NUMERICAL FUNCTIONAL ANALYSIS AND OPTIMIZATION
http://dx.doi.org/10.1080/01630563.2017.1291520

Best Simultaneous Local Approximation in the L^p Norms

D. E. Ferreyra[a,b] and F. E. Levis[b]

[a]Departamento de Matemática, FCEyN, Universidad Nacional de La Pampa, La Pampa, Argentina; [b]Departamento de Matemática, FCEFQyN, Universidad Nacional de Rio Cuarto, Rio Cuarto, Argentina

ABSTRACT
We study the behavior of the best simultaneous approximation to two functions from a convex set in L^p spaces, $2 < p < \infty$, on a finite union of intervals when its measure tends to zero. In particular, we give sufficient conditions over the differentiability of two functions to assure existence of the best simultaneous local approximation from the class of algebraic polynomials of a fixed degree. These conditions are weaker than the ordinary differentiability given in previous works. More precisely, we consider differentiable functions in the sense L^p.

ARTICLE HISTORY
Received 13 September 2016
Revised 2 February 2017
Accepted 2 February 2017

KEYWORDS
Convex sets; L^p spaces; simultaneous approximation

AMS CLASSIFICATION
41A28; 41A10

1. Introduction

Let $x_j \in \mathbb{R}, 1 \leq j \leq k, k \in \mathbb{N}$, and let B_j be disjoint pairwise closed intervals centered at x_j and radius $r > 0$. Let $\mathcal{M}_0$ be the space of equivalence class of Lebesgue measurable real functions on $I := \cup_{j=1}^{k} B_j$.

Let $2 < p < \infty$ and $L^p(I)$ be the space of functions $h \in \mathcal{M}_0$ such that $\int_I |h|^p \frac{1}{|I|} < \infty$, where $|I|$ is the Lebesgue measure of I. If $h \in L^p(I)$, we consider the L^p norm

$$\|h\|_p := \left(\int_I |h|^p \frac{1}{|I|} \right)^{\frac{1}{p}}.$$

For each $0 < \epsilon < 1$, we also put $\|h\|_{p,\epsilon} = \|h^\epsilon\|_p$, where $h^\epsilon(t) = h(\epsilon(t - x_j) + x_j)$, $t \in B_j$. Therefore,

$$\|h\|_{p,\epsilon} = \left(\int_{I_\epsilon} |h|^p \frac{1}{|I_\epsilon|} \right)^{\frac{1}{p}},$$

where $I_\epsilon = \bigcup_{j=1}^{k} B_{j,\epsilon}$ with $B_{j,\epsilon} = \left[x_j - r\epsilon, x_j + r\epsilon \right]$.

Let K be a closed convex subset of $L^p(I)$, and let $f_1, f_2 \in L^p(I)$. A function $h_{p,\epsilon} = h_{p,\epsilon}(f_1, f_2) \in K$ is called the best simultaneous approximation (b.s.a.) to f_1 and f_2 from K in $L^p(I_\epsilon)$ (L^p-b.s.a.), if

$$E_{p,\epsilon} := \inf_{h \in K} \left(\|f_1 - h\|_{p,\epsilon}^p + \|f_2 - h\|_{p,\epsilon}^p \right)^{\frac{1}{p}} = \left(\|f_1 - h_{p,\epsilon}\|_{p,\epsilon}^p + \|f_2 - h_{p,\epsilon}\|_{p,\epsilon}^p \right)^{\frac{1}{p}}.$$

CONTACT F. E. Levis flevis@exa.unrc.edu.ar Departamento de Matemática, FCEFQyN, Universidad Nacional de Rio Cuarto, Rio Cuarto 5800, Argentina.

Existence and uniqueness theorems for the b.s.a. are given in [20, Corollary 3.5].

If the net $\{h_{p,\epsilon}\}$ has a limit in K as $\epsilon \to 0$, this limit is called the *best simultaneous local approximation (b.s.l.a.) in L^p to f_1 and f_2 from K on* $\{x_1, \ldots, x_k\}$ *(L^p-b.s.l.a.).*

For $n \in \mathbb{N} \cup \{0\}$, we will denote by Π^n the class of algebraic polynomials of degree at most n.

In 1934, Walsh proved in [21] that the Taylor polynomial of degree n for an analytic function f can be obtained by taking the limit as $\epsilon \to 0$ of the best Chebyshev approximation from Π^n to f on the disk $|z| \leq \epsilon$. The concept of best local approximation has been introduced and developed by Chui et al. in [4] for a single function. Later, several authors [2, 3, 12, 16, 22] have studied this problem.

On the other hand, the subject of simultaneous approximation also has been extensively treated. Existence, uniqueness, and characterization theorems can be seen in [11, 15, 19].

In [13], the authors proved that the L^2-b.s.a. to two functions is identical with the best L^2-approximation to the mean value of the functions. It is well known that the L^p-b.s.a., in general, does not match with the best L^p-approximation to the mean of the functions [14]. However, it is useful to know whether they are close when we have a small enough domain.

In [10], the authors have studied the asymptotic behavior of a net of b.s.a. on the intervals $[-\epsilon, \epsilon]$ to N functions from Π^n, respect to the norm $\sum_{j=1}^{N} \|f_j - h\|_{p,\epsilon}$, $1 \leq p < \infty$, as $\epsilon \to 0$. They showed that if the functions f_j are sufficiently differentiable, the set of cluster points of the net is a convex compact set and it is contained in the convex hull of the Taylor polynomials of the functions at zero.

The problem of best simultaneous L^p-approximation to two functions from Π^n it was considered again in [6, 7]. The authors proved that the L^p-b.s.a. to two functions on an interval converges to the Taylor polynomial of degree n of the mean value of functions when the measure of interval tends to zero. Best simultaneous L^p-approximation for many intervals was also considered in [7, 8]. In these papers, interpolation theorems for the L^p-b.s.a. to two functions were given. As a consequence, they obtained that a net of L^p-b.s.a. is uniformly bounded on compact sets, when the measure of the domain tends to zero. Moreover, the authors proved that the set of cluster points of the net is contained in the set of solutions of a discrete minimization problem. More results about these topics can be seen in [9]. In all these works, the ordinary differentiability of functions is assumed.

In this paper, we generalize several results presented in [6]–[10] relative to b.s.l.a. We study the asymptotic behavior of a net of b.s.a. to two functions from a convex set in $L^p(I)$, $2 < p < \infty$, as $\epsilon \to 0$. In the particular case, where $K = \Pi^n$, we given some results about the existence and characterization of the b.s.l.a. in L^p to two functions, under weaker conditions of differentiability. More precisely, we consider differentiable functions in the sense L^p.

We remark that it is important to find the limit of the b.s.a., since as such it provides useful qualitative and analytic information concerning the b.s.a. on small regions, which is difficult to obtain from a strictly numerical treatment.

2. Asymptotic behavior of the b.s.a. from a convex set

In this section, we study the asymptotic behavior of the net $\{h_{p,\epsilon}\}$ of L^p-b.s.a. to two functions f_1 and f_2 from a convex set K on I, as $\epsilon \to 0$.

We consider the function $G_p : \mathbb{R} \to \mathbb{R}$ and $H_p : \mathbb{R}^2 \to [0, +\infty)$ defined by $G_p(t) = |t|^{p-1} sgn(t)$ and

$$H_p(x,y) = \begin{cases} |x|^{p-2} \dfrac{1 - \left|\frac{y}{x}\right|^{p-1} sgn(x)sgn(y)}{1 - \left|\frac{y}{x}\right| sgn(x)sgn(y)} & \text{if } |x| > |y|, \\ \left|y\right|^{p-2} \dfrac{1 - \left|\frac{x}{y}\right|^{p-1} sgn(x)sgn(y)}{1 - \left|\frac{x}{y}\right| sgn(x)sgn(y)} & \text{if } |x| \le |y|,\ x \ne y, \\ \left|y\right|^{p-2}(p-1) & \text{if } x = y. \end{cases} \tag{1}$$

It is easily seen that for all $x, y \in \mathbb{R}$,

$$G_p(x) - G_p(y) = (x - y)H_p(x, y). \tag{2}$$

Let $\alpha_p, \beta_p : [0, 1] \to [0, +\infty)$ be given by

$$\alpha_p(t) = \begin{cases} \dfrac{1 - t^{p-1}}{1 - t} & \text{if } t \ne 1 \\ p - 1 & \text{if } t = 1 \end{cases} \quad \text{and} \quad \beta_p(t) = \frac{1 + t^{p-1}}{1 + t}.$$

A trivial verification shows that

$$1 \le \alpha_p(t) \le p - 1 \quad \text{and} \quad \frac{1}{2} \le \beta_p(t) \le 1, \quad t \in [0, 1]. \tag{3}$$

To prove the next lemma, we use the following property of real numbers.

$$|x + y| + |x - y| = 2\max\{|x|, |y|\}, \quad x, y \in \mathbb{R}. \tag{4}$$

Lemma 2.1. *It verifies that*

$$\frac{1}{2}\left(\left|\frac{x+y}{2}\right| + \left|\frac{x-y}{2}\right|\right)^{p-2} \le H_p(x,y) \le (p-1)\left(\left|\frac{x+y}{2}\right| + \left|\frac{x-y}{2}\right|\right)^{p-2}, \quad x, y \in \mathbb{R}. \tag{5}$$

Proof. From (1) and (3), we have

$$\frac{1}{2}\max\{|x|, |y|\}^{p-2} \le H_p(x,y) \le (p-1)\max\{|x|, |y|\}^{p-2}.$$

So, (5) it is valid by (4). □

The following theorem is an immediate consequence of [18, Theorem 1.6].

Theorem 2.2. *Let K be a closed convex subset of $L^p(I)$. Assume $f_1, f_2 \in L^p(I)$ such that $E_{p,\epsilon} \neq 0$ for some ϵ. Then $h_{p,\epsilon}$ is the unique element in K satisfying*

$$\int_{I_\epsilon} \left(\left| \frac{f_1 - h_{p,\epsilon}}{E_{p,\epsilon}} \right|^{p-1} sgn\left(\frac{f_1 - h_{p,\epsilon}}{E_{p,\epsilon}} \right) + \left| \frac{f_2 - h_{p,\epsilon}}{E_{p,\epsilon}} \right|^{p-1} sgn\left(\frac{f_2 - h_{p,\epsilon}}{E_{p,\epsilon}} \right) \right)$$
$$\times (h_{p,\epsilon} - h) \frac{1}{|I_\epsilon|} \geq 0, \quad h \in K.$$

For $E_{p,\epsilon} \neq 0$, we will denote by $\nu_{p,\epsilon}$ the weight function given by

$$\nu_{p,\epsilon} := \frac{1}{E_{p,\epsilon}^{p-2}} H_p(f_1 - h_{p,\epsilon}, h_{p,\epsilon} - f_2). \tag{6}$$

We consider the seminorm on $L^p(I)$ defined by

$$\|h\|_{\nu_{p,\epsilon}} := \left(\int_{I_\epsilon} |h|^2 \frac{\nu_{p,\epsilon}}{|I_\epsilon|} \right)^{\frac{1}{2}}, \quad h \in L^p(I).$$

Given $g, h \in L^p(I)$, we will denote by $\gamma^+_{\nu_{p,\epsilon}}(g,h)$ the one-sided Gateaux derivative of the seminorm $\| \cdot \|_{\nu_{p,\epsilon}}$ at g in the direction h, i.e.,

$$\gamma^+_{\nu_{p,\epsilon}}(g,h) = \int_{I_\epsilon} gh \frac{\nu_{p,\epsilon}}{|I_\epsilon|}.$$

Lemma 2.3. *It verifies that*

$$\left\| \nu_{p,\epsilon} \right\|_{\frac{p}{p-2},\epsilon} \leq (p-1) 2^{\frac{(p-2)^2}{p}}. \tag{7}$$

Proof. By the monotonicity of norm $\| \cdot \|_{\frac{p}{p-2},\epsilon}$, Lemma 2.1 and the Minkowski inequality, we have

$$E_{p,\epsilon}^{p-2} \left\| \nu_{p,\epsilon} \right\|_{\frac{p}{p-2},\epsilon} \leq (p-1) \left\| \left(\left| \frac{f_1 + f_2}{2} - h_{p,\epsilon} \right| + \left| \frac{f_1 - f_2}{2} \right| \right)^{p-2} \right\|_{\frac{p}{p-2},\epsilon}$$
$$= (p-1) \left\| \left| \frac{f_1 + f_2}{2} - h_{p,\epsilon} \right| + \left| \frac{f_1 - f_2}{2} \right| \right\|_{p,\epsilon}^{p-2}. \tag{8}$$

By the inequalities of Minkowski and Clarkson, respectively, we have

$$\left\| \left| \frac{f_1 + f_2}{2} - h_{p,\epsilon} \right| + \left| \frac{f_1 - f_2}{2} \right| \right\|_{p,\epsilon}$$
$$\leq \left\| \frac{f_1 + f_2}{2} - h_{p,\epsilon} \right\|_{p,\epsilon} + \left\| \frac{f_1 - f_2}{2} \right\|_{p,\epsilon}$$

$$\leq \left(2^{p-1}\left(\left\|\frac{f_1+f_2}{2}-h_{p,\epsilon}\right\|_{p,\epsilon}^p+\left\|\frac{f_1-f_2}{2}\right\|_{p,\epsilon}^p\right)\right)^{\frac{1}{p}}$$

$$\leq \left(2^{p-2}\left(\|f_1-h_{p,\epsilon}\|_{p,\epsilon}^p+\|f_2-h_{p,\epsilon}\|_{p,\epsilon}^p\right)\right)^{\frac{1}{p}} = 2^{\frac{p-2}{p}}E_{p,\epsilon}. \tag{9}$$

From (8) and (9), we obtain (7). □

We observe that if $E_{p,\epsilon} = 0$, then $f_1 = f_2$ on I_ϵ, and hence $h_{p,\epsilon}$ is the best approximation to $\frac{f_1+f_2}{2}$ in the L^p norm on I_ϵ. We can now establish a relationship between the b.s.a. to two functions and a weighted best approximation to the mean value of the functions whenever $E_{p,\epsilon} \neq 0$.

Theorem 2.4. *Let K be a closed convex subset of $L^p(I)$. Assume $f_1, f_2 \in L^p(I)$ such that $E_{p,\epsilon} \neq 0$. Then $h_{p,\epsilon}$ is a best approximation to $\frac{f_1+f_2}{2}$ from K with respect to $\|\cdot\|_{\nu_{p,\epsilon}}$, i.e.,*

$$\left\|\frac{f_1+f_2}{2}-h_{p,\epsilon}\right\|_{\nu_{p,\epsilon}} \leq \left\|\frac{f_1+f_2}{2}-h\right\|_{\nu_{p,\epsilon}}, \quad h \in K.$$

Proof. Let $h \in K$. According to (2), the definition of $\nu_{p,\epsilon}$, and Theorem 2.2, we get

$$\gamma^+_{\nu_{p,\epsilon}}\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}, h_{p,\epsilon}-h\right) = \int_{I_\epsilon}\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)(h_{p,\epsilon}-h)\frac{\nu_{p,\epsilon}}{|I_\epsilon|}$$

$$\geq \int_{I_\epsilon}\frac{1}{2E_{p,\epsilon}^{p-2}}\left(G_p(f_1-h_{p,\epsilon})-G_p(h_{p,\epsilon}-f_2)\right)(h_{p,\epsilon}-h)\frac{1}{|I_\epsilon|}$$

$$= \int_{I_\epsilon}\left(G_p\left(\frac{f_1-h_{p,\epsilon}}{E_{p,\epsilon}}\right)-G_p\left(\frac{h_{p,\epsilon}-f_2}{E_{p,\epsilon}}\right)\right)(h_{p,\epsilon}-h)\frac{E_{p,\epsilon}}{2|I_\epsilon|}$$

$$= \frac{E_{p,\epsilon}}{2}\int_{I_\epsilon}\left(\left|\frac{f_1-h_{p,\epsilon}}{E_{p,\epsilon}}\right|^{p-1} sgn\left(\frac{f_1-h_{p,\epsilon}}{E_{p,\epsilon}}\right)\right.$$

$$\left.+\left|\frac{f_2-h_{p,\epsilon}}{E_{p,\epsilon}}\right|^{p-1} sgn\left(\frac{f_2-h_{p,\epsilon}}{E_{p,\epsilon}}\right)\right)(h_{p,\epsilon}-h)\frac{1}{|I_\epsilon|} \geq 0. \tag{10}$$

Now, the proof immediately follows from (10) and Theorem 2.2. □

The following result provide us a useful and important property for a red $\{h_{p,\epsilon}\}$ of b.s.a. to two functions from a convex set in $L^p(I_\epsilon)$. It will be used to study the asymptotic behavior of the net of b.s.a.

Theorem 2.5. *Let K be a closed convex subset of $L^p(I)$. Assume $f_1, f_2 \in L^p(I)$ such that $E_{p,\epsilon} \neq 0$. Then*

$$\|h-h_{p,\epsilon}\|_{\nu_{p,\epsilon}} \leq (p-1)^{\frac{1}{2}}2^{\frac{(p-2)^2}{2p}}\left\|\frac{f_1+f_2}{2}-h\right\|_{p,\epsilon}, \quad h \in K.$$

Proof. Let $h \in K$. From (10), we have $\int_{I_\epsilon}\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)(h_{p,\epsilon}-h)\frac{v_{p,\epsilon}}{|I_\epsilon|} \geq 0$, and so

$$\left\|h-h_{p,\epsilon}\right\|^2_{v_{p,\epsilon}} \leq \int_{I_\epsilon}\left(\frac{f_1+f_2}{2}-h\right)(h_{p,\epsilon}-h)\frac{v_{p,\epsilon}}{|I_\epsilon|}.$$

By the Hölder inequality, it follows that

$$\begin{aligned}\left\|h-h_{p,\epsilon}\right\|^2_{v_{p,\epsilon}} &\leq \int_{I_\epsilon}\left(\left|\frac{f_1+f_2}{2}-h\right|\frac{v^{\frac{1}{2}}_{p,\epsilon}}{|I_\epsilon|^{\frac{1}{2}}}\right)\left((h-h_{p,\epsilon})\frac{v^{\frac{1}{2}}_{p,\epsilon}}{|I_\epsilon|^{\frac{1}{2}}}\right)\\ &\leq \left\|\frac{f_1+f_2}{2}-h\right\|_{v_{p,\epsilon}}\left\|h-h_{p,\epsilon}\right\|_{v_{p,\epsilon}}. \qquad (11)\end{aligned}$$

Since $\frac{2}{p}+\frac{p-2}{p}=1$, applying the Hölder inequality, we obtain

$$\begin{aligned}&\left\|\frac{f_1+f_2}{2}-h\right\|^2_{v_{p,\epsilon}}\\ &= \int_{I_\epsilon}\left|\frac{f_1+f_2}{2}-h\right|^2\frac{v_{p,\epsilon}}{|I_\epsilon|} = \int_{I_\epsilon}\left(\left|\frac{f_1+f_2}{2}-h\right|^2\frac{1}{|I_\epsilon|^{\frac{2}{p}}}\right)\left(\frac{v_{p,\epsilon}}{|I_\epsilon|^{\frac{p-2}{p}}}\right)\\ &\leq \left\|\frac{f_1+f_2}{2}-h\right\|^2_{p,\epsilon}\left\|v_{p,\epsilon}\right\|_{\frac{p}{p-2},\epsilon}.\end{aligned}$$

In consequence, from (11) and Lemma 2.3, we see that

$$\left\|h-h_{p,\epsilon}\right\|_{v_{p,\epsilon}} \leq \left\|\frac{f_1+f_2}{2}-h\right\|_{v_{p,\epsilon}} \leq (p-1)^{\frac{1}{2}}2^{\frac{(p-2)^2}{2p}}\left\|\frac{f_1+f_2}{2}-h\right\|_{p,\epsilon}.$$ □

An immediate consequence of this result is the following corollary.

Corollary 2.6. *Let K be a closed convex subset of $L^p(I)$. Assume $f_1, f_2 \in L^p(I)$ such that $E_{p,\epsilon} \neq 0$, $0 < \epsilon < \epsilon_0$. If there are $c \in \mathbb{N}\cup\{0\}$ and $h \in K$ such that $\left\|\frac{f_1+f_2}{2}-h\right\|_{p,\epsilon} = O(\epsilon^c)$ as $\epsilon \to 0$, then $\left\|h-h_{p,\epsilon}\right\|_{v_{p,\epsilon}} = O(\epsilon^c)$ as $\epsilon \to 0$.*

Remark 2.7. We observe that if $f_1 = f_2$ on I_{ϵ_0} for some $0 < \epsilon_0 < 1$, then $f_1 = f_2$ on I_ϵ and $h_{p,\epsilon}$ is the best approximation to $\frac{f_1+f_2}{2}$ in the L^p norm, $0 < \epsilon \leq \epsilon_0$. In consequence, if there are $c \in \mathbb{N}\cup\{0\}$ and $h \in K$ such that $\left\|\frac{f_1+f_2}{2}-h\right\|_{p,\epsilon} = O(\epsilon^c)$, then $\left\|h-h_{p,\epsilon}\right\|_{p,\epsilon} = O(\epsilon^c)$, as $\epsilon \to 0$.

Next, we give a result about behavior of the error. The proof is mutatis mutandis the same as for [9, Theorem 4.1].

Theorem 2.8. *Let K be a closed convex subset of $L^p(I)$. Assume $f_1, f_2 \in L^p(I)$. If there are $c \in \mathbb{N} \cup \{0\}$ and $h \in K$ such that $\left\|\frac{f_1+f_2}{2} - h\right\|_{p,\epsilon} = O(\epsilon^c)$ as $\epsilon \to 0$, then*

$$E_{p,\epsilon} = 2^{\frac{1}{p}} \left\|\frac{f_1 - f_2}{2}\right\|_{p,\epsilon} + O(\epsilon^c) \quad as \quad \epsilon \to 0.$$

Furthermore, $E_{p,\epsilon} = O(1)$ as $\epsilon \to 0$.

Next, we state our main result about asymptotic behavior of the net $\{h_{p,\epsilon}\}$ of L^p-b.s.a., as $\epsilon \to 0$.

Theorem 2.9. *Let K be a closed convex subset of $L^p(I)$. Assume $f_1, f_2 \in L^p(I)$. If there are $c \in \mathbb{N} \cup \{0\}$ and $h \in K$ such that $\liminf_{\epsilon \to 0} \epsilon^{-c} \left\|f_1 - f_2\right\|_{p,\epsilon} \neq 0$ and $\left\|\frac{f_1+f_2}{2} - h\right\|_{p,\epsilon} = O(\epsilon^c)$ as $\epsilon \to 0$, then*

$$\left(\int_{I_\epsilon} \left|h - h_{p,\epsilon}\right|^2 \left|f_1 - f_2\right|^{p-2} \frac{1}{|I_\epsilon|}\right)^{\frac{1}{2}} = O\left(\epsilon^c \left\|f_1 - f_2\right\|_{p,\epsilon}^{\frac{p}{2}-1}\right) \quad as \quad \epsilon \to 0. \tag{12}$$

Proof. We claim that

$$\int_{I_\epsilon} \left|h - h_{p,\epsilon}\right|^2 \left|\frac{f_1 - f_2}{2}\right|^{p-2} \frac{1}{|I_\epsilon|} \leq (p-1) 2^{\frac{(p-2)^2}{p}} E_{p,\epsilon}^{p-2} \left\|\frac{f_1 + f_2}{2} - h\right\|_{p,\epsilon}^2. \tag{13}$$

Indeed, since $\liminf_{\epsilon \to 0} \epsilon^{-c} \left\|f_1 - f_2\right\|_{p,\epsilon} \neq 0$, there exists $\epsilon_0 > 0$ such that $E_{p,\epsilon} \neq 0$ and $\left\|f_1 - f_2\right\|_{p,\epsilon} \neq 0$, $0 < \epsilon < \epsilon_0$. From Lemma 2.1 and Theorem 2.5, we obtain

$$\begin{aligned}
&\left(\int_{I_\epsilon} \left|h - h_{p,\epsilon}\right|^2 \frac{1}{2E_{p,\epsilon}^{p-2}} \left|\frac{f_1 - f_2}{2}\right|^{p-2} \frac{1}{|I_\epsilon|}\right)^{\frac{1}{2}} \\
&\leq \left\|h - h_{p,\epsilon}\right\|_{v_{p,\epsilon}} \leq (p-1)^{\frac{1}{2}} 2^{\frac{(p-2)^2}{2p}+1} \left\|\frac{f_1 + f_2}{2} - h\right\|_{p,\epsilon},
\end{aligned}$$

and so we get (13).

By Theorem 2.8, we have $\frac{E_{p,\epsilon}}{\left\|f_1 - f_2\right\|_{p,\epsilon}} = O(1)$ as $\epsilon \to 0$. Now, according to (13) and the hypothesis, we complete the proof of (12). □

Remark 2.10. Corollary 2.6 and Theorems 2.8 and 2.9 remain valid if $O(\epsilon^c)$ is replaced by $o(\epsilon^{c-1})$ as $\epsilon \to 0$, everywhere.

3. Existence of b.s.l.a. in L^p spaces from Π^n

In this section, we study the behavior of net $\{h_{p,\epsilon}\}$ of b.s.a. to two functions f_1 and f_2 in $L^p(I_\epsilon)$ when $K = \Pi^n$, $n \in \mathbb{N} \cup \{0\}$, as $\epsilon \to 0$. We will show that under some suitable conditions on smoothness of f_1 and f_2, the b.s.l.a. exists, is unique, and is characterized as the solution of a certain optimization problem involving only the values of the functions and its derivatives up to a order depending on n and k at the points $x_1, \dots, x_k$.

We recall the following pointwise smoothness condition which was introduced by Calderón and Zygmund in [1]. Let $m \in \mathbb{N} \cup \{0\}$, $f \in L^p(I)$ and let a be an interior point of I. We say that the function $f \in t_m^p(a)$ if there is $h \in \Pi^m$ for which

$$\left(\frac{1}{2\epsilon}\int_{a-\epsilon}^{a+\epsilon} |f-h|^p\right)^{\frac{1}{p}} = o(\epsilon^m) \quad \text{as} \quad \epsilon \to 0. \tag{14}$$

The number $h^{(m)}(a) \in \mathbb{R}$ is called the m-th L^p-derivative of f at a and denoted by $f_p^{(m)}(a)$. When $m = 0$, then $h = h(a)$ is also called L^p-limit of f at a. If $f_p^{(m)}(a)$ exists, then it is unique. Moreover, all the derivatives $f_p^{(s)}(a)$, $0 \le s \le m$, exist, and

$$\left(\frac{1}{2\epsilon}\int_{a-\epsilon}^{a+\epsilon} \left|f - \sum_{r=0}^{s} \frac{f_p^{(r)}(a)}{r!}(\cdot - a)^r\right|^p\right)^{\frac{1}{p}} = o(\epsilon^s) \quad \text{as} \quad \epsilon \to 0, \quad 0 \le s \le m. \tag{15}$$

To prove the next lemma, we use the following property of real numbers.

$$\big||x|^q - |z|^q\big| \le \begin{cases} |x-z|^q & \text{if } 0 < q \le 1 \\ q|x-z|\,||x|^{q-1} + |z|^{q-1}| & \text{if } 1 < q < \infty \end{cases}, \quad x, z \in \mathbb{R}. \tag{16}$$

Lemma 3.1. *Let $m \in \mathbb{N} \cup \{0\}$ and $f \in t_m^p(a)$. The following conditions are equivalent.*

(a) *(i) $m = 0$ or (ii) $m > 0$ and $f_p^{(i)}(a) = 0$, $0 \le i \le m-1$;*

(b) $\frac{1}{2\epsilon}\int_{a-\epsilon}^{a+\epsilon} \left| |f|^q - \left|\frac{f_p^{(m)}(a)}{m!}(\cdot - a)^m\right|^q \right| = o\left(\epsilon^{mq}\right)$ *as $\epsilon \to 0$, for all $q \in \mathbb{R}$, $0 < q \le p$;*

(c) $\lim_{\epsilon \to 0} \frac{1}{2\epsilon}\int_{a-\epsilon}^{a+\epsilon} \left|\frac{f}{\epsilon^m}\right|^q = |y|^q$, *for all $q \in \mathbb{R}$, $0 < q \le p$, where $y = (mq+1)^{-\frac{1}{q}}\frac{f_p^{(m)}(a)}{m!}$.*

Proof.

(a) $\Rightarrow$ (b) Let $q \in \mathbb{R}$ be such that $0 < q \le p$, and we write $B_{a,\epsilon} = [a-\epsilon, a+\epsilon]$. From (15) and Lemma [17, Lemma 1.12.3], it follows that

$$\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}} \left|f - \frac{f_p^{(m)}(a)}{m!}(\cdot - a)^m\right|^q\right)^{\frac{1}{q}} = o(\epsilon^m) \quad \text{as} \quad \epsilon \to 0. \tag{17}$$

If $0 < q \leq 1$, from (16), it follows immediately that

$$\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\left|\frac{f}{\epsilon^m}\right|^q-\left|\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|^q\right| \leq \epsilon^{-mq}\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|f-\frac{f_p^{(m)}(a)}{m!}(\cdot-a)^m\right|^q\right),$$

and so (b) holds by (17). Now assume $1 < q < \infty$. According to (16), the Hölder inequality and the Minkowski inequality we have

$$\begin{aligned}
&\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\left|\frac{f}{\epsilon^m}\right|^q-\left|\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|^q\right| \\
&\leq \frac{1}{2\epsilon}\int_{B_{a,\epsilon}} q\left|\frac{f}{\epsilon^m}-\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|\left|\left|\frac{f}{\epsilon^m}\right|^{q-1}+\left|\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|^{q-1}\right| \\
&\leq q\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\frac{f}{\epsilon^m}-\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|^q\right)^{\frac{1}{q}} \\
&\quad\times\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\left|\frac{f}{\epsilon^m}\right|^{q-1}+\left|\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|^{q-1}\right|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \\
&\leq q\,\epsilon^{-m}\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|f-\frac{f_p^{(m)}(a)}{m!}(\cdot-a)^m\right|^q\right)^{\frac{1}{q}} \\
&\quad\times\left(\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\frac{f}{\epsilon^m}\right|^q\right)^{\frac{q-1}{q}}+\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|^q\right)^{\frac{q-1}{q}}\right).
\end{aligned}$$

Since

$$\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|^q\right)^{\frac{1}{q}} = (mq+1)^{-\frac{1}{q}}\left|\frac{f_p^{(m)}(a)}{m!}\right| = |y|$$

and

$$\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\frac{f}{\epsilon^m}\right|^q\right)^{\frac{1}{q}} \leq \epsilon^{-m}\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|f-\frac{f_p^{(m)}(a)}{m!}(\cdot-a)^m\right|^q\right)^{\frac{1}{q}}+|y|,$$

there is $\epsilon_0 > 0$ that satisfies $\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\frac{f}{\epsilon^m}\right|^q\right)^{\frac{1}{q}} < |y|+1$, $0<\epsilon<\epsilon_0$. Therefore,

$$\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\left|\frac{f}{\epsilon^m}\right|^q - \left|\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|^q\right|$$
$$\leq 2\,q(1+|y|)^{q-1}\epsilon^{-m}\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|f-\frac{f_p^{(m)}(a)}{m!}(\cdot-a)^m\right|^q\right)^{\frac{1}{q}},$$

$0<\epsilon<\epsilon_0$, and thus (b) holds by (17).

$(b) \Rightarrow (c)$ It is clear from the inequality

$$\left|\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\frac{f}{\epsilon^m}\right|^q\right) - |y|^q\right|$$
$$= \left|\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\frac{f}{\epsilon^m}\right|^q\right) - \left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|^q\right)\right|$$
$$\leq \frac{1}{2\epsilon}\int_{B_{a,\epsilon}}\left|\left|\frac{f}{\epsilon^m}\right|^q - \left|\frac{f_p^{(m)}(a)}{m!}\frac{(\cdot-a)^m}{\epsilon^m}\right|^q\right|.$$

$(c) \Rightarrow (a)$ Suppose $m>0$. By the hypothesis $\left(\frac{1}{2\epsilon}\int_{B_{a,\epsilon}}|f|^p\right)^{\frac{1}{p}} = o(\epsilon^{m-1})$ as $\epsilon\to 0$. Therefore, from (15), we have $f_p^{(i)}(a)=0$, $0\leq i\leq m-1$. □

From now on, by simplicity, we put $r=1$ and we make the assumption

$$n+1 = kc+d, \quad c\in\mathbb{N}\cup\{0\}, \quad 0\leq d<k.$$

Lemma 3.2. *Let $m\in\mathbb{N}\cup\{0\}$ be such that $m\leq c$. Assume $f\in t_m^p(x_j)$, $1\leq j\leq k$. The following conditions are equivalent.*
(a) *(i) $m=0$ or (ii) $m>0$ and $f_p^{(i)}(x_j)=0$, $0\leq i\leq m-1$, $1\leq j\leq k$;*
(b) $\lim_{\epsilon\to 0}\epsilon^{-m}\|f\|_{p,\epsilon} = \frac{(mp+1)^{-\frac{1}{p}}}{m!k^{\frac{1}{p}}}\left(\sum_{j=1}^{k}\left|f_p^{(m)}(x_j)\right|^p\right)^{\frac{1}{p}} =: \Lambda.$

Proof. We observe that

$$\epsilon^{-m}\|f\|_{p,\epsilon} = \frac{1}{k^{\frac{1}{p}}}\left(\sum_{j=1}^{k}\frac{1}{2\epsilon}\int_{B_{j,\epsilon}}\left|\frac{f}{\epsilon^m}\right|^p\right)^{\frac{1}{p}}.$$

$(a) \Rightarrow (b)$ From Lemma 3.1 we deduce that $\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{B_{j,\epsilon}}\left|\frac{f}{\epsilon^m}\right|^p = (mp+1)^{-1}\left|\frac{f_p^{(m)}(x_j)}{m!}\right|^p$, and so (b) holds.

$(b) \Rightarrow (a)$ Suppose $m > 0$. By the hypothesis $\|f\|_{p,\epsilon} = o(\epsilon^{m-1})$ as $\epsilon \to 0$. So,

$$\left(\frac{1}{2\epsilon}\int_{B_{j,\epsilon}} |f|^p\right)^{\frac{1}{p}} \le \left(\sum_{j=1}^{k}\frac{1}{2\epsilon}\int_{B_{j,\epsilon}} |f|^p\right)^{\frac{1}{p}} = o(\epsilon^{m-1}) \quad \text{as} \quad \epsilon \to 0.$$

In consequence, from (15), we obtain $f_p^{(i)}(x_j) = 0, 0 \le i \le m-1, 1 \le j \le k$. $\square$

As an immediate consequence from Lemma 3.2 and Theorem 2.8 we have the following remark.

Remark 3.3. Let $m \in \mathbb{N} \cup \{0\}$ be such that $m \le c$. Assume $f_1, f_2 \in t_m^p(x_j)$, $1 \le j \le k$. If (i) $m = 0$ or (ii) $m > 0$ and $(f_1 - f_2)_p^{(i)}(x_j) = 0$, $0 \le i \le m-1$, $1 \le j \le k$, then $E_{p,\epsilon} = O(\epsilon^m)$ as $\epsilon \to 0$.

For $h \in \Pi^n$ and $f_1, f_2 \in t_m^p(x_j)$, $1 \le j \le k$, we write

$$d_{p,j}^m(f_1, f_2, h) := \min_{g\in\Pi^{m-1}} \sum_{l=1}^{2}\int_{B_j}\left|\frac{(f_l - h)_p^{(m)}(x_j)}{m!}(\cdot - x_j)^m - g\right|^p.$$

Here, $\Pi^{-1} = \{0\}$. We consider a basis of Π^n, $\{u_{sv}\}_{\substack{1\le v\le k\\ 0\le s\le c-1}} \cup \{w_e\}_{1\le e\le d}$ which satisfies

$$u_{sv}^{(i)}(x_j) = \delta_{(i,j)(s,v)}, \quad w_e^{(i)}(x_j) = 0, \quad 0 \le i \le c-1, \quad 1 \le j \le k,$$

where δ is the Krönecker delta function. Let A be the cluster point set of the net $\{h_{p,\epsilon}\}$.

Theorem 3.4. *Let $m \in \mathbb{N} \cup \{0\}$ be such that $m \le c$. Assume $f_1, f_2 \in t_m^p(x_j)$, $1 \le j \le k$. If (i) $m = 0$ or (ii) $m > 0$ and $(f_1 - f_2)_p^{(i)}(x_j) = 0$, $0 \le i \le m-1$, $1 \le j \le k$, then A is contained in the set $\mathcal{M}(f_1, f_2)$ of solutions of the following minimization problem:*

$$\min_{h\in\Pi^n} \sum_{j=1}^{k} d_{p,j}^m(f_1, f_2, h) := E \tag{18}$$

with the constrains $h^{(i)}(x_j) = \frac{(f_1+f_2)_p^{(i)}(x_j)}{2}$, $0 \le i \le m-1$, $1 \le j \le k$.

If $m = 0$, no constrain on Π^n is assumed.

Proof. Let $h_0 \in A$, then there exist some sequence $\epsilon \downarrow 0$ such that $h_{p,\epsilon}$ converges to h_0. Set $1 \le j \le k$ and $1 \le l \le 2$. From the Minkowski inequality, we have

$$\left(\frac{1}{2\epsilon}\int_{B_{j,\epsilon}}\left|h_{p,\epsilon} - \sum_{i=0}^{m}\frac{(f_l)_p^{(i)}(x_j)}{i!}(\cdot - x_j)^i\right|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{2\epsilon}\int_{B_{j,\epsilon}} |f_l - h_{p,\epsilon}|^p\right)^{\frac{1}{p}}$$
$$+ \left(\frac{1}{2\epsilon}\int_{B_{j,\epsilon}} \left|f_l - \sum_{i=0}^{m} \frac{(f_l)_p^{(i)}(x_j)}{i!}(\cdot - x_j)^i\right|^p\right)^{\frac{1}{p}}. \tag{19}$$

We observe that $\left(\frac{1}{2\epsilon}\int_{B_{j,\epsilon}} |f_l - h_{p,\epsilon}|^p\right)^{\frac{1}{p}} = O(E_{p,\epsilon})$. If (i) or (ii) holds, according to Remark 3.3, we get $E_{p,\epsilon} = O(\epsilon^m)$ as $\epsilon \to 0$. In consequence, (15), (19) and the change of variable $x - x_j = \epsilon(y - x_j)$, $y \in B_j$ imply that

$$\left(\int_{B_j}\left|h_{p,\epsilon}^{\epsilon} - \sum_{i=0}^{m} \frac{(f_l)_p^{(i)}(x_j)}{i!}\epsilon^i(\cdot - x_j)^i\right|^p\right)^{\frac{1}{p}} = O(\epsilon^m), \quad \text{as} \quad \epsilon \to 0.$$

From the equivalence of the norms on Π^n, there exists a constant $M' > 0$ such that $\max_{0\leq i\leq n} |h^{(i)}(x_j)| \leq M'\left(\int_{B_j}|h|^p\right)^{\frac{1}{p}}$, $h \in \Pi^n$. Hence $\left|(f_l - h_{p,\epsilon})_p^{(i)}(x_j)\epsilon^{i-m}\right| = O(1)$ as $\epsilon \to 0$, and so

$$h_{p,\epsilon}^{(m)}(x_j) = O(1) \quad \text{and} \quad (f_l - h_{p,\epsilon})_p^{(i)}(x_j) = O(\epsilon^{m-i}),$$
$$0 \leq i \leq c-1, \quad \text{as} \quad \epsilon \to 0. \tag{20}$$

Since

$$(f_1 - f_2)_p^{(i)}(x_j) = 0, \quad 0 \leq i \leq m-1, \tag{21}$$

we deduce that $\lim_{\epsilon\to 0}(f_l - h_{p,\epsilon})_p^{(i)}(x_j)\epsilon^{i-m} = d_{ij}$, $0 \leq i \leq m-1$, for some subsequence, which we again denote in the same way. Thus,

$$\lim_{\epsilon\to 0}\sum_{i=0}^{m} \frac{(f_l - h_{p,\epsilon})_p^{(i)}(x_j)}{i!}\epsilon^{i-m}(\cdot - x_j)^i =: h_{lj}, \quad \text{uniformly on } B_j, \tag{22}$$

where $h_{lj} = \frac{(f_l - h_0)_p^{(m)}(x_j)}{c!}(\cdot - x_j)^m + \sum_{i=0}^{m-1}\frac{d_{ij}}{i!}(\cdot - x_j)^i$. Expanding $(h_{p,\epsilon})^{\epsilon}$ by its Taylor polynomial at x_j up to order $m-1$, we obtain

$$\epsilon^{-m}(f_l - h_{p,\epsilon})^{\epsilon}(t)$$
$$= \epsilon^{-m}(f_l)^{\epsilon}(t) - \sum_{i=0}^{m-1}\frac{h_{p,\epsilon}^{(i)}(x_j)}{i!}\epsilon^{i-m}(t - x_j)^i - \frac{h_{p,\epsilon}^{(m)}(\xi_j(t))}{i!}(t - x_j)^m$$
$$= \epsilon^{-m}(f_l)^{\epsilon}(t) - \sum_{i=0}^{m}\frac{(f_l)_p^{(i)}(x_j)}{i!}\epsilon^{i-m}(t - x_j)^i$$

$$+\sum_{i=0}^{m}\frac{\left(f_l-h_{p,\epsilon}\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-m}(t-x_j)^i$$
$$+\frac{h_{p,\epsilon}^{(m)}(x_j)}{m!}(t-x_j)^m-\frac{h_{p,\epsilon}^{(m)}(\xi_j(t))}{m!}(t-x_j)^m, \tag{23}$$

$t\in B_j$, where $\xi_j(t)\in B_{j,\epsilon}$. Set

$$\lambda_{l\epsilon}=\left(\sum_{j=1}^{k}\int_{B_j}\left|\epsilon^{-m}\left(f_l\right)^{\epsilon}-\sum_{i=0}^{m}\frac{\left(f_l\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-m}(\cdot-x_j)^i\right|^p\right)^{\frac{1}{p}}$$

and

$$\varrho_\epsilon=\left(\sum_{j=1}^{k}\int_{B_j}\left|\frac{h_{p,\epsilon}^{(m)}(x_j)-h_{p,\epsilon}^{(m)}(\xi_j(\cdot))}{m!}(\cdot-x_j)^m\right|^p\right)^{\frac{1}{p}}.$$

By (23) and the Minkowski inequality, we have

$$\left|\epsilon^{-m}\left\|f_l-h_{p,\epsilon}\right\|_{p,\epsilon}-\left(\sum_{j=1}^{k}\int_{B_j}\left|\sum_{i=0}^{m}\frac{\left(f_l-h_{p,\epsilon}\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-m}(\cdot-x_j)^i\right|^p\right)^{\frac{1}{p}}\right|$$
$$\leq\lambda_{l\epsilon}+\varrho_\epsilon. \tag{24}$$

Clearly, $\lim_{\epsilon\to0}\lambda_{l\epsilon}=0$ by (15), and $\lim_{\epsilon\to0}\varrho_\epsilon=0$. Therefore, (22) and (24) yield

$$\lim_{\epsilon\to0}\epsilon^{-mp}\left\|f_l-h_{p,\epsilon}\right\|_{p,\epsilon}^p$$
$$=\sum_{j=1}^{k}\int_{B_j}\left|\frac{\left(f_l-h_0\right)_p^{(m)}(x_j)}{m!}(\cdot-x_j)^m+\sum_{i=0}^{m-1}\frac{d_{ij}}{i!}(\cdot-x_j)^i\right|^p. \tag{25}$$

On the other hand, let $h\in\Pi^n$ be such that $h^{(i)}(x_j)=\left(f_1\right)_p^{(i)}(x_j)$, $0\leq i\leq m-1$, $1\leq j\leq k$. Then there exist two sets of real numbers (independent of ϵ), say $\{c_{sv}\}_{\substack{1\leq v\leq k\\ m\leq s\leq c-1}}$ and $\{b_e\}_{1\leq e\leq d}$, that satisfies

$$h=\sum_{v=1}^{k}\sum_{s=0}^{m-1}\left(f_1\right)_p^{(s)}(x_v)u_{sv}+\sum_{e=1}^{d}b_ew_e+\sum_{v=1}^{k}\sum_{s=m}^{c-1}c_{sv}u_{sv}.$$

We choose $\{c_{sv}\}_{\substack{1\leq v\leq k\\ 0\leq s\leq m-1}}$ such that $g_j=\sum_{i=0}^{m-1}\frac{c_{ij}}{i!}(\cdot-x_j)^i$ verifies that

$$d_{p,j}^m(f_1,f_2,h)=\sum_{l=1}^{2}\int_{B_j}\left|\frac{(f_l-h)_p^{(m)}(x_j)}{m!}(\cdot-x_j)^m-g_j\right|^p.$$

We consider the following net of polynomials in Π^n,

$$g_\epsilon = \sum_{v=1}^{k}\sum_{s=0}^{m-1}\left(\left(f_1\right)_p^{(s)}(x_v) - c_{sv}\epsilon^{m-s}\right)u_{sv} + \sum_{e=1}^{d} b_e w_e + \sum_{v=1}^{k}\sum_{s=m}^{c-1} c_{sv}u_{sv}.$$

From (21) we observe that $g_\epsilon^{(i)}(x_j) = \left(f_l\right)_p^{(i)}(x_j) - c_{ij}\epsilon^{m-i}$, $1 \le j \le k$, $0 \le i \le m-1$, $1 \le l \le 2$. Expanding g_ϵ^ϵ by its Taylor polynomial at x_j up to order $m-1$, we get

$$\begin{aligned}\epsilon^{-m}\left(f_l - g_\epsilon\right)^\epsilon(t) &= \epsilon^{-m}\left(f_l\right)^\epsilon(t) - \sum_{i=0}^{m-1}\frac{g_\epsilon^{(i)}(x_j)}{i!}\epsilon^{i-m}(t-x_j)^i \\ &\quad - \frac{g_\epsilon^{(m)}(\eta_j(t))}{i!}(t-x_j)^m \\ &= \epsilon^{-m}\left(f_l\right)^\epsilon(t) - \sum_{i=0}^{m}\frac{\left(f_l\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-m}(t-x_j)^i \\ &\quad + g_j + \frac{\left(f_l\right)_p^{(m)}(x_j)}{m!}(t-x_j)^m - \frac{g_\epsilon^{(m)}(\eta_j(t))}{m!}(t-x_j)^m,\end{aligned}$$

where $\eta_j(t) \in B_{j,\epsilon}$. By the Minkowski inequality, we obtain

$$\left|\epsilon^{-m}\left\|f_l - g_\epsilon\right\|_{p,\epsilon} - \left(\sum_{j=1}^{k}\int_{B_j}\left|\frac{\left(f_l\right)_p^{(m)}(x_j) - g_\epsilon^{(m)}(\eta_j(\cdot))}{m!}(\cdot - x_j)^m + g_j\right|^p\right)^{\frac{1}{p}}\right| \le \lambda_{l\epsilon}.$$

As g_ϵ converges to h uniformly on I, as $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0}\epsilon^{-mp}\left\|f_l - g_\epsilon\right\|_{p,\epsilon}^p = \sum_{j=1}^{k}\int_{B_j}\left|\frac{\left(f_l - h\right)_p^{(m)}(x_j)}{m!}(\cdot - x_j)^m + g_j\right|^p. \tag{26}$$

Since $h_{p,\epsilon}$ is the b.s.a. to f_1 and f_2 from Π^n in $L^p(I_\epsilon)$, (25), and (26) leads to

$$\begin{aligned}\sum_{j=1}^{k} d_{p,j}^m(f_1, f_2, h_0) &\le \sum_{j=1}^{k}\sum_{l=1}^{2}\int_{B_j}\left|\frac{\left(f_l - h_0\right)_p^{(m)}(x_j)}{m!}(\cdot - x_j)^m + \sum_{i=0}^{m-1}\frac{d_{ij}}{i!}(\cdot - x_j)^i\right|^p \\ &= \lim_{\epsilon\to 0}\epsilon^{-mp}\sum_{l=1}^{2}\left\|f_l - h_{p,\epsilon}\right\|_{p,\epsilon}^p \le \lim_{\epsilon\to 0}\epsilon^{-mp}\sum_{l=1}^{2}\left\|f_l - g_\epsilon\right\|_{p,\epsilon}^p \\ &= \sum_{j=1}^{k}\sum_{l=1}^{2}\int_{B_j}\left|\frac{\left(f_l - h\right)_p^{(m)}(x_j)}{m!}(\cdot - x_j)^m + g_j\right|^p \\ &= \sum_{j=1}^{k} d_{p,j}^m(f_1, f_2, h),\end{aligned}$$

for all $h \in \Pi^n$ such that $h^{(i)}(x_j) = (f_1)_p^{(i)}(x_j)$, $0 \le i \le m-1$, $1 \le j \le k$. According to (20), we get $h_0^{(i)}(x_j) = (f_1)_p^{(i)}(x_j)$, $0 \le i \le m-1$, $1 \le j \le k$, and so (21) shows that $h_0 \in \mathcal{M}(f_1,f_2)$. □

Next, we establish a result which be need later.

Lemma 3.5. *Let $m \in \mathbb{N} \cup \{0\}$ be such that $m \le c$, and let $f_1, f_2 \in t_m^p(x_j)$, $1 \le j \le k$. Assume (i) $m = 0$ or (ii) $m > 0$ and $(f_1 - f_2)_p^{(i)}(x_j) = 0$, $0 \le i \le m-1$, $1 \le j \le k$. If $h_0, h_1 \in \mathcal{M}(f_1,f_2)$, then*

$$h_0^{(i)}(x_j) = h_1^{(i)}(x_j), \quad 0 \le i \le m, \quad 1 \le j \le k. \tag{27}$$

In addition, if $m \le c-1$ we have

$$h_0^{(i)}(x_j) = \left(\frac{f_1 + f_2}{2}\right)_p^{(i)}(x_j), \quad 0 \le i \le m, \quad 1 \le j \le k. \tag{28}$$

Proof. Let $h_0, h_1 \in \mathcal{M}(f_1,f_2)$. Then there exist $(g_{01}, \ldots, g_{0k}), (g_{11}, \ldots, g_{1k}) \in \Pi^{m-1} \times \cdots \times \Pi^{m-1}$ such that

$$E = \sum_{l=1}^{2} \left(\sum_{j=1}^{k} \int_{B_j} \left| \frac{(f_l)_p^{(m)}(x_j)}{m!} (\cdot - x_j)^m - \left(\frac{h_s^{(m)}(x_j)}{m!} (\cdot - x_j)^m - g_{sj} \right) \right|^p \right),$$
$$0 \le s \le 1.$$

Let $Y = \Pi^n \times \Pi^{m-1} \times \cdots \times \Pi^{m-1}$, $Z = \Pi^m \times \cdots \times \Pi^m$ and let $\rho : Z \to [0, +\infty)$ be the norm defined by $\rho(z_1, \ldots, z_k) = \left(\sum_{j=1}^k \int_{B_j} |z_j|^p \right)^{\frac{1}{p}}$. Let Δ, Γ be the convex sets in Y and Z given by

$$\Delta = \left\{ (h, g_1, \ldots, g_k) \in Y : h^{(i)}(x_j) = \frac{(f_1 + f_2)_p^{(i)}(x_j)}{2}, 0 \le i \le m-1, 1 \le j \le k \right\} \text{ and}$$

$$\Gamma = \left\{ \left(\frac{h^{(m)}(x_1)}{m!} (\cdot - x_1)^m - g_1, \ldots, \frac{h^{(m)}(x_k)}{m!} (\cdot - x_k)^m - g_k \right) : (h, g_1, \ldots, g_k) \in \Delta \right\}.$$

It is easy to see that the norm $\nu(v_1, v_2) = \left(\rho(v_1)^p + \rho(v_2)^p \right)^{\frac{1}{p}}$ is strictly convex on $Z \times Z$. Set $v_1 = \left(\frac{(f_1)_p^{(m)}(x_1)}{m!} (\cdot - x_1)^m, \ldots, \frac{(f_1)_p^{(m)}(x_k)}{m!} (\cdot - x_k)^m \right)$, $v_2 = \left(\frac{(f_2)_p^{(m)}(x_1)}{m!} (\cdot - x_1)^m, \ldots, \frac{(f_2)_p^{(m)}(x_k)}{m!} (\cdot - x_k)^m \right)$ and $u_s = \left(\frac{h_s^{(m)}(x_1)}{m!} (\cdot - x_1)^m - g_{s1}, \ldots, \frac{h_s^{(m)}(x_k)}{m!} (\cdot - x_k)^m - g_{sk} \right) \in \Gamma$. We observe that (u_s, u_s) is a best approximation to (v_1, v_2) from the convex set $\{(u,u) : u \in \Gamma\}$ respect to ν. Since this problem has a unique solution [18, Theorem 1.14], we have $\frac{h_0^{(m)}(x_j)}{m!} (\cdot - x_j)^m - g_{0j} = \frac{h_1^{(m)}(x_j)}{m!} (\cdot - x_j)^m - g_{1j}$ $1 \le j \le k$, and so (27) holds.

Now, assume $m \le c-1$ and let $g_0 \in \mathcal{M}(f_1,f_2)$. Clearly, there exists $h_3 \in \Pi^n$ such that $h_3^{(i)}(x_j) = \left(\frac{f_1+f_2}{2}\right)_p^{(i)}(x_j)$, $0 \le i \le m$, $1 \le j \le k$. An straightforward computation shows that

$$d_{p,j}^m(f_1,f_2,h_3) = \sum_{l=1}^{2} \int_{B_j} \left| \frac{(-1)^{l+1}}{m!} \left(\frac{f_l-f_2}{2}\right)_p^{(m)}(x_j)(\cdot - x_j)^m \right|^p = \frac{2}{m!} \int_{B_j} \left| \left(\frac{f_l-f_2}{2}\right)_p^{(m)}(x_j)(\cdot - x_j)^m \right|^p,$$

and hence

$$\sum_{j=1}^{k} d_{p,j}^m(f_1,f_2,h_3) = \frac{2}{m!} \sum_{j=1}^{k} \int_{B_j} \left| \left(\frac{f_l-f_2}{2}\right)_p^{(m)}(x_j)(\cdot - x_j)^m \right|^p \le E.$$

So, $h_3 \in \mathcal{M}(f_1,f_2)$, and consequently (28) holds by (27). □

For $c \in \mathbb{N}$ and $f_1,f_2 \in t_{c-1}^p(x_j)$, $1 \le j \le k$, we will denote by

$$A_j = \left\{ i : 0 \le i \le c-1 \text{ and } (f_1-f_2)_p^{(i)}(x_j) \ne 0 \right\}.$$

If $A_j \ne \emptyset$, we write $m_j = \min A_j - 1$, otherwise $m_j = c-1$. We define $\overline{m} = \overline{m}(f_1,f_2) := \min\{m_j : 1 \le j \le k\}$. If $c = 0$ and $f_1,f_2 \in t_0^p(x_j)$, $1 \le j \le k$, we put $\overline{m} = -1$. Observe that $-1 \le \overline{m} \le c-1$, and if $\overline{m} > -1$, then

$$(f_1-f_2)_p^{(i)}(x_j) = 0, \quad 0 \le i \le \overline{m}, \quad 1 \le j \le k.$$

We complete the study by considering four cases:
(1) $\overline{m} = c-1$;
(2) $\overline{m} = c-2$ and $d = 0$;
(3) $\overline{m} = c-2$ and $d > 0$;
(4) $\overline{m} \le c-3$.

Cases (1) or (2)

We will now show the existence and characterization for the L^p-b.s.l.a. to two functions f_1 and f_2 from Π^n, in the case (1) or (2).

Theorem 3.6. *Assume $f_1,f_2 \in L^p(I)$. Consider the following condition:*
(a) *$f_1,f_2 \in t_c^p(x_j)$, $1 \le j \le k$, and $c = 0$;*
(b) *$f_1,f_2 \in t_{c-1}^p(x_j)$, $1 \le j \le k$, $\overline{m} = c-2$, and $d = 0$;*
(c) *$f_1,f_2 \in t_c^p(x_j)$, $1 \le j \le k$, $\overline{m} = c-1$, and $d > 0$.*

If (a), (b) or (c) holds, then there exists the L^p-b.s.l.a. to f_1 and f_2 from Π^n on $\{x_1, \ldots, x_k\}$, and it is the unique solution of the minimization problem given in (18).

Proof. Let $1 \le j \le k$ and $1 \le l \le 2$. As in the proof of Theorem 3.4, we have

$$h_{p,\epsilon}^{(\overline{m}+1)}(x_j) = O(1) \quad \text{and}$$
$$\left(f_l - h_{p,\epsilon}\right)_p^{(i)}(x_j) = O(\epsilon^{\overline{m}+1-i}), \quad 0 \le i \le c-1, \quad \text{as} \quad \epsilon \to 0. \tag{29}$$

Therefore, if (a), (b), or (c) holds, then the net $\{h_{p,\epsilon}\}$ is uniformly bounded on I, and so $A \neq \emptyset$. Now, we observe that under our assumptions, the problem (18) has a unique solution by Lemma 3.5, and so we conclude that there exists the L^p-b.s.l.a. to f_1 and f_2 from Π^n on $\{x_1, \ldots, x_k\}$, and it is the solution of (18). □

Cases (3) or (4)

For $f_1, f_2 \in t_{c-1}^p(x_j)$, $1 \le j \le k$, we observe that the $(\overline{m}+1)$-th L^p-derivative of f_1 and f_2 at x_j exist, because $\overline{m}+1 \le c-1$ in the cases (3) or (4).

Next, we establish two results which be need later.

Lemma 3.7. *Assume $f_1, f_2 \in t_{c-1}^p(x_j)$, $1 \le j \le k$. Consider the family of measurable sets given by*

$$C_{j,\epsilon} := \left\{ t \in B_j \backslash \left[x_j - \frac{1}{8}, x_j + \frac{1}{8} \right] : \left| \frac{(f_1 - f_2)^\epsilon(t)}{\epsilon^{\overline{m}+1}} \right| \ge \xi_j \right\}, \quad 0 < \epsilon < 1,$$

where $\xi_j = \frac{\left|(f_1-f_2)_p^{(\overline{m}+1)}(x_j)\right|}{2(\overline{m}+1)!8^{\overline{m}+1}}$. Then there exists $\epsilon_0 > 0$ such that $|C_{j,\epsilon}| \ge \frac{1}{2}$, $0 < \epsilon < \epsilon_0$.

Proof. If $(f_1 - f_2)_p^{(\overline{m}+1)}(x_j) = 0$, then $|C_{j,\epsilon}| = \frac{3}{4}$, $0 < \epsilon < 1$. Suppose that $(f_1 - f_2)_p^{(\overline{m}+1)}(x_j) \neq 0$. By Lemma 3.1, we deduce that

$$\left| \left| \frac{(f_1 - f_2)^\epsilon}{\epsilon^{\overline{m}+1}} \right| - \left| \frac{(f_1 - f_2)_p^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!} (\cdot - x_j)^{\overline{m}+1} \right| \right|$$

converges in measure to 0 on B_j as $\epsilon \to 0$.

Set

$$A_{j,\epsilon} := \left\{ t \in B_j \backslash \left[x_j - \frac{1}{8}, x_j + \frac{1}{8} \right] : \left| \left| \frac{(f_1 - f_2)^\epsilon(t)}{\epsilon^{\overline{m}+1}} \right| - \left| \frac{(f_1 - f_2)_p^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!} (t - x_j)^{\overline{m}+1} \right| \right| \le \xi_j \right\}.$$

It follows immediately that $\lim_{\epsilon\to 0}|A_{j,\epsilon}| = \frac{3}{4}$. Since $A_{j,\epsilon} \subset C_{j,\epsilon}$, the proof is complete. □

Lemma 3.8. *Assume $f_1, f_2 \in t^p_{c-1}(x_j)$, $1 \le j \le k$. Consider the family $\{C_{j,\epsilon}\}_{0<\epsilon<\epsilon_0}$ of Lemma 3.7. Then there exists a constant $s > 0$, non depending on ϵ, such that*

$$\left|\left\{t \in C_{j,\epsilon} : |h(t)| \ge \frac{\|h\|_{\infty,C_{j,\epsilon}}}{s}\right\}\right| \ge \frac{3}{8}, \quad 0 < \epsilon < \epsilon_0, \quad h \in \Pi^n. \tag{30}$$

Proof. We observe that the statement is obvious for constant polynomials. For $0 \ne h(t) = \sum_{s=0}^n c_s t^s$, we denote $g(t) = \frac{h(t)}{\max\limits_{0\le s\le n}|c_s|}$. From Lemma 3.7, $|C_{j,\epsilon}| \ge \frac{1}{2}$, $0 < \epsilon < \epsilon_0$. By the continuity of the measure, there is $\beta = \beta(C_{j,\epsilon}, g) > 0$ such that

$$\left|\left\{t \in C_{j,\epsilon} : |g(t)| > \frac{\|g\|_{\infty,C_{j,\epsilon}}}{\beta}\right\}\right| = \frac{3}{8}. \tag{31}$$

From the equivalence of the norms on Π^n, there exist constants $M, M' > 0$ such that

$$0 < M \le \|g\|_{\infty,B_j} \le M', \tag{32}$$

thus using (31), we obtain

$$\left|\left\{t \in C_{j,\epsilon} : |g(t)| > \frac{M'}{\beta}\right\}\right| \le \frac{3}{8}. \tag{33}$$

Suppose that $\{\beta\}$ is not bounded. Then there are subsequences $\{C_{j,\epsilon_l}\}$ and $\{g_l\} \subset \Pi^n$ such that $\beta_l = \beta(C_{j,\epsilon_l}, g_l) \to \infty$ as $l \to \infty$. From (32), there is a subsequence of $\{g_l\}$, that we denote in the same way, which converges to a polynomial $g_0 \in \Pi^n\backslash\{0\}$ on B_j. Let $0 < s < \|g_0\|_{\infty,B_j}$ verifying

$$\left|\left\{t \in B_j : |g_0(t)| > s\right\}\right| \ge \frac{15}{16}. \tag{34}$$

Denote $C = \left\{t \in B_j : |g_0(t)| > s\right\}$. Clearly, there exists a nonnegative integer l_0 such that

$$\frac{M'}{\beta_l} < \frac{s}{2} \quad \text{and} \quad \big||g_0(t)| - |g_l(t)|\big| < \frac{s}{2}, \quad l \ge l_0, \quad t \in B_j.$$

Then we get

$$C \cap C_{j,\epsilon} \subset \left\{t \in C_{j,\epsilon} : |g_l(t)| > \frac{M'}{\beta_l}\right\}, \quad l \ge l_0. \tag{35}$$

Since $|C_{j,\epsilon}\backslash C| \le |B_j\backslash C| \le \frac{1}{16}$ by (34), according to (33) and (35), we have

$$\frac{3}{8} \ge |C \cap C_{j,\epsilon}| = |C_{j,\epsilon}| - |C_{j,\epsilon}\backslash C| \ge \frac{1}{2} - \frac{1}{16} = \frac{7}{16},$$

which is a contradiction. Therefore, the set $\{\beta\}$ is bounded. So, from (31) we obtain (30) with $s = \sup\{\beta\}$. □

Let $f_1, f_2 \in t^p_{c-1}(x_j)$, $1 \le j \le k$. Since $-1 \le \overline{m} \le c-2$, we see that the set

$$\tau(f_1, f_2) := \left\{ j : 1 \le j \le k \text{ and } (f_1 - f_2)_p^{(\overline{m}+1)}(x_j) \ne 0 \right\}$$

is nonempty. Now, we establish a main result of this section.

Theorem 3.9. *Assume $f_1, f_2 \in L^p(I)$.*
(a) *If $f_1, f_2 \in t^p_c(x_j)$, $1 \le j \le k$, and $d > 0$, then for every $0 \le i \le c-1$, we have*

$$h^{(c)}_{p,\epsilon}(x_j) = O(1), \quad \begin{array}{l} \left(\frac{f_1+f_2}{2} - h_{p,\epsilon}\right)_p^{(i)}(x_j) = O(\epsilon^{c-i}), \\ j \in \tau(f_1, f_2) \\ \left(\frac{f_1+f_2}{2} - h_{p,\epsilon}\right)_p^{(i)}(x_j) = O(\epsilon^{\overline{m}+1-i}), \\ j \notin \tau(f_1, f_2) \end{array}, \qquad as \quad \epsilon \to 0. \tag{36}$$

(b) *If $f_1, f_2 \in t^p_{c-1}(x_j)$, $1 \le j \le k$, and $d = 0$, then for every $0 \le i \le c-1$, we have*

$$\begin{array}{ll} \left(\frac{f_1+f_2}{2} - h_{p,\epsilon}\right)_p^{(i)}(x_j) = o(\epsilon^{c-1-i}), & j \in \tau(f_1, f_2) \\ \left(\frac{f_1+f_2}{2} - h_{p,\epsilon}\right)_p^{(i)}(x_j) = O(\epsilon^{\overline{m}+1-i}), & j \notin \tau(f_1, f_2) \end{array}, \qquad as \quad \epsilon \to 0. \tag{37}$$

Proof. We prove (a); the proof of (b) is similar, mutatis mutandis. Let $j \in \tau(f_1, f_2)$ and let $\{C_{j,\epsilon}\}_{0<\epsilon<\epsilon_0}$ be the family of Lemma 3.8. Since $\frac{f_1+f_2}{2} \in t^p_c(x_j)$, $1 \le j \le k$, there exists $g \in \Pi^n$ such that

$$\left\| \frac{f_1+f_2}{2} - g \right\|_{p,\epsilon} = O(\epsilon^c) \quad \text{as} \quad \epsilon \to 0, \tag{38}$$

and

$$g^{(i)}(x_j) = \left(\frac{f_1+f_2}{2}\right)_p^{(i)}(x_j), \quad 0 \le i \le c-1, \quad 1 \le j \le k.$$

As $(g - h_{p,\epsilon})^\epsilon \in \Pi^n$ on B_j, Lemma 3.8 implies that there exists a constant $s > 0$, non depending on ϵ, such that

$$\left| \left\{ t \in C_{j,\epsilon} : |(g - h_{p,\epsilon})^\epsilon(t)| \ge \frac{\|(g - h_{p,\epsilon})^\epsilon\|_{\infty, C_{j,\epsilon}}}{s} \right\} \right| \ge \frac{3}{8}, \quad 0 < \epsilon < \epsilon_0.$$

According to the change of variable $x-x_j=\epsilon(y-x_j)$, $y\in B_j$, from Lemma 3.7, we have

$$\begin{aligned}\int_{I_\epsilon}\left|g-h_{p,\epsilon}\right|^2\left|\frac{f_1-f_2}{\epsilon^{\overline{m}+1}}\right|^{p-2}\frac{1}{|I_\epsilon|}&=\frac{1}{k}\sum_{j=1}^{k}\int_{B_j}\left|(g-h_{p,\epsilon})^\epsilon\right|^2\left|\frac{(f_1-f_2)^\epsilon}{\epsilon^{\overline{m}+1}}\right|^{p-2}\\&\geq\frac{\xi_j^{p-2}}{k}\int_{C_{j,\epsilon}}\left|(g-h_{p,\epsilon})^\epsilon\right|^2\quad 0<\epsilon<\epsilon_0.\\&\geq\frac{3\,\xi_j^{p-2}}{8\,k\,s^2}\|(g-h_{p,\epsilon})^\epsilon\|^2_{\infty,C_{j,\epsilon}},\end{aligned}\tag{39}$$

By Lemma 3.2, $\lim_{\epsilon\to0}\epsilon^{-(\overline{m}+1)}\|f_1-f_2\|_{p,\epsilon}\neq0$ and $\lim_{\epsilon\to0}\epsilon^{-c}\left\|f_1-f_2\right\|_{p,\epsilon}=+\infty$. So, (38), (39), and Theorem 2.9 imply that

$$\|(g-h_{p,\epsilon})^\epsilon\|_{\infty,C_{j,\epsilon}}=O(\epsilon^c)\quad\text{as}\quad\epsilon\to0.\tag{40}$$

Let $i_1=n+1$ and $\overline{B}_1=\{t-x_j:t\in B_j\}$. Since $\left|\overline{B}_1\right|=1$, from [5, Theorem 1.3], there exists a constant $\gamma>0$, depending on n and $\overline{B}_1$ such that

$$\frac{\left|h^{(i)}(0)\right|}{i!}\leq\frac{\gamma}{|E|^n}\|h\|_{\infty,E},\quad 0\leq i\leq n,\quad h\in\Pi^n,\quad E\subset\overline{B}_1\text{ with }|E|>0,$$

Since $h=(g-h_{p,\epsilon})(\cdot\epsilon+x_j)\in\Pi^n$ and $\widetilde{E}=\{t-x_j:t\in C_{j,\epsilon}\}\subset\overline{B}_1$ with $|\widetilde{E}|=|C_{j,\epsilon}|\geq\frac{1}{2}$, we obtain

$$\begin{aligned}\epsilon^i\frac{\left|(g-h_{p,\epsilon})^{(i)}(x_j)\right|}{i!}&\leq\frac{\gamma}{|\widetilde{E}|^n}\|h\|_{\infty,\widetilde{E}}=\frac{\gamma}{|C_{j,\epsilon}|^n}\|(g-h_{p,\epsilon})^\epsilon\|_{\infty,C_{j,\epsilon}}\\&\leq2^n\gamma\,\|(g-h_{p,\epsilon})^\epsilon\|_{\infty,C_{j,\epsilon}},\end{aligned}$$

$0\leq i\leq n$, $0<\epsilon<\epsilon_0$. From (40) we obtain $\left|(g-h_{p,\epsilon})^{(i)}(x_j)\right|=O(\epsilon^{c-i})$ as $\epsilon\to0$, $0\leq i\leq c$, and so

$$\begin{aligned}&h_{p,\epsilon}^{(c)}(x_j)=O(1)\quad\text{and}\\&\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)_p^{(i)}(x_j)=O(\epsilon^{c-i}),\quad 0\leq i\leq c-1,\quad\text{as}\quad\epsilon\to0.\end{aligned}$$

According to (29), we have $h_{p,\epsilon}^{(\overline{m}+1)}(x_j)=O(1)$ and $\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)_p^{(i)}(x_j)=O(\epsilon^{\overline{m}+1-i})$, $0\leq i\leq c-1$, $j\notin\tau(f_1,f_2)$, as $\epsilon\to0$. Since $\overline{m}+1\leq c-1$, the proof is complete. □

Let us mention an important consequence of Theorem 3.9. We prove the existence and characterization of the L^p-b.s.l.a. when $k=1$.

Theorem 3.10. *Assume $f_1, f_2 \in t_n^p(x_1)$. Then there exists the L^p-b.s.l.a. to f_1 and f_2 from Π^n on $\{x_1\}$, and it is the unique $g \in \Pi^n$ defined by the $n+1$ interpolation conditions*

$$g^{(i)}(x_1) = \left(\frac{f_1 + f_2}{2}\right)_p^{(i)}(x_1), \quad 0 \le i \le n.$$

Proof. As $k = 1$, then $n + 1 = c$, $d = 0$, and $\tau(f_1, f_2) = \{1\}$. According to Theorem 3.9 (b), we have $\left(\frac{f_1+f_2}{2} - h_{p,\epsilon}\right)_p^{(i)}(x_1) = o(\epsilon^{n-i})$, $1 \le i \le n$, as $\epsilon \to 0$. This completes the proof. □

Let $h_0 \in A$. Under the same hypotheses of Theorem 3.9, if (3) or (4) holds, from (28), (36), and (37) it follows that

$$h_0^{(i)}(x_j) = \left(\tfrac{f_1+f_2}{2}\right)_p^{(i)}(x_j), \quad 0 \le i \le c-1, \quad j \in \tau(f_1, f_2),$$
$$h_0^{(i)}(x_j) = \left(\tfrac{f_1+f_2}{2}\right)_p^{(i)}(x_j), \quad 0 \le i \le \overline{m}+1, \quad j \notin \tau(f_1, f_2).$$

If $n+1 = kc$, $\overline{m}+1 \le c-2$, and $\#(\tau(f_1, f_2)) < k$, the above conditions do not uniquely determine the polynomial h_0. Does it remain valid the last equality for $\overline{m}+1 < i \le c-1$ and $j \notin \tau(f_1, f_2)$? The answer is no, as shown in the following example.

Example 3.11. Let $p = 4$ and $n = 3$. Consider $x_j = (-1)^j$, $j = 1, 2$, $f_1(x) = (x-1)^4$, and $f_2(x) = (x-1)^3$. We observe that $k = c = 2$, $d = 0$, $\overline{m} = -1$, and $2 \notin \tau(f_1, f_2)$. Using Wolfram Mathematica software with the above information, we illustrate in Table 1 the asymptotic behavior of a net $\{h_{4,l^{-1}}\}$ of b.s.a. to f_1 and f_2 from Π^3 in $L^4\left(I_{l^{-1}}\right)$ as $l \to \infty$, $l \in \mathbb{N}$.

Table 1. b.s.a. in $L^4\left(I_{l^{-1}}\right)$.

l	$h_{4,l^{-1}}(x) = a_l x^3 + b_l x^2 + c_l x + d_l$
2	$-2.3469x^3 + 2.0082x^2 + 0.8135x + 0.4454$
3	$-2.3299x^3 + 1.8377x^2 + 0.5457x + 0.3818$
10	$-2.3188x^3 + 1.6970x^2 + 0.3388x + 0.3242$
100	$-2.2998x^3 + 1.7004x^2 + 0.3000x + 0.2999$
1000	$-2.3000x^3 + 1.7000x^2 + 0.3000x + 0.3000$

b.s.a.: best simultaneous approximation.

We see that $h_{4,l^{-1}}^{(1)}(1)$ does not converge to $0 = \left(\frac{f_1+f_2}{2}\right)_p^{(1)}(1)$ as $l \to \infty$, and so the L^4-b.s.l.a. does not interpolate to the mean value of f_1 and f_2 at x_j up to order 1, $j = 1, 2$.

Now, we will show the existence and characterization for the L^p-b.s.l.a. to two functions f_1 and f_2 from Π^n when $\#(\tau(f_1, f_2)) = k$.

If $n+1=kc$, the following theorem can be proved in a similar way to Theorem 3.10.

Theorem 3.12. *Assume $d=0$, $f_1,f_2\in t^p_{c-1}(x_j)$, $1\le j\le k$. If $\#(\tau(f_1,f_2))=k$, then there exists the L^p-b.s.l.a. to f_1 and f_2 from Π^n on $\{x_1,\dots,x_k\}$, and it is the unique $g\in\Pi^n$ defined by the $n+1$ interpolation conditions*

$$g^{(i)}(x_j)=\left(\frac{f_1+f_2}{2}\right)^{(i)}_p(x_j),\quad 0\le i\le c-1,\quad 1\le j\le k.$$

For the case $d>0$, we need the following lemma.

Lemma 3.13. *Assume $d>0$ and $f_1,f_2\in t^p_c(x_j)$, $1\le j\le k$. Let ω_p be the function given by*

$$\omega_p:=2^{\frac{(1-p)(p-2)}{p}}\Lambda^{p-2}\sum_{j=1}^{k}\left|\frac{(f_1-f_2)^{(\overline{m}+1)}_p(x_j)}{2(\overline{m}+1)!}(\cdot-x_j)^{\overline{m}+1}\right|^{p-2}\chi_{B_j},$$

where Λ is the number defined in Lemma 3.2. If $\#(\tau(f_1,f_2))=k$, then $v^{\epsilon}_{p,\epsilon}$ converges weakly to ω_p in $L^{\frac{p}{p-2}}(I)$, for some sequence $\epsilon\downarrow 0$, where $v_{p,\epsilon}$ was introduced in (6).

Proof. By (36), there exists a sequence $\epsilon\downarrow 0$ and $h_0\in\Pi^n$ such that $h_{p,\epsilon}$ converges to h_0, uniformly on I,

$$\lim_{\epsilon\to 0}h^{(c)}_{p,\epsilon}(x_j)=h^{(c)}_0(x_j)\quad\text{and}\quad\lim_{\epsilon\to 0}\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)^{(i)}_p(x_j)\epsilon^{i-c}=d_{ij},$$
$$0\le i\le c-1,\quad 1\le j\le k. \tag{41}$$

Thus,

$$\lim_{\epsilon\to 0}\sum_{i=0}^{c}\frac{\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)^{(i)}_p(x_j)}{i!}\epsilon^{i-c}(\cdot-x_j)^i=h_j,\quad\text{uniformly on } B_j, \tag{42}$$

where $h_j=\sum_{i=0}^{c-1}\frac{d_{ij}}{i!}(\cdot-x_j)^i+\frac{\left(\frac{f_1+f_2}{2}-h_0\right)^{(c)}_p(x_j)}{c!}(\cdot-x_j)^c$. From (15) it follows that

$$\epsilon^{-c}\left(\frac{f_1+f_2}{2}\right)^{\epsilon}-\sum_{i=0}^{c}\frac{\left(\frac{f_1+f_2}{2}\right)^{(i)}_p(x_j)}{i!}\epsilon^{i-c}(\cdot-x_j)^i\text{ converges in measure to 0 on } B_j, \tag{43}$$

as $\epsilon \to 0$. Expanding $(h_{p,\epsilon})^\epsilon$ by its Taylor polynomial at x_j up to order c, we obtain

$$\begin{aligned}
&\epsilon^{-c}\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)^{\epsilon}(t)\\
&=\epsilon^{-c}\left(\frac{f_1+f_2}{2}\right)^{\epsilon}(t)-\sum_{i=0}^{c-1}\frac{h_{p,\epsilon}^{(i)}(x_j)}{i!}\epsilon^{i-c}(t-x_j)^i\\
&\quad-\frac{h_{p,\epsilon}^{(c)}(\xi_j(t))}{i!}(t-x_j)^c=\epsilon^{-c}\left(\frac{f_1+f_2}{2}\right)^{\epsilon}(t)\\
&\quad-\sum_{i=0}^{c}\frac{\left(\frac{f_1+f_2}{2}\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-c}(t-x_j)^i\\
&\quad+\sum_{i=0}^{c}\frac{\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-c}(t-x_j)^i\\
&\quad+\frac{h_{p,\epsilon}^{(c)}(x_j)}{c!}(t-x_j)^c-\frac{h_{p,\epsilon}^{(c)}(\xi_j(t))}{c!}(t-x_j)^c,
\end{aligned}\tag{44}$$

$t \in B_j$, where $\xi_j(t) \in B_{j,\epsilon}$. According to (41)–(43), we get

$$\lim_{\epsilon\to 0}\epsilon^{-c}\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)^{\epsilon}=h_j,\quad \text{a.e. on}\quad B_j,\tag{45}$$

for some subsequence, that we again denote by ϵ. From (15), we deduce that

$$\lim_{\epsilon\to 0}\frac{(f_1-f_2)^\epsilon}{\epsilon^{\overline{m}+1}}=\frac{(f_1-f_2)_p^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(\cdot-x_j)^{\overline{m}+1},\quad \text{a.e. on}\quad B_j,\tag{46}$$

for some subsequence, which we again denote the same way. Since $-1 \le \overline{m} \le c-2$, (45) and (46) imply that

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{(f_1-h_{p,\epsilon})^\epsilon}{\epsilon^{\overline{m}+1}}&=\lim_{\epsilon\to 0}\frac{(h_{p,\epsilon}-f_2)^\epsilon}{\epsilon^{\overline{m}+1}}\\
&=\frac{(f_1-f_2)_p^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!}(\cdot-x_j)^{\overline{m}+1},\quad \text{a.e. on}\quad B_j,
\end{aligned}\tag{47}$$

By (2) and mean value theorem, we have

$$\begin{aligned}
v_{p,\epsilon}^\epsilon&=\left(\frac{E_{p,\epsilon}}{\epsilon^{\overline{m}+1}}\right)^{2-p}H_p\left(\frac{(f_1-h_{p,\epsilon})^\epsilon}{\epsilon^{\overline{m}+1}},\frac{(h_{p,\epsilon}-f_2)^\epsilon}{\epsilon^{\overline{m}+1}}\right)\\
&=\left(\frac{E_{p,\epsilon}}{\epsilon^{\overline{m}+1}}\right)^{2-p}(p-1)\,|\eta_\epsilon|^{p-2},\quad \text{on}\quad B_j,
\end{aligned}$$

with $\eta_\epsilon(t)$ in the segment of extremes $\frac{(f_1-h_{p,\epsilon})^\epsilon(t)}{\epsilon^{\overline{m}+1}}$ and $\frac{(h_{p,\epsilon}-f_2)^\epsilon(t)}{\epsilon^{\overline{m}+1}}$. Now (47), Theorem 2.8 and Lemma 3.2 show that

$$\lim_{\epsilon\to 0} v_{p,\epsilon}^\epsilon = 2^{\frac{(1-p)(p-2)}{p}}\Lambda^{p-2}\left|\frac{(f_1-f_2)_p^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!}(\cdot - x_j)^{\overline{m}+1}\right|^{p-2}, \quad \text{a.e. on} \quad B_j,$$

Finally, from Lemma 2.3, we deduce that $v_{p,\epsilon}^\epsilon$ converges weakly to ω_p in $L^{\frac{p}{p-2}}(I)$. The proof is complete. □

Theorem 3.14. *Assume $d > 0, f_1, f_2 \in t_c^p(x_j)$, $1 \le j \le k$. If $\#(\tau(f_1, f_2)) = k$, then there exists the L^p-b.s.l.a. to f_1 and f_2 from Π^n on $\{x_1, \dots, x_k\}$, and it is the unique solution of the following minimization problem in $\mathbb{R}^k$:*

$$\begin{aligned}&\min_{h\in\Pi^n}\sum_{j=1}^k \left|\left(\tfrac{f_1+f_2}{2}-h\right)_p^{(c)}(x_j)\right|^2\left|(f_1-f_2)_p^{(\overline{m}+1)}(x_j)\right|^{p-2}\\ &\text{with the constrains } h^{(i)}(x_j) = \left(\tfrac{f_1+f_2}{2}\right)_p^{(i)}(x_j),\ 0\le i\le c-1,\ 1\le j\le k.\end{aligned} \tag{48}$$

Proof. Let

$$u_\epsilon = \left|\sum_{i=0}^c \frac{\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-c}(\cdot - x_j)^i\right|^2 v_{p,\epsilon}^\epsilon.$$

From (36) and Lemma 3.13, there exists a sequence $\epsilon \downarrow 0$ and $h_0 \in \Pi^n$ such that $h_{p,\epsilon}$ converges to h_0, uniformly on I, and

$$\lim_{\epsilon\to 0} u_\epsilon = |h_j|^2\,\omega_p, \ \text{a.e. on } B_j,$$

$$\text{where } h_j = \frac{\left(\frac{f_1+f_2}{2}-h_0\right)_p^{(c)}(x_j)}{c!}(\cdot - x_j)^c + \sum_{i=0}^{c-1}\frac{d_{ij}}{i!}(\cdot - x_j)^i. \tag{49}$$

Set

$$\lambda_\epsilon = \left(\sum_{j=1}^k\int_{B_j}\left|\epsilon^{-c}\left(\frac{f_1+f_2}{2}\right)^\epsilon - \sum_{i=0}^c\frac{\left(\frac{f_1+f_2}{2}\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-c}(\cdot - x_j)^i\right|^2 v_{p,\epsilon}^\epsilon\right)^{\frac12}$$

and

$$\varrho_\epsilon = \left(\sum_{j=1}^k\int_{B_j}\left|\frac{h_{p,\epsilon}^{(c)}(x_j) - h_{p,\epsilon}^{(c)}(\xi_j(\cdot))}{i!}(\cdot - x_j)^c\right|^2 v_{p,\epsilon}^\epsilon\right)^{\frac12}.$$

Since $\frac{p-2}{p}+\frac{2}{p}=1$, Hölder inequality shows that

$$\lambda_\epsilon \le \left(\int_{B_j}\sum_{j=i}^{k}\left|\epsilon^{-c}\left(\frac{f_1+f_2}{2}\right)^{\epsilon}-\sum_{i=0}^{c}\frac{\left(\frac{f_1+f_2}{2}\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-c}(\cdot-x_j)^i\right|^p\right)^{\frac{1}{p}} \times \left\|v_{p,\epsilon}\right\|^{\frac{1}{2}}_{\frac{p}{p-2},\epsilon},$$

In the same manner, we can see that

$$\varrho_\epsilon \le \left(\sum_{j=1}^{k}\int_{B_j}\left|\frac{h_{p,\epsilon}^{(c)}(x_j)-h_{p,\epsilon}^{(c)}(\xi_j(\cdot))}{i!}(\cdot-x_j)^c\right|^p\right)^{\frac{1}{p}}\left\|v_{p,\epsilon}\right\|^{\frac{1}{2}}_{\frac{p}{p-2},\epsilon},$$

where $\xi_j(t)$ is the number given in (44). By Minkowski inequality and (44), it follows that

$$\begin{aligned}
&\epsilon^{-c}\left\|\frac{f_1+f_2}{2}-h_{p,\epsilon}\right\|_{v_{p,\epsilon}} \\
&= \left(\sum_{j=1}^{k}\int_{B_j}\left|\epsilon^{-c}\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)^{\epsilon}\right|^2 v_{p,\epsilon}^{\epsilon}\right)^{\frac{1}{2}} \\
&\ge \left(\sum_{j=1}^{k}\int_{B_j}\left|\sum_{i=0}^{c}\frac{\left(\frac{f_1+f_2}{2}-h_{p,\epsilon}\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-c}(\cdot-x_j)^i\right|^2 v_{p,\epsilon}^{\epsilon}\right)^{\frac{1}{2}} \\
&- \varrho_\epsilon-\lambda_\epsilon = \left(\sum_{j=1}^{k}\int_{B_j}u_\epsilon\right)^{\frac{1}{2}}-\varrho_\epsilon-\lambda_\epsilon
\end{aligned} \tag{50}$$

On the other hand, let $h\in\Pi^n$ be such that $h^{(i)}(x_j)=\left(\frac{f_1+f_2}{2}\right)_p^{(i)}(x_j)$, $0\le i\le c-1$, $1\le j\le k$. Then there exists a set of real numbers (independent of ϵ), say $\{b_e\}_{1\le e\le d}$, that satisfies

$$h=\sum_{v=1}^{k}\sum_{s=0}^{c-1}\left(\frac{f_1+f_2}{2}\right)_p^{(s)}(x_v)u_{sv}+\sum_{e=1}^{d}b_e w_e.$$

Set $\kappa_j=\min_{q\in\Pi^{c-1}}\|(\cdot-x_j)^c-q\|_{(j)}$, where $\|f\|_{(j)}^2=\int_{B_j}|f|^2|(\cdot-x_j)|^{(\overline{m}+1)(p-2)}$ and let $\{c_{sv}\}_{\substack{1\le v\le k\\0\le s\le c-1}}$ be such that $\sum_{i=0}^{c-1}\frac{c_{ij}}{i!}(\cdot-x_j)^i$ is the best approximation to $\frac{\left(\frac{f_1+f_2}{2}-h\right)^{(c)}(x_j)}{c!}(\cdot-x_j)^c$ from Π^{c-1} respect to $\|\cdot\|_{(j)}$. We consider the following net of polynomials in Π^n,

$$g_\epsilon = \sum_{v=1}^{k}\sum_{s=0}^{c-1}\left(\left(\frac{f_1+f_2}{2}\right)_p^{(s)}(x_v) - c_{sv}\epsilon^{c-s}\right)u_{sv} + \sum_{e=1}^{d} b_e w_e.$$

We observe that $g_\epsilon^{(i)}(x_j) = \left(\frac{f_1+f_2}{2}\right)_p^{(i)}(x_j) - c_{ij}\epsilon^{c-i}$, $1 \le j \le k$, $0 \le i \le c-1$. Expanding g_ϵ^ϵ by its Taylor polynomial at x_j up to order c, we get

$$\begin{aligned}
\epsilon^{-c}\left(\frac{f_1+f_2}{2} - g_\epsilon\right)^\epsilon(t) &= \epsilon^{-c}\left(\frac{f_1+f_2}{2}\right)^\epsilon(t) - \sum_{i=0}^{c-1}\frac{g_\epsilon^{(i)}(x_j)}{i!}\epsilon^{i-c}(t-x_j)^i \\
&\quad - \frac{g_\epsilon^{(c)}(\eta_j(t))}{i!}(t-x_j)^c \\
&= \epsilon^{-c}\left(\frac{f_1+f_2}{2}\right)^\epsilon(t) - \sum_{i=0}^{c}\frac{\left(\frac{f_1+f_2}{2}\right)_p^{(i)}(x_j)}{i!}\epsilon^{i-c}(t-x_j)^i \\
&\quad + \sum_{i=0}^{c-1}\frac{c_{ij}}{i!}(t-x_j)^i + \frac{\left(\frac{f_1+f_2}{2}\right)_p^{(c)}(x_j)}{c!}(t-x_j)^c \\
&\quad - \frac{g_\epsilon^{(c)}(\eta_j(t))}{c!}(t-x_j)^c, \qquad (51)
\end{aligned}$$

where $\eta_j(t) \in B_{j,\epsilon}$. Since $h_{p,\epsilon}$ is the b.s.a. to f_1 and f_2 from Π^n in $L^p(I_\epsilon)$, by Theorem 2.4, Minkowski inequality, (50) and (51), we obtain

$$\begin{aligned}
\left(\sum_{j=1}^{k}\int_{B_j} u_\epsilon\right)^{\frac{1}{2}} - \varrho_\epsilon - \lambda_\epsilon &\le \epsilon^{-c}\left\|\frac{f_1+f_2}{2} - h_{p,\epsilon}\right\|_{\nu_{p,\epsilon}} \le \epsilon^{-c}\left\|\frac{f_1+f_2}{2} - g_\epsilon\right\|_{\nu_{p,\epsilon}} \\
&\le \lambda_\epsilon + \|f_\epsilon\|_{\nu_{p,\epsilon}}, \qquad (52)
\end{aligned}$$

where $f_\epsilon = \sum_{i=0}^{c-1}\frac{c_{ij}}{i!}(\cdot - x_j)^i + \frac{\left(\frac{f_1+f_2}{2}\right)_p^{(c)}(x_j)}{c!}(\cdot - x_j)^c - \frac{g_\epsilon^{(c)}(\xi_j(\cdot))}{c!}(\cdot - x_j)^c$. Since $\lim_{\epsilon\to 0} g_\epsilon = h$, uniformly on I, we have

$$\lim_{\epsilon\to 0} f_\epsilon = \frac{\left(\frac{f_1+f_2}{2} - h\right)_p^{(c)}(x_j)}{c!}(\cdot - x_j)^c + \sum_{i=0}^{c-1}\frac{c_{ij}}{i!}(\cdot - x_j)^i =: \varphi_j, \quad \text{uniformly on} \quad B_j.$$

By the inequalities of Minkowski and Hölder and Lemma 2.3, we get

$$\begin{aligned}
&\left|\int_{B_j}|f_\epsilon|^2\nu_{p,\epsilon}^\epsilon - \int_{B_j}|\varphi_j|^2\omega_p\right| \\
&\quad = \left|\int_{B_j}(|f_\epsilon|^2 - |\varphi_j|^2)\nu_{p,\epsilon}^\epsilon - \int_{B_j}|\varphi_j|^2(\omega_p - \nu_{p,\epsilon}^\epsilon)\right|
\end{aligned}$$

$$\leq \left|\int_{B_j}(|f_\epsilon|^2-|\varphi_j|^2)\nu^\epsilon_{p,\epsilon}\right| + \left|\int_{B_j}|\varphi_j|^2\omega_p - \int_{B_j}|\varphi_j|^2\nu^\epsilon_{p,\epsilon}\right|$$
$$\leq \left(\int_{B_j}||f_\epsilon|^2-|\varphi_j|^2|^{\frac{p}{2}}\right)^{\frac{2}{p}}\left(\int_{B_j}|\nu^\epsilon_{p,\epsilon}|^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}} + \left|\int_{B_j}|\varphi_j|^2\omega_p - \int_{B_j}|\varphi_j|^2\nu^\epsilon_{p,\epsilon}\right|$$
$$\leq (p-1)2^{\frac{(p-2)^2}{p}}\left(\int_{B_j}||f_\epsilon|^2-|\varphi_j|^2|^{\frac{p}{2}}\right)^{\frac{2}{p}} + \left|\int_{B_j}|\varphi_j|^2\omega_p - \int_{B_j}|\varphi_j|^2\nu^\epsilon_{p,\epsilon}\right|,$$

and so $\lim_{\epsilon\to 0}\|f_\epsilon\|_{\nu_{p,\epsilon}} = \left(\sum_{j=1}^k\int_{B_j}|\varphi_j|^2\omega_p\right)^{\frac{1}{2}}$ by Lemma 3.13. According to Lemma 2.3 and (15), we see that $\lim_{\epsilon\to 0}\varrho_\epsilon = \lim_{\epsilon\to 0}\lambda_\epsilon = 0$. Therefore, the Fatou Lemma, (49) and (52) lead to

$$\tau^{\frac{1}{2}}\left(\sum_{j=1}^k\kappa_j^2\left|\frac{\left(\frac{f_1+f_2}{2}-h_0\right)_p^{(c)}(x_j)}{c!}\right|^2\left|\frac{(f_1-f_2)_p^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!}\right|^{p-2}\right)^{\frac{1}{2}}$$
$$\leq \left(\sum_{j=1}^k\int_{B_j}\left|\frac{\left(\frac{f_1+f_2}{2}-h_0\right)_p^{(c)}(x_j)}{c!}(\cdot-x_j)^c+\sum_{i=0}^{c-1}\frac{d_{ij}}{i!}(\cdot-x_j)^i\right|^2\right.$$
$$\left.\times\,\tau\left|\frac{(f_1-f_2)_p^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!}(\cdot-x_j)^{\overline{m}+1}\right|^{p-2}\right)^{\frac{1}{2}}$$
$$\leq \left(\sum_{j=1}^k\int_{B_j}\left|\frac{\left(\frac{f_1+f_2}{2}-h\right)_p^{(c)}(x_j)}{c!}(\cdot-x_j)^c+\sum_{i=0}^{c-1}\frac{c_{ij}}{i!}(\cdot-x_j)^i\right|^2\right.$$
$$\left.\times\,\tau\left|\frac{(f_1-f_2)_p^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!}(\cdot-x_j)^{\overline{m}+1}\right|^{p-2}\right)^{\frac{1}{2}}$$
$$= \left(\sum_{j=1}^k\tau\left|\frac{(f_1-f_2)_p^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!}\right|^{p-2}\right.$$
$$\left.\times\int_{B_j}\left|\frac{\left(\frac{f_1+f_2}{2}-h\right)_p^{(c)}(x_j)}{c!}(\cdot-x_j)^c+\sum_{i=0}^{c-1}\frac{c_{ij}}{i!}(\cdot-x_j)^i\right|^2|\cdot-x_j|^{(\overline{m}+1)(p-2)}\right)^{\frac{1}{2}}$$

$$= \tau^{\frac{1}{2}} \left(\sum_{j=1}^{k} \kappa_j^2 \left| \frac{\left(\frac{f_1+f_2}{2} - h\right)_p^{(c)}(x_j)}{c!} \right|^2 \left| \frac{(f_1 - f_2)_p^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!} \right|^{p-2} \right)^{\frac{1}{2}},$$

where $\tau = 2^{\frac{(1-p)(p-2)}{p}} \Lambda^{p-2}$. As $\kappa_j = \min_{q \in \Pi^{c-1}} \left(\int_{-1}^{1} |y^c - q|^2 |y|^{(\overline{m}+1)(p-2)} dy \right)^{\frac{1}{2}}$, $1 \leq j \leq k$, we have

$$\sum_{j=1}^{k} \left| \left(\frac{f_1+f_2}{2} - h_0 \right)_p^{(c)} (x_j) \right|^2 |(f_1 - f_2)_p^{(\overline{m}+1)}(x_j)|^{p-2}$$
$$\leq \sum_{j=1}^{k} \left| \left(\frac{f_1+f_2}{2} - h \right)_p^{(c)} (x_j) \right|^2 |(f_1 - f_2)_p^{(\overline{m}+1)}(x_j)|^{p-2},$$

for all $h \in \Pi^n$ such that $h^{(i)}(x_j) = \left(\frac{f_1+f_2}{2}\right)_p^{(i)}(x_j)$, $0 \leq i \leq c-1$, $1 \leq j \leq k$. So, h_0 is a solution of (48).

Finally, we observe that the problem (48) is equivalent to find the best approximation to $\left(\left(\frac{f_1+f_2}{2}\right)_p^{(c)}(x_1), \ldots, \left(\frac{f_1+f_2}{2}\right)_p^{(c)}(x_k) \right)$ from the convex set

$$\Gamma = \left\{ (h^{(c)}(x_1), \ldots, h^{(c)}(x_k)) : h \in \Pi^n \text{ and } h^{(i)}(x_j) = \left(\frac{f_1+f_2}{2}\right)_p^{(i)}(x_j), \right.$$
$$\left. 0 \leq i \leq c-1,\ 1 \leq j \leq k \right\}$$

respect to $l^2_\mu(\mathbb{R}^k)$-norm, where $\mu = \left(\left|(f_1 - f_2)_p^{(\overline{m}+1)}(x_1)\right|^{p-2}, \ldots, \left|(f_1 - f_2)_p^{(\overline{m}+1)}(x_k)\right|^{p-2} \right)$ (If $c = 0$, no constrain on Π^n is assumed). Since this problem has a unique solution [18, Theorem 1.14], we conclude that there exists the L^p-b.s.l.a. to f_1 and f_2 from Π^n on $\{x_1, \ldots, x_k\}$, and it is the solution of (48). □

Remark 3.15. We observe that if $\overline{m} \leq c-2$, the limit of $v^{\epsilon}_{p,\epsilon}$ as $\epsilon \to 0$ does not depend of d_{ij}, given in (41), which guarantee a unique minimization problem. However, from (45) and (46), it does not occur if $\overline{m} = c-1$. So Theorem 3.14 cannot be used for this case.

Acknowledgments

We deeply thank to our colleague and friend Professor Cuenya for all his help and wise counsel throughout our education.

Funding

This work was partially supported by Universidad Nacional de Río Cuarto and CONICET, FCEFQyN, Río Cuarto, Argentina.

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