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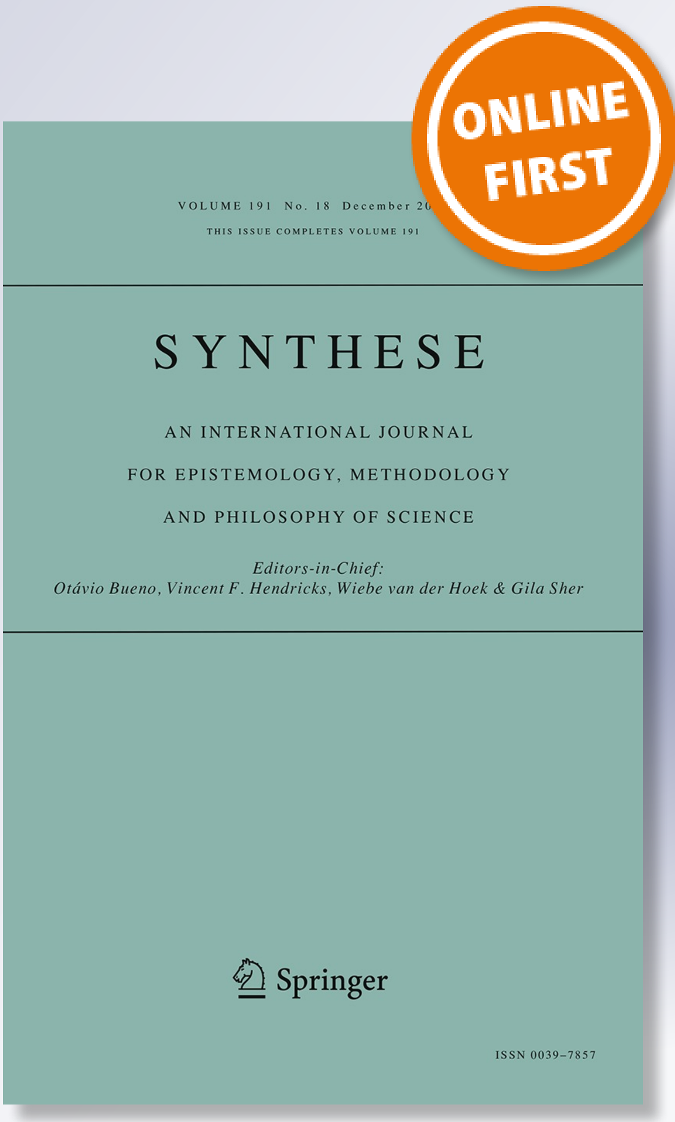

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A formal framework for the study of the notion of undefined particle number in quantum mechanics

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Abstract It is usually stated that quantum mechanics presents problems with the identity of particles, the most radical position—supported by E. Schrödinger—asserting that *elementary particles are not individuals*. But the subject goes deeper, and it is even possible to obtain states with an undefined particle number. In this work we present a set theoretical framework for the description of undefined particle number states in quantum mechanics which provides a precise logical meaning for this notion. This construction goes in the line of solving a problem posed by Y. Manin, namely, *to incorporate quantum mechanical notions at the foundations of mathematics*. We also show that our system is capable of representing quantum superpositions.

Keywords Quantum mereology · Set theory · Undefined particle number · Quantum indistinguishability · Quantum superpositions

1 Introduction

Quantum mechanics (QM) in both of its versions, relativistic and non-relativistic, is considered as one of the most important physical theories of our time, giving rise to spectacular technological developments and experimental predictions. Yet, interpreta-

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tion of QM still gives rise to difficult problems, which are far from finding a definitive solution. This is, perhaps, one of the most interesting features of QM, and poses important philosophical questions. In particular, while classical extensional mereology is widely investigated in important philosophical textbooks (see for example Simons (1987) for a complete study), the development of a quantum mereology (i.e., a mereology based on objects obeying the laws of QM) is still lacking. And this is an important issue for ontological considerations, because it is expected that a quantum mereology will be quite different than classical extensional mereology (at least, if we follow the standard interpretation of QM and many other interpretations as well).

The development of formal systems in which mereological properties (or features) of a given ontology are rigorously expressed is a helpful goal. This is the case in Leśniewski's Mereology (based on his "Calculus of Names") or the "Calculus of Individuals" of Leonard and Goodman (Simons 1987).

In this work we will develop a formal framework which captures important features of the quantum formalism, namely,

- *undefined particle number* and
- *undefined properties* (as the ones appearing in quantum superpositions).

By capturing these quantum features, our system may be helpful for the task of developing a quantum mereology in a rigorous way. This is an important issue for any philosopher interested in the development of an ontology based on QM.

We will present a construction which goes in the direction of solving a problem posed by Y. Manin (see French and Krause (2006) for a complete discussion of Manin's problem and an alternative proposal of solution for it). In his words (Manin 1976, 1977)

> We should consider the possibilities of developing a totally new language to speak about infinity. Set theory is also known as the theory of the 'infinite'. Classical critics of Cantor (Brouwer et al.) argued that, say, the general choice axiom is an illicit extrapolation of the finite case.
> I would like to point out that this is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the 'set' of grains of sand. In general, a highly probabilistic 'physical infinity' looks considerably more complicated and interesting than a plain infinity of 'things'. (Manin 1976)

Thus, Manin suggests the development of set theories (Manin 2010; Halmos 1963; Kunen 1980; Brignole and Costa 1971) incorporating the novel features of quantum entities, which depart radically from our every day concepts.[1] In this line, many alternatives where developed, most of them grounded in non-reflexive logics (Costa 1980; Costa and Bueno 2009). In particular, it is possible to incorporate in a Zermelo–Frenkel (ZF) set theory the notion of quantum non-individuality (French and Krause 2006; Dalla Chiara and Toraldo di Francia 1995; Dalla Chiara et al. 1998; Santorelli et al.

[1] Although Manin has seemingly changed his position regarding this subject Manin (2010), the problem posed above still seems interesting to us and we will take it as a basis for our work.

2005; Krause 2003) and this was done by introducing indistinguishability "right at the start" (Post 1963). According to the interpretation of E. Schrödinger an elementary particle cannot be considered as an individual entity

> I mean this: that the elementary particle is not an individual; it cannot be identified, it lacks 'sameness'. The fact is known to every physicist, but is rarely given any prominence in surveys readable by nonspecialists. In technical language it is covered by saying that the particles 'obey' a newfangled statistics, either Einstein–Bose or Fermi–Dirac statistics. [...] The implication, far from obvious, is that the unsuspected epithet 'this' is not quite properly applicable to, say, an electron, except with caution, in a restricted sense, and sometimes not at all. E. Schrödinger (Schrödinger 1998, p. 197)

Similarly, Michael Redhead and Paul Teller claim in Redhead and Teller (1991, 1992) that:

> Interpreters of QM largely agree that classical concepts do not apply without alteration or restriction to quantum objects. In Bohr's formulation this means that one cannot simultaneously apply complementary concepts, such as position and momentum, without restriction. In particular, this means that one cannot attribute classical, well defined trajectories to quantum systems. But in a more fundamental respect it would seem that physicists, including Bohr, continue to think of quantum objects classically as individual things, capable, at least conceptually, of bearing labels. It is this presumption and its implications which we need to understand and critically examine. M. Redhead and P. Teller (Redhead and Teller 1992, p. 202)

It is important to mention that, besides the conception of quantum entities as non-individuals, the validity of the principle of identity of indiscernibles (PII) in QM was also questioned (see for example French and Krause 2006; French and Redhead 1988; Butterfield 1993). Principle of identity of indiscernibles can be written as follows: it is not possible for two *individuals* to possess all the same attributes in common French and Redhead 1988. As remarked in French and Redhead (1988), if quanta were not individuals, "PII would not be either true or false, but simply inapplicable". Thus, violation of PII and non-individuality of quanta are not equivalent and should not be confused.

In the last years, a different perspective on the problem of quantum indistinguishability was developed Saunders (2003, 2006). In Muller and Saunders (2008), Muller and Seevinck (2009) (see also Muller 2014) it is claimed that according to quantum theory, indistinguishable particles are not utterly indiscernible, but obey a weaker form of discernibility, namely, *weak discernibility*. This weak form of discernment is achieved by a relational symmetric and non reflexive relation between the relata. Different grades of discernibility in standard model theory and its logical relations and links with philosophical problems are discussed in Caulton and Butterfield (2012a) and Ladyman et al. (2012) (see also Caulton and Butterfield 2012b).

Though these works are very compelling, the success in their application to the problem of distinguishability of elementary particles is far from being conclusive.

In the first place, the results presented in Muller and Saunders (2008), Muller and Seevinck (2009) were criticized in Caulton (2013)[2], because the properties used to discern (weakly) were unphysical, a perspective to which we adhere[3]. But the solution proposed in Caulton (2013) is not very attractive either: particles are weakly discerned by using an observable based on their (squared) relative positions in space. But one may wonder how is it possible to discern something in this way, given that it is widely known how difficult is to assign definite positions to particles previous to any measurement. It seems that the only thing achieved here is numerical distinctness of space-time points, something which in principle should not be equated with discernibility of the particles involved (unless cumbersome interpretational moves are made)[4]. A similar observation applies to observables different than position. Indeed, a similar problem seems to appear in Muller (2014), where the case of two entangled bosons is discussed.

It may be argued that the relation of weak discernibility holding between two electrons, can probably ensure that the number of objects is indeed two. But it seems that it falls short of separating them in such a way that they can be successfully identified. Indeed, in Bigaj (2013) it is pointed out that:

> One sense of discerning involves recognizing some qualitative differences (whether in the form of different properties or different relations) between the objects considered. When we discern objects in this sense, we should (at least in principle) be able to pick out one of them but not the other. Being able to discern objects in that way seems to be a prerequisite for making successful reference, or giving a unique name, to each individual object. But by discerning we can also mean recognizing objects as numerically distinct. In this sense of the word, discernment is a process by which, using some qualitative features of the objects, we make sure that there are indeed two entities and not one.

In Ladyman and Bigaj (2010), the notion of witness-discernibility is used to argue against the use of weak discernibility as a means to rehabilitate PII in QM. Even the very applicability of the notion of weak discernibility in the quantum framework was criticized in Dieks et al. (2010). Taking into account the different criticisms

[2] In the Concluding remarks of Caulton (2013), Caulton claims that the approaches of Muller, Saunders and Seevinck "...have been seen to fail, due to their surreptitious use of mathematical predicates that can be given no physical interpretation."

[3] Similarly, in Dieks et al. (2010) it is claimed that (our emphasis): "All evidence points into the same direction: 'identical quantum particles' behave like money units in a bank account rather than like Blackean spheres. It does not matter what external standards we introduce, they will always possess the same relations to all (hypothetically present) entities. The irreflexive relations used by Saunders and others to argue that identical quantum particles are weakly discernible individuals *lack the physical significance required to make them suitable for the job*."

[4] Related to this observation, see also Dieks et al. (2010) where a similar argument can be found for spins and the following observation is made regarding position measurements in QM: "To see how this complicates matters, think of a one-particle position measurement carried out on a many-particles system described by such a symmetrized state. The result found in such a measurement (for example, the click of a Geiger counter or a black spot on a photographic plate) is not linked to one of the 'particle labels'; it is, in symmetrical fashion, linked to all of them. This already demonstrates how the classical limit of QM does not simply connect the classical particle concept to individual indices in the quantum formalism".

mentioned above, the conclusion that weak discernibility entails a recovering of PII and discernibility of quanta is too hasty. Put in the words of Dieks et al. (2010):

> The analogy between quantum mechanical systems of "identical particles" and classical collections of weakly discernible objects is only superficial. There is no sign within standard quantum mechanics that "identical particles" are things at all: there is no ground for the supposition that relations between the indices in the formalism possess physical significance in the sense that they connect actual objects. Consequently, the irreflexivity of these relations is not important either. Conventional wisdom appears to have it right after all. Dieks et al. (2010)

So, even if the approach based on weak discernibility could be developed in the future in order to provide a more attractive solution to the problem of discerning elementary particles, none of the results presented up to now is conclusive. The plausibility of non-individuals was defended in Arenhart (2013a) and Arenhart and Krause (2014), the validity of PII was questioned in Arenhart (2013b), French and Redhead (1988), Butterfield (1993) and different criticisms against weak discernibility are presented in Bigaj (2013), Ladyman and Bigaj (2010), Dieks et al. (2010), Hawley (2006), Hawley (2009), Fraassen and Peschard (2008). Furthermore, even from the perspective of weak discernibility approach, quanta are not individuals in the sense that they cannot be absolutely discerned by qualitative physical properties. In this way, the question regarding individuality or non-individuality of quanta remains unsettled.

In a similar vein, the usual assumption that a *definite particle number* can be always obtained was also criticized. This conclusion is grounded in the well known result that it is not possible to assign in general, previous to measurement, definite values to observables in superposition states Mittelstaedt (1998). Thus, a new turn of the Manin's problem was presented in Holik (2006, 2010), Domenech and Holik (2007); Holik (2011), Domenech et al. (2008a, b).

In this work we will follow the interpretation of QM which denies that quantum systems can be always considered as singular unities (a quantum system as a "one"), or collections of them (a quantum system as a "many").

It is important to remark here that there are other interpretations which deny the existence of systems with undefined particle number. In such interpretations, states which involve superpositions with different particle number are usually interpreted as ordinary mixtures. Another possibility may be to consider P. Teller's notion of non-supervenient relations in order to describe superpositions in particle number. Regarding this last possibility, we quote Teller (1986):

> Supervenience provides an attractive answer to this question, attractive because the answer is consistent with the absence of explicit reductions or definitions of the non-physical in terms of the physical. For example, a physicalist might claim that mental states supervene on brain or other bodily states, in the sense that two physically identical bodily states would exhibit the same mental states, even though these mental states might well not be definable in terms of the bodily states.

We see that for Teller, the 'attractiveness' of the approach based on supervenience lies in the fact that there can be no explicit reductions or definitions in terms of the relata. But from the point of view of *relational holism* (Teller 1986),[5] it is plausible that there exist collections of objects having physical relations which do not supervene on the non-relational physical properties of the parts. This would be the case for entangled states in QM, such as those violating Bell's inequalities. By continuing this, one may try to explain states with undefined particle number as a kind non-supervenient relation between the particles involved in the terms of the superposition. But undefined particle number should not be confused with entanglement; it is an undefined property of the system *as a whole*: the superposition describes a state of affairs in which one of the properties of *the whole collection* is *undefined*, in this case, particle number. While undefined particle number states may present non-local correlations (i.e., they violate some kind of Bell inequality), these two effects should not be confused. This distinction suggests that undefined particle number could not be described as a non-supervenient relational property between the particles involved, simply because it is not a well defined property at all. These considerations are very probably not sufficient to rule out a description of undefined particle number as a non-supervenient relation, but this is not determinant for our concerns in this article.

Our interest in this work is not to settle the question about which is the correct interpretation. We focus on the development of a framework for studying the consequences of assuming that undefined particle number states actually exist. Notwithstanding, it is very important to remark here that the formal framework presented in this work contains a copy of the standard approach to mathematics (see Sect. 3.1). This implies that *any mereological construction which can be attained in a standard set theoretical framework can also be attained with ours*. Thus, our mereological framework *has the advantage of being able to cope with different interpretations of quantum phenomena*. In particular, the approaches of Muller and Saunders (2008), Muller and Seevinck (2009) or a possible description of superpositions in particle number in terms of non-supervenient relations (in case they can be accommodated within standard formal frameworks) can be perfectly described in our framework.

The considerations mentioned above point in the direction that a non-standard mereology is worth to be developed. Firstly, because metaphysical underdetermination does not single out a unique interpretation for quantum theory, and as we have mentioned above, the different alternatives remain inconclusive. In particular, the standard interpretation of QM—asserting that superpositions represent states of affairs in which no definite values can be assigned to the superposed property—remains strong. Secondly, because in order to discuss about different interpretations, it is important to have at hand formal frameworks in order to cope with them, trying to capture (or to describe) in a precise (rigorous) way the essence of the intuitive notions involved. Thus, we present here a mereological framework powerful enough to describe different interpretations of the quantum formalism oriented to the problem of undefined particle number.

We will face the problems linked to *undefined particle number* and—going in the line of the Manin's problem—we will develop a formal set theoretical framework

[5] See also Teller (1989) and Morganti (2009) for a development of this notion and the problems posed by Teller.

capable of incorporating such a quantum mechanical feature. We will also see that our framework is capable of describing undefined properties arising from quantum superpositions. This is the reason why our system could be considered as a solution to a generalization of the problem possed by Manin (see also French and Krause (2006) for an alternative solution considering non-individuality of quanta). We believe that the formal setting presented in this work could be a concrete step—for interpretational purposes—to give a precise logical meaning to what is meant by "undefined particle number" by incorporating this notion into a set theoretical framework. And also that it constitutes in itself an interesting structure for the possible development of new non-standard mathematics, which in turn, could be the basis for new formal frameworks with potential applications to physics. As an example of this procedure see Domenech et al. (2008b). At the same time, the developments presented in this work constitute a concrete step in order to develop a quantum mereology.

Before entering into the content of the article, it is important to mention that there is another important branch of formal developments induced by QM, namely, a vast family of quantum logics. Since the seminal paper of Birkhoff and von Newmann Birkhoff and Neumann (1936), several investigations were motivated in the fields of logic, algebraic logic, and the foundations of physics. Besides these developments, some authors have claimed that according to the logical structure of QM, we should *abandon classical logic* (see for example Putnam (1968)). On the other hand, the nowadays dominant interpretation of the quantum logical formalism developed by Birkhoff and von Neumann considers it as the study of algebraic structures linked to QM, and by no means is considered as an alternative to classical logic. Notwithstanding, it is important to remark that there are several examples of modifications of classical logic in the following sense. Even if it is a subtle matter to define exactly what classical logic is, it is possible to consider it as having two levels:

- (1) an *elementary level*, which is essentially first order predicate calculus, with or without identity, and
- (2) a *non elementary level*, which could be a set theory, a category theory, or a theory of logical types.

It is then possible to modify level 2 in order to develop a family of logics which can be considered non-classical. Indeed, the system presented in this paper in non-classical in the sense mentioned above. It is also possible to modify level 1, as shown in Pavičić and Megill (2008). Of course, the existence of these possibilities does not suffices to settle the question about the adequacy or non adequacy of classical logic. Thought we will not discuss this subject in detail in this paper, we remark that it is a matter of fact that the influence of QM in the development of formal systems gave rise to a considerable proliferation of investigations (Mackey 1957; Jauch 1968; Piron 1976; Kalmbach 1983, 1986; Varadarajan 1968, 1970; Greechie 1981; Gudder 1978; Giuntini 1991; Pták and Pulmannova 1991; Beltrametti and Cassinelli 1981; Dalla Chiara et al. 2004; Dvurečenskij and Pulmannová 2000; Engesser et al. 2009; Aerts and Daubechies 1979a, b; Randall and Foulis 1981; Domenech et al. 2010; Holik et al. 2012), including the development of "quantum set theories" (Takeuti 1981; Titani and Kozawa 2003).

The article is organized as follows. In Sect. 2, we discuss the meaning of superpositions of particle number eigenstates in Fock-space, introducing the interpretation which supports the existence of undefined particle number states.[6] In Sect. 3 we present the preliminary notions of our set theoretical framework by introducing its specific axioms. After doing this, we are ready to show how our framework solves the problem of incorporating undefined particle number in Sect. 4, and also that it is capable of describing quantum superpositions. We will also present in this Section some special features of our axiomatic and general remarks about our construction, which could be useful for further developments. Finally, we pose our conclusions in 5.

2 Undefined particle number: an overview

QFT requires an understanding of states with no definite particle number and, as explained above, we shall attempt to construct a formal framework accommodating that notion. In order that a superposition of states with different particle number occur, it is necessary to have a space which includes states with different particle number. This is provided by the Fock-space formalism (*FSF*). The *FSF* is used, for example, in the *second quantization formalism*, and we find a version of it both in relativistic and non-relativistic QM. It can be shown that the *FSF* may be used as an alternative approach to non relativistic QM (Robertson 1973). This can be seen by using the heuristic approach presented in elementary expositions like Ballentine (1998), Robertson (1973) (but see for example Clifton and Halvorson 2001; Bratteli and Robinson 1997; de la Harpe and Jones 1995 for a mathematically rigorous presentation). For an important introduction to the philosophical problems of quantum field theory (in which the FSF and particle number superpositions are discussed) we refer to Huggett (2000).

We will concentrate here on coherent states of the electromagnetic field in order to make the exposition simpler. But it is important to remark that there are other more involved examples of undetermined particle number, as is the case of Rindler quanta (Clifton and Halvorson 2001) or the BCS state of Bose–Einstein condensates (Ballentine 1998), but we will not treat them here.

The second quantization approach to QM has its roots in considering the Schrödinger's equation as a *classical field equation*, and its solution $\Psi_n(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ as a *classical field to be quantized*. This alternative view was originally adopted by P. Jordan (Schroer 2003; Duncan and Janssen 2008), one of the foundation fathers of QM, and spread worldwide after the Dirac's paper (Dirac 1927). And it is a standard way of dealing with relativistic QM (canonical quantization). The space in which these quantized fields operate is the Fock-space.

It is important to remark that the n particle Schrödinger wave equation is not completely equivalent to its analogue in the Fock-space formalism. Only solutions of the Fock-space equation which are eigenvectors of the particle number operator with particle number n can be solutions of the corresponding n particle Schrödinger wave equation. And the other way around, not all the solutions of the n particle Schrödinger

[6] The Fock-space formulation is also discussed with great detail in French and Krause (2006), Chapter 9. See also Domenech et al. (2008b) and Domenech et al. (2009).

wave equation can be solutions of the Fock equation, only those which are symmetrized do. *Then, both conditions, definite particle number and symmetrization, must hold in order that both formalisms yield equivalent predictions.*

The hamiltonian of the mth mode of a quantized electromagnetic field can be written in terms of the creation and annihilation operators $a_k^\dagger$ and a_k as follows

$$H_n = \hbar\omega(a_k^\dagger a_k + \frac{1}{2}) \tag{1}$$

and so, each $a_m^\dagger$ (a_m) creates (annihilates) a photon in mode m. Then, a fock space state (with definite particle number) can be expressed as

$$|n_1, n_2, \ldots, n_m, \ldots\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \ldots \otimes |n_m\rangle \otimes \ldots \tag{2}$$

with n_i the number of photons present in each mode of the field. If for simplicity we concentrate in only one frequency mode of the field, we can create any normalized superposition of states, and in particular, the famous *coherent state*

$$|z\rangle = \exp(-\frac{1}{2}|z|^2)\sum_{n=0}^{\infty}\frac{z^n}{(n!)^{\frac{1}{2}}}|n\rangle \tag{3}$$

which can be realized in laboratory (Ballentine 1998). State (3) is clearly a superposition of different photon number states and thus is not an eigenstate of the particle number operator. It follows that, according to the standard interpretation, it represents a physical system formed by an undefined number of photons. It is important to remark that there are -at least- two interpretations of (3)

- 1-Equation (3) represents an statistical mixture of states with definite particle number.
- 2-Equation (3) represents an state which has no definite particle number.

The orthodox interpretation of QM points in the direction of the second option and the first one is very difficult to sustain unless involved hypotheses are made (Mittelstaedt 1998. Regardless the interpretational debate, it will suffice for us that *there exists at least one interpretation compatible with QM in which particle number is undefined.* Thus, given that systems in states like (3) are predicted by QM and can indeed be reproduced in the laboratory, we are going to propose below a formalism in order to incorporate physical systems in such states in a set theoretical framework.

3 Preliminaries and primitive symbols

We will work with a variant of Zermelo–Frenkel (*ZF*) set theory (Brignole and Costa 1971) with physical things (*PTs*). We will denote this theory by *ZF**. The underlying logic of *ZF** is the classical first order predicate calculus with equality (identity). The primitive symbols of ZF^* are the following

- Those of classical first order predicate calculus using only identity and the membership symbol "$\in$"
- the unary predicate symbol "$\mathcal{C}()\ldots$" (such that "$\mathcal{C}(x)$" reads "x is a set")
- a binary predicate symbol "$\sqsubset$" whose meaning will be clear below, when we give the general mereological axioms used in our framework

ZF^* concerns sets and PTs (which are not sets), and so, it is involved with a kind of mereology. PTs are meant to represent physical objects. Depending on the particular interpretation of our framework, PTs may represent fields, particles, strings or any collection of physical objects whose interpretation is compatible with the ontology intended for our framework. In particular, we will consider the system represented by a state such as the one of (3), *as formed by an undefined number of photons*.

Definitions of formulas, sentences (formulas without free variables), bound variables, free variables, etc., are the standard ones. As usual, we write "$\exists_{\mathcal{C}} x(F(x))$" instead of "$\exists x(\mathcal{C}(x) \wedge F(x))$" and "$\forall_{\mathcal{C}} x(F(x))$" instead of "$\forall x(C(x) \longrightarrow F(x))$".

ZF^* possesses axioms of two different kinds: the ones concerned with sets and the ones concerned with PTs. Let us begin by listing the set theoretical axioms.

3.1 Set theoretical axioms

The following postulates constitute an adaptation of those of Zermelo–Frenkel set theory (see Brignole and Costa 1971 for details).

Axiom 3.1 (*Extensionality*)

$$(\forall_{\mathcal{C}} x)(\forall_{\mathcal{C}} y)((\forall z)(z \in x \longleftrightarrow z \in y) \longrightarrow x = y)$$

Axiom 3.2 (*Union*)

$$(\forall x)(\forall y)(\exists_{\mathcal{C}} t)(\forall z)(z \in t \longleftrightarrow (z \in x \vee z \in y))$$

Axiom 3.3 (*Power set*)

$$(\forall_{\mathcal{C}} x)(\exists_{\mathcal{C}} y)(\forall_{\mathcal{C}} t)(t \in y \longleftrightarrow t \subseteq x)$$

If $F(x)$ is a formula, x, y and z are distinct variables and y does not occur free in $F(x)$, we have

Axiom 3.4 (*Separation*)

$$(\forall_{\mathcal{C}} z)(\exists_{\mathcal{C}} y)(\forall x)(x \in y \longleftrightarrow F(x) \wedge x \in z)$$

Axiom 3.5 (*Empty set*)

$$(\exists_{\mathcal{C}} t)(\forall x)(x \notin t)$$

Axiom 3.6 (*Amalgamation*)

$$(\forall_{\mathcal{C}} x)((\forall y)(y \in x \longrightarrow \mathcal{C}(y)) \longrightarrow (\exists_{\mathcal{C}} z)(\forall t)(t \in z \longleftrightarrow (\exists v)(v \in x \wedge t \in v)))$$

If $F(x, y)$ is a formula and the variables satisfy evident conditions we have:

Axiom 3.7 (*Replacement*)

$$(\forall x)(\exists! y)(F(x, y)) \longrightarrow (\forall_{\mathcal{C}} u)(\exists_{\mathcal{C}} v)(\forall y)(y \in v \longleftrightarrow (\exists x)(x \in u \wedge F(x, y)))$$

Axiom 3.8 (*Infinity*)

$$(\exists_{\mathcal{C}} z)(\emptyset \in z \wedge (\forall x)(x \in z \longrightarrow x \cup \{x\} \in z))$$

Axiom 3.9 (*Choice*)

$$(\forall_{\mathcal{C}} x)\{(\forall y)(y \in x \longrightarrow \mathcal{C}(y)) \wedge (\forall y)(\forall z)(y \in x \wedge z \in x \longrightarrow (y \cap z = \emptyset \wedge y \neq \emptyset)) \longrightarrow (\exists_{\mathcal{C}} u)(\forall y)(\exists v)(y \in x \longrightarrow (y \cap u = \{v\}))\}$$

Axiom 3.10 (*Foundation*)

$$(\forall_{\mathcal{C}} x)(x \neq \emptyset \wedge (\forall y)(y \in x \longrightarrow \mathcal{C}(y))) \longrightarrow (\exists z)(z \in x \wedge z \cap x = \emptyset)$$

3.2 Axioms for PTs

Now we list the axioms for PTs. We will use small Greek letters for variables restricted to PTs. Informally, the symbol "$\sqsubset$" will express the "being part of" relation. Thus, "$\alpha \sqsubset \beta$" means that "α and β are PTs and α is a part of β". We start with some preliminary definitions.

Definition 3.11 (*Disjointness*)

$$\alpha | \beta := \neg \exists \gamma (\gamma \sqsubset \alpha \wedge \gamma \sqsubset \beta)$$

$\alpha | \beta$ is interpreted as "α and β are PTs which share no part in common"; a possible definition of indistinguishability could be given as follows (though we will not use it in this work).

Definition 3.12 (*Indiscernibility*)

$$\alpha \equiv \beta := \alpha \sqsubset \beta \wedge \beta \sqsubset \alpha$$

$\alpha \equiv \beta$ means that α and β are indistinguishable, in the sense that they cannot be discerned by any physical means.

Definition 3.13 (*PT*)

$$T(x) := \neg\mathcal{C}(x)$$

$T(x)$ reads "x is not a set", and thus, it is a PT.

Definition 3.14 (*Sum of parts*)

$$\mathcal{S}(x,\alpha) := \mathcal{C}(x) \wedge \forall y(y \in x \longrightarrow T(y)) \longrightarrow \forall\gamma(\gamma|\alpha \longleftrightarrow \forall\beta(\beta \in x \longrightarrow \beta|\gamma))$$

The explanation of $\mathcal{S}(x,\alpha)$ is that if x is a set such that all its elements are PTs, then for every γ which satisfies being disjoint to α, then it will also be disjoint to any element β in x and viceversa. Intuitively, the only PT α which has this property is the *physical sum* of all the PTs belonging to x.

We now formulate a general axiomatic for PTs. These axioms may encompass a general class of entities, ranging from field quanta to non relativistic particles. But it is important to remark that all these entities need more specific axioms in order to be fully characterized; we are concentrating here in their general mereological porperties.

We start by stating that every thing is a part of itself

Axiom 3.15

$$(\forall\alpha)(\alpha \sqsubset \alpha)$$

It is reasonable to assume transitivity of the relationship "$\sqsubset$"

Axiom 3.16

$$(\forall\alpha)(\forall\beta)(\forall\gamma)(\alpha \sqsubset \beta \wedge \beta \sqsubset \gamma \longrightarrow \alpha \sqsubset \gamma)$$

We will postulate that there exists the sum of any non empty set of PTs

Axiom 3.17

$$(\forall x)(\exists\alpha)(\mathcal{S}(x,\alpha))$$

4 Things with undefined number of parts

We will use the following notation

Definition 4.1

$$\exists\{x \mid F(x)\} := (\exists y)(\forall x)(x \in y \longleftrightarrow F(x))$$

and the following definition will allow us to present a possible solution to the problem posed in Sect. 1

Definition 4.2

$$Cant(\alpha) := \exists\{\beta \mid \beta \sqsubset \alpha\}$$

If $Cant(\alpha)$ we will say that α *is Cantorian*.[7] The above definition says that if a PT α is cantorian, then, all parts of α form a set (and vice versa). Thus, it is possible to assign a cardinal to any Cantorian thing α by assigning a cardinal number to its set of parts in the usual way (using choice Axiom 3.9). Notice that it is straightforward to show that if α is Cantorian, then there exists *only one* set satisfying the equality of Definition 4.2.

For any x such that $\mathcal{C}(x)$, denote $\sharp(x)$ the cardinal assigned in the usual way using the *ZF* axiomatic (and we can use it for sets, because the axiomatic of ZF^* includes that of *ZF*). Thus we define

Definition 4.3 If $Cant(\alpha)$, let z be the only set satisfying the equality of definition 4.2. Then we define the *cardinal* of α (abreviated as $\sharp(\alpha)$) as

$$\sharp(\alpha) := \sharp(z)$$

Any PT α will be cantorian or not. If α is not Cantorian (i.e., if $\neg(Cant(\alpha))$), then, there is no means for ensuring that its parts form a set using the above axioms. Because of this, there is no way in which we can assign to α a cardinal using *ZF* axioms, and from this point of view, it is reasonable to interpret a non Cantorian PT as having no cardinal. In this way, we find that the axiomatic framework presented in this work is useful to represent PTs with undefined number of constituents as the ones presented in Sect. 2. But once this general solution is presented, new problems may be posed. We list them below:

1. We provided a general axiomatic for PTs. But it is clear that each theory and spatio-temporal setting will have its own and characteristic ontological features implying its particular axiomatic. Which should be the specific axioms for non relativistic QM and relativistic QM respectively?
2. How to represent a physical thing which is in a superposition state like the one represented by equation (3)?
3. How to represent a physical superposition in general?
4. Related to (1) and (2), how to represent entanglement?

In this work, we presented a possible solution for question 2. Systems formed of an undefined particle number are represented by non-cantorian things. But -up to now- our formalism does not distinguishes the state $a_1|n\rangle + a_2|m\rangle$ from $a'_1|n\rangle + a'_2|m\rangle$ (with $a'_1 \neq a_1$ or $a'_2 \neq a_2$). In future works, we will essay possible solutions for the problems posed above.

Notwithstanding, something can be said about superpositions using non-cantorian sets right now (thus providing a partial answer to question 3). The following construction, shows that non-cantorian sets possess unexpected properties, which are

[7] We use "Cantorian" in analogy with the system NF of Quine (1953), Rosser (1953). But this should not lead to any confusion: the analogy is not too deep.

capable to yield non-standard mathematics and can represent physical situations at the same time. Suppose that α is such that $\neg(Cant(\alpha))$. Then, given a formula $F(x)$, it is impossible—with the above axioms—to grant the existence of the set

$$\alpha_F = \{\beta \sqsubset \alpha \mid F(\beta)\} \tag{4}$$

The separation axiom *cannot be applied*, because the parts of α do not confirm necessarily a set! But in a standard set theory (like *ZF*), "properties" are usually expressed as the membership to given set. For example, if we want to state that the number 4 is even, we can express this by the formula $4 \in \{x \in \mathbb{N} \mid \exists y(x = 2 \times y \wedge y \in \mathbb{N})\}$. But if we want to interpret our formula $F(x)$ as representing a physical property in *ZF** (defined by extension as the set of all PTs possessing that property), we will face a problem. We cannot grant the existence of the set formed by the parts of α possessing the property defined by $F(x)$. This is a direct consequence of $\neg(Cant(\alpha))$. This situation could be interpreted as follows: "if α is not Cantorian, we cannot assert that its parts possess the property defined by $F(x)$ or that they do not possess it". This fact, does not constitutes a real problem for our framework, but an unexpected advantage: *this kind of undetermination in the possession of a property can be interpreted as being in a superposition state*. Indeed, a key feature of a quantum mechanical superposition is the lack of meaning in asserting or denying the possession of a given property.

When we face a superposition—say, in a system of spin $\frac{1}{2}$—such as $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$, we are not capable of asserting that the system has spin up nor spin down: this is a key aspect of superpositions, captured by our framework. Thus, our framework is also capable of giving a precise logical meaning to superpositions (at least of a special kind). In order to make thinks clearer, think of α as formed by the photons of a state of the electro magnetic field such as (3). As it is a superposition in particle number, its energy is also undefined, and thus, the set of photons possessing a definite energy value will inherit the non-Cantorianity of α.

Taking into account the above discussions, it would be interesting to provide a definition of what should be considered classical and quantum PTs within our framework. We give definitions below trying to capture such notions.

Definition 4.4 (*Irreducible Part*)

$$\mathcal{I}(\alpha, \beta) := \alpha \sqsubset \beta \wedge (\forall \gamma)(\gamma \sqsubset \alpha \longrightarrow \gamma \equiv \alpha)$$

$\mathcal{I}(\alpha, \beta)$ will be interpreted as "α is an irreducible part of β", and this means that α is a part of β and that any part of α will be indistinguishable of α itself. It is straightforward to show that if α is cantorian, then there exists the set of all irreducible parts (hint: use separation). We remark that this set may be the empty set. Now we will define the important notions of *classical part* and *quantum part* with respect to a well formed formula $F(x)$. If α is a PT and $F(x)$ is a formula, we define

Definition 4.5 $Cant_F(\alpha) := \exists\{\beta \sqsubset \alpha \mid F(\beta)\}$

If $Cant_F(\alpha)$ we will say that α *has a cantorian subset of parts satisfying* $F(x)$. If $\neg Cant_F(\alpha)$, we will interpret this as: "some parts of α are in a superposition state with respect to the property $F(x)$". Thus, given a formula $F(x)$, we will say that

Definition 4.6 (*Quantum Part*)

$$\mathcal{QP}_F(\alpha) := \neg Cant_F(\alpha)$$

and interpret this as: "α is quantal with respect to $F(x)$".

Definition 4.7 (*Classical Part*)

$$\mathcal{CP}_F(\alpha) := Cant_F(\alpha)$$

and interpret this as: "α is classical with respect to $F(x)$".

We conclude this Section by adding a list of general remarks which could be useful to consider in further developments of a mereology involving quantum entities.

1. As remarked above, different axioms could be added to the above framework in order to capture different kinds of PTs. The specific form of these axioms will depend on the particular physical theory but also—and strongly—on the interpretation of that theory.
2. It should be clear that the spatio-temporal setting in which the theory is developed (v.g., Galilean space time for non-relativistic QM and Minkowski space-time for QFT) have a crucial influence in the mereological properties of the corresponding physical objects. This implies that, in order to develop a more specific framework, axioms containing specific space time notions should be added to the axiomatic presented in this work.
3. We may represent a general physical system as a triplet $< P, M, S >$, where P is a set representing PTs, M is the corresponding space-time differential manifold of the theory and S is a mathematical structure involving mathematical objects, some of which are built with the help of M. For example, non-relativistic QM may be represented as a set, endowed with Galilean manifold and the axiomatic of von Neumann written in the mathematical language of functional analysis. A unitary transformation will thus be a mathematical concept linked to the space-time notion of Galilean symmetry transformation. It is important to remark that the explicit inclusion of the space time manifold, while necessary for experimental verification of the theory, does not implies necessarily that the entities involved has well defined spatio-temporal properties, as is the case in the orthodox interpretation of QM.
4. It is easy to show that if in our system there are Cantorian sets, then, the totality for PTs will not be a set.
5. If one wants to quit identity of our system (in order to consider indistinguishable objects as in French and Krause (2006)), it suffices replace identity "=" for a new

symbol "≡", postulating that it is an equivalence relation with extra conditions (chosen in a suitable way in order to capture the desired physical features).

In future works, we will address these questions by developing a new system, namely Z^{**}, capable of incorporating all these features, and thus, providing a complete quantum mereology. The development of a quantum mereology is still an open problem, and the formal framework presented here is a concrete step in this direction. In particular, the formal approach to quantum features of our system is not present in previous mereological discussions (as for example, in Simons 1987; Darby and Watson 2010; Borghini and Lando 2011).

5 Conclusions

In this work we presented a solution for what can be considered a generalization of the Manin's problem, namely, the problem of *incorporating in a set theoretical framework the quantum mechanical notion of undefined particle number*. Furthermore, *our system recovers the interesting feature of possessing undefined properties representing quantum superpositions*. Although our proposal is a valid solution for the problems posed in Holik (2006, 2010), Domenech and Holik (2007), Holik (2011), Domenech et al. (2008a) a lot of questions arise and remain unsolved. In particular, it would be interesting to search for other axiomatic systems capturing quantum entanglement.

By incorporating these quantum features, *our framework is a concrete step for the development of a rigorous quantum mereology*. This is an important issue for those philosophers interested in the development of any ontology which takes QM as a fundamental theory.

Of course, many other constructions could be envisaged, and they may depend on the particular interpretation of the quantum formalism. For example, it would be interesting to look for the specific implications that the spatio-temporal setting has for the mereological axiomatic capturing the properties of the physical systems of different theories. In particular, a quantum relativistic and non-relativistic mereology is lacking, and we think that the development of set theoretical frameworks like the one presented in this work could be useful for that purpose.

The characterization of undefined particle number and more general quantum superpositions presented in this work, could be used in different—and perhaps, more sophisticated—frameworks. We note that the proposed logical system presented in this paper can be used as a basis for all non relativistic QM; we shall discuss this question in a forthcoming paper.

Another interesting question to look at would be that of the implications for mathematics of systems like the one presented here. How would it be a mathematics not based on our every day concepts, but on QM? Such a question was partially answered French and Krause 2006, but our system opens a new door to such a research program. In particular, the system presented above, constitutes a novel example of non-standard mathematics, which gives a precise logical meaning to the—up to now—intuitive notion of what physicists mean by "undefined particle number".

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