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The Cesàro maximal operator in dimension greater than one

A.L. Bernardis [a,*,1] and F.J. Martín-Reyes [b,2]

[a] *IMAL, Güemes 3450, (3000) Santa Fe, Argentina*
[b] *Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain*

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Abstract

We consider a maximal operator defined on $\mathbb{R}^n$ which is related to the Cesàro α continuity of functions. We characterize the weights w for which the maximal operator is of weak type, strong type and restricted weak type (p, p) with respect to the measure $w(x)\,dx$.

1. Introduction

The Lebesgue's differentiation theorem in the real line establishes that if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ then

$$\lim_{R\to 0^+} \frac{1}{|I(x,R)|} \int_{I(x,R)} \big|f(y) - f(x)\big|\,dy = 0 \tag{1.1}$$

for almost every x, where $I(x,R) = [x - R, x + R]$. We can interpret the above limit as Cesàro $(C,1)$ continuity of f at x (see [3]). In general, for $\alpha > -1$, we say that f is $(C, 1+\alpha)$ continuous at x if

* Corresponding author.
E-mail addresses: bernard@ceride.gov.ar (A.L. Bernardis), martin@anamat.cie.uma.es (F.J. Martín-Reyes).
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$$\lim_{R\to 0^+}\frac{1}{|I(x,R)|^{1+\alpha}}\int_{I(x,R)}\bigl|f(y)-f(x)\bigr|d\bigl(y,\partial I(x,R)\bigr)^{\alpha}\,dy=0,$$

where $\partial I(x,R)$ is the border of $I(x,R)$, i.e., the set $\{x-R,x+R\}$ and $d(y,\partial I(x,R))=\min\{x+R-y,y-(x-R)\}$.

In dimension greater than one, a version of the Lebesgue's differentiation theorem consists of replacing in (1.1) the intervals $I(x,R)$ by the cubes $Q(x,R)=[x-R,x+R]^n$. Following this idea we say that $f:\mathbb{R}^n\to\mathbb{R}$ is $(C,1+\alpha)$ continuous at x, $\alpha>-1$, if

$$\lim_{R\to 0^+}\frac{1}{|Q(x,R)|^{1+\alpha/n}}\int_{Q(x,R)}\bigl|f(y)-f(x)\bigr|d\bigl(y,\partial Q(x,R)\bigr)^{\alpha}\,dy=0,\tag{1.2}$$

where $d(y,\partial Q(x,R))=\min_{1\leqslant i\leqslant n}\{x_i+R-y_i,y_i-(x_i-R)\}$ is the distance in the infinity norm from y to the border of $Q(x,R)$. It is easy to see that the $(C,1+\alpha)$ continuity of f at x implies the $(C,1+\beta)$ continuity of f at x for all $\beta>\alpha>-1$.

In order to study the above limit, it is natural to consider the following maximal operator:

$$M_\alpha f(x)=\sup_{R>0}\frac{1}{R^{n+\alpha}}\int_{Q(x,R)}\bigl|f(y)\bigr|d\bigl(y,\partial Q(x,R)\bigr)^{\alpha}\,dy,\quad \alpha>-1.$$

It follows from the results in [3] that M_α, $\alpha>-1$, is of restricted weak type $(1/(1+\alpha),1/(1+\alpha))$ and, consequently, it is of strong type (p,p) for $p>1/(1+\alpha)$. In this paper we are interested in the characterization of the weights w such that M_α are of weak, strong and restricted weak type (p,p) with respect to w. If $\alpha\geqslant 0$, the operator M_α is pointwise equivalent to the Hardy–Littlewood maximal operator. For that reason we shall only consider negative values of α. We remark that the boundedness with weights for the operator M_α in one dimension can be obtained from the corresponding results for the one sided versions studied in [4] (see also [2]).

Throughout this paper α will be a number such that $-1<\alpha<0$ and cube means a cube with sides parallel to the axis. By $|A|$ and $w(A)$ we denote the Lebesgue measure of A and the integral $\int_A w(s)\,ds$, respectively. If $1<p<\infty$ then p' will denote its conjugate exponent, i.e., $1/p+1/p'=1$. By σ we denote the function $w^{1-p'}$. The letter C will mean a positive constant not necessarily the same at each occurrence and if $x\in\mathbb{R}^n$ we shall write $x=(x_1,\dots,x_n)$.

2. Weighted weak type inequalities

The first result of the paper characterizes the weighted weak type inequalities for M_α by means of a Muckenhoupt type condition.

Theorem 2.1. *Let w be a nonnegative measurable function on $\mathbb{R}^n$ and let $-1<\alpha<0$. If $1<p<\infty$ then the following are equivalent*:

(i) *M_α is of weak type (p, p) with respect to $w(x)dx$, i.e., there exists C such that*

$$w\big(\{M_\alpha f > \lambda\}\big) \leqslant C\lambda^{-p} \int |f|^p w$$

for all $\lambda > 0$ and all $f \in L^p(w)$.

(ii) *w satisfies $A_{p,\alpha}$, i.e., there exists C such that for any cube Q,*

$$\left(\int_Q w\right)^{1/p} \left(\int_Q \sigma(y) d(y, \partial Q)^{\alpha p'} dy\right)^{1/p'} \leqslant C|Q|^{1+\alpha/n}.$$

Remark 2.2. Observe that if w satisfies $A_{p,\alpha}$ then w is in the Muckenhoupt $A_p = A_{p,0}$ class. Therefore, the weights in $A_{p,\alpha}$ are doubling weights. On the other hand, if $-1 < \alpha < 0$ and $p(1+\alpha) > 1$ then the Muckenhoupt class $A_{p(1+\alpha)}$ is contained in $A_{p,\alpha}$ (the proof is similar to the one of Proposition 6.1 in [1]).

In order to prove the theorem we introduce $2n$ noncentred maximal operators, which are pointwise bounded by the operators M_α. Given $z \in \mathbb{R}^n$, $R > 0$ and $i \in \{1, \ldots, n\}$ we define the maximal operators

$$N^-_{\alpha,i} f(x) = \sup_{x \in U_i(z,R)} \frac{1}{R^{n+\alpha}} \int_{V_i(z,R)} |f(y)| \big(y_i - (z_i - R)\big)^\alpha \, dy$$

and

$$N^+_{\alpha,i} f(x) = \sup_{x \in V_i(z,R)} \frac{1}{R^{n+\alpha}} \int_{U_i(z,R)} |f(y)| (z_i + R - y_i)^\alpha \, dy,$$

where

$$U_i(z, R) = \mathcal{K}_i(z, R) \cap \{y\colon\ y_i \geqslant z_i\}, \qquad V_i(z, R) = \mathcal{K}_i(z, R) \cap \{y\colon\ y_i \leqslant z_i\}$$

and

$$\mathcal{K}_i(z, R) = \big\{y \in Q(z, R)\colon\ |y_j - z_j| \leqslant |y_i - z_i|,\ j = 1, \ldots, n\big\}.$$

Notice that the kernels in $N^-_{\alpha,i} f(x)$ and $N^+_{\alpha,i} f(x)$ are equal to $d(y, \partial Q(x, R))^\alpha$.

Proposition 2.3. *Let $-1 < \alpha < 0$. There exists a positive constant C depending only on α and n such that*

$$N^-_{\alpha,i} f(x) \leqslant C M_\alpha f(x) \quad \text{and} \quad N^+_{\alpha,i} f(x) \leqslant C M_\alpha f(x)$$

for all $i = 1, \ldots, n$ and all measurable function f.

Proof. We shall only prove that $N^-_{\alpha,i} f(x) \leqslant C M_\alpha f(x)$ because the other inequality follows in a similar way. Given $z \in \mathbb{R}^n$ and $R > 0$, let $x \in U_i(z, R)$ and $S = R + x_i - z_i$. Clearly $R \leqslant S \leqslant 2R$. It is easy to see that $V_i(z, R) \subset V_i(x, S)$ and $y_i - (z_i - R) = d(y, \partial Q(x, S))$ for all $y \in V_i(z, R)$. Then we get that

$$\frac{1}{R^{n+\alpha}} \int\limits_{V_i(z,R)} |f(y)|\big(y_i-(z_i-R)\big)^{\alpha}\,dy = \frac{1}{R^{n+\alpha}} \int\limits_{V_i(z,R)} |f(y)|d\big(y,\partial Q(x,S)\big)^{\alpha}\,dy$$
$$\leqslant \frac{C}{S^{n+\alpha}} \int\limits_{Q(x,S)} |f(y)|d\big(y,\partial Q(x,S)\big)^{\alpha}\,dy \leqslant CM_\alpha f(x).$$

Taking supremum on $R>0$ we are done. □

The following lemma shows necessary conditions on the weight w for the operators $N^-_{\alpha,i}$ and $N^+_{\alpha,i}$ to be of weak type (p,p) with respect to w.

Lemma 2.4. *Let w be a nonnegative measurable function on $\mathbb{R}^n$, let $-1<\alpha<0$ and $1<p<\infty$. The following statements hold for all $i\in\{1,\dots,n\}$:*

(i) *If $N^-_{\alpha,i}$ is of weak type (p,p) with respect to $w(x)\,dx$, then $w\in A^-_{p,\alpha,i}$, i.e., there exists C such that for all $z\in\mathbb{R}^n$ and $R>0$,*

$$\Bigg(\int\limits_{U_i(z,R)} w\Bigg)^{1/p}\Bigg(\int\limits_{V_i(z,R)} \sigma(y)\big(y_i-(z_i-R)\big)^{\alpha p'}\Bigg)^{1/p'} \leqslant CR^{n+\alpha}.$$

(ii) *If $N^+_{\alpha,i}$ is of weak type (p,p) with respect to $w(x)\,dx$, then $w\in A^+_{p,\alpha,i}$, i.e., there exists C such that for all $z\in\mathbb{R}^n$ and $R>0$,*

$$\Bigg(\int\limits_{V_i(z,R)} w\Bigg)^{1/p}\Bigg(\int\limits_{U_i(z,R)} \sigma(y)(z_i+R-y_i)^{\alpha p'}\Bigg)^{1/p'} \leqslant CR^{n+\alpha}.$$

Proof. We only prove (i) since (ii) is similar. Let $z\in\mathbb{R}^n$ and $R>0$. If we consider for every $n\in\mathbb{N}$ the function

$$f(y)=\big(w(y)+1/n\big)^{1-p'}\Big[\min\big\{\big(y_i-(z_i-R)\big)^{\alpha},n\big\}\Big]^{p'-1}\chi_{V_i(z,R)}(y),$$

then for all $x\in U_i(z,R)$,

$$N^-_{\alpha,i}f(x)\geqslant \frac{1}{R^{n+\alpha}}\int\limits_{V_i(z,R)} f(y)\big(y_i-(z_i-R)\big)^{\alpha}\,dy \equiv \lambda.$$

This means that $U_i(z,R)\subset\{N^-_{\alpha,i}f\geqslant\lambda\}$. Then (i) follows by a standard argument, that is, applying the weak type inequality for $N^-_{\alpha,i}$ and letting n tend to infinity. □

Before proving Theorem 2.1 we need the following lemma.

Lemma 2.5. *Let w be a nonnegative measurable function on $\mathbb{R}^n$ and let $-1<\alpha<0$. If $1<p<\infty$ then the following statements are equivalent*:

(i) *w satisfies $A_{p,\alpha}$.*
(ii) $w \in \bigcap_{i=1}^{n}(A^{-}_{p,\alpha,i} \cap A^{+}_{p,\alpha,i})$.

Proof. Given $z \in \mathbb{R}^n$ and $R > 0$, let $Q = Q(z, R)$, $U_i = U_i(z, R)$ and $V_i = V_i(z, R)$.

(i) ⇒ (ii) Notice that

$$\int_{V_i} \sigma(y)\big(y_i - (z_i - R)\big)^{\alpha p'}\,dy = \int_{V_i} \sigma(y) d(y, \partial Q)^{\alpha p'}\,dy \leqslant \int_{Q} \sigma(y) d(y, \partial Q)^{\alpha p'}\,dy.$$

Then it is easy to see that $w \in A_{p,\alpha}$ implies that $w \in A^{-}_{p,\alpha,i}$. With a similar argument we obtain that $A_{p,\alpha} \subset A^{+}_{p,\alpha,i}$.

(ii) ⇒ (i) By (ii) we get that

$$w(U_i) \leqslant C w(V_i) \quad \text{and} \quad w(V_i) \leqslant C w(U_i). \tag{2.1}$$

On the other hand, by making the dyadic partition of the cube Q we obtain 2^n cubes Q_j. If we apply the above inequalities to the cubes Q_j we get that

$$w(Q) \leqslant C w(U_i) \quad \text{and} \quad w(Q) \leqslant C w(V_i) \tag{2.2}$$

for all $i = 1, \ldots, n$. Now, since

$$\int_{Q} \sigma(y) d(y, Q)^{\alpha p'}\,dy$$
$$= \sum_{i=1}^{n} \int_{V_i} \sigma(y)\big(y_i - (z_i - R)\big)^{\alpha p'}\,dy + \sum_{i=1}^{n} \int_{U_i} \sigma(y)(z_i + R - y_i)^{\alpha p'}\,dy,$$

the inequalities in (2.2) and $w \in \bigcap_{i=1}^{n}(A^{-}_{p,\alpha,i} \cap A^{+}_{p,\alpha,i})$ imply that $w \in A_{p,\alpha}$. □

Proof of Theorem 2.1. Implication (i) ⇒ (ii) follows directly from Proposition 2.3, Lemmas 2.4 and 2.5.

(ii) ⇒ (i) Given $x \in \mathbb{R}^n$ and $R > 0$, let $Q = Q(x, R)$ be any cube with centre x. By the Hölder inequality and the $A_{p,\alpha}$ condition we obtain

$$\int_{Q} \big|f(y)\big| d(y, Q)^{\alpha}\,dy \leqslant \left(\int_{Q} |f|^p w\right)^{1/p} \left(\int_{Q} \sigma(y) d(y, Q)^{\alpha p'}\,ds\right)^{1/p'}$$
$$\leqslant C\left(\int_{Q} |f|^p w\right)^{1/p} \left(\int_{Q} w\right)^{-1/p} |Q|^{1+\alpha/n}.$$

Therefore,

$$M_\alpha f(x) \leqslant C\big[\mathcal{M}_w\big(|f|^p\big)\big]^{1/p}(x),$$

where

$$\mathcal{M}_w g(x) = \sup_{R>0}\left[\frac{1}{w(Q(x, R))} \int_{Q(x,R)} |g| w\right].$$

Now (i) follows from the above inequality and the well-known fact that $\mathcal{M}_w$ is of weak type $(1,1)$ with respect to $w(x)\,dx$. □

3. Weighted strong type inequalities

The strong type (p, p) for the operator M_α is characterized also by $A_{p,\alpha}$.

Theorem 3.1. *Let $-1 < \alpha < 0$ and $1 < p < \infty$. Let w be a nonnegative measurable function on $\mathbb{R}^n$. The following statements are equivalent*:

(i) *M_α is of strong type (p, p) with respect to w, i.e., there exists C such that*

$$\int |M_\alpha f|^p w \leqslant C \int |f|^p w$$

for all $f \in L^p(w)$.

(ii) *w satisfies $A_{p,\alpha}$.*

In order to prove the theorem we need to give a suitable characterization of the condition $A_{p,\alpha}$. This characterization appears in Proposition 3.3 and it is given in terms of the Muckenhoupt A_p condition with respect to a general Borel measure (see [5]). First we state the definition and then the proposition.

Definition 3.2. If μ is a Borel measure finite on compact sets, it is said that a nonnegative measurable function w satisfies $A_p(\mu)$, $1 < p < \infty$, if there exists a positive constant C such that

$$\left(\int_Q w\,d\mu\right)^{1/p}\left(\int_Q w^{1-p'}\,d\mu\right)^{1/p'} \leqslant C\mu(Q)$$

for all cubes Q.

Proposition 3.3. *Let $-1 < \alpha < 0$ and $1 < p < \infty$. Let w be a nonnegative measurable function. The following statements are equivalent*:

(a) *w satisfies $A_{p,\alpha}$.*

(b) *There exists C such that for any cube Q with centre in x and all $i = 1, \ldots, n$,*

$$\left(\int_Q w\right)^{1/p}\left(\int_Q \sigma(y)|x_i - y_i|^{\alpha p'}\,dy\right)^{1/p'} \leqslant C|Q|^{1+\alpha/n}.$$

(c) *For all $i = 1, \ldots, n$, the functions $y \to w(y)|h - y_i|^{-\alpha}$ satisfy $A_p(\mu_{h,i})$ with a constant independent of $h \in \mathbb{R}$ where $d\mu_{h,i} = |h - y_i|^\alpha\,dy$, i.e., there exists C such that for any cube Q, all $h \in \mathbb{R}$ and all $i = 1, \ldots, n$,*

$$\left(\int_Q w\right)^{1/p}\left(\int_Q \sigma(y)|h-y_i|^{\alpha p'}\,dy\right)^{1/p'} \leqslant C\int_Q |h-y_i|^{\alpha}\,dy.$$

As a corollary of Proposition 3.3, we get that the classes $A_{p,\alpha}$ are left open.

Corollary 3.4. *Let $-1 < \alpha < 0$, $1 < p < \infty$, and let w be a nonnegative measurable function on $\mathbb{R}^n$. If w satisfies $A_{p,\alpha}$ then there exists $\epsilon > 0$, $0 < \epsilon < p-1$, such that w satisfies $A_{p-\epsilon,\alpha}$.*

It is clear that Theorem 3.1 follows from this corollary, Theorem 2.1 and Marcinkiewicz interpolation theorem. Therefore, the proof of Theorem 3.1 will be complete as soon as we prove Proposition 3.3 and Corollary 3.4.

Proof of Proposition 3.3. Given $x \in \mathbb{R}^n$ and $R > 0$, Q will denote the cube $Q = Q(x, R)$. For every i, $1 \leqslant i \leqslant n$, let e_i be the point of $\mathbb{R}^n$ with all the coordinates equal to zero except the ith coordinate which is equal to 1.

(a) $\Rightarrow$ (b). For fixed i, $1 \leqslant i \leqslant n$, let us define $\tilde{Q} = Q(\tilde{x}, 2R)$ and $\bar{Q} = Q(\bar{x}, 2R)$, where $\tilde{x} = x - 2Re_i$ and $\bar{x} = x + 2Re_i$. It is clear that the set $\{y \in Q\colon\ y_i \leqslant x_i\}$ is contained in $U_i(\tilde{x}, 2R)$ and the set $\{y \in Q\colon\ y_i \geqslant x_i\}$ is contained in $V_i(\bar{x}, 2R)$. On the other hand, $|x_i - y_i| = d(y, \partial\tilde{Q})$ for all $y \in U_i(\tilde{x}, 2R)$ and $|x_i - y_i| = d(y, \partial\bar{Q})$ for all $y \in V_i(\bar{x}, 2R)$. Then,

$$\int_Q \sigma(y)|x_i - y_i|^{\alpha p'}\,dy \leqslant \int_{U_i(\tilde{x},2R)} \sigma(y)d(y,\partial\tilde{Q})^{\alpha p'}\,dy + \int_{V_i(\bar{x},2R)} \sigma(y)d(y,\partial\bar{Q})^{\alpha p'}\,dy.$$

Now (b) follows by using that w is a doubling weight and the conditions $A^+_{p,\alpha,i}$ and $A^-_{p,\alpha,i}$ (see Lemma 2.5).

(b) $\Rightarrow$ (c) Let us fix $i \in \{1,\dots,n\}$. Assume first that $|x_i - h| \leqslant R$. For fixed i, $1 \leqslant i \leqslant n$, let $\tilde{Q} = Q(\tilde{x}, 2R)$, where $\tilde{x} = x + (h - x_i)e_i$. Then

$$\int_Q \sigma(y)|h-y_i|^{\alpha p'}\,dy \leqslant \int_{\tilde{Q}} \sigma(y)|\tilde{x}_i - y_i|^{\alpha p'}\,dy.$$

Now (c) follows from this inequality, (b) and the fact that $|Q|^{1+\alpha/n} \leqslant C\int_Q |h-y_i|^{\alpha}\,dy$.

Now, we shall assume that $|x_i - h| > R$. If $h > x_i + R$ (the other case is similar) then

$$\int_Q \sigma(y)|h-y_i|^{\alpha p'}\,dy = \int_Q \sigma(y)(x_i + R - y_i)^{\alpha p'} g(y_i)^{\alpha p'}\,dy,$$

where

$$g(y_i) = \frac{h - y_i}{x_i + R - y_i}.$$

Since g is an increasing function we get that

$$\int_Q \sigma(y)|h-y_i|^{\alpha p'}\,dy \leqslant \left(\frac{(h-(x_i-R))}{2R}\right)^{\alpha p'} \int_Q \sigma(y)(x_i+R-y_i)^{\alpha p'}\,dy.$$

Hence, by using the first part of the proof with $h = x_i + R$, we get that

$$\left(\int_Q w\right)^{1/p}\left(\int_Q \sigma(y)|h-y_i|^{\alpha p'}\right)^{1/p'} \leqslant CR^n\big(h-(x_i-R)\big)^{\alpha} \leqslant C\int_Q |h-y_i|^{\alpha}\,dy.$$

(c) ⇒ (a) The implication follows from the following inequalities:

$$\begin{aligned}
&\int_Q \sigma(y)d(y,\partial Q)^{\alpha p'}\,dy \\
&\quad= \sum_{i=1}^{n}\int_{V_i} \sigma(y)\big(y_i-(x_i-R)\big)^{\alpha p'}\,dy + \sum_{i=1}^{n}\int_{U_i} \sigma(y)(x_i+R-y_i)^{\alpha p'}\,dy \\
&\quad\leqslant \sum_{i=1}^{n}\int_{Q} \sigma(y)\big|y_i-(x_i-R)\big|^{\alpha p'}\,dy + \sum_{i=1}^{n}\int_{Q} \sigma(y)|x_i+R-y_i|^{\alpha p'}\,dy,
\end{aligned}$$

by using (c) with $h = x_i - R$ and $h = x_i + R$. □

Proof of Corollary 3.4. We know by Proposition 3.3 that $w(y)|h-y_i|^{-\alpha}$ satisfies $A_p(\mu_{h,i})$ with an $A_p(\mu_{h,i})$-constant independent of h. Furthermore, the measures $\mu_{h,i}$ are doubling measures with the same doubling constant. Then (see [5, p. 5]) there exists $\epsilon > 0$ depending only on the $A_p(\mu_{h,i})$-constant such that $w(y)|h-y_i|^{-\alpha}$ satisfies $A_{p-\epsilon}(\mu_{h,i})$, where the $A_{p-\epsilon}(\mu_{h,i})$-constant depends only on the $A_p(\mu_{h,i})$-constant and ϵ. Applying again Proposition 3.3 we obtain that w satisfies $A_{p-\epsilon,\alpha}$. □

4. Restricted weak type inequalities

As we said above, the operator M_α is not of weak type $(1/(1+\alpha), 1/(1+\alpha))$ with respect to Lebesgue measure if $\alpha < 0$ but it is of restricted weak type $(1/(1+\alpha), 1/(1+\alpha))$; in other words, M_α satisfies the weak type $(1/(1+\alpha), 1/(1+\alpha))$ inequality for characteristic functions or, equivalently, M_α maps the Lorentz space $L(1/(1+\alpha), 1)(dx)$ into the Lorentz space $L(1/(1+\alpha), \infty)(dx)$. Therefore, it is interesting to study the weights w such that $w(\{x\colon M_\alpha\chi_E(x) > \lambda\}) \leqslant C\lambda^{-p}w(E)$ for all $\lambda > 0$ and all measurable sets $E \subset \mathbb{R}^n$.

Theorem 4.1. *Let w be a nonnegative measurable function on $\mathbb{R}^n$ and let $-1 < \alpha \leqslant 0$. If $1 \leqslant p < \infty$ then the following are equivalent*:

(i) *M_α is of restricted weak type (p, p) with respect to $w(x)\,dx$, i.e., there exists C such that $w(\{x\colon M_\alpha\chi_E(x) > \lambda\}) \leqslant C\lambda^{-p}w(E)$ for all $\lambda > 0$ and all measurable $E \subset \mathbb{R}^n$.*

(ii) *w satisfies $RA_{p,\alpha}$, i.e., there exists C such that for every cube Q and all measurable $E \subset \mathbb{R}^n$,*

$$\left(\int_Q w\right)\left(\int_{E\cap Q} d(y,\partial Q)^\alpha\,dy\right)^p \leqslant C|Q|^{(n+\alpha)p}\int_{E\cap Q} w.$$

The proof of the theorem is similar to the proof of Theorem 2.1 and we omit it.

References

[1] A.L. Bernardis, F.J. Martín-Reyes, Weighted inequalities for a maximal function on the real line, Proc. Roy. Soc. Edinburgh Sect. A 131 (2001) 267–277.

[2] A.L. Bernardis, F.J. Martín-Reyes, Two weighted inequalities for convolution maximal operators, Publ. Mat. 46 (2002) 119–138.

[3] W. Jurkat, J. Troutman, Maximal inequalities related to generalized a.e. continuity, Trans. Amer. Math. Soc. 252 (1979) 49–64.

[4] F.J. Martín-Reyes, A. de la Torre, Some weighted inequalities for general one-sided maximal operators, Studia Math. 122 (1997) 1–14.

[5] J.-O. Strömberg, A. Torchinsky, Weighted Hardy Inequalities, in: Lecture Notes in Mathematics, Vol. 1381, Springer-Verlag, Berlin, 1989.