_n + c_n \end{pmatrix} \xrightarrow[n\to\infty]{} 0 \ , \tag{10}$$

So, the sequences $a_n$ and $d_n$ converge to $\operatorname{tr}(T)/2 = u$. On the other hand, following essentially the same lines as in Ando-Yamazaki's proof, we get $0 < m = \inf_n (b_n - c_n)^2 = \lim_{n\to\infty}(b_n - c_n)^2$. Hence, $b_n - c_n$ must converge to $m^{1/2}$ or $-m^{1/2}$. Moreover, since $b_n + c_n \xrightarrow[n\to\infty]{} 0$ by formula (10), then $m^{1/2} = 2v$, for each $\lambda \in (0,1)$. Therefore

$$\Gamma(\lambda, T) = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} = \Gamma(1/2, T) \quad \text{or} \quad \Gamma(\lambda, T) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} .$$

But $\Gamma$ is continuous on $\lambda$, so $\Gamma(\lambda, T) = \Gamma(1/2, T)$ for every $\lambda \in (0,1)$. □

**Remark 4.10.** With similar arguments to those used in the proofs of the previous two theorems, it can be proved that the map $\Gamma$ is jointly continuous.

**Example 4.11.** If $T \in \mathcal{M}_2(\mathbb{C})$ has eigenvalues with different moduli, then the map $\lambda \mapsto \Delta_\lambda^\infty(T)$ does not seem to be constant, in general. For example, if $T = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}$, numerical computations show that

$$\Delta_{0.3}^\infty(T) \cong \begin{pmatrix} 2.22738 & 0.973807 \\ 0.973807 & 1.77262 \end{pmatrix} \quad \text{while}$$

$$\Delta_{0.7}^\infty(T) \cong \begin{pmatrix} 1.37162 & -0.777907 \\ -0.777907 & 2.62838 \end{pmatrix} .$$

Nevertheless, for many other matrices $T$ with different modulus eigenvalues, the map $\lambda \mapsto \Delta_\lambda^\infty(T)$ seems to be constant. △

## The Jordan structure of Aluthge transforms

In this subsection, we study some properties of the Jordan structure of the iterated Aluthge transforms. We show a reduction of the conjecture on the convergence of the sequence $\{\Delta_\lambda^m(T)\}$ for $T \in \mathcal{M}_n(\mathbb{C})$, to the invertible case. We also study the behavior of the angles between the spectral subspaces of iterates of the Aluthge transform for $T \in \mathcal{M}_n(\mathbb{C})$.

The following result states a simple relation between the null spaces of polynomials in $T$ and in $\Delta_\lambda(T)$. This relation has some consequences regarding multiplicity and Jordan structure of eigenvalues of $T$ and $\Delta_\lambda(T)$. We denote by $\mathbb{C}[x]$ the set of complex polynomials.

**Lemma 4.12.** *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\lambda \in (0,1)$.*

1. *Given $p \in \mathbb{C}[x]$, then $\dim N(p(T)) \leqslant \dim N(p(\Delta_\lambda(T)))$.*
2. *For $n \in \mathbb{N}$, $n \geqslant 2$, $\dim N(T^n) = \dim N(\Delta_\lambda(T)^{n-1})$.*

*Proof.* Assume first that $p(0) \neq 0$. In this case $N(T) \cap N(p(T)) = \{0\}$. Hence

$$\dim |T|^\lambda \big(N(p(T))\big) = \dim N(p(T)) \ ,$$

because $N(T) = N(|T|) = N(|T|^\lambda)$. Using Proposition 3.4, we know that $p(\Delta_\lambda(T))|T|^\lambda = |T|^\lambda p(T)$, so that

$$|T|^\lambda(N(p(T)) \subseteq N(p(\Delta_\lambda(T)) \ .$$

If $p(0) = 0$, Note that $N(T) \subseteq N(p(T))$ and also $N(T) \subseteq N(p(\Delta_\lambda(T)))$. Denote by $\mathcal{S} = N(p(T)) \ominus N(T)$. Then $\dim |T|^\lambda(\mathcal{S}) = \dim \mathcal{S}$ and $|T|^\lambda(\mathcal{S}) \subseteq N(T)^\perp$. On the other hand, we get that $|T|^\lambda(\mathcal{S}) \subseteq N(p(\Delta_\lambda(T)))$ as before. Then

$$\begin{aligned} \dim N(p(T)) &= \dim N(T) + \dim \mathcal{S} \\ &= \dim N(T) + \dim |T|^\lambda(\mathcal{S}) \\ &= \dim \big[N(T) \oplus |T|^\lambda(\mathcal{S})\big] \leqslant \dim N(p(\Delta_\lambda(T))). \end{aligned}$$

Finally, note that if $n \geqslant 2$ we have

$$N(\Delta_\lambda(T)^{n-1}\, |T|^\lambda) = N(|T|^\lambda T^{n-1}) = N(T^n).$$

Let $\mathcal{S} = N(\Delta_\lambda(T)^{n-1}) \ominus N(T)$. Since $|T|^\lambda$ operates bijectively on $N(T)^\perp$, there is a subspace $\mathcal{M} \subseteq N(T)^\perp$ such that $\dim \mathcal{M} = \dim \mathcal{S}$ and $|T|^\lambda(\mathcal{M}) = \mathcal{S}$. Hence

$$N(\Delta_\lambda(T)^{n-1}\, |T|^\lambda) = \big\{x \in \mathbb{C}^n : |T|^\lambda(x) \in N(\Delta_\lambda(T)^{n-1})\ \big\} = N(T) \oplus \ \mathcal{M}.$$

So that $\dim N(\Delta_\lambda(T)^{n-1}) = \dim N(\Delta_\lambda(T)^{n-1}\, |T|^\lambda) = \dim N(T^n)$. □

**Definition 4.13.** Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\mu \in \sigma(T)$. We denote

1. $m(T,\mu)$ the *algebraic multiplicity* of the eigenvalue $\mu$ for $T$.
2. $m_0(T,\mu) = \dim N(T - \mu I)$, the *geometric multiplicity* of the eigenvalue $\mu$ for $T$.
3. $r(T,\mu) = \min\{k \in \mathbb{N} : \dim N(T-\mu I)^k = m(T,\mu)\}$, usually called the *index* of $\mu$. Note that $r(T,\mu)$ is the size of the biggest Jordan block of $T$ associated to $\mu$.

We say that the Jordan structure of $T$ for the eigenvalue $\mu$ is *trivial* if $m(T,\mu) = m_0(T,\mu)$, or equivalently, if $r(T,\mu) = 1$. △

**Proposition 4.14.** *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\lambda \in (0,1)$.*

1. *Suppose that $0 \in \sigma(T)$. Then*

$$m(T,0) = m_0(\Delta_\lambda^{r(T,0)-1}(T)\,, 0) = \dim N(\Delta_\lambda^{r(T,0)-1}(T)).$$

*Therefore, after $r(T,0)-1$ iterations of the Aluthge transform, we get a matrix whose Jordan structure for the eigenvalue $0$ is trivial.*

2. *If $\mu \in \sigma(T)/\{0\}$, then*

$$m_0(T,\mu) \leqslant m_0(\Delta_\lambda(T),\mu) \qquad \text{and} \qquad r(T,\mu) \geqslant r(\Delta_\lambda(T),\mu)\ .$$

*Proof.* 1. Denote $r(T,0)=r$. If $r \geq 2$, by Lemma 4.12,

$$m(T,0) = \dim N(T^r) = \dim N(\Delta_\lambda(T)^{r-1}) = \dim N(\Delta_\lambda^2(T)^{r-2}) = \dots$$

$$\dots = \dim N(\Delta_\lambda^{r-2}(T)^2) = \dim N(\Delta_\lambda^{r-1}(T))\ .$$

If $r=1$, then $\Delta_\lambda^{r-1}(T) = \Delta_\lambda^0(T) = T$ by definition, and

$$m(T,0) = m_0(T,0) = \dim(\Delta_\lambda^{r-1}(T)).$$

2. Consider $P_m(x) = (x-\mu)^m$, $m \in \mathbb{N}$. Taking $m=1$, by Lemma 4.12,

$$m_0(T,\mu) = \dim N(T-\mu I) \leqslant \dim N(\Delta_\lambda(T) - \mu I) = m_0(\Delta_\lambda(T),\mu).$$

Taking $m = r(T,\mu)$, again by Lemma 4.12, we have that

$$\begin{aligned} m(T,\mu) &= \dim N((T-\mu I)^{r(T,\mu)}) \\ &\leqslant \dim N((\Delta_\lambda(T)-\mu I)^{r(T,\mu)}) \leqslant m(\Delta_\lambda(T),\mu)\ . \end{aligned}$$

Since $m(\Delta_\lambda(T),\mu) = m(T,\mu)$, we get that $r(T,\mu) \geqslant r(\Delta_\lambda(T),\mu)$.

□

**Remark 4.15.** In particular, Proposition 4.14 shows that if $T$ is nilpotent of order $n$ then $\Delta_\lambda^{n-1}(T) = 0$. This result was proved by Jung, Ko and Pearcy in [16].

**Corollary 4.16.** *Let $\lambda \in (0,1)$. If the sequence $\{\Delta_\lambda^m(S)\}$ converges for every* **invertible** *matrix $S \in \mathcal{M}_n(\mathbb{C})$ and every $n \in \mathbb{N}$, then the sequence $\{\Delta_\lambda^m(T)\}$ converges for all $T \in \mathcal{M}_n(\mathbb{C})$ and every $n \in \mathbb{N}$.*

*Proof.* Let $T \in \mathcal{M}_n(\mathbb{C})$. By Lemma 4.14, we can assume that $m(T,0) = m_0(T,0)$. Note that, in this case, $N(\Delta_\lambda(T)) = N(T)$, because $N(T) \subseteq N(\Delta_\lambda(T))$ and $m_0(\Delta_\lambda(T),0) = m(T,0)$. On the other hand, $R(\Delta_\lambda(T)) \subseteq R(|T|)$ so that $R(\Delta_\lambda(T))$ and $N(\Delta_\lambda(T))$ are orthogonal subspaces. Thus, there exists a unitary matrix $U$ such that

$$U\Delta_\lambda(T)U^* = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$$

where $S \in M_s(\mathbb{C})$ is invertible ($s = n - m(T,0)$ ). Since for every $m \geqslant 2$

$$\Delta_\lambda^m(T) = U^* \begin{pmatrix} \Delta_\lambda^{m-1}(S) & 0 \\ 0 & 0 \end{pmatrix} U\ ,$$

the sequence $\{\Delta_\lambda^m(T)\}$ converges, because the sequence $\{\Delta_\lambda^{m-1}(S)\}$ converges by hypothesis.
□

**Remark 4.17.** If $T \in \mathcal{M}_n(\mathbb{C})$ is invertible, then $|T|^\lambda$ is invertible for every $\lambda \in (0,1)$, and

$$\Delta_\lambda(T) = |T|^\lambda \, T \, |T|^{-\lambda}. \tag{11}$$

Therefore, $T$ and $\Delta_\lambda^m(T)$ are similar matrices, for every $m \in \mathbb{N}$. That is, $\Delta_\lambda^m(T)$ and $T$ have the same Jordan structure. This shows that the geometric multiplicity of non-zero eigenvalues do not increases in general. On the other hand, Proposition 4.14 implies that for non-invertible operators $T$, $\Delta_\lambda(T)$ and $T$ may be not similar. In particular, the Jordan structure of $T$ and $\Delta_\lambda(T)$ may be different.

Numerical experiences show that the rate of convergence of the sequence $\{\Delta_\lambda^m(T)\}$ is smaller for non-diagonalizable $T$, than for diagonalizable examples. △

**Definition 4.18.** Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\mu \in \sigma(T)$.

1. Denote $\mathcal{H}_{\mu,T} = N((T-\mu I)^{r(T,\mu)})$. Note that $\mathbb{C}^n = \bigoplus_{\gamma \in \sigma(T)} \mathcal{H}_{\gamma,T}$ .

2. Denote $Q_{\mu,T} \in \mathcal{M}_n(\mathbb{C})$ the oblique projection with

$$R(Q_{\mu,T}) = \mathcal{H}_{\mu,T} \quad \text{and} \quad N(Q_{\mu,T}) = \bigoplus_{\gamma \neq \mu} \mathcal{H}_{\gamma,T} \ .$$

**Proposition 4.19.** *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\lambda \in (0,1)$. Then, for every $\mu \in \sigma(T)$ ,*

$$\|Q_{\mu,\Delta_\lambda^m(T)}\| \xrightarrow[m\to\infty]{} 1.$$

*Proof.* Let $f_\mu \in \operatorname{Hol}(T)$ be an analytic map which takes the value 1 in a neighborhood of $\mu$, and the value 0 in a neighborhood of $\sigma(T) \setminus \{\mu\}$. Then it is known that $f_\mu(T) = Q_{\mu,T}$. Moreover, since $\sigma(\Delta_\lambda^m(T)) = \sigma(T)$, we have that $Q_{\mu,\Delta_\lambda^m(T)} = f_\mu(\Delta_\lambda^m(T))$ , $m \in \mathbb{N}$, $\mu \in \sigma(T)$. Then, by Proposition 3.14,

$$\|Q_{\mu,\Delta_\lambda^m(T)}\| \geqslant \|Q_{\mu,\Delta_\lambda^{m+1}(T)}\| \ , \quad m \in \mathbb{N}, \ \mu \in \sigma(T) \ .$$

On the other hand, there exists a subsequence $\Delta_\lambda^{m_k}(T) \xrightarrow[k\to\infty]{} L$ for some normal matrix $L \in \mathcal{M}_n(\mathbb{C})$, with $\sigma(L) = \sigma(T)$. Then, by Proposition 2.1,

$$\|Q_{\mu,\Delta_\lambda^{m_k}(T)}\| = \|f_\mu(\Delta_\lambda^{m_k}(T))\| \xrightarrow[k\to\infty]{} \|f_\mu(L)\| = \|Q_{\mu,L}\| = 1,$$

because the spectral projections of normal operators are selfadjoint (i.e., orthogonal). □

**Remark 4.20.** Given two subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathbb{C}^n$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$, the **angle** between $\mathcal{M}$ and $\mathcal{N}$ is the angle in $[0, \pi/2]$ whose cosine is defined by

$$\begin{aligned} c\,[\mathcal{M}, \mathcal{N}\,] &= \sup\big\{ \, |\langle x, y\rangle| : \ x \in \mathcal{M}, \ y \in \mathcal{N} \text{ and } \|x\| = \|y\| = 1 \big\} \\ &= \|P_{\mathcal{M}} P_{\mathcal{N}}\| \ , \end{aligned} \tag{12}$$

where $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$. The *sine* of this angle is $s\,[\,\mathcal{M},\,\mathcal{N}\,] = \left(1 - c\,[\,\mathcal{M},\,\mathcal{N}\,]^2\right)^{1/2}$. If $\mathcal{M} \oplus \mathcal{N} = \mathbb{C}^n$ and $Q$ is the oblique projection with range $\mathcal{M}$ and null space $\mathcal{N}$, it is known that

$$\begin{aligned} \|Q\| &= \left(1 - \|P_{\mathcal{M}}\, P_{\mathcal{N}}\|^2\right)^{-1/2} = \left(1 - c\,[\,\mathcal{M},\,\mathcal{N}\,]^2\right)^{-1/2} \\ &= s\,[\,\mathcal{M},\,\mathcal{N}\,]^{-1} \ . \end{aligned}$$

For proofs of these results, the reader is referred to Gohberg and Krein [13], Deutsch [11], or Ben-Israel and Greville [5].

Now we can see that Proposition 4.19 is equivalent to the following statement: given $\mu \in \sigma(T)$, the angle between the spectral subspaces $\mathcal{H}_{\mu,\Delta_\lambda^m(T)}$ and $\mathcal{N}_\mu = \bigoplus_{\gamma \neq \mu} \mathcal{H}_{\gamma,\Delta_\lambda^m(T)}$ converges to $\pi/2$. Given $\mu \neq \gamma \in \sigma(T)$, since $\mathcal{H}_{\gamma,\Delta_\lambda^m(T)} \subseteq \mathcal{N}_\mu$, it is easy to see that

$$c\left[\,\mathcal{H}_{\mu,\Delta_\lambda^m(T)},\, \mathcal{H}_{\gamma,\Delta_\lambda^m(T)}\,\right] \leqslant\; c\left[\,\mathcal{H}_{\mu,\Delta_\lambda^m(T)},\, \mathcal{N}_\mu\,\right] \xrightarrow[m\to\infty]{} 0 \ .$$

Therefore, also the angle between $\mathcal{H}_{\mu,\Delta_\lambda^m(T)}$ and $\mathcal{H}_{\gamma,\Delta_\lambda^m(T)}$ converges to $\pi/2$. Another description of this fact is that

$$P_{\mathcal{H}_{\mu,\Delta_\lambda^m(T)}} P_{\mathcal{H}_{\gamma,\Delta_\lambda^m(T)}} \xrightarrow[m\to\infty]{} 0 \ .$$

This also follows from equation (12). $\triangle$

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