Weighted projections and Riesz frames

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Abstract

Let $\mathcal{H}$ be a (separable) Hilbert space and $\{e_k\}_{k\geq 1}$ a fixed orthonormal basis of $\mathcal{H}$. Motivated by many papers on scaled projections, angles of subspaces and oblique projections, we define and study the notion of compatibility between a subspace and the abelian algebra of diagonal operators in the given basis. This is used to refine previous work on scaled projections, and to obtain a new characterization of Riesz frames.

Keywords: scaled projection, weighted projection, compatibility, angles, frames, Riesz frames.
AMS classification: Primary 47A30, 47B15

1 Introduction

Weighted projections (also called scaled projections) play a relevant role in a variety of least-square problems. As a sample of their applications and of their relatives, namely, weighted pseudoinverses, they have been used in optimization (feasibility theory, interior point methods), statistics (linear regression, weighted estimation), and signal processing (noise reduction).

Frequently, weighted pseudoinverses take the forms $(ADA^t)^{-1}AD$, $(ADA^t)^{\dagger}AD$, $(ADA^*)^{-1}AD$ or $(ADA^*)^{\dagger}AD$, according to the field (real or complex) which is involved in the problem and to different hypothesis of invertibility. Analogous formulas hold for the corresponding weighted projections. In general D is a positive definite matrix and A is a full column rank matrix.

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In a series of papers, Stewart [32], O'Leary [29], Ben-Tal and Taboulle [4], Hanke and Neumann [23], Forsgren [19], Gonzaga and Lara [22], Forsgren and Spörre [20], and Wei [36], [35], [34], [33] have studied and computed quantities of the type

$$\sup_{D \in \Gamma} \|\gamma(D, A)\|,$$

where Γ denotes a certain subset of positive definite invertible matrices and $\gamma(D, A)$ is any of the weighted pseudoinverses mentioned above. The reader is referred to the papers by Forsgren and Spörre [19], [20] for excellent surveys on the history and motivations of the problem of estimating the supremum above. It should be said, however, that the references mentioned above only deal with the finite dimensional context. In order to deal with increasing dimensions or arbitrarily large data sets, we present the problem in an infinite dimensional Hilbert space.

Moreover, we present a different approach to this theory, valid also in the finite dimensional context, based on techniques and results on generalized selfadjoint projections. Recall that, if D is a selfadjoint operator on a complex (finite or infinite dimensional) Hilbert space $\mathcal{H}$, another operator C on $\mathcal{H}$ is called D-selfadjoint if C is Hermitian with respect to the Hermitian sesquilinear form

$$\langle \xi, \eta \rangle_D = \langle D\xi, \eta \rangle \quad (\xi, \eta \in \mathcal{H}),$$

i.e. if $DC = C^*D$. We say that a closed subespace $\mathcal{S}$ of $\mathcal{H}$ is *compatible* with D (or that the pair $(D, \mathcal{S})$ is compatible) if there exists an D-selfadjoint projection Q in $\mathcal{H}$ with image $\mathcal{S}$. It is well known [11] that in finite dimensional spaces, every subspace is compatible with any positive semidefinite operator D. In infinite dimensional spaces this is not longer true; however, every (closed) subspace is compatible with any positive invertible operator and, in general, compatibility can be characterized in terms of angles between certain closed subspaces of $\mathcal{H}$, e. g., the angle between $\mathcal{S}$ and $(D\mathcal{S})^\perp$.

If the pair $(D, \mathcal{S})$ is compatible, the set of D-selfadjoint projections onto $\mathcal{S}$ may be infinite; nonetheless, a distinguished one denoted by $P_{D,\mathcal{S}}$, can be defined and computed (see [11] or section 2.2 below).

In the finite dimensional case, from the point of view of D-selfadjoint projections, the study of weighted projections allows us to obtain simpler proofs of some known results. Another advantage this perspective offers is that these proofs can be easily extended to more general settings which are also important in applications. These applications include projections with complex weights and the infinite dimensional case. Moreover, this approach establishes the relationships among the quantities that have appeared throughout the study on weighted projection (usually, operator norms, vector norms and angles).

A well known result due to Ben-Tal and Teboulle states that the solutions to weighted least squares problems lies in the convex hull of solutions to some non-singular square subsystems. We refer the reader to Ben-Tal and Teboulle's paper [4], or [19], [36] for the following formulation: let A be an $m \times n$ matrix of full rank. Denote by $J(A)$ the set of all $m \times m$ orthogonal diagonal projections such that $QA : \mathbb{C}^n \to R(Q)$ is bijective. Then, for every $m \times m$ positive diagonal matrix D,

$$A(A^*DA)^{-1}A^*D = \sum_{Q \in J(A)} \left(\frac{\det(D_Q)|\det(A_Q)|^2}{\sum_{P \in J(A)} \det(D_P)|\det(A_P)|^2} \right) A(QA)^{-1}Q \tag{1}$$

where A_Q (resp. D_Q) is QA (resp. QD) considered as a square submatrix of A (resp. D).

In section 3 we show that, if $\mathcal{S} = R(A)$, then for every $D \in \mathcal{D}^+$ and $Q \in J(A)$ the following identities hold:

$$A(A^*DA)^{-1}A^*D = P_{D,\mathcal{S}} \quad \text{and} \quad A(QA)^{-1}Q = P_{Q,\mathcal{S}},$$

where $P_{D,\mathcal{S}}$ and $P_{Q,\mathcal{S}}$ denote the distinguished projections onto $\mathcal{S}$ which are D-selfadjoint and Q-selfadjoint, respectively.

Then, Ben-Tal and Teboulle's formula (1) can be rewritten in the following way: if $R(A) = \mathcal{S}$ and for every $D \in \mathcal{D}^+$,

$$P_{D,\mathcal{S}} \in \operatorname{co}\{P_{Q,\mathcal{S}} : Q \in J(A)\}.$$

This implies, in particular, that $\sup\limits_{D\in\mathcal{D}_n^+} \|P_{D,\mathcal{S}}\| \leq \max\limits_{Q\in J(A)} \|P_{Q,\mathcal{S}}\|$. The same inequality was proved independently by O'Leary in [29], while the reverse inequality was initially proved by Stewart [32]. A slight generalization of Stewart's result is proved in this section. Another application of the projections techniques provides an easy proof of a result of Gonzaga and Lara [22] about scaled projections, even for complex weights.

In section 4, we extend the notion of compatibility of a closed subspace, with respect to certain subsets of $L(\mathcal{H})^+$. Given $\Gamma \subseteq L(\mathcal{H})^+$ and a closed subspace $\mathcal{S}$, we say that $\mathcal{S}$ is compatible with Γ if $(D, \mathcal{S})$ is compatible for every $D \in \Gamma$ and it satisfies Stewart's condition:

$$\sup_{D\in\Gamma} \|P_{D,\mathcal{S}}\| < \infty.$$

For a fixed orthonormal basis $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$, we denote by $\mathcal{D}$ the diagonal algebra with respect to $\mathcal{B}$, i.e. $D \in \mathcal{D}$ if $De_n = \lambda_n e_n$ $(n \in \mathbb{N})$ for a bounded sequence (λ_n) of complex numbers. Next, we consider compatibility of $\mathcal{S}$ with respect to

1. $\mathcal{D}^+$, the set of positive invertible elements of $\mathcal{D}$ (i.e. all $\lambda_n > \varepsilon$, for some $\varepsilon > 0$);
2. $\mathcal{P}(\mathcal{D})$, the set of projections in $\mathcal{D}$ (i.e. all $\lambda_n = 0$ or 1);
3. $\mathcal{P}_0(\mathcal{D})$, the set of elements in $\mathcal{P}(\mathcal{D})$ with finite rank, and
4. $\mathcal{P}_{0,\mathcal{S}}(\mathcal{D})$, the set of elements $Q \in \mathcal{P}_0(\mathcal{D})$ such that $R(Q) \cap \mathcal{S} = \{0\}$

For a closed subspace $\mathcal{S}$, we show that compatibility with any of these sets is equivalent. In the first case, we say that $\mathcal{S}$ is compatible with the basis $\mathcal{B}$ (or $\mathcal{B}$-compatible).

This notion is very restrictive. Nevertheless, the class of subspaces which are compatible with a given basis $\mathcal{B}$ has its own interest. Indeed, as we show in section 5, if $\dim \mathcal{S}^\perp = \infty$, then $\mathcal{S}$ is $\mathcal{B}$-compatible if and only if the class of frames whose preframe operators (in terms of the basis $\mathcal{B}$) have nullspace $\mathcal{S}$, consists of Riesz frames (see section 5 for definitions or Casazza [7], Christensen [9], [10] for modern treatments of Riesz frame theory and applications). We completely characterize compatible subspaces with $\mathcal{B}$ in terms of Friedrichs angles (see Definition 2.1) and we obtain an analogue of Stewart-O'Leary identity. Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. For $J \subseteq \mathbb{N}$, denote by $\mathcal{H}_J$ the closed span of the set $\{e_n : n \in J\}$ and P_J the orthogonal projector onto $\mathcal{H}_J$. In the case that $J = \{1, \ldots, n\}$, we denote $\mathcal{H}_n$ and P_n instead of $\mathcal{H}_J$ and P_J. Then, the main results of this paper are:

1. The following conditions are equivalent:

   (a) $\mathcal{S}$ is compatible with $\mathcal{D}^+$;

   (b) $\sup\{c\,[\,\mathcal{S},\,\mathcal{H}_J\,] : J \subseteq \mathbb{N}\} < 1$, were $c\,[\,\mathcal{T},\,\mathcal{M}\,]$ denotes the Friedrichs angle between the closed subspaces $\mathcal{T}$ and $\mathcal{M}$;

   (c) $\sup\{c\,[\,\mathcal{S},\,\mathcal{H}_J\,] : J \subseteq \mathbb{N}$ and J is finite $\} < 1$;

   (d) all pairs $(P_J, \mathcal{S})$ are compatible and $\sup\{\|P_{P_J,\mathcal{S}}\| : J \subseteq \mathbb{N}\} < \infty$.

   In this case

$$\sup\left\{\|P_{D,\mathcal{S}}\| : D \in \mathcal{D}^+\right\} = \sup\left\{\|P_{P_J,\mathcal{S}}\| : J \subseteq \mathbb{N}\right\} = \Big(1 - \sup_{Q\in\mathcal{P}(\mathcal{D})} c\,[\,\mathcal{S},\,R(Q)\,]^2\Big)^{-1/2}.$$

2. $\mathcal{S}$ is compatible with $\mathcal{D}^+$ if and only if

   (a) $\mathcal{S} = \overline{\cup_{n\in\mathbb{N}}\ \mathcal{S}\cap\mathcal{H}_n}$ and

   (b) for every $n \in \mathbb{N}$, the subspace $\mathcal{S}\cap\mathcal{H}_n$ is compatible with $\mathcal{B}$ and there exists $M > 0$ such that $\sup\{\|P_{P_J,\mathcal{S}\cap\mathcal{H}_n}\| : J \subseteq \mathbb{N}\} \leq M$ for every $n \in \mathbb{N}$.

3. If $\dim \mathcal{S} < \infty$, then $\mathcal{S}$ is compatible with $\mathcal{B}$ if and only if there exists $n \in \mathbb{N}$ such that $\mathcal{S} \subseteq \mathcal{H}_n$.

2 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space and $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. For an operator $A \in L(\mathcal{H})$, we denote by $R(A)$ the range or image of A, $N(A)$ the nullspace of A, $\sigma(A)$ the spectrum of A, A^* the adjoint of A, $\rho(A)$ the spectral radius of A, $\|A\|$ the usual norm of A and, if $R(A)$ is closed, $A^\dagger$ the Moore-Penrose pseudoinverse of A.

Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, we denote by $P_\mathcal{S}$ the orthogonal (i.e. selfadjoint) projection onto $\mathcal{S}$. If $B \in L(\mathcal{H})$ satisfies $P_\mathcal{S} B P_\mathcal{S} = B$, we consider the compression of B to $\mathcal{S}$, (i.e. the restriction of B to $\mathcal{S}$ as a map from $\mathcal{S}$ to $\mathcal{S}$), and we say that B is considered as *acting* on $\mathcal{S}$.

Given a subspace $\mathcal{S}$ of $\mathcal{H}$, its unit ball is denoted by $(\mathcal{S})_1$, and its closure by $\overline{\mathcal{S}}$. The distance between two subsets S_1 and S_2 of $\mathcal{H}$ is

$$\mathrm{d}\,(S_1,\,S_2) = \inf\{\|x-y\| :\ x \in S_1\ \ y \in S_2\}.$$

Along this note we use the fact that every subspace $\mathcal{S}$ of $\mathcal{H}$ induces a representation of elements of $L(\mathcal{H})$ by 2×2 block matrices. We shall identify each $A \in L(\mathcal{H})$ with a 2×2-matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{array}{l} \mathcal{S} \\ \mathcal{S}^\perp \end{array}$, which we write to emphasize the decomposition which induces it. Observe that $\begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix}$ is the matrix which represents A^*.

2.1 Angle between subspaces

Among different notions of angle between subspaces in a Hilbert space, we consider two definitions due to Friedrichs and Dixmier (see [16] and [21]).

Definition 2.1 (Friedrichs). Given two closed subspaces $\mathcal{M}$ and $\mathcal{N}$, the *angle* between $\mathcal{M}$ and $\mathcal{N}$ is the angle in $[0, \pi/2]$ whose cosine is defined by

$$c\,[\,\mathcal{M},\,\mathcal{N}\,] = \sup\{\,|\,\langle\xi,\,\eta\rangle\,|:\ \xi \in \mathcal{M}\ominus(\mathcal{M}\cap\mathcal{N}),\ \eta\in\mathcal{N}\ominus(\mathcal{M}\cap\mathcal{N}) \text{ and } \|\xi\| = \|\eta\| = 1\}.$$

Then, the sine of this angle is

$$s\,[\,\mathcal{M},\,\mathcal{N}\,] = (1 - c\,[\,\mathcal{M},\,\mathcal{N}\,])^{1/2} = \mathrm{d}\,(\,(\mathcal{M})_1,\,\mathcal{N}\ominus(\mathcal{M}\cap\mathcal{N}))\,.$$

The last equality follows from the definition by direct computations.

Definition 2.2 (Dixmier). Given two closed subspaces $\mathcal{M}$ and $\mathcal{N}$, the *minimal angle* between $\mathcal{M}$ and $\mathcal{N}$ is the angle in $[0, \pi/2]$ whose cosine is defined by

$$c_0\,[\,\mathcal{M},\,\mathcal{N}\,] = \sup\{\,|\,\langle\xi,\,\eta\rangle\,|:\ \xi \in \mathcal{M},\ \eta\in\mathcal{N} \text{ and } \|\xi\| = \|\eta\| = 1\}$$

The reader is referred to the excellent survey by F. Deutsch [15] and the book of Ben-Israel and Greville [3] which also have complete references. The next propositions collect the results about angles which are relevant to our work.

Proposition 2.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be to closed subspaces of $\mathcal{H}$. Then*

1. $0 \le c\,[\,\mathcal{M},\,\mathcal{N}\,] \le c_0\,[\,\mathcal{M},\,\mathcal{N}\,] \le 1$
2. $c\,[\,\mathcal{M},\,\mathcal{N}\,] = c_0\,[\,\mathcal{M}\ominus(\mathcal{M}\cap\mathcal{N}),\,\mathcal{N}\,] = c_0\,[\,\mathcal{M},\,\mathcal{N}\ominus(\mathcal{M}\cap\mathcal{N})\,]$
3. $c\,[\,\mathcal{M},\,\mathcal{N}\,] = c\left[\,\mathcal{M}^\perp,\,\mathcal{N}^\perp\,\right]$
4. $c_0\,[\,\mathcal{M},\,\mathcal{N}\,] = \|P_\mathcal{M}P_\mathcal{N}\| = \|P_\mathcal{M}P_\mathcal{N}P_\mathcal{M}\|^{1/2}$
5. $c\,[\,\mathcal{M},\,\mathcal{N}\,] = \|P_\mathcal{M}P_\mathcal{N} - P_{(\mathcal{M}\cap\mathcal{N})^\perp}\| = \|P_\mathcal{M}P_\mathcal{N}P_{(\mathcal{M}\cap\mathcal{N})^\perp}\|$
6. *The following statements are equivalent*
   - *i.* $c\,[\,\mathcal{M},\,\mathcal{N}\,] < 1$
   - *ii.* $\mathcal{M}+\mathcal{N}$ *is closed*
   - *iii.* $\mathcal{M}^\perp+\mathcal{N}^\perp$ *is closed*

Proposition 2.4. *([6], [27]) Given $A, B \in L(\mathcal{H})$, then $R(AB)$ is closed if and only if $c\,[\,R(B),\,N(A)\,] < 1$.*

Proposition 2.5. *Let P and Q be two orthogonal projections defined on $\mathcal{H}$. Then,*

$$\|(PQ)^k - P \wedge Q\| = c\,[\,R(P),\,R(Q)\,]^{2k-1}$$

where $P \wedge Q$ is the orthogonal projection onto $R(P) \cap R(Q)$.

Proposition 2.6 (Ljance-Ptak [31]). *Let Q be a projection with range $\mathcal{R}$ and with nullspace $\mathcal{N}$. Then*

$$\|Q\| = \frac{1}{\left(1 - \|P_{\mathcal{R}}\, P_{\mathcal{N}}\|^2\right)^{1/2}} = \frac{1}{\left(1 - c\,[\,\mathcal{R},\,\mathcal{N}\,]^2\right)^{1/2}} = s\,[\,\mathcal{R},\,\mathcal{N}\,]^{-1}$$

2.2 D-selfadjoint projections and compatibility

Any selfadjoint operator $D \in L(\mathcal{H})$ defines a bounded Hermitian sesquilinear form $\langle \xi, \eta \rangle_D = \langle D\xi, \eta \rangle$, $\xi,\ \eta \in \mathcal{H}$. The D-orthogonal subspace of a subset $\mathcal{S}$ of $\mathcal{H}$ is $\mathcal{S}^{\perp_D} := \{\xi :\ \langle D\xi, \eta \rangle = 0\ \ \forall \eta \in \mathcal{S}\ \} = D^{-1}(\mathcal{S}^{\perp}) = D(\mathcal{S})^{\perp}$.

We say that $C \in L(\mathcal{H})$ is *D-selfadjoint* if $DC = C^*D$. Consider the set of D-selfadjoint projections whose range is exactly $\mathcal{S}$:

$$\mathcal{P}(D, \mathcal{S}) = \{Q \in \mathcal{Q} : R(Q) = \mathcal{S}, DQ = Q^*D\}.$$

A pair $(D, \mathcal{S})$ is called *compatible* if $\mathcal{P}(D, \mathcal{S})$ is not empty. Sometimes we say that D is $\mathcal{S}$-compatible or that $\mathcal{S}$ is D-compatible.

Remark 2.7. It is known (see Douglas [17]), that if $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are Hilbert spaces, $B \in L(\mathcal{H}_3, \mathcal{H}_2)$ and $C \in L(\mathcal{H}_1, \mathcal{H}_2)$, then the following conditions are equivalent:
a) $R(B) \subseteq R(C)$;
b) there exists a positive number λ such that $BB^* \leq \lambda CC^*$ and
c) there exists $A \in L(\mathcal{H}_1, \mathcal{H}_3)$ such that $B = CA$.
Moreover, there exists a unique operator A which satisfies the conditions $B = CA$ and $R(A) \subseteq \overline{R(C^*)}$. In this case, $N(A) = N(B)$ and $\|A\|^2 = \inf\{\lambda : BB^* \leq \lambda CC^*\}$; A is called the *reduced* solution of the equation $CX = B$. If $R(C)$ is closed, then $A = C^\dagger B$.

In the following theorem we present several results about compatibility, taken from [11] and [12].

Theorem 2.8. *If $D \in L(\mathcal{H})$ is selfadjoint, and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, we denote* $D = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}$. *Then:*

1. *$(D, \mathcal{S})$ is compatible if and only if $R(b) \subseteq R(a)$ if and only if $\mathcal{S} + D^{-1}(\mathcal{S}^{\perp}) = \mathcal{H}$.*

2. In this case, if $d \in L(\mathcal{S}^\perp, \mathcal{S})$ is the reduced solution of the equation $ax = b$ then

$$P_{D,\mathcal{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix} \in \mathcal{P}(D, \mathcal{S})$$

and, if $\mathcal{N} = D^{-1}(S^\perp) \cap \mathcal{S}$, then $N(P_{D,\mathcal{S}}) = D^{-1}(\mathcal{S}^\perp) \ominus \mathcal{N}$.

3. If $D \in L(\mathcal{H})^+$ then $\mathcal{N} = N(D) \cap \mathcal{S}$ and, for every $Q \in \mathcal{P}(D, \mathcal{S})$, there is $z \in L(\mathcal{S}^\perp, \mathcal{N})$ such that

$$Q = P_{D,\mathcal{S}} + z = \begin{array}{c} \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \mathcal{S} \ominus \mathcal{N} \\ \mathcal{N} \\ \mathcal{S}^\perp \end{array} . \tag{2}$$

Observe that $\mathcal{P}(D, \mathcal{S})$ has a unique element (namely, $P_{D,\mathcal{S}}$) if and only if $N(D) \cap \mathcal{S} = \{0\}$.

4. $P_{D,\mathcal{S}}$ has minimal norm in $\mathcal{P}(D, \mathcal{S})$, i.e. $\|P_{D,\mathcal{S}}\| = \min\{ \|Q\| : Q \in \mathcal{P}(D, \mathcal{S})\}$.

The reader is referred to [11], [12] and [13] for several applications of $P_{D,\mathcal{S}}$ (see also Hassi and Nordström [24]).

From now on, we shall suppose that $D \in L(\mathcal{H})^+$, in which case $D^{1/2}$ denotes the positive square root of D.

Remark 2.9. Under additional hypothesis on D, other characterizations of compatibility can be used. We mention a sample of these, taken from [11] and [12]:

1. If $R(PDP)$ is closed (or, equivalently, if $R(PD^{1/2})$ or $D^{1/2}(\mathcal{S})$ are closed), then $(D, \mathcal{S})$ is compatible. In this case, if $D = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, then $P_{D,\mathcal{S}} = \begin{pmatrix} 1 & a^\dagger b \\ 0 & 0 \end{pmatrix}$, since $a = PDP$ has closed range, and $a^\dagger b$ is the reduced solution of $ax = b$.
2. If D has closed range then the pair $(D, \mathcal{S})$ is compatible $\iff$ $R(PDP)$ is closed $\iff$ $R(DP)$ is closed $\iff$ $c[N(D), \mathcal{S}] < 1$.
3. If P, Q are orthogonal projections with $R(P) = \mathcal{S}$, then $(Q, \mathcal{S})$ is compatible $\iff$ $R(QP)$ is closed $\iff$ $c[N(Q), \mathcal{S}] < 1$. Moreover, if $(Q, \mathcal{S})$ is compatible, then $\mathcal{H} = \mathcal{S} + Q^{-1}(\mathcal{S}^\perp) = \mathcal{S} + (R(Q) \cap \mathcal{S}^\perp) + N(Q)$ and, if $\mathcal{N} = N(Q) \cap \mathcal{S}$ and $\mathcal{M} = \mathcal{S} \ominus \mathcal{N}$, then $\mathcal{M} \oplus (N(Q) \oplus (R(Q) \cap \mathcal{S}^\perp)) = \mathcal{H}$, and $P_{Q,\,\mathcal{M}}$ is the projection onto $\mathcal{M}$ given by this decomposition. In particular, if $\mathcal{S} \oplus N(Q) = \mathcal{H}$, then $P_{Q,\mathcal{S}}$ is the projection onto $\mathcal{S}$ given by this decomposition. Observe that $P_{Q,\mathcal{S}} = P_\mathcal{N} + P_{Q,\,\mathcal{M}}$. It follows that

$$\|P_{Q,\mathcal{S}}\| = \|P_{Q,\,\mathcal{M}}\| = (1 - \|(1-Q)P_\mathcal{M}\|^2)^{-1/2} = (1 - c\,[\,N(Q),\, \mathcal{S}\,]^2)^{-1/2} = s\,[\,N(Q),\, \mathcal{S}\,]^{-1} .$$

Observe that, in finite dimensional spaces, every pair $(D, \mathcal{S})$ is compatible because every subspace, a fortiori $R(PDP)$, is closed.

We end this section with the following technical result, which we shall need in what follows:

Proposition 2.10. *Suppose that $\mathcal{S} \subseteq \mathcal{T}$ are closed subspaces of $\mathcal{H}$ and $P_{\mathcal{T}}D = DP_{\mathcal{T}}$. If $R(P_{\mathcal{S}}DP_{\mathcal{S}})$ is closed and $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{matrix} \mathcal{T} \\ \mathcal{T}^{\perp} \end{matrix}$, then $P_{D,\mathcal{S}} = \begin{pmatrix} P_{D_1,\mathcal{S}} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{T} \\ \mathcal{T}^{\perp} \end{matrix}$, where we consider $P_{D_1,\mathcal{S}}$ as acting on $\mathcal{T}$.*

Proof. Let $D_1 = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{T} \ominus \mathcal{S} \end{matrix}$. Then $P_{D_1,\mathcal{S}} = \begin{pmatrix} 1 & a^{\dagger}b \\ 0 & 0 \end{pmatrix}$. On the other hand, if

$$D = \begin{pmatrix} a & b & 0 \\ b^* & c & 0 \\ 0 & 0 & D_2 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{T} \ominus \mathcal{S} \\ \mathcal{T}^{\perp} \end{matrix}, \text{ then } P_{D,\mathcal{S}} = \begin{pmatrix} 1 & a^{\dagger}b & a^{\dagger}0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{T} \ominus \mathcal{S} \\ \mathcal{T}^{\perp} \end{matrix} = \begin{pmatrix} P_{D_1,\mathcal{S}} & 0 \\ 0 & 0 \end{pmatrix}.$$

■

Remark 2.11. Proposition 2.10 is still valid if the assumption that "$R(P_{\mathcal{S}}DP_{\mathcal{S}})$ is closed" is replaced by "the pair $(D, \mathcal{S})$ is compatible". The proof follows the same lines but is a little bit more complicated, because it uses the more general notion of reduced solutions (see Remark 2.7) instead of Moore-Penrose pseudoinverses. One must also show that the pair $(D_1, \mathcal{S})$ is compatible in $L(\mathcal{T})$.

3 Scaled projections in finite dimensional spaces

In this section we study scaled projections in finite dimensional Hilbert spaces from the viewpoint of D-selfadjoint projections. This is a new geometrical approach to the widely studied subject of weighted projections which may be helpful in the applications. In the next section we shall use this approach to extend some of these results to infinite dimensional spaces. Additionally, we shall study the projections with *complex weights* as those considered by M. Wei ([35], [34], [36] and E. Bobrovnikova and S. Vavasis [5] and we shall prove generalizations of some well known results about classical scaled projections to the complex case.

Throughout this section, $\mathcal{D}_n$ denotes the abelian algebra of diagonal $n \times n$ complex matrices, $\mathcal{D}_n^+$ denotes the set of positive invertible matrices of $\mathcal{D}_n$ and $\mathcal{P}(\mathcal{D}_n)$ denotes the set of projections in $\mathcal{D}_n$.

Scaled projections are connected with scaled pseudo-inverses which appear in weighted least squares problems of the form

$$\min\{\|D^{1/2}(\beta - A\xi)\|^2 : \xi \in \mathbb{C}^n\},$$

where $m \geq n$, A is an $m \times n$ matrix of full rank, $\beta \in \mathbb{C}^m$ and $D \in \mathcal{D}_m^+$. It is well known that the solution to this problem is

$$\xi = (A^*DA)^{-1}A^*D\beta.$$

The operator $A_D^{\dagger} = (A^*DA)^{-1}A^*D$ is called a *weighted pseudo-inverse* of A.

In some situations it is useful to have a bound for the norms of the scaled pseudo-inverses $A_D^\dagger$. In order to study this problem, G. W. Stewart [32] used the oblique projections $P_D = A(A^*DA)^{-1}A^*D$ and proved that

$$M_A = \sup\{\|A(A^*DA)^{-1}A^*D\| : \ D \in \mathcal{D}_m^+\} < \infty. \tag{3}$$

For $A \in \mathbb{R}^{n\times n}$ and $I \subseteq \{1, \dots, n\}$ let m_I denote the minimal non-zero singular value of the submatrix corresponding to the rows indexed by I of a matrix U whose columns form an orthonormal basis of $N(A)^\perp$.

Stewart [32] proved that

$$M_A^{-1} \leq \min\{m_I : I \subseteq \{1, 2, ..., n\}\}, \tag{4}$$

and O'Leary [29] proved that both numbers actually coincide.

Independently, A. Ben Tal and M. Teboulle [4] proved the next theorem, which refines Stewart-O'Leary's result:

Theorem 3.1. *Let A be an $m\times n$ matrix of full rank and let $J(A)$ be the set of all $Q \in \mathcal{P}(\mathcal{D}_m)$ such that $QA : \mathbb{C}^n \to R(Q)$ is bijective. Then, for every $D \in \mathcal{D}_m^+$ it holds that*

$$A(A^*DA)^{-1}A^*D = \sum_{Q\in J(A)} \left(\frac{\det(D_Q)|\det(A_Q)|^2}{\sum_{P\in J(A)} \det(D_P)|\det(A_P)|^2} \right) A(QA)^{-1}Q$$

where A_Q (resp. D_Q) is QA (resp. QD) considered as a square submatrix of A (resp. D). In particular

$$P_D \in co\left\{A(QA)^{-1}Q : \ Q \in J(A)\right\}$$

The reader will find illustrative surveys in the papers by Forsgren [19], Forsgren and Spörre [20] and in Ben-Israel and Greville's book [3] nice surveys on these matters. See also the papers by Hanke and Neumann ([23]), Gonzaga and Lara [22], Wei [33], [34], [35] and Wei and de Pierro [36].

Given a fixed positive diagonal matrix $D \in \mathcal{D}_m$, the solution of

$$\min\{\|D^{1/2}(\beta - \xi)\|^2 : \xi \in R(A)\}$$

is given by $\xi = P_D\,\beta$. Observe that $\|D^{1/2}\cdot\|$ is the norm induced by the inner product $\langle D\cdot, \cdot\rangle$, and therefore, P_D is the (unique) projection onto $R(A)$ that is orthogonal with respect to the inner product $\langle D\cdot, \cdot\rangle$. Therefore, under the notations of section 2.2, $P_D = P_{D,\,R(A)}$. It is natural to ask if $A(QA)^{-1}Q$ coincides with $P_{Q,\,R(A)}$ for every Q under the conditions of Ben Tal and Teboulle's Theorem 3.1. The answer to this question is the goal of the next Proposition.

Definition 3.2 (Dixmier [16]). *Two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ are in position P' if it holds that $\mathcal{T}^\perp \cap \mathcal{S} = \mathcal{T} \cap \mathcal{S}^\perp = \{0\}$. In this case, we write $\mathcal{S} \veebar \mathcal{T}$.*

Proposition 3.3. *Given an $m \times n$ matrix A of full rank, let $D \in \mathcal{D}_m^+$ and let Q be a diagonal projection. Then*

1. $P_{D,\,R(A)} = A(A^*DA)^{-1}A^*D$.
2. $QA : \mathbb{C}^n \to R(Q)$ *is bijective if and only if* $R(Q) \veebar R(A)$.
3. *If* $R(Q) \veebar R(A)$ *then* $P_{Q,\,R(A)} = A(QA)^{-1}Q$.

Proof.

1. It suffices to observe the coincidence of the range (resp. nullspace) of both projections $P_{D,\,R(A)}$ and $A(A^*DA)^{-1}A^*D$.
2. $QA : \mathbb{C}^n \to R(Q)$ is bijective if and only if $QA : \mathbb{C}^n \to R(Q)$ and $A^*Q : R(Q) \to \mathbb{C}^n$ are injective. As $QA : \mathbb{C}^n \to R(Q)$ is injective if and only if $R(A) \cap N(Q) = \{0\}$ and $A^*Q : R(Q) \to \mathbb{C}^n$ is injective if and only if $R(Q) \cap R(A)^\perp = R(Q) \cap N(A^*) = \{0\}$, it follows that $QA : \mathbb{C}^n \to R(Q)$ is bijective if and only if $R(Q) \veebar R(A)$.
3. Clearly, $A(QA)^{-1}Q$ is a projection whose range is $R(A)$ and whose nullspace is $N(Q)$. But, $P_{Q,\,R(A)}$ is also a projector with the same range and nullspace as $A(QA)^{-1}Q$. In fact, by item 2 in Theorem 2.8,

$$R(P_{Q,\,R(A)}) = R(A) \quad \text{and} \quad N(P_{Q,\,R(A)}) = Q^{-1}(R(A)^\perp) \ominus (N(Q) \cap R(A)) = N(Q).$$

Hence $A(QA)^{-1}Q = P_{Q,\,R(A)}$. ■

Using Proposition 3.3 we can restate Theorem 3.1 in the following way:

Theorem 3.4. *Let $\mathcal{S}$ be a subspace of $\mathbb{C}^n$ and let $D \in \mathcal{D}_n^+$. Then*

$$P_{D,\mathcal{S}} \in co\{P_{Q,\mathcal{S}} : Q \in \mathcal{P}(\mathcal{D}_n) \ \text{and} \ R(Q) \veebar \mathcal{S}\}. \tag{5}$$

In particular,

$$\sup_{D \in \mathcal{D}_n^+} \|P_{D,\mathcal{S}}\| \leq \max\Big\{\|P_{Q,\mathcal{S}}\| : Q \in \mathcal{P}(\mathcal{D}_n) : \ R(Q) \veebar \mathcal{S}\Big\}. \tag{6}$$

Remark 3.5. Inequality (6) is actually an equality. The converse inequality was proved by Stewart and it is also a consequence of the next Proposition which follows essentially Stewart's ideas.

Proposition 3.6. *Let $\mathcal{S}$ be a subspace of $\mathbb{C}^n$ and denote by $\mathcal{D}_{0,n}^+$ the set of all diagonal positive semidefinite $n \times n$ matrices. Then*

$$\sup_{D \in \mathcal{D}_n^+} \|P_{D,\mathcal{S}}\| = \sup_{D \in \mathcal{D}_{0,n}^+} \|P_{D,\mathcal{S}}\|. \tag{7}$$

Proof. Let $D \in \mathcal{D}_{0,n}^+$ and consider the sequence of invertible positive operators $\{D_k\}_{k\geq 1}$ defined by

$$D_k = D + \frac{1}{k}\, I.$$

If $\mathcal{N} = \mathcal{S} \cap N(D)$, then

$$D = \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ b^* & 0 & c \end{pmatrix} \begin{matrix} \mathcal{S} \ominus \mathcal{N} \\ \mathcal{N} \\ \mathcal{S}^\perp \end{matrix} \qquad \text{and} \qquad D_k = \begin{pmatrix} a + \frac{1}{k} I & 0 & b \\ 0 & \frac{1}{k} I & 0 \\ b^* & 0 & c + \frac{1}{k} I \end{pmatrix} \begin{matrix} \mathcal{S} \ominus \mathcal{N} \\ \mathcal{N} \\ \mathcal{S}^\perp \end{matrix},$$

where a, and therefore $a + \dfrac{1}{k} I$ are invertible. Hence, by by Theorem 2.8,

$$P_{D_k, \mathcal{S}} = \begin{pmatrix} I & 0 & (a + \frac{1}{k} I)^{-1} b \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad P_{D, \mathcal{S}} = \begin{pmatrix} I & 0 & a^{-1} b \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So we obtain

$$\|P_{D,\mathcal{S}}\| = \lim_{k\to\infty} \|P_{D_k,\mathcal{S}}\| \leq \sup_{D' \in \mathcal{D}^+} \|P_{D',\mathcal{S}}\|$$

which proves one inequality. The other inequality is a consequence of equation (6). ∎

Corollary 3.7. *Let $\mathcal{S}$ be a subspace of $\mathbb{C}^n$. Then*

$$\sup_{D \in \mathcal{D}_n^+} \|P_{D,\mathcal{S}}\| = \max \Big\{ \|P_{Q,\mathcal{S}}\| : Q \in \mathcal{P}(\mathcal{D}_n) : \ R(Q) \veebar \mathcal{S} \Big\}. \tag{8}$$

Remark 3.8. This Corollary and the equation (4) are connected in the following sense. Observe that, by Theorem 3.6, the maximum can be taken either over all projections in position P' with $\mathcal{S}$ or over all the diagonal projections. So, if $I \subseteq \{1, \ldots, n\}$ and Q_I is the orthogonal projection onto the diagonal subspace spanned by $\{e_i : \ i \in I\}$, then

$$\sup_{D \in \mathcal{D}_n^+} \|P_{D,\mathcal{S}}\| = \max \Big\{ \|P_{Q_I,\mathcal{S}}\| : I \subseteq \{1, \ldots, n\} \Big\}$$

Given a fixed $I \subseteq \{1, \ldots, n\}$, by Proposition 2.6 it is easy to see that for a given diagonal projection Q

$$\|P_{Q_I,\mathcal{S}}\|^{-2} = s\,[\,\mathcal{S},\ N(Q_I)\,] = \min\{\langle Q_I x,\, x\rangle : x \in \mathcal{S} \ominus (\mathcal{S} \cap N(Q_I)) \text{ and } \|x\| = 1\}$$

Since $(\mathcal{S} \cap N(Q_I)) \oplus \mathcal{S}^\perp$ is the null space of $P_\mathcal{S} Q_I P_\mathcal{S}$, the previous equality can be rewritten as

$$\|P_{Q_I,\mathcal{S}}\|^{-2} = \min\{\lambda \in \sigma\,(P_\mathcal{S} Q_I P_\mathcal{S}) : \ \lambda \neq 0\} = \min\{\lambda \in \sigma\,((Q_I P_\mathcal{S} Q_I) : \ \lambda \neq 0\}$$

Consequently, if U is a matrix whose columns form an orthonormal basis of $\mathcal{S}$, then we get $Q_I P_\mathcal{S} Q_I = Q_I U U^* Q_I$. Therefore

$$m_I^2 = \min\{\lambda \in \sigma\,(Q_I P_\mathcal{S} Q_I) : \ \lambda \neq 0\}$$

which shows that equations (4) and (8) are equivalent. In particular, note that if A is a full rank matrix whose range is $\mathcal{S}$ and the subspaces $R(Q_I)$ and $\mathcal{S}$ are in position P', then

$$\|P_{Q_I,\mathcal{S}}\| = \|A(Q_I A)^{-1} Q_I\| = m_I^{-1}.$$

Next, we consider projections with complex weights. These projections are studied by Bobrovnikova and Vavasis in [5], who define, for each positive real number μ, the sets

$$C_\mu = \{z \in \mathbb{C} : \ |\operatorname{Im} z| \leq \mu \operatorname{Re} z \text{ and } z \neq 0\} \quad \text{and} \quad \mathcal{D}_\mu = \{D : \ D \in \mathcal{D} \text{ with entries in } C_\mu\},$$

and they prove that

$$\overline{\chi}_{A,\,\mu} = \sup_{D \in \mathcal{D}_\mu} \|A(A^*DA)^{-1}A^*D\| < \infty.$$

Let Z be an $m \times (m-n)$ matrix, such that its columns form a basis of $R(A)^\perp$. If the weights are positive, i.e. if $\mu = 0$, then Gonzaga and Lara [22] prove that $\overline{\chi}_{A,0} = \overline{\chi}_{Z,0}$. In the next Proposition, we generalize this result for $\mu \neq 0$.

Proposition 3.9. *Let A be an $m \times n$ matrix of full rank, and let Z be an $m \times (m-n)$ matrix whose columns form a basis of $R(A)^\perp$. Then,*

$$\overline{\chi}_{A,\,\mu} = \overline{\chi}_{Z,\,\mu} \qquad \forall \mu > 0$$

Proof. Fix $D \in \mathcal{D}_\mu$. On one hand, $A(A^*DA)^{-1}A^*D$ is idempotent and

$$\begin{aligned} R(A(A^*DA)^{-1}A^*D) &= R(A) =: R \\ N(A(A^*DA)^{-1}A^*D) &= N(A^*D) = R(D^*A)^\perp = \Big[D^*(R(A))\Big]^\perp =: N \end{aligned}$$

On the other hand, $Z(Z^*D^{-1}Z)^{-1}Z^*D^{-1}$ is also idempotent and

$$\begin{aligned} R(Z(Z^*D^{-1}Z)^{-1}Z^*D^{-1}) &= R(Z) = R(A)^\perp = R^\perp \\ N(Z(Z^*D^{-1}Z)^{-1}Z^*D^{-1}) &= N(Z^*D^{-1}) = R(D^{*-1}Z)^\perp = \Big[D^{*-1}(R(A)^\perp)\Big]^\perp \\ &= D(R(A)) = N^\perp \end{aligned}$$

Using the fact that $\|P_R P_N\| = c\,[\,R,\, N\,] = c\,[\,R^\perp,\, N^\perp\,] = \|P_{R^\perp} P_{N^\perp}\|$, and Ljance-Ptak's formula (Proposition 2.6), we obtain

$$\begin{aligned} \|A^*(ADA^*)^{-1}AD\| &= (1 - \|P_R P_N\|^2)^{-1/2} \\ &= (1 - \|P_{R^\perp} P_{N^\perp}\|^2)^{-1/2} = \|Z^*(ZD^{-1}Z^*)^{-1}ZD^{-1}\| \end{aligned}$$

Finally, since the map $D \to D^{-1}$ is a bijection of the set $\mathcal{D}_\mu$, the result follows just by taking supremum over all positive definite diagonal matrices. ■

4 Compatibility of subspaces and orthonormal basis

Definitions and main results

Throughout this section, $\mathcal{H}$ is a separable Hilbert space with a fixed orthonormal basis $\mathcal{B} = \{e_k\}_{k \in \mathbb{N}}$. Consider the abelian algebra $\mathcal{D}$ of all operators which are diagonal with respect to $\mathcal{B}$, i.e. $C \in L(\mathcal{H})$ belongs to $\mathcal{D}$ if there exists a bounded sequence of complex numbers $\{c_n\}$ such that $Ce_n = c_n e_n$ $(n \in \mathbb{N})$. Denote by $\mathcal{D}^+$ the set of all positive invertible operators of $\mathcal{D}$ and by $\mathcal{P}(\mathcal{D})$ the set of all projections of $\mathcal{D}$.

Let us extend the definition of compatibility to the context of the diagonal algebra $\mathcal{D}$.

Definition 4.1. *A closed subspace $\mathcal{S}$ of $\mathcal{H}$ is **compatible** with $\mathcal{B}$ (or $\mathcal{B}$-**compatible**) if*

$$\sup_{Q\in\mathcal{P}(\mathcal{D})} c\,[\,\mathcal{S},\, R(Q)\,] := c\,[\,\mathcal{S},\, \mathcal{D}\,] < 1. \tag{9}$$

In this case, we define

$$K\,[\,\mathcal{S},\, \mathcal{D}\,] = (1 - c\,[\,\mathcal{S},\, \mathcal{D}\,]^2)^{-1/2} = \left(\inf_{Q\in\mathcal{P}(\mathcal{D})} s\,[\,\mathcal{S},\, R(Q)\,]\right)^{-1}$$

Remarks 4.2.

1. Since $c\,[\,\mathcal{S},\, \mathcal{T}\,] = c\left[\,\mathcal{S}^\perp,\, \mathcal{T}^\perp\,\right]$ for every pair of closed subspaces $\mathcal{S}$ and $\mathcal{T}$, a subspace $\mathcal{S}$ is $\mathcal{B}$-compatible if and only if $\mathcal{S}^\perp$ is $\mathcal{B}$-compatible. Moreover, $c\,[\,\mathcal{S},\, \mathcal{D}\,] = c\left[\,\mathcal{S}^\perp,\, \mathcal{D}\,\right]$.

2. If the dimension of the Hilbert space is finite, every subspace is compatible with every orthonormal basis.

Using Remark 2.9 we can give an alternative characterization of compatibility.

Theorem 4.3. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, $\mathcal{S}$ is $\mathcal{B}$-compatible if and only if*

$$\sup_{Q\in\mathcal{P}(\mathcal{D})} \|P_{Q,\mathcal{S}}\| < \infty. \tag{10}$$

Moreover, in this case,

$$\sup_{Q\in\mathcal{P}(\mathcal{D})} \|P_{Q,\mathcal{S}}\| = K[\mathcal{S}, \mathcal{D}] \quad .$$

Proof. Given a projection Q, by Remark 2.9 we know that

$$\|P_{Q,\mathcal{S}}\| = (1 - c\,[\,\mathcal{S},\, N(Q)\,]^2)^{-1/2}.$$

If $\mathcal{S}$ is compatible with $\mathcal{B}$, then $c\,[\,\mathcal{S},\, \mathcal{D}\,] < 1$, and therefore

$$\sup_{Q\in\mathcal{P}(\mathcal{D})} \|P_{Q,\mathcal{S}}\| \le (1 - c\,[\,\mathcal{S},\, \mathcal{D}\,]^2)^{-1/2} = K\,[\,\mathcal{S},\, \mathcal{D}\,] < \infty.$$

Conversely, if (10) holds, there exist $M > 1$ such that $\sup_{Q\in\mathcal{P}(\mathcal{D})} \|P_{Q,\mathcal{S}}\| \le M$. Therefore,

$$\sup_{Q\in\mathcal{P}(\mathcal{D})} c\,[\,\mathcal{S},\, R(Q)\,] = (1 - M^{-2})^{-1/2} < 1.$$

■

The main result of this section is the following theorem which is the natural generalization of Theorem 3.4 (or, more precisely, Corollary 3.7) to the infinite dimensional setting.

Theorem 4.4. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, the following statements are equivalent*

1. *$\mathcal{S}$ is compatible with $\mathcal{B}$;*

2. $\sup\limits_{D\in\mathcal{D}^+} \|P_{D,\mathcal{S}}\| < \infty$.

In this case, it holds

$$\sup_{D\in\mathcal{D}^+} \|P_{D,\mathcal{S}}\| = K\,[\,\mathcal{S},\,\mathcal{D}\,] = \sup_{Q\in\mathcal{P}(\mathcal{D})} \|P_{Q,\mathcal{S}}\|.$$

The proof of this Theorem will be divided in several parts. We start with a technical result.

Lemma 4.5. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and suppose that $\sup_{D\in\mathcal{D}^+} \|P_{D,\mathcal{S}}\| < \infty$. Then, for every projection $Q \in \mathcal{P}(\mathcal{D})$ the pair $(Q,\mathcal{S})$ is compatible, so that the oblique projection $P_{Q,\mathcal{S}}$ is well defined for all $Q \in \mathcal{P}(\mathcal{D})$.*

Proof. Let $Q \in \mathcal{P}(\mathcal{D})$ and consider the sequence of invertible positive operators $\{D_k\}_{k\geq 1}$ defined by $D_k = Q+\frac{1}{k}I$. Since D_k is invertible, the projection $P_{D_k,\mathcal{S}}$ is well defined. Moreover, by hypothesis we know that $\sup_{k\geq 1} \|P_{D_k,\mathcal{S}}\| < \infty$. Therefore, the sequence $\{P_{D_k,\mathcal{S}}\}$ has a limit point P in the weak operator topology (WOT) of $L(\mathcal{H})$, because the unit ball of $L(\mathcal{H})$ is WOT-compact (see 5.1.3 of [28]). Moreover, if $\mathcal{H}$ is separable, the ball is metrizable for the weak operator topology. Therefore, we can suppose that $P_{D_k,\mathcal{S}} \xrightarrow[n\to\infty]{\text{W.O.T.}} P$.

We shall prove that $P \in \mathcal{P}(Q,\,\mathcal{S})$, that is, $P^2 = P$, $R(P) = \mathcal{S}$ and $QP = P^*Q$. The first two conditions follow from the fact that, for every $k \in \mathbb{N}$,

$$P_{D_k,\mathcal{S}} = \begin{pmatrix} 1 & x_k \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \quad , \quad \text{so that} \quad P = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \quad ,$$

where x is the WOT-limit of the sequence $x_k = P_\mathcal{S} D_k(1 - P_\mathcal{S})$. On the other hand, for each $k \in \mathbb{N}$,

$$D_k P_{D_k,\mathcal{S}} = P^*_{D_k,\mathcal{S}} D_k \ .$$

An easy $\frac{\varepsilon}{2}$ argument shows that $D_k P_{D_k,\mathcal{S}} \xrightarrow[n\to\infty]{\text{W.O.T.}} QP$, so, taking limit in the above equality and using the fact that the involution is continuous in the weak operator topology, we obtain $QP = P^*Q$. ■

The next result, which can be proved in the same way as Proposition 3.6, by using Lemma 4.5, provides the easier inequality in Theorem 4.4:

Proposition 4.6. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and suppose that $\sup_{D\in\mathcal{D}^+} \|P_{D,\mathcal{S}}\| < \infty$. Then, $\mathcal{S}$ is $\mathcal{B}$-compatible and*

$$K\,[\,\mathcal{S},\,\mathcal{D}\,] = \sup_{Q\in\mathcal{P}(\mathcal{D})} \|P_{Q,\mathcal{S}}\| \leq \sup_{D\in\mathcal{D}^+} \|P_{D,\mathcal{S}}\| \tag{11}$$

The other inequality of Theorem 4.4 is more complicated, so we first need to prove a particular case of it. For each $n \in \mathbb{N}$, denote $\mathcal{H}_n = \operatorname{span}\{e_1, \ldots, e_n\}$, Q_n the orthogonal projection onto $\mathcal{H}_n$, and for any closed subspace $\mathcal{S}$, denote $\mathcal{S}_n = \mathcal{S} \cap \mathcal{H}_n$. Recall that, given two orthogonal projections P and Q, $P \wedge Q$ denotes the orthogonal projection onto $R(P) \cap R(Q)$.

Proposition 4.7. *Let $\mathcal{S}$ be a finite dimensional subspace of $\mathcal{H}$ such that, for some $n \in \mathbb{N}$, $\mathcal{S} \subseteq \mathcal{H}_n$. Then $\mathcal{S}$ is $\mathcal{B}$-compatible. Moreover, if $E \in \mathcal{P}_0(\mathcal{D})$ satisfies $P_{\mathcal{S}} \leq E$, then*

$$\{P_{D,\mathcal{S}} : D \in \mathcal{D}^+\} \subseteq co\{P_{Q,\mathcal{S}} : Q \in \mathcal{P}(\mathcal{D}),\ Q \leq E \text{ and } R(Q) \veebar \mathcal{S}\}.$$

In particular,

$$\sup_{D \in \mathcal{D}^+} \|P_{D,\mathcal{S}}\| \leq \sup\{\|P_{Q,\mathcal{S}}\| : Q \in \mathcal{P}(\mathcal{D}),\ Q \leq E \text{ and } R(Q) \veebar \mathcal{S}\} = K\,[\,\mathcal{S},\, \mathcal{D}\,]\,.$$

Proof. Let $E \in \mathcal{P}_0(\mathcal{D})$ be such that $P_{\mathcal{S}} \leq E$. Denote $\mathcal{T} = R(E)$. Given $D \in \mathcal{D}$, $D \geq 0$, the subspace $\mathcal{T}$ induces a matrix decomposition of D,

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{array}{c} \mathcal{T} \\ \mathcal{T}^\perp \end{array} .$$

If the pair $(D, \mathcal{S})$ is compatible, it is easy to see that the pair $(D_1, \mathcal{S})$ is compatible in $L(\mathcal{T})$ and, by Proposition 2.10,

$$P_{D,\mathcal{S}} = \begin{pmatrix} P_{D_1,\mathcal{S}} & 0 \\ 0 & 0 \end{pmatrix} \begin{array}{c} \mathcal{T} \\ \mathcal{T}^\perp \end{array} , \tag{12}$$

where $P_{D_1,\mathcal{S}}$ is considered as an operator in $L(\mathcal{T})$. Since $\dim \mathcal{T} < \infty$, we deduce that $\mathcal{S}$ is $\mathcal{B}$-compatible. The other statements follow from Theorems 3.4 and 4.3. ■

Lemma 4.8. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that*

$$c \ := \ \sup\,\{c\,[\,\mathcal{S},\, \mathcal{H}_n\,] : n \in \mathbb{N}\} < 1. \tag{13}$$

Then

$$\overline{\bigcup_{n=1}^{\infty} \mathcal{S}_n} = \mathcal{S}\ .$$

Proof. The assertion of the Lemma is equivalent to

$$P_{\mathcal{S}} \wedge Q_n \underset{n\to\infty}{\overset{\text{SOT}}{\nearrow}} P_{\mathcal{S}}\ .$$

Let $\xi \in \mathcal{H}$ be a unit vector and let $\varepsilon > 0$. Let $k \in \mathbb{N}$ such that $c^{2k-1} \leq \dfrac{\varepsilon}{2}$. By Proposition 2.5, for every $n \geq 1$ it holds that

$$\left\|(P_{\mathcal{S}}Q_n)^k - P_{\mathcal{S}} \wedge Q_n\right\| \leq \frac{\varepsilon}{2}.$$

On the other hand, since $Q_n P_{\mathcal{S}} \xrightarrow[n\to\infty]{\text{S.O.T.}} P_{\mathcal{S}}$ and the function $f(x) = x^k$ is SOT-continuous on bounded sets (see, for example, 2.3.2 of [30]), there exists $n_0 \geq 1$ such that, for every $n \geq n_0$,

$$\left\| \left[(Q_n P_{\mathcal{S}})^k - P_{\mathcal{S}} \right] \xi \right\| < \frac{\varepsilon}{2} .$$

Therefore, for every $n \geq n_0$,

$$\|(P_{\mathcal{S}} - P_{\mathcal{S}} \wedge Q_n)\ \xi\| \leq \left\| \left[P_{\mathcal{S}} - (P_{\mathcal{S}} Q_n)^k \right] \xi \right\| + \left\| \left((P_{\mathcal{S}} Q_n)^k - P_{\mathcal{S}} \wedge Q_n \right) \xi \right\| < \varepsilon .$$

■

Observe that, using Proposition 4.7 and Lemma 4.8, we get the following characterization of finite dimensional subspaces which are $\mathcal{B}$-compatible:

Corollary 4.9. *Let $\mathcal{S}$ be a finite dimensional subspace of $\mathcal{H}$. Then $\mathcal{S}$ is $\mathcal{B}$-compatible if and only if there exists $n \in \mathbb{N}$ such that $\mathcal{S} \subseteq \mathcal{H}_n$.*

Lemma 4.10. *Let $\mathcal{S}$ be a $\mathcal{B}$-compatible subspace and $\mathcal{S}_n = \mathcal{S} \cap \mathcal{H}_n$, $n \in \mathbb{N}$. Then*

$$K[\mathcal{S}_n, \mathcal{D}] = \sup_{Q \in \mathcal{P}(\mathcal{D})} \|P_{Q, \mathcal{S}_n}\| \leq \sup_{Q \in \mathcal{P}(\mathcal{D})} \|P_{Q, \mathcal{S}}\| = K[\mathcal{S}, \mathcal{D}]$$

for every $n \geq 1$.

Proof. Using Corollary 4.7 and Theorem 4.6, we get

$$K[\mathcal{S}_n, \mathcal{D}] = \sup\{\|P_{Q, \mathcal{S}_n}\| : Q \in \mathcal{P}(\mathcal{D}),\ Q \leq Q_n \text{ and } R(Q) \veebar \mathcal{S}_n\}.$$

Thus, it suffices to prove the inequality for every $Q \in \mathcal{P}(\mathcal{D})$ such that $R(Q) \veebar \mathcal{S}_n$ and $Q \leq Q_n$. For each such Q consider $\widehat{Q} = Q + (1 - Q_n) \in \mathcal{P}(\mathcal{D})$. Then $N(\widehat{Q}) = N(Q) \cap R(Q_n)$, and

$$\begin{aligned} c[N(Q), \mathcal{S}_n] &= \sup\{ |\langle \xi, \eta \rangle| : \ \xi \in N(Q),\ \eta \in \mathcal{S}_n \text{ and } \|\xi\| = \|\eta\| = 1\} \\ &= \sup\{ |\langle \xi, \eta \rangle| : \ \xi \in N(Q) \cap R(Q_n),\ \eta \in \mathcal{S}_n \text{ and } \|\xi\| = \|\eta\| = 1\} \\ &\leq \sup\{ |\langle \xi, \eta \rangle| : \ \xi \in N(Q) \cap R(Q_n),\ \eta \in \mathcal{S} \text{ and } \|\xi\| = \|\eta\| = 1\} \\ &= \sup\{ |\langle \xi, \eta \rangle| : \ \xi \in N(\widehat{Q}),\ \eta \in \mathcal{S} \text{ and } \|\xi\| = \|\eta\| = 1\} \\ &= c\left[N(\widehat{Q}), \mathcal{S} \right]. \end{aligned}$$

Therefore, using Remark 2.9, we obtain

$$\|P_{Q, \mathcal{S}_n}\| = s[N(Q), \mathcal{S}_n]^{-1} \leq s[N(Q), \mathcal{S}]^{-1} = s\left[N(\widehat{Q}), \mathcal{S} \right]^{-1} = \|P_{\widehat{Q}, \mathcal{S}}\|,$$

which proves the desired inequality. ■

Remark 4.11. The statement of Lemma 4.10 can be rewritten as

$$K\left[\mathcal{S}_n,\, \mathcal{D}\right] \leq \left(\inf_{E\in\mathcal{P}(\mathcal{D})} s\left[\mathcal{S},\, R(E)\right]\right)^{-1} .$$

Actually, we proved that it suffices to take the infimum over the projections $E \in \mathcal{P}_0(\mathcal{D})$. Indeed, it is enough to consider $E = 1 - \widehat{Q}$, where $\widehat{Q}$ are the projections which appear in the proof of Lemma 4.10.

We can now complete the proof of Theorem 4.4.

Proposition 4.12. *Let $\mathcal{S}$ be a $\mathcal{B}$-compatible subspace of $\mathcal{H}$. Then*

$$\sup_{D\in\mathcal{D}^+} \|P_{D,\mathcal{S}}\| \leq K\left[\mathcal{S},\, \mathcal{D}\right]. \tag{14}$$

Proof. Fix $D \in \mathcal{D}^+$. Recall that $\|\cdot\|_D$ denotes the norm defined by $\xi \to \|D^{1/2}\xi\|$. Since D is invertible, $\|\cdot\|_D$ is equivalent to $\|\cdot\|$; thus, the union of the subspaces $\mathcal{S}_n$ is dense in $\mathcal{S}$ under both norms $\|\cdot\|_D$ and $\|\cdot\|$. Since $P_{D,\mathcal{S}}$ (resp. $P_{D,\mathcal{S}_n}$) is the D-orthogonal projection onto the subspace $\mathcal{S}$ (resp. $\mathcal{S}_n$), then for every unit vector $\xi \in \mathcal{H}$

$$\|(P_{D,\mathcal{S}_n} - P_{D,\mathcal{S}})\,\xi\|_D \xrightarrow[n\to\infty]{} 0.$$

Using again the equivalence between the norms $\|\cdot\|_D$ and $\|\cdot\|$, we get

$$\|(P_{D,\mathcal{S}_n} - P_{D,\mathcal{S}})\,\xi\| \xrightarrow[n\to\infty]{} 0.$$

On the other hand, using Lemma 4.10 and Proposition 4.7, it holds that, for each $n \in \mathbb{N}$,

$$\|P_{D,\mathcal{S}_n}\,\xi\| \leq \|P_{D,\mathcal{S}_n}\| \leq K\left[\mathcal{S}_n,\, \mathcal{D}\right] \leq K\left[\mathcal{S},\, \mathcal{D}\right].\ \sup_{Q\in\mathcal{P}(\mathcal{D})} \|P_{Q,\mathcal{S}}\|.$$

Thus,

$$\|P_{D,\mathcal{S}}\,\xi\| = \lim_{n\to\infty} \|P_{D,\mathcal{S}_n}\,\xi\| \leq K\left[\mathcal{S},\, \mathcal{D}\right],$$

which completes the proof. ∎

Alternative characterizations of compatibility

In this section we add some characterizations of compatibility which involve only finite dimensional diagonal subspaces.

Let us begin with a Proposition whose proof is connected with the proof of Theorem 4.4. We use the notations $\mathcal{H}_n$, Q_n and $\mathcal{S}_n$ (for a closed subspace $\mathcal{S}$) as before.

Proposition 4.13. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, the following statements are equivalent*

1. *$\mathcal{S}$ is $\mathcal{B}$-compatible;*

2. *a.* $\bigcup_{n=1}^{\infty} \mathcal{S}_n$ *is dense in* $\mathcal{S}$*;*

b. there exists $M > 0$ *such that* $K[\mathcal{S}_n, \mathcal{D}] = \sup_{Q\in\mathcal{P}(\mathcal{D})} \|P_{Q,\mathcal{S}_n}\| \leq M$ *for every* $n \in \mathbb{N}$.

Proof.

$2 \Rightarrow 1$. It is a consequence of lemmas 4.8 and 4.10.

$1 \Rightarrow 2$. Following the same argument as in Proposition 4.12, we obtain that

$$\sup_{D\in\mathcal{D}^+} \|P_{D,\mathcal{S}}\| < \infty,$$

and, by Proposition 4.6, $\mathcal{S}$ is $\mathcal{B}$-compatible. ■

Given $T \in L(\mathcal{H})$, its reduced minimum modulus (see, e.g., [26]), is defined by

$$\gamma(T) = \inf\{\|T\xi\| : \xi \in N(T)^\perp, \|\xi\| = 1\}.$$

It is easy to see that $\gamma(T) > 0$ if and only if $R(T)$ is closed. By Proposition 2.4, if $A, B \in L(\mathcal{H})$ have closed ranges, then

$$\gamma(AB) > 0 \iff c[N(A), R(B)] < 1.$$

The following proposition describes a useful relation between angles and the reduced minimum modulus of an operator.

Proposition 4.14. *Let* $T \in L(\mathcal{H})$ *and let* $P \in L(\mathcal{H})$ *be a projection with* $R(P) = \mathcal{S}$*. Suppose that* $\gamma(T) > 0$*. Then*

$$\gamma(T)(1 - c\,[\,N(T), \mathcal{S}\,]^2\,)^{1/2} \leq \gamma(TP) \leq \|T\|(1 - c\,[\,N(T), \mathcal{S}\,]^2\,)^{1/2}. \tag{15}$$

Proof. Note that $c[N(T), \mathcal{S}] = c_0[N(T), \mathcal{S} \ominus (N(T) \cap \mathcal{S})] = \|P_{N(T)} P_{\mathcal{S}\ominus(N(T)\cap\mathcal{S})}\|$, by Proposition 2.3. On the other hand,

$$N(T)P = P^{-1}(N(T)) = P^{-1}(N(T) \cap \mathcal{S}) = N(P) \oplus (N(T) \cap \mathcal{S}),$$

so that

$$(N(T)P)^\perp = \mathcal{S} \ominus (N(T) \cap \mathcal{S}) \subseteq \mathcal{S} = R(P).$$

If $\xi \in (N(T)P)^\perp$, $\|TP\xi\| = \|T\xi\| = \|T(P_{N(T)^\perp}\xi)\|$. Therefore, for every $\xi \in N(TP)^\perp$,

$$\gamma(T)\|P_{N(T)^\perp}\xi\| \leq \|TP\xi\| \leq \|T\|\|P_{N(T)^\perp}\xi\|.$$

Now, if $\|\xi\| = 1$, then

$$\|P_{N(T)^\perp}\xi\|^2 = 1 - \|P_{N(T)}\xi\|^2 \geq 1 - \|P_{N(T)}P_{\mathcal{S}\ominus(N(T)\cap\mathcal{S})}\|^2 = 1 - c[N(T), \mathcal{S}]^2,$$

since $\xi \in \mathcal{S} \ominus (N(T) \cap \mathcal{S})$. This shows that $\gamma(T)(1 - c[N(T), \mathcal{S}]\ )^{1/2} \leq \gamma(TP)$.

In order to prove the second inequality, consider a sequence ξ_n of unit vectors in $\mathcal{S} \ominus (N(T) \cap \mathcal{S}) = (N(T)P)^{\perp}$ such that $\|P_{N(T)}\xi_n\| \to \|P_{N(T)}P_{\mathcal{S}\ominus(N(T)\cap\mathcal{S})}\| = c[N(T), \mathcal{S}]$; then,

$$\gamma(TP)^2 \leq \|TP\xi_n\|^2 \leq \|T\|^2\|P_{N(T)^{\perp}}\xi_n\|^2 = \|T\|^2(1 - \|P_{N(T)}\xi_n\|^2) \to \|T\|^2(1 - c[N(T), \mathcal{S}]^2).$$

■

Recall the notations $\mathcal{P}_0(\mathcal{D}) = \{Q \in \mathcal{P}(\mathcal{D}) : Q \text{ has finite rank}\}$ and $\mathcal{P}_{0,\mathcal{S}}(\mathcal{D}) = \{Q \in \mathcal{P}_0(\mathcal{D}) : R(Q) \cap \mathcal{S} = \{0\}\ \}$.

Proposition 4.15. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, the following conditions are equivalent:*

1. *$\mathcal{S}$ is $\mathcal{B}$-compatible, that is* $\sup\limits_{Q\in\mathcal{P}(\mathcal{D})} c\,[\,\mathcal{S},\, R(Q)\,] < 1$;

2. $\sup\limits_{Q\in\mathcal{P}_0(\mathcal{D})} c\,[\,\mathcal{S},\, R(Q)\,] < 1$;

3. $\sup\limits_{Q\in\mathcal{P}_{0,\mathcal{S}}(\mathcal{D})} c\,[\,\mathcal{S},\, R(Q)\,] < 1$.

Proof. It is clear that $1 \Rightarrow 2 \Rightarrow 3$. In order to prove that $3 \Rightarrow 2$, let $T \in L(\mathcal{H})$ such that $N(T) = \mathcal{S}$ and $\gamma(T) > 0$. Given $Q \in \mathcal{P}_0(\mathcal{D})$ there exists $E \in \mathcal{P}_{0,\mathcal{S}}(\mathcal{D})$ such that $E \leq Q$ and $R(TQ) = R(TE)$. So, using equation (15), we obtain

$$\begin{aligned}\gamma(T)^2(1 - c\,[\,\mathcal{S},\, R(E)\,]^2\ ) &\leq \gamma(TE)^2 = \gamma(TET^*) \leq \gamma(TQT^*) = \gamma(TQ)^2 \\ &\leq \|T\|^2(1 - c\,[\,\mathcal{S},\, R(Q)\,]^2\ ).\end{aligned}$$

Therefore,

$$0 < \frac{\gamma(T)^2}{\|T\|^2} \inf_{Q\in\mathcal{P}_{0,\mathcal{S}}(\mathcal{D})}(1 - c\,[\,\mathcal{S},\, R(E)\,]^2\ ) \leq 1 - c\,[\,\mathcal{S},\, R(Q)\,]^2\,.$$

Since we have chosen an arbitrary projection of $\mathcal{P}_0(\mathcal{D})$ it holds that

$$0 < \inf_{Q\in\mathcal{P}_0(\mathcal{D})}(1 - c\,[\,\mathcal{S},\, R(Q)\,]^2).$$

which is equivalent to $\sup\limits_{Q\in\mathcal{P}_0(\mathcal{D})} c\,[\,\mathcal{S},\, R(Q)\,] < 1$.

Finally, if condition 2 holds, we have that, in particular

$$\sup\ \{c\,[\,\mathcal{S},\, \mathcal{H}_n\,] : n \in \mathbb{N}\} < 1,$$

so $\overline{\bigcup\limits_{n=1}^{\infty} \mathcal{S}_n} = \mathcal{S}$. On the other hand, by Remark 4.11, there exists $M \in \mathbb{R}$ such that

$$K\,[\,\mathcal{S}_n,\, \mathcal{D}\,] < M\ , \qquad \forall\, n \in \mathbb{N}.$$

Therefore, by Proposition 4.13, $\mathcal{S}$ is $\mathcal{B}$-compatible. ■

5 An application to Riesz frames

Recall that a sequence $\{\xi_n\}$ of elements of $\mathcal{H}$ is called a $frame$ if there exist positive constants A, B such that

$$A\|\xi\|^2 \leq \sum |\langle \xi, \xi_n \rangle|^2 \leq B\|\xi\|^2 \tag{16}$$

for all $\xi \in \mathcal{H}$. The theory of frames, introduced by Duffin and Schafter [18] in their study of non harmonic Fourier series, has grown enormously after Daubechies, Grassmann and Meyer [14] emphasized their relevance in time-frequency analysis. The reader is referred to the books by Young [37] and Chistensen [8], and the surveys by Heil and Walmut [25] Casazza [7] and Christensen [9] for modern treatments of frame theory and applications.

It is well known that $\{\xi_n\}$ is a frame of $\mathcal{H}$ if and only if there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ and an epimorphism (i.e. surjective) $T \in L(\mathcal{H})$ such that $\xi_n = Te_n$, $n \in \mathbb{N}$. If the basis $\{e_n\}_{n\in\mathbb{N}}$ is fixed, the operator T is called the *analysis* operator and T^*, given by $T^*\xi = \sum_n \langle \xi, \xi_n \rangle e_n$, is called the *synthesis* operator of the frame. The positive inversible operator $S = TT^*$ (given by $S\xi = \sum_n \langle \xi, \xi_n \rangle \xi_n$) is called the $frame$ operator. In this case, the optimal constants for equation (16) are $B = \|S\| = \|T\|^2$ and $A = \|S^{-1}\|^{-1} = \gamma(T)^2$.

The frame $\{\xi_n\}$ is called a $Riesz\ frame$ (see [10]) if there exists $C > 0$ such that, for every $J \subseteq \mathbb{N}$, the sequence $\{\xi_n\}_{n\in J}$ is a frame (with constants A_J and B_J) of the space $\mathcal{H}_J = \overline{\text{span}\{\xi_n : n \in J\}}$ and $A_J \geq C$.

Consider $P_J = P_{\mathcal{H}_J} \in \mathcal{P}(\mathcal{D})$. It is easy to see that $\{\xi_n\}$ is a Riesz frame if and only if there exists $c > 0$ such that $c \leq \gamma(TP_J)$ for every $J \subseteq \mathbb{N}$. We prove now that this condition is equivalent to the fact that $N(T)$ is compatible with the basis $\{e_n\}_{n\in\mathbb{N}}$:

Theorem 5.1. *Given an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$ and an epimorphism $T \in L(\mathcal{H})$, then $(Te_n)_{n\in\mathbb{N}}$ is a Riesz frame if and only if $N(T)$ is compatible with respect to the basis $(e_n)_{n\in\mathbb{N}}$.*

Proof. Fix $J \subseteq \mathbb{N}$. Then $R(TP_J)$ is closed if and only if $c\,[\,N(T),\,\mathcal{H}_J\,] < 1$ (that is, $\gamma(TP_J) \neq 0$) and, in this case, $T|_{\mathcal{H}_J} : \mathcal{H}_J \to R(TP_J)$ defines a frame with constants $A_J = \gamma(TP_J)^2$ and $B_J = \|TP_J\|^2$. Now, using Proposition 4.14, the statement becomes clear, because the frame defined by T is a Riesz frame if and only if $\inf_{J\subseteq\mathbb{N}} \gamma(TP_J) > 0$, which is equivalent to $\sup_{J\subseteq\mathbb{N}} c\,[\,N(T),\,\mathcal{H}_J\,] = c\,[\,N(T),\,\mathcal{D}\,] < 1$. ■

Corollary 5.2. *Given an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ and an epimorphism $T \in L(\mathcal{H})$ such that $N(T)$ has finite dimension, then $\{Te_n\}_{n\in\mathbb{N}}$ is a Riesz frame if and only if there exists $n \in \mathbb{N}$ such that $N(T) \subseteq span\{e_1, \ldots, e_n\}$.*

Remark 5.3. Further applications of the relationship between Riesz frames and compatible subspaces will be developed in a forthcoming paper.

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