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Addendum to "Vertex adjacencies in the set covering polyhedron" [Discrete Appl. Math. 218 (2017) 40–56]

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ABSTRACT

We study the relationship between the vertices of an up-monotone polyhedron R and those of the polytope P obtained by truncating R with the unit hypercube. When R has binary vertices, we characterize the vertices of P in terms of the vertices of R, show their integrality, and prove that the 1-skeleton of R is an induced subgraph of the 1-skeleton of P. We conclude by applying our findings to settle a claim in the original paper.

1. Introduction

In [1] we studied vertex adjacency in the (unbounded version of the) set covering polyhedron associated with a binary matrix A:

$$Q^*(A) = \operatorname{conv}(\{x \in \mathbb{Z}^n \mid Ax \geq \mathbf{1}, x \geq \mathbf{0}\}), \tag{1.1}$$

where $\mathbf{0}$ and $\mathbf{1}$ denote vectors of appropriate dimension with all zeros and all ones components respectively, and $\operatorname{conv}(X)$ denotes the convex hull of the set $X \subset \mathbb{R}^n$. This polyhedron is the dominant of the set covering polytope associated with A:

$$\overline{Q^*(A)} = \operatorname{conv}(\{x \in \mathbb{Z}^n \mid Ax \geq \mathbf{1}, \mathbf{1} \geq x \geq \mathbf{0}\}), \tag{1.2}$$

that is, $Q^*(A) = \overline{Q^*(A)} + \{x \in \mathbb{R}^n \mid x \geq \mathbf{0}\}$ where $+$ denotes the Minkowski sum of subsets of $\mathbb{R}^n$.

Immediately after stating Theorem 2.1 in [1], we made the following claim:

Claim 1.1. *It can be proved that for any binary matrix A two vertices of $Q^*(A)$ are adjacent if and only if they are adjacent in $\overline{Q^*(A)}$.*

Although this result may seem quite natural, we would like to observe that it is no longer true if we replace $Q^*(A)$ by its linear relaxation,

$$Q(A) = \{x \in \mathbb{R}^n \mid Ax \geq \mathbf{1}, x \geq \mathbf{0}\}, \tag{1.3}$$

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and $\overline{Q^*(A)}$ by the corresponding bounded version,

$$\overline{Q(A)} = Q(A) \cap [0, 1]^n. \tag{1.4}$$

This may be seen by considering the circulant matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \tag{1.5}$$

In this case, the vertices of $\overline{Q(A)}$ are

$$(1, 1, 0),\ (0, 1, 1),\ (1, 0, 1),\ (1, 1, 1),\ (1/2, 1/2, 1/2),$$

and $\xi = (1, 1, 0)$ and $\eta = (0, 1, 1)$ are adjacent in $\overline{Q(A)}$ but not in $Q(A)$. Furthermore, as is readily verified, in this example ξ and η are adjacent in $Q^*(A)$, which means that in general $Q(A)$ does not have the *Trubin property* with respect to $Q^*(A)$.[1]

This is rather surprising since in the special case in which A has precisely two ones per row, i.e., the case in which A is the edge-node incidence matrix of a graph G, $\overline{Q(A)}$ has the Trubin property with respect to $\overline{Q^*(A)}$.[2]

One of the aims of this paper is to prove the validity of Claim 1.1. Along the road we will establish relationships between the vertices of an up-monotone polyhedron R and those of a polyhedron $Q \subseteq R$ such that the vertices of R belong to Q. The results here do not depend on those in [1], and we think they are interesting by themselves.

This addendum is organized as follows. In Section 2 we introduce some notation and present basic results concerning vertices and their adjacency in an up-monotone polyhedron. Section 3 is the core of the paper, where we study the effect of cutting with the unit hypercube an up-monotone polyhedron having only binary vertices, first characterizing the vertices of the new polytope (Corollary 3.5) and proving their integrality (Corollary 3.6), and then studying the adjacency of the vertices of the larger polyhedron in the new polytope (Theorem 3.7). We conclude by relating our findings to the original article [1] in Section 4.

2. Some properties of vertices in up-monotone polyhedra

In this section we introduce notation which perhaps is not quite established in the literature, and state a few basic results that are either simple to prove or well-known, and so we will omit most of the proofs.

Let us start with the notation, part of which we have already used.

The set $\{1, \dots, n\}$ is denoted by $\mathcal{I}_n$, the family of subsets of $\mathcal{I}_n$ by $\mathscr{P}$, the ith vector of the canonical base of $\mathbb{R}^n$ by $\mathbf{e}_i$, and the scalar product in $\mathbb{R}^n$ by a dot: $x \cdot y = \sum_{i=1}^n x_i y_i$.

For x and y in $\mathbb{R}^n$, $[x, y] = \operatorname{conv}(\{x, y\})$ represents the (closed) segment joining them, and we write $x \geq y$ (resp. $x > y$) if $x_i \geq y_i$ (resp. $x_i > y_i$) for all $i \in \mathcal{I}_n$ (notice that $x \gneqq y$, i.e., $x \geq y$ and $x \neq y$, does not imply $x > y$).

Given a polyhedron S, the set of its vertices is denoted by $\mathrm{V}(S)$.

Throughout the paper we will assume that $R \subset \{x \in \mathbb{R}^n \mid x \geq \mathbf{0}\}$ is a non-empty polyhedron which is *up-monotone*,[3] that is, it satisfies any of the following equivalent conditions:

- $x \in R$ and $y \geq x$ imply $y \in R$,
- $x \in R$ if and only if $x = y + \mu$ with $y \in \operatorname{conv}(\mathrm{V}(R))$ and $\mu \geq \mathbf{0}$.

Our first result relates vertices and minimality in R.

Lemma 2.1. *Assuming ξ and η are distinct vertices of R and $x \in R$, we have:*

(a) *If $x \leq \xi$ then $x = \xi$.*
(b) *If $x \in [\xi, \eta]$ and $\mu \geq \mathbf{0}$ is such that $x + \mu \in [\xi, \eta]$, then $\mu = \mathbf{0}$.*

The following proposition is fundamental to our work.

Proposition 2.2. *If $\mathrm{V}(R) = \{\xi = \zeta^1, \eta = \zeta^2, \dots, \zeta^r\}$, then the following are equivalent:*

(a) *ξ and η are adjacent in R, that is, there exist $c \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $c \cdot x \geq b$ for all $x \in R$, with equality if and only if $x \in [\xi, \eta]$.*
(b) *There exist $c \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $c > \mathbf{0}$ and*

$$c \cdot x \geq b \text{ for all } x \in \operatorname{conv}(\mathrm{V}(R)), \text{ with equality if and only if } x \in [\xi, \eta]. \tag{2.1}$$

[1] Let us recall that a polyhedron R has the Trubin property with respect to a polyhedron P contained in R if the 1-skeleton of P is an induced subgraph of the 1-skeleton of R, see [3].

[2] Since the linear relaxation FRAC(G) of the stable set polytope STAB(G) has the Trubin property with respect to STAB(G) (see [2]), and the function $x \to \mathbf{1} - x$ affinely maps $\overline{Q(A)}$ to FRAC(G).

[3] Or *upper comprehensive* in the nomenclature of some authors.

(c) *If $x+\mu \in [\xi,\eta]$, where $\mu \geq \mathbf{0}$ and x is a convex combination of the form $\sum_{k=1}^{r} \lambda_k \zeta^k$, then $\lambda_k = 0$ for $k = 3, \dots, r$ (so $x \in [\xi,\eta]$ and $\mu = \mathbf{0}$).*

Proof. It is easy to show that (a) implies (b) and that (b) implies (c). Thus, we next show only that (c) implies (a). We do this by contradiction, so assume (c) holds but ξ and η are not adjacent in R. Then, the minimal face of R containing ξ and η has dimension at least 2. It follows that there exist two points $y', y'' \in R$ and $\lambda \in \mathbb{R}$ such that $0 < \lambda < 1$, $\lambda y' + (1-\lambda) y'' \in [\xi,\eta]$ and neither y' nor y'' belong to $[\xi,\eta]$. Since R is up-monotone, we can find x' and x'' in conv(V(R)) and $\mu', \mu'' \geq \mathbf{0}$ such that $y' = x' + \mu'$ and $y'' = x'' + \mu''$. Writing $x = \lambda x' + (1-\lambda)x''$ and $\mu = \lambda\mu' + (1-\lambda)\mu''$, we have $x \in$ conv(V(R)), $\mu \geq \mathbf{0}$, and $x + \mu = \lambda y' + (1-\lambda) y'' \in [\xi,\eta]$, so by (c) we conclude that $\mu = \mathbf{0} = \mu' = \mu''$, that is, $y' = x'$ and $y'' = x''$. Since x' and x'' are in conv(V(R)), it follows that

$$y' = x' = \sum_{k=1}^{r} \tau'_k \zeta^k, \quad y'' = x'' = \sum_{k=1}^{r} \tau''_k \zeta^k, \quad x = \sum_{k=1}^{r} \big(\lambda\tau'_k + (1-\lambda)\tau''_k\big)\zeta^k,$$

and again by (c) we must have

$$\lambda\tau'_k + (1-\lambda)\tau''_k = 0 \quad \text{for } k \neq 1, 2,$$

i.e., $\tau'_k = \tau''_k = 0$ for $k \neq 1, 2$. Hence, y' and y'' are convex combinations of ξ and η, that is, they are in $[\xi,\eta]$, contradicting the way they have been chosen above. □

3. Bounding with the unit hypercube

We now turn our attention to studying the relationship between the vertices of the up-monotone polyhedron R and those of $R \cap [0,1]^n$.

We omit the proof of the following simple result relating the vertices of two polyhedra in a somewhat more general setting.

Lemma 3.1. *Let S and T be polyhedra such that $S \subset T$. We have:*

(a) *If $\xi \in$ V(T) $\cap S$ then $\xi \in$ V(S).*
(b) *If ξ and η are distinct points in V(T) $\cap S$ which are adjacent in T, then they are also adjacent in S.*

In the remainder of this section, we will assume that R is an up-monotone polyhedron satisfying

$$\mathrm{V}(R) \subset [0,1]^n, \tag{3.1a}$$

and P is defined by

$$P = R \cap [0,1]^n, \tag{3.1b}$$

so that V(R) $\subset P$.

We will find it convenient to consider the function $\varphi : \mathscr{P} \times \mathbb{R}^n \to \mathbb{R}^n$ defined component-wise by

$$\varphi(I,x)_i = \begin{cases} 1 & \text{if } i \in I, \\ x_i & \text{otherwise,} \end{cases} \tag{3.2}$$

that is, a projection for each $I \in \mathscr{P}$. Notice that $\varphi(I,x) = x$ if I is empty, and that if $x \in P = R \cap [0,1]^n$ then $x \leq \varphi(I,x)$ and $\varphi(I,x) \in P$ because R is up-monotone.

The next result says that any vertex of P can be obtained by "lifting" a vertex of R via φ.

Theorem 3.2. *If φ is defined by (3.2), then*

$$\mathrm{V}(P) \subset \{\varphi(I,\zeta) \mid I \in \mathscr{P}, \zeta \in \mathrm{V}(R)\}. \tag{3.3}$$

Proof. Observe that $P = R \cap \{x \in \mathbb{R}^n \mid x \leq \mathbf{1}\}$ because we have assumed $R \subset \{x \in \mathbb{R}^n \mid x \geq \mathbf{0}\}$. Then, to prove the theorem it is enough to show that any polyhedron in the sequence: $P_0 = R$ and $P_k = P_{k-1} \cap \{x \in \mathbb{R}^n \mid x_k \leq 1\}$ for $k \in \mathcal{I}_n$, satisfies (3.3). We do this by induction. Since $\varphi(I,x) = x$ if I is empty, it is obvious that P_0 satisfies (3.3). So assume now that P_{k-1} satisfies (3.3). Note that the vertices of P_k which are not vertices of P_{k-1} coincide with the intersections consisting of a single point of the hyperplane $\{x \in \mathbb{R}^n \mid x_k = 1\}$ with the relative interior of edges of P_{k-1}. Besides, observe that the relative interior of no bounded edge of P_{k-1} can intersect $\{x \in \mathbb{R}^n \mid x_k = 1\}$ in a single point because that would imply $\xi_k > 1$ for some vertex ξ of P_{k-1}, contradicting that $V(P_{k-1}) \subset [0,1]^n$ (which follows from the fact that P_{k-1} satisfies (3.3) and R satisfies (3.1a)). Thus, any vertex of P_k which is not a vertex of P_{k-1} is given by the intersection of the relative interior of an unbounded edge of P_{k-1} with $\{x \in \mathbb{R}^n \mid x_k = 1\}$. Since R is up-monotone, any unbounded edge of P_{k-1} is of the form $\{\xi + \gamma\mathbf{e}_h \mid \gamma \geq 0\}$, where ξ is a vertex of P_{k-1} and $h \in \{k, \dots, n\}$. This completes the proof, because when the intersection of $\{\xi + \gamma\mathbf{e}_h \mid \gamma \geq 0\}$ with $\{x \in \mathbb{R}^n \mid x_k = 1\}$ is not empty (i.e., when $h = k$), it consists of a point which can be obtained replacing the hth component of ξ by a one. □

The following is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Suppose R and P verify (3.1) and S is a polyhedron verifying* $\mathrm{V}(R) \subset S \subset R$ *and*

$$\varphi(I, \zeta) \in S \text{ for all } I \in \mathscr{P} \text{ and } \zeta \in \mathrm{V}(R).$$

Then $\mathrm{V}(P) \subset S$, *that is,* $P = R \cap [0, 1]^n \subset S$.

So far we have not assumed the integrality of the vertices of R, and, for instance, Theorem 3.2 may be applied to $R = Q(A)$ (defined in (1.3)) and $P = \overline{Q(A)}$ (defined in (1.4)).

Before studying the case $\mathrm{V}(R) \subset \mathbb{B}^n$, where $\mathbb{B} = \{0, 1\}$ denotes the set of binary numbers, let us state without proof some simple properties relating φ, binary points and vertices of R and $P = R \cap [0, 1]^n$.

Lemma 3.4. *In the following we assume $I \in \mathscr{P}$.*

(a) *If $x \in \mathbb{B}^n$ then $\varphi(I, x) \in \mathbb{B}^n$.*
(b) *If $x \in P \cap \mathbb{B}^n$ then $x \in \mathrm{V}(P)$.*
(c) *If $x \in R \cap \mathbb{B}^n$ then $\varphi(I, x) \in \mathrm{V}(P)$.*

The following result characterizes the vertices of P when the vertices of R are binary.

Corollary 3.5. *If* $\mathrm{V}(R) \subset \mathbb{B}^n$, $P = R \cap [0, 1]^n$, *and φ is defined by (3.2), then*

$$\mathrm{V}(P) = \{\varphi(I, \zeta) \mid I \in \mathscr{P}, \zeta \in \mathrm{V}(R)\}.$$

Proof. One inclusion is given by Lemma 3.4(c), and the other one by Theorem 3.2. □

Using Lemma 3.4, it is easy to see now that all vertices of $R \cap [0, 1]^n$ are binary.

Corollary 3.6. *If* $\mathrm{V}(R) \subset \mathbb{B}^n$ *and* $P = R \cap [0, 1]^n$, *then* $\mathrm{V}(P) \subset \mathbb{B}^n$.

We come now to the main result of this work.

Theorem 3.7. *Assume* $\mathrm{V}(R) \subset \mathbb{B}^n$, $P = R \cap [0, 1]^n$, *and ξ and η are distinct vertices of R.*
Then, ξ and η are adjacent in P if and only if they are adjacent in R.

Proof. Since $\mathrm{V}(R) \subset R \cap \mathbb{B}^n \subset R \cap [0, 1]^n = P \subset R$, one implication is given by Lemma 3.1(b).

For the other, if ξ and η are adjacent in P there exist $c \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$c \cdot x \geq b \text{ for all } x \in P, \text{ with equality if and only if } x \in [\xi, \eta]. \tag{3.4}$$

If $c > \mathbf{0}$ the result follows by Proposition 2.2(b) because $\mathrm{conv}(\mathrm{V}(R)) \subset P$. So let us assume that the set

$$I = \{i \in \mathcal{I}_n \mid c_i \leq 0\} \tag{3.5}$$

is not empty. In this case, to prove that ξ and η are adjacent in R, we show that Proposition 2.2(c) is satisfied. Then, setting

$$\mathrm{V}(R) = \{\xi = \zeta^1, \eta = \zeta^2, \ldots, \zeta^r\},$$

let a convex combination of the vertices of R of the form

$$z = \sum_{k=1}^{r} \lambda_k \zeta^k \tag{3.6}$$

and $\mu \geq \mathbf{0}$ be such that

$$z + \mu \in [\xi, \eta]. \tag{3.7}$$

Notice that for any $x \in [\xi, \eta] \subset P$, since $\varphi(I, x) \in P$ for such x and $c_i \leq 0$ for $i \in I$, by (3.4) we have

$$b = c \cdot x = \sum_{i \notin I} c_i x_i + \sum_{i \in I} c_i x_i \geq \sum_{i \notin I} c_i x_i + \sum_{i \in I} c_i = c \cdot \varphi(I, x) \geq b,$$

and so $c \cdot \varphi(I, x) = b$. It follows that $\varphi(I, x) \in [\xi, \eta]$ by (3.4), and then that $x = \varphi(I, x)$ by Lemma 2.1(b). In particular, we conclude that

$$x_i = 1 \text{ for all } x \in [\xi, \eta] \text{ and all } i \in I. \tag{3.8}$$

Letting

$$y = \sum_{k=1}^{r} \lambda_k \varphi(I, \zeta^k), \tag{3.9}$$

and defining τ component-wise by

$$\tau_i = \begin{cases} 0 & \text{if } i \in I, \\ \mu_i & \text{otherwise,} \end{cases} \tag{3.10}$$

by checking the components and using (3.8) and (3.7), we see that $y + \tau = z + \mu$. Moreover, as $\tau_i = 0$ for $i \in I$ and $c_i > 0$ for $i \notin I$, by (3.4) we obtain

$$b = c \cdot (z + \mu) = c \cdot (y + \tau) \geq c \cdot y \geq b,$$

and therefore $\tau_i = 0$ for $i \notin I$, that is, $\tau = \mathbf{0}$. Thus, we have $y = z + \mu \in [\xi, \eta]$.

By Lemma 3.4(c), $\varphi(I, \zeta^k) \in \mathrm{V}(P)$ for all k, and since $y \in [\xi, \eta]$, (3.9) and the adjacency of ξ and η in P imply now that for each $k = 1, \ldots, r$, either $\varphi(I, \zeta^k) \in \{\xi, \eta\}$ or $\lambda_k = 0$. Using Lemma 2.1(a) and the fact that $\zeta^k \leq \varphi(I, \zeta^k)$, we see that $\varphi(I, \zeta^k) \in \{\xi, \eta\}$ implies $\zeta^k \in \{\xi, \eta\}$. Thus, in (3.6) we must have either $\zeta^k \in \{\xi, \eta\}$ or $\lambda_k = 0$, that is, Proposition 2.2(c) is satisfied. □

Corollary 3.8. *Suppose* $\mathrm{V}(R) \subset \mathbb{B}^n$, *S is a polyhedron such that* $\mathrm{V}(R) \subset S \subset R$ *and* $\varphi(I, \zeta) \in S$ *for all* $I \in \mathscr{P}$ *and* $\zeta \in \mathrm{V}(R)$, *and* ξ *and* η *are distinct vertices of R (and hence of S by Lemma 3.1(a)).*

Then ξ *and* η *are adjacent in S if and only if they are adjacent in R.*

Proof. Let us start by assuming that ξ and η are adjacent in S. By Corollary 3.3, $P = R \cap [0, 1]^n \subset S$, and adjacency in S implies adjacency in P by Lemma 3.1(b) because $\{\xi, \eta\} \subset \mathrm{V}(S) \cap P$ and $P \subset S$. So, by Theorem 3.7, ξ and η are adjacent in R.

On the other hand, if ξ and η are adjacent in R, their adjacency in S follows again from Lemma 3.1(b), as $\mathrm{V}(R) \subset S \subset R$. □

The conclusion of relevance in Theorem 3.7 is that vertices in the up-monotone polyhedron R which are adjacent in $P = R \cap [0, 1]^n$ are also adjacent in R, provided that $\mathrm{V}(R) \subset \mathbb{B}^n$. As we have seen in the Introduction, we cannot discard this hypothesis: if A is given by (1.5), then $R = Q(A)$ has just one fractional vertex, and the vertices $(1, 1, 0)$ and $(0, 1, 1)$ are adjacent in $P = \overline{Q(A)}$ but not in R.

4. The claim in the original article

The polyhedron

$$R = Q^*(A) = \mathrm{conv}(\{x \in \mathbb{Z}^n \mid Ax \geq \mathbf{1}, x \geq \mathbf{0}\}),$$

is up-monotone and $\mathrm{V}(R) \subset \mathbb{B}^n$, and it is simple to see that

$$S = \overline{Q^*(A)} = \mathrm{conv}(\{x \in \mathbb{Z}^n \mid Ax \geq \mathbf{1}, \mathbf{1} \geq x \geq \mathbf{0}\})$$

satisfies the hypothesis in Corollary 3.8, proving Claim 1.1.

Notice that Corollary 3.3 implies $Q^*(A) \cap [0, 1]^n \subset \overline{Q^*(A)}$. On the other hand, clearly $\{x \in \mathbb{Z}^n \mid Ax \geq \mathbf{1}, \mathbf{1} \geq x \geq \mathbf{0}\}$ is contained in both $Q^*(A)$ and $[0, 1]^n$, so

$$\overline{Q^*(A)} = Q^*(A) \cap [0, 1]^n,$$

and actually Theorem 3.7 may be applied directly.

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