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On the assumed inherent stability of semi-active control systems

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A R T I C L E I N F O

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A B S T R A C T

Vibration control systems are usually classified into: passive, active and semi-active. Semi-active control systems are based on formerly passive mechanical devices, such as springs and dampers, whose characteristics are adjusted in real-time by active means. The attractiveness of semi-active control systems mainly relies on their assumed "inherent stability", which makes them almost as reliable and fault-tolerant as passive control systems.

The present paper shows that these assumptions are only partially true, by applying passivity formalism and bounded-input bounded-output stability definitions. Based on this study, semi-active control devices are rationally classified into three classes with two subclasses each: (1.1) non-negative variable-damping dampers, (1.2) possibly-negative variable-damping dampers, (2.1) independently-variable-stiffness springs, (2.2) resettable-stiffness springs, (3.1) independently-variable-inertance inerters, and (3.2) resettable-inertance inerters. It is found that a control system using any of the semi-active control devices of type (1.2), (2.1) or (3.1) is not inherently stable, as it is assumed in some previous papers; because those devices are "active" from the perspective of the passivity formalism. Interestingly, hybrid combinations of independently-variable-inertance inerters with non-negative variable-damping dampers can be designed to produce inherently-stable control systems. Following this framework, several published works on semi-active control systems are reviewed and classified.

The presented methodology is useful when developing new devices. This is demonstrated by proposing a novel control device, which is classified and assessed in terms of inherent passivity. Moreover, this passivity assessment is conveniently used to propose a control law for the device. Finally, a frame structure controlled by the device is numerically simulated through a number of scenarios including instability and a countermeasure for its mitigation.

1. Introduction

Structural vibrations are detrimental to the performance of many engineering applications and, therefore, several methods of vibration control have been proposed to reduce them. These methods are generally classified into: Passive Control (PC) [1,2], Active Control (AC) [3,4], or Semi-Active Control (SAC) [5,6]; although hybrid combinations (HC) are also common [3]. In a SAC system, the properties of *formerly* passive devices (e.g., viscous dampers, springs, pendula) are conveniently adjusted in real-time through auxiliary actuators (e.g., valves, motors) according to a control law [7]. SAC is attractive since it offers the reliability of PC; while approaching the adaptability and effectiveness of AC, without imposing high power demands. Furthermore, it is common to assume that SAC systems are "inherently stable" [6].

An important benefit of "inherent stability" is a guaranteed stability irrespectively of control-law design, modelling errors, and failures in miscellaneous hardware of the SAC system, i.e. sensors, transmission equipment, control computers, auxiliary actuators, and power supplies (see Fig. 1). The fault-tolerance of these subsystems is particularly important in three cases: (1) applications that remain in standby for many years until their operation is needed, as mitigation of seismic vulnerability in civil structures [8]; (2) applications under harsh environments, such as smart suspension systems for vehicles [9]; and (3) applications deployed in remote locations, such as artificial satellites and other space structures [10]. AC systems, which are not inherently stable, can yield a dangerously large structural response if any fault is present in the hardware or if its design is inadequate.

An additional benefit of "inherent stability" is that researchers and practicing engineers that are not familiar with non-linear control theory can use and/or propose new semi-active devices without destabilization risk. Moreover, new control laws can be proposed without stability analysis; e.g. heuristically as in [11]. This advantage of the assumed "inherent stability" is important since vibration control design is a multidisciplinary task that involves not only control engineers but also

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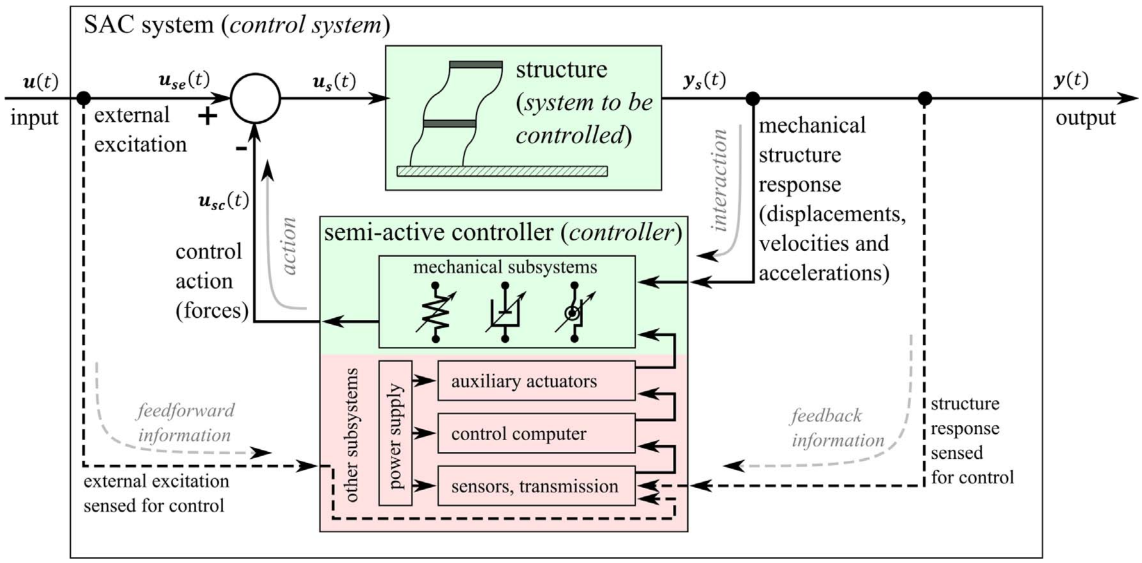


Fig. 1. General semi-active control system.

mechanical, electronics and civil engineers.

Many investigations [6,8,12–20] appeal to the "inherent stability" of SAC systems and justify it on the "passivity" of the devices used to implement them; although it has also been claimed that "inherent stability" cannot be generalized [21,22]. This inconsistency arises from the use of ambiguous definitions. Since their introduction in the 1970s [7], many new semi-active devices have been proposed; e.g. variable-damping dampers [7], variable-stiffness springs [23] and, recently, variable-inertance inerters [24,25]. As a consequence, the definition of SAC is often adapted in order to encompass the new devices, which leads to an increasing risk of misunderstanding.

The purpose of this paper is to clarify the definitions of "semi-active", "stability", and "inherent stability", in order to formally address the issue of the "inherent stability of semi-active control systems" within a general framework. The approach proposed is based on the passivity formalism [26], as suggested by Hrovat [27] for classifying "active" and "passive" suspension systems. Among the many available definitions of "stability" [28], this study considers bounded-input bounded-output (BIBO), which can be deduced from the passivity theorem [29] and is appropriate for systems under forced excitation. Finally, this paper denotes the stability as "inherent" when it depends exclusively on the mechanical subsystems (the green blocks in Fig. 1) of the controller.

2. Definitions and nomenclature

2.1. Mathematical preliminaries

In order to establish the boundedness of vector-valued functions, such as displacements, velocities, accelerations and forces, the L_2 norm of a vector function x is defined as:

$$\|\boldsymbol{x}\|_2 = \sqrt{\langle \boldsymbol{x},\boldsymbol{x}\rangle} \tag{1}$$

where $\langle\cdot,\cdot\rangle$, the inner product of vector functions, is defined as:

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle = \int_0^\infty \boldsymbol{x}'(t)\boldsymbol{y}(t)dt \tag{2}$$

where $'$ is the transpose operator. Note that $\|x\|_2$ is a scalar denoting the norm of the vector-valued function x; which must not be confused with $\|x(t)\|_2 = \sqrt{x'(t)x(t)}$.

Similarly, the truncated inner product of x and y over the interval $[0,T]$, is defined as [30]:

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle_T = \int_0^T \boldsymbol{x}'(t)\boldsymbol{y}(t)dt \tag{3}$$

where T is a particular instant of time; and the truncated L_2 norm of x as follows [30]:

$$\|\boldsymbol{x}\|_{2,T} = \sqrt{\langle \boldsymbol{x},\boldsymbol{x}\rangle_T}. \tag{4}$$

Thus, a function x lies in the L_2 space, i.e. $x \in L_2$, if $\|x\|_2 < \infty$ [28]. Correspondingly, a function x lies in the extended- L_2 space, i.e. $x \in L_{2e}$, if $\|x\|_{2,T} < \infty \ \forall \ T$ [28].

Below, some useful inequalities are summarized [26,31]:

$$\lambda_{\min}(\boldsymbol{A})\boldsymbol{x}'(t)\boldsymbol{x}(t) \leqslant \boldsymbol{x}'(t)\boldsymbol{A}\boldsymbol{x}(t) \leqslant \lambda_{\max}(\boldsymbol{A})\boldsymbol{x}'(t)\boldsymbol{x}(t). \tag{5}$$

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle_T \leqslant \|\boldsymbol{x}\|_{2,T}\|\boldsymbol{y}\|_{2,T}. \tag{6}$$

$$\|\boldsymbol{x}+\boldsymbol{y}\|_{2,T} \leqslant \|\boldsymbol{x}\|_{2,T} + \|\boldsymbol{y}\|_{2,T}. \tag{7}$$

where $\lambda_{\min}(\boldsymbol{A})$ and $\lambda_{\max}(\boldsymbol{A})$ are the smallest and largest eigenvalues of $\boldsymbol{A}$, Eq. (6) is the Cauchy–Bunyakovsky–Schwarz (CBS) inequality, and Eq. (7) is the triangle inequality.

From the perspective of vibration control engineering, L_2 norm ($\|\cdot\|_2$) measures a response function in *RMS sense*, which is an evaluation criteria commonly used in that field [32]. Other important criterion is the measurement of responses in a *peak sense*. In this regard, the L_∞ norm and its truncated version are defined as:

$$\|\boldsymbol{x}\|_\infty = \sup_{t\in[0,\infty)} |\boldsymbol{x}(t)| \tag{8}$$

$$\|\boldsymbol{x}\|_{\infty,T} = \sup_{t\in[0,T]} |\boldsymbol{x}(t)| \tag{9}$$

Thus, a function $\boldsymbol{x}$ lies in the L_∞ space, i.e. $\boldsymbol{x} \in L_\infty$, if $\|\boldsymbol{x}\|_\infty < \infty$ [28]. Correspondingly, a function $\boldsymbol{x}$ lies in the extended-L_∞ space, i.e. $\boldsymbol{x} \in L_{\infty e}$, if $\|\boldsymbol{x}\|_{\infty,T} < \infty \ \forall \ T$ [28].

2.2. Definition of "stable"

A system is BIBO stable when the norm of the system output is finite for any input with finite norm. Formally, a system whose input is $\boldsymbol{u}$ and output is $\boldsymbol{y}$ is L_2-stable if [28]:

$$\boldsymbol{u} \in L_2 \Rightarrow \boldsymbol{y} \in L_2. \tag{10}$$

Moreover, if there exists a finite constant $\gamma > 0$ such that:

$$\boldsymbol{u} \in L_2 \Rightarrow \|\boldsymbol{y}\|_2 \leqslant \gamma\|\boldsymbol{u}\|_2, \tag{11}$$

The system is said to be L_2-stable with finite gain (γ) and zero bias.

This latter is a condition more useful than Eq. (10) since it not only guarantees boundedness but also provides the bound value to be compared with others. For simplicity, systems satisfying either condition, EqS. (10), (11) or other similar relation, are referred to as BIBO stable in the present paper.

2.3. Definition of "inherently stable"

In this paper, a *control system* is defined as the combination of: (1) a *system to be controlled* (e.g. a structure) and (2) a *controller* (e.g. a semi-active controller). This is illustrated in Fig. 1 for the particular case of a SAC system. In turn, a *controller* consists of several subsystems, of which some are mechanical.

Based on this layout, if the *stability* of the *control system* can be shown by only considering the *system to be controlled* and the mechanical subsystems of the *controller*, then, that *control system* is defined as *inherently stable*.

Such mechanical subsystems can be any combination of variable-damping dampers [33,34], variable-stiffness springs [35,36], and variable-inertance inerters [19]. Note that variable-inertia mass is the particular case of a variable-inertance inerter with a fixed end [37–39].

2.4. Definitions of "passive", "active" and "semi-active"

A system is *passive* if it cannot generate[1] energy. Formally, a system whose input is $\boldsymbol{u}$ and output is $\boldsymbol{y}$, for which it is defined a lower bounded (Lyapunov-like) storage function V, is said to be *passive* if it satisfies the following equation:

$$\dot{V}(t) = \boldsymbol{u}'(t)\boldsymbol{y}(t) - g(t), \tag{12}$$

with $g(t) \geqslant 0 \; \forall \; t \in [0,T]$ [26].

By integrating Eq. (12) and recalling that $V(t) \geqslant 0$, it is obtained the following equivalent condition for *passivity*: the system is *passive* if there exists a constant $\beta = V(0) \geqslant 0$ such that [30]:

$$\langle \boldsymbol{u}, \boldsymbol{y} \rangle_T \geqslant -\beta \; \forall \; T, \tag{13}$$

Also, the system is *strictly-output passive* if there is a constant $\varepsilon > 0$ such that:

$$\langle \boldsymbol{u}, \boldsymbol{y} \rangle_T \geqslant \varepsilon \|\boldsymbol{y}\|_{2,T}^2 - \beta \; \forall \; T \tag{14}$$

From a physical point of view, $V(t)$ is the stored energy, $\boldsymbol{u}'(t)\boldsymbol{y}(t)$ is the external power input, $g(t)$ is the dissipated power, and $\langle \boldsymbol{u}, \boldsymbol{y} \rangle_T$ is the external energy supplied to the system.

When the term "semi-active control" is assessed from the perspective of this *passivity formalism* [26], the first cause of misunderstanding can be identified. Namely, definitions of *active* and *passive* are mutually exclusive, so the term "semi-active" involves some abuse of notation (as warned in [40]). Further, a system is stated to be *active* or *passive* depending on the definition of its inputs, outputs and state variables.

In light of this concept, the terms "active control" (AC), "passive control" (PC) and "semi-active control" (SAC), which are broadly used in vibration control literature, are precisely defined below based on the notion of Control-Structure Interaction (CSI) [41] and under the formal definitions of *passive* and *active* (*passivity formalism*).

AC: the controller is composed of *active* devices that are able to act on the structure independently of the structure response; i.e., CSI is only an accidental phenomenon that can be taken into account to improve performance.

PC: the controller is composed of *passive* devices that act on the structure exclusively from the interaction with the structure response.

SAC: (see Fig. 1) the controller is composed of (formerly) *passive* devices that act on the structure exclusively from the interaction with the structure response; but their characteristic parameters are modified by auxiliary actuators (*active* subsystems). This latter, however, can remove the (former) *passivity* of the devices, as shown in Section 4.2.

[1] "generated energy" actually refers to energy supplied to the mechanical subsystems by the auxiliary actuators (see Fig. 1).

3. System description

From the scheme of Fig. 1, a general mathematical model is developed in this section for a SAC system. The mathematical model is "general" in the sense that it is compatible to the vast majority of works found in the SAC literature. The key for accomplishing this task consists in modelling only the structure and the mechanical subsystems of the SAC controller (the green blocks in Fig. 1).

3.1. Structure

An n-Degree-Of-Freedom (n-DOF) structure can be modelled by the following linear equation of motion [42]:

$$\mathbf{M_s}\ddot{\boldsymbol{q}}_s(t) + \mathbf{C_s}\dot{\boldsymbol{q}}_s(t) + \mathbf{K_s}\boldsymbol{q}_s(t) = \boldsymbol{f}_s(t), \tag{15}$$

in which: M_s, C_s and K_s are the mass, damping and stiffness matrices, all of them positive-definite; $q_s(t) = [q_{s1} \cdots q_{si} \cdots q_{sn}]^T$ is the displacement vector; and $f_s(t) = f_{se}(t) - f_{sc}(t)$ is the vector of forces exerted on the structure, where $f_{se}(t)$ contains external forces and $f_{sc}(t)$ the forces exerted by the controller.

Eq. (15) can be recast as the following state-space representation of a Linear Time-Invariant (LTI) system [6]:

$$\dot{\boldsymbol{x}}_s(t) = \boldsymbol{A}_s\boldsymbol{x}_s(t) + \boldsymbol{B}_s\boldsymbol{u}_s(t), \tag{16}$$

$$\boldsymbol{y}_s(t) = \boldsymbol{C}_s\boldsymbol{x}_s(t) + \boldsymbol{D}_s\boldsymbol{u}_s(t), \tag{17}$$

where $\boldsymbol{x}_s(t)$ is the vector of state variables, $\boldsymbol{A}_s$ is the state matrix, $\boldsymbol{B}_s$ is the input matrix, $\boldsymbol{C}_s$ is the output matrix, $\boldsymbol{D}_s$ is the direct-transfer matrix, $\boldsymbol{y}_s(t)$ is the output vector, and $\boldsymbol{u}_s(t)$ is the input vector. Considering that the system output is composed of those structure responses that can interact with the mechanical subsystems of the semi-active controller (i.e. displacements, velocities and accelerations), the following vectors and matrices are defined [6,43]:

$$\boldsymbol{x}_s'(t) = \begin{bmatrix} \boldsymbol{q}_s'(t) & \dot{\boldsymbol{q}}_s'(t) \end{bmatrix} \in \mathbb{R}^{2n\times 1} \tag{18}$$

$$\boldsymbol{y}_s'(t) = \begin{bmatrix} \boldsymbol{q}_s'(t) & \dot{\boldsymbol{q}}_s'(t) & \ddot{\boldsymbol{q}}_s'(t) \end{bmatrix} \in \mathbb{R}^{3n\times 1} \tag{19}$$

$$\boldsymbol{u}_s(t) = \boldsymbol{u}_{se}(t) - \boldsymbol{u}_{sc}(t) = \begin{bmatrix} 0_{n\times 1} \\ \boldsymbol{f}_s(t) \\ 0_{n\times 1} \end{bmatrix} = \begin{bmatrix} 0_{n\times 1} \\ \boldsymbol{f}_{se}(t) \\ 0_{n\times 1} \end{bmatrix} - \begin{bmatrix} 0_{n\times 1} \\ \boldsymbol{f}_{sc}(t) \\ 0_{n\times 1} \end{bmatrix} \in \mathbb{R}^{3n\times 1} \tag{20}$$

$$\boldsymbol{A}_s = \begin{bmatrix} 0_{n\times n} & \boldsymbol{I}_{n\times n} \\ -\mathbf{M_s}^{-1}\mathbf{K_s} & -\mathbf{M_s}^{-1}\mathbf{C_s} \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \tag{21}$$

$$\boldsymbol{B}_s = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \mathbf{M_s}^{-1} & 0_{n\times n} \end{bmatrix} \in \mathbb{R}^{2n\times 3n} \tag{22}$$

$$\boldsymbol{C}_s = \begin{bmatrix} \boldsymbol{I}_{n\times n} & 0_{n\times n} \\ & \boldsymbol{A}_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \boldsymbol{I}_{n\times n} \\ -\mathbf{M_s}^{-1}\mathbf{K_s} & -\mathbf{M_s}^{-1}\mathbf{C_s} \end{bmatrix} \in \mathbb{R}^{3n\times 2n} \tag{23}$$

$$\boldsymbol{D}_s = \begin{bmatrix} 0_{n\times 3n} \\ \boldsymbol{B}_s \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \mathbf{M_s}^{-1} & 0_{n\times n} \end{bmatrix} \in \mathbb{R}^{3n\times 3n} \tag{24}$$

where $\boldsymbol{I}$ and 0 are the identity and zero matrices, respectively.

Thus the system input is defined in such a way that the operation $\langle \boldsymbol{u}_s, \boldsymbol{y}_s \rangle_T$ is possible and has physical meaning: it equals the mechanical energy supplied to the structure in the time interval $[0,T]$, since

$\boldsymbol{u}_s'(t)\boldsymbol{y}_s(t)$ is the mechanical power supplied to the structure.

3.2. Semi-active controller

As can be inferred from Fig. 1, a semi-active controller can be modelled by considering only its mechanical subsystems composed of dampers, springs or inerters; since the other subsystems (red blocks) do not act on the structure but merely modify parameters of the devices that compose the mechanical subsystems.

In this study, no particular control law is considered; but, stiffness, damping and inertance are arbitrary functions of time (as in [44]). So, the controller can be modelled as a direct-transfer time-varying system whose input is $\boldsymbol{y}_s(t)$ and output is $\boldsymbol{u}_{sc}(t)$, i.e.:

$$\boldsymbol{u}_{sc}(t) = \widehat{\boldsymbol{B}}_{sc}\boldsymbol{D}_c(t)\widehat{\boldsymbol{B}}_{cs}\boldsymbol{y}_s(t). \tag{25}$$

In Eq. (25), matrix $\boldsymbol{D}_c(t)$ arranges the time-varying parameters of the devices (which can be up to p devices of each type) as follows:

$$\boldsymbol{D}_c(t) = [\mathbf{K}_c(t) \quad \mathbf{C}_c(t) \quad \mathbf{M}_c(t)] \in \mathbb{R}^{p\times 3p} \tag{26}$$

where $\mathbf{K}_c(t)$, $\mathbf{C}_c(t)$ and $\mathbf{M}_c(t)$ are $p\times p$ diagonal matrices containing the variable stiffness $k_{ci}(t)$, variable damping $c_{ci}(t)$ and variable inertance $m_{ci}(t)$ parameters. Matrices $\widehat{\boldsymbol{B}}_{sc}$ and $\widehat{\boldsymbol{B}}_{cs}$ state the relation between the responses of the structure $\boldsymbol{y}_s(t)$ and the relative-displacements $\boldsymbol{q}_c(t)$, -velocities $\dot{\boldsymbol{q}}_c(t)$ and -accelerations $\ddot{\boldsymbol{q}}_c(t)$ across the devices as follows:

$$\widehat{\boldsymbol{B}}_{cs} = \begin{bmatrix} \boldsymbol{B}_{sc}' & 0_{p\times n} & 0_{p\times n} \\ 0_{p\times n} & \boldsymbol{B}_{sc}' & 0_{p\times n} \\ 0_{p\times n} & 0_{p\times n} & \boldsymbol{B}_{sc}' \end{bmatrix} \in \mathbb{R}^{3p\times 3n} \tag{27}$$

$$\widehat{\boldsymbol{B}}_{sc} = \begin{bmatrix} 0_{n\times p} \\ \boldsymbol{B}_{sc} \\ 0_{n\times p} \end{bmatrix} \in \mathbb{R}^{3n\times p} \tag{28}$$

where $\boldsymbol{B}_{sc} \in \mathbb{R}^{n\times p}$ is a matrix containing pairs of (or singleton) direction cosines such that the vector of relative displacements across the devices is $\boldsymbol{q}_c = \boldsymbol{B}_{sc}'\boldsymbol{q}_s$. Note that, the product $\boldsymbol{B}_{sc}\mathbf{M}_c(t)\boldsymbol{B}_{sc}'$ is a symmetric matrix in the more general case of variable-inertance inerters and it is a diagonal matrix in the particular case of variable-inertia masses.

Although this model is algebraic, semi-active *controllers* having internal dynamics (e.g. Tuned Mass Dampers (TMD) [45], Semi-Active Friction Tendons (SAFT) [11], or Magnetorheological (MR) dampers modelled as proposed by Gamota and Filisko [46]) can be modelled by considering their internal DOFs as part of the *structure*.

Furthermore, the developed model can be easily extended to encompass nonlinear devices (e.g. MR dampers [46] or large-angle pendula [47]) by means of local linearization. In general, devices are characterized by the following nonlinear functions: $f_{ki}(q_{ci},t)$, for variable-stiffness springs; $f_{ci}(q_{ci},\dot{q}_{ci},t)$, for variable-damping dampers; and $f_{mi}(\ddot{q}_{ci}(t),\dot{q}_{ci},t)$, for variable-inertance inerters. Hence, equivalent linear parameters can be calculated simply as:

$$k_{ci}(q_{ci}(t),\alpha_{ki}(t)) = \begin{cases} \frac{f_{ki}(q_{ci}(t),t)}{q_{ci}(t)}, & q_{ci}(t) \neq 0 \\ 0, & q_{ci}(t) = 0 \end{cases} \tag{29}$$

$$c_{ci}(q_{ci}(t),\dot{q}_{ci}(t),\alpha_{ci}(t)) = \begin{cases} \frac{f_{ci}(q_{ci}(t),\dot{q}_{ci}(t),t)}{\dot{q}_{ci}(t)}, & \dot{q}_{ci}(t) \neq 0 \\ 0, & \dot{q}_{ci}(t) = 0 \end{cases} \tag{30}$$

$$m_{ci}(\dot{q}_{ci}(t),\alpha_{mi}(t)) = \begin{cases} \frac{f_{mi}(\ddot{q}_{ci}(t),\dot{q}_{ci}(t),t)}{\ddot{q}_{ci}(t)}, & \ddot{q}_{ci}(t) \neq 0 \\ 0, & \ddot{q}_{ci}(t) = 0 \end{cases} \tag{31}$$

where $\alpha_{ki}(t)$, $\alpha_{ci}(t)$ and $\alpha_{mi}(t)$ are control parameters which, in normal conditions, depend on the control law. Without loss of generality, these control parameters are assumed to be related with the corresponding equivalent parameters as:

$$\frac{\partial k_{ci}}{\partial \alpha_{ki}} \geqslant 0, \frac{\partial c_{ci}}{\partial \alpha_{ci}} \geqslant 0, \frac{\partial m_{ci}}{\partial \alpha_{mi}} \geqslant 0. \tag{32}$$

In the most general case, parameters can vary subjected to the following design constraints:

$$k_{ci\min} \leqslant k_{ci}(t) \leqslant k_{ci\max}, \tag{33}$$

$$k_{ri\min} \leqslant \dot{\alpha}_{ki}(t) \leqslant k_{ri\max}, \tag{34}$$

$$c_{ci\min} \leqslant c_{ci}(t) \leqslant c_{ci\max}, \tag{35}$$

$$c_{ri\min} \leqslant \dot{c}_{ci}(t) \leqslant c_{ri\max}, \tag{36}$$

$$m_{ci\min} \leqslant m_{ci}(t) \leqslant m_{ci\max}, \tag{37}$$

$$m_{ri\min} \leqslant \dot{\alpha}_{mi}(t) \leqslant m_{ri\max}, \tag{38}$$

where k_{cimin}, k_{cimax}, c_{cimin}, c_{cimax}, m_{cimin}, and m_{cimax} are referred to as *range characteristic parameters*; and k_{rimin}, k_{rimax}, c_{rimin}, c_{rimax}, m_{rimin}, and m_{rimax} as *rate characteristic parameters*. The suitability of the definitions given by Eqs. (34) and (38) becomes apparent in Section 4.2.

3.3. Semi-active control system

As shown comprehensively in Fig. 1 and summarized in Fig. 2, the *SAC system* is a feedback combination (closed-loop) between the *structure* and the *semi-active controller*.

4. Passivity properties and stability results

The *passivity theorem* [29] allows showing the *stability* of a closed-loop control system based on the *passivity* of the systems belonging to the corresponding loop (e.g. a *system to be controlled* and a *controller*).

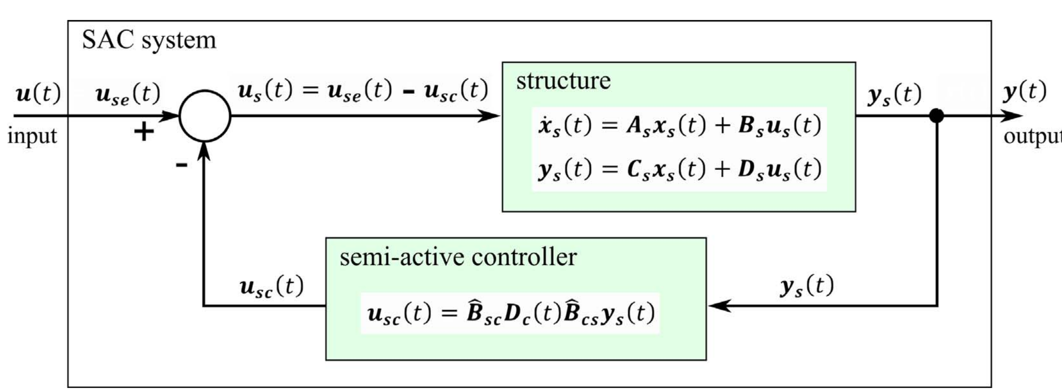


Fig. 2. Simplified scheme of the semi-active control system.

Although more general versions of this theorem are available, a particular case that is suitable for the system of Fig. 2 is addressed below. This is done in three steps: (1) study of the *passivity* of the structure, (2) study of the *passivity* of the semi-active controller and (3) application of the *passivity theorem* to the *SAC system* as a whole for studying its *stability*.

4.1. Passivity and stability properties of the structure

In this subsection, it is shown that the structure is *passive*, and, furthermore, *strictly-velocity passive* (i.e. strictly-output passive if output is considered to be composed of velocities only).

By considering the following storage function (mechanical energy):

$$V_s(t) = \frac{1}{2}\begin{bmatrix} \boldsymbol{q}_s'(t) & \dot{\boldsymbol{q}}_s'(t)\end{bmatrix}\begin{bmatrix}\mathbf{K_s} & 0\\ 0 & \mathbf{M_s}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_s(t)\\ \dot{\boldsymbol{q}}_s(t)\end{bmatrix} \tag{39}$$

its first derivative results in:

$$\dot{V}_s(t) = \dot{\boldsymbol{q}}_s'(t)\boldsymbol{f}_s(t) - \dot{\boldsymbol{q}}_s'(t)\mathbf{C_s}\dot{\boldsymbol{q}}_s(t), \tag{40}$$

where $\ddot{q}_s(t)$ has been substituted by $M_s^{-1}(f_s(t)-C_s\dot{q}_s(t)-K_s q_s(t))$ (solved from Eq. (15)).

On the other hand, the external power input is:

$$\boldsymbol{u}_s'(t)\boldsymbol{y}_s(t) = \boldsymbol{f}_s'(t)\dot{\boldsymbol{q}}_s(t). \tag{41}$$

Replacing Eqs. (40) and (41) into Eq. (12), the following expression for dissipated power is obtained:

$$g_s(t) = \dot{\boldsymbol{q}}_s'(t)\mathbf{C_s}\dot{\boldsymbol{q}}_s(t). \tag{42}$$

Now, replacing Eqs. (41) and (42) into Eq. (12) and integrating in the interval $[0,T]$,

$$V_s(T) - V_s(0) = \langle \boldsymbol{f}_s, \dot{\boldsymbol{q}}_s\rangle_T - \int_0^T \dot{\boldsymbol{q}}_s'(t)\mathbf{C_s}\dot{\boldsymbol{q}}_s(t)dt. \tag{43}$$

Finally, assuming $V_s(0) = 0$, recalling $V_s(T) \geqslant 0\ \forall\ T$, and appealing to Eq. (5), it is concluded that:

$$\langle \boldsymbol{u}_s, \boldsymbol{y}_s\rangle_T = \langle \boldsymbol{f}_s, \dot{\boldsymbol{q}}_s\rangle_T \geqslant \varepsilon_s\|\dot{\boldsymbol{q}}_s\|_{2,T}^2 \geqslant 0 \tag{44}$$

Since $\mathbf{C_s}$ is a positive-definite matrix, $\varepsilon_s = \lambda_{\min}(\mathbf{C_s}) > 0$; therefore, the structure is *passive* (compare Eqs. (44) and (13)). Moreover, the structure is *strictly-velocity passive* (compare Eqs. (44) and (14)). This *passivity* property of the structure is used bellow to show that it is also BIBO stable.

Invoking the CBS inequality in Eq. (44) yields:

$$\|\boldsymbol{f}_s\|_{2,T}\|\dot{\boldsymbol{q}}_s\|_{2,T} \geqslant \varepsilon_s\|\dot{\boldsymbol{q}}_s\|_{2,T}^2 \tag{45}$$

from which $\|\dot{\boldsymbol{q}}_s\|_{2,T}$ can be solved to obtain:

$$\|\dot{\boldsymbol{q}}_s\|_{2,T} \leqslant \gamma_s\|\boldsymbol{f}_s\|_{2,T} \tag{46}$$

where $\gamma_s = \varepsilon_s^{-1}$. Additionally, if $T \to \infty$ (lemma 6.1.24 in [28]):

$$\|\dot{\boldsymbol{q}}_s\|_2 \leqslant \gamma_s\|\boldsymbol{f}_s\|_2, \tag{47}$$

i.e., the structure is *bounded-input bounded-velocity stable* with finite gain γ_s (see Eq. (11)).

Regarding displacements, their boundedness is shown below in a *peak sense* using Eq. (46). Note that recalling $g_s(t) \geqslant 0$, $V_s(0) = 0$, and CBS inequality enables rewriting Eq. (43) as follows:

$$V_s(T) \leqslant \langle \boldsymbol{f}_s, \dot{\boldsymbol{q}}_s\rangle_T \leqslant \|\boldsymbol{f}_s\|_{2,T}\|\dot{\boldsymbol{q}}_s\|_{2,T}, \tag{48}$$

where Eqs. (39), (5) and (46) can be introduced to yield:

$$\frac{1}{2}\lambda_{\min}(\mathbf{K_s})\boldsymbol{q}_s'(T)\boldsymbol{q}_s(T) + \frac{1}{2}\lambda_{\min}(\mathbf{M_s})\dot{\boldsymbol{q}}_s'(T)\dot{\boldsymbol{q}}_s(T) \leqslant \gamma_s\|\boldsymbol{f}_s\|_{2,T}^2. \tag{49}$$

From this expression, since all the terms in the left-hand side are non-negative, the following results are derived:

$$|q_{si}(T)| \leqslant \sqrt{\frac{2\gamma_s}{\lambda_{\min}(\mathbf{K_s})}}\|\boldsymbol{f}_s\|_{2,T}\, i = 1,\ldots,n \tag{50}$$

$$|\dot{q}_{si}(T)| \leqslant \sqrt{\frac{2\gamma_s}{\lambda_{\min}(\mathbf{M_s})}}\|\boldsymbol{f}_s\|_{2,T}\, i = 1,\ldots,n \tag{51}$$

These are upper bounds for peak velocities and displacements at the time instant T. As with $\|\boldsymbol{f}_s\|_{2,T}$, these bounds are monotonically increasing with T; so they are valid for every $t \in [0,T]$. Therefore:

$$\|\boldsymbol{q}_s\|_{\infty,T} \leqslant \sqrt{\frac{2\gamma_s}{\lambda_{\min}(\mathbf{K_s})}}\|\boldsymbol{f}_s\|_{2,T}, \tag{52}$$

$$\|\dot{\boldsymbol{q}}_s\|_{\infty,T} \leqslant \sqrt{\frac{2\gamma_s}{\lambda_{\min}(\mathbf{M_s})}}\|\boldsymbol{f}_s\|_{2,T}. \tag{53}$$

This means that peak displacements and peak velocities of the structure are bounded and the corresponding upper bounds are directly related to the RMS force acting on the structure. This result, while evident, will be useful when compared to the case of the whole SAC system.

In terms of BIBO stability, Eqs. (52) and (53) imply that: $f_s \in L_2 \Rightarrow q_s, \dot{q}_s \in L_\infty$; by lemma 6.1.24 in [28].

4.2. Passivity properties of the semi-active controller

In this subsection, it is shown that a semi-active controller is *passive* in certain cases and is *active* in others.

First, consider a storage function that is equal to the potential energy E_p plus the kinetic energy E_k stored in the *semi-active controller*:

$$V_c(t) = \int_0^{\boldsymbol{q}_c(t)} \boldsymbol{q}_c'\mathbf{K_c}(\boldsymbol{q}_c,\boldsymbol{\alpha}_k(t))d\boldsymbol{q}_c + \int_0^{\dot{\boldsymbol{q}}_c(t)} \dot{\boldsymbol{q}}_c'\mathbf{M_c}(\dot{\boldsymbol{q}}_c,\boldsymbol{\alpha}_m(t))d\dot{\boldsymbol{q}}_c, \tag{54}$$

which is lower bounded ($V_c \geqslant 0$) if only the cases in which $k_{cimin} \geqslant 0$ and $m_{cimin} \geqslant 0$ are considered. Note that both, $\mathbf{K_c}$ and $\mathbf{M_c}$, have double dependence: on the one hand, they vary due to intrinsic nonlinearities of the devices; and, on the other hand, it can vary because of control parameters. Below, it is shown that only the latter has influence on *passivity*.

By inspecting integrals in Eq. (54), it is found that V_c is a memoryless function depending only on current state (displacements and velocities) and current values of control parameters, i.e.:

$$V_c(t) = E_p(\boldsymbol{q}_c(t),\boldsymbol{\alpha}_k(t)) + E_k(\dot{\boldsymbol{q}}_c(t),\boldsymbol{\alpha}_m(t)). \tag{55}$$

Therefore, its first time derivative can be calculated as follows[2]:

$$\dot{V}_c(t) = \frac{\partial E_p(\boldsymbol{q}_c,\boldsymbol{\alpha}_k)}{\partial \boldsymbol{q}_c}\dot{\boldsymbol{q}}_c(t) + \frac{\partial E_p(\boldsymbol{q}_c,\boldsymbol{\alpha}_k)}{\partial \boldsymbol{\alpha}_k}\dot{\boldsymbol{\alpha}}_k(t) + \frac{\partial E_k(\dot{\boldsymbol{q}}_c,\boldsymbol{\alpha}_m)}{\partial \dot{\boldsymbol{q}}_c}\ddot{\boldsymbol{q}}_c(t) + \frac{\partial E_k(\dot{\boldsymbol{q}}_c,\boldsymbol{\alpha}_m)}{\partial \boldsymbol{\alpha}_m}\dot{\boldsymbol{\alpha}}_m \tag{56}$$

which, due to Eq. (54), results in:

$$\dot{V}_c(t) = \boldsymbol{q}_c'(t)\mathbf{K}_c(t)\dot{\boldsymbol{q}}_c(t) + \boldsymbol{e}_{k\alpha}'(\boldsymbol{q}_c(t),\boldsymbol{\alpha}_k(t))\dot{\boldsymbol{\alpha}}_k(t) + \dot{\boldsymbol{q}}_c'(t)\mathbf{M}_c(t)\ddot{\boldsymbol{q}}_c(t) + \boldsymbol{e}_{m\alpha}'(\dot{\boldsymbol{q}}_c(t),\boldsymbol{\alpha}_m(t))\dot{\boldsymbol{\alpha}}_m \tag{57}$$

where the row vectors of coefficients $\boldsymbol{e}_{k\alpha}'(\boldsymbol{q}_c,\boldsymbol{\alpha}_k)$ and $\boldsymbol{e}_{m\alpha}'(\dot{\boldsymbol{q}}_c,\boldsymbol{\alpha}_m)$, which are time-independent and characterize the control devices, can be calculated as follows:

$$\boldsymbol{e}_{k\alpha}'(\boldsymbol{q}_c,\boldsymbol{\alpha}_k) = \frac{\partial}{\partial \boldsymbol{\alpha}_k}\sum_{i=1}^{p}\int_0^{q_{ci}} q_{ci}k_{ci}(q_{ci},\alpha_{ki})dq_{ci} = \begin{bmatrix}\cdots & \int_0^{q_{ci}} q_{ci}\frac{\partial k_{ci}(q_{ci},\alpha_{ki})}{\partial \alpha_{ki}}dq_{ci} & \cdots\end{bmatrix} \tag{58}$$

[2] assuming numerator layout convention.

$$\boldsymbol{e}'_{m\alpha}(\dot{\boldsymbol{q}}_c,\boldsymbol{\alpha}_m) = \frac{\partial}{\partial \boldsymbol{\alpha}_m}\sum_{i=1}^{p}\int_0^{\dot{q}_{ci}}\dot{q}_{ci}m_{ci}(\dot{q}_{ci},\alpha_{mi})d\dot{q}_{ci} = \left[\cdots \int_0^{\dot{q}_{ci}}\dot{q}_{ci}\frac{\partial m_{ci}(\dot{q}_{ci},\alpha_{mi})}{\partial \alpha_{mi}}d\dot{q}_{ci} \cdots\right] \tag{59}$$

Therefore, recalling Eq. (32), the following assumptions can be made:

$$\boldsymbol{e}'_{k\alpha}\begin{cases} = 0 & \text{if} \boldsymbol{q}_c = 0 \\ \geqslant 0 & \text{if} \boldsymbol{q}_c \neq 0 \end{cases} \tag{60}$$

$$\boldsymbol{e}'_{m\alpha}\begin{cases} = 0 & \text{if} \dot{\boldsymbol{q}}_c = 0 \\ \geqslant 0 & \text{if} \dot{\boldsymbol{q}}_c \neq 0 \end{cases} \tag{61}$$

On the other hand, the external power input is (see Eqs. (25)–(28)):

$$\boldsymbol{y}'_s(t)\boldsymbol{u}_{sc}(t) = \dot{\boldsymbol{q}}'_c(t)(\mathbf{K_c}(t)\boldsymbol{q}_c(t) + \mathbf{C_c}(t)\dot{\boldsymbol{q}}_c(t) + \mathbf{M_c}(t)\ddot{\boldsymbol{q}}_c(t)) \tag{62}$$

Replacing Eqs. (57) and (62) into Eq. (12), the dissipated power can be solved as follows:

$$g_c(t) = \dot{\boldsymbol{q}}'_c(t)\mathbf{C_c}(t)\dot{\boldsymbol{q}}_c(t) - \boldsymbol{e}'_{k\alpha}(\boldsymbol{q}_c,\boldsymbol{\alpha}_k)\dot{\boldsymbol{\alpha}}_k(t) - \boldsymbol{e}'_{m\alpha}(\dot{\boldsymbol{q}}_c,\boldsymbol{\alpha}_m)\dot{\boldsymbol{\alpha}}_m(t), \tag{63}$$

which can be of either sign depending on the design constraints defined in Eqs. (33)–(38), and the relative-displacements and -velocities across the control devices.

Under assumptions (60)–(61), Eq. (63) is very important since it shows the three mechanisms by which energy can be *dissipated* (or *generated* 1). This is due to the existence of: (1) relative velocities across dampers with *positive* (or *negative*) damping, (2) relative displacements across springs whose stiffness is *decreasing* (or *increasing*) due to its control parameter, and (3) relative velocities across inerters whose inertance is *decreasing* (or *increasing*) due to its control parameter. The suitability of definitions stated in Eqs. (34) and (38) for rate characteristic parameters is evident.

In the particular cases of time-varying linear-springs and -inerters, i.e. $\mathbf{K_c}(t) = \mathrm{diag}(\boldsymbol{\alpha}_k(t))$ and $\mathbf{M_c}(t) = \mathrm{diag}(\boldsymbol{\alpha}_m(t))$, Eqs. (63), (58) and (59) reduce to:

$$g_c(t) = \sum_{i=1}^{p}\dot{q}_{ci}^2(t)c_{ci}(t) - \frac{1}{2}\sum_{i=1}^{p}q_{ci}^2(t)\dot{k}_{ci}(t) - \frac{1}{2}\sum_{i=1}^{p}\dot{q}_{ci}^2(t)\dot{m}_{ci}(t), \tag{64}$$

which can be used for the particular case of linear devices or for conceptual purposes.

From the definition of *passivity* based on Eq. (12), it can be stated that the semi-active controller is *passive* if $g_c(t) \geqslant 0\ \forall\ t \in [0,T]$, otherwise it is *active*. Formally:

$$g_c(t) \geqslant 0\ \forall\ t \in [0,T] \Rightarrow \langle \boldsymbol{y}_s,\boldsymbol{u}_{sc}\rangle_T = \langle \dot{\boldsymbol{q}}_s\boldsymbol{f}_{sc}\rangle_T \geqslant 0 \tag{65}$$

which can be shown by integration as in the case of the structure. Therefore, a semi-active controller can be *passive* or *active* depending on its design parameters (Eqs. (33)–(38)).

A relevant aspect of *semi-active controllers* is that their *passivity* can be guaranteed, irrespective of their non-mechanical subsystems, simply by appropriately choosing their design parameters (e.g. $k_{rimin} = k_{rimax} = m_{rimin} = m_{rimax} = 0\ \forall\ i = 1,\ldots,p$ and $c_{imin} \geqslant 0\forall\ i = 1,\ldots,p$). In such cases, i.e. when $g_c(t) \geqslant 0\ \forall\ t \in [0,T]$, a semi-active controller is defined, in this paper, as *inherently passive*.

4.3. Stability results for the semi-active control system

In this subsection, a particular case of the *passivity theorem* [29] is applied to show that when the semi-active *controller* is *inherently passive,* the *SAC system* (as a whole) presents a *BIBO stability*.

For this special case of SAC systems, three useful equations are recalled:

$$\boldsymbol{f}_s(t) = \boldsymbol{f}_{se}(t) - \boldsymbol{f}_{sc}(t), \tag{66}$$

$$\langle \boldsymbol{f}_s,\dot{\boldsymbol{q}}_s\rangle_T \geqslant_s ||\dot{\boldsymbol{q}}_s||_{2,T}^2 b \tag{67}$$

$$\langle \dot{\boldsymbol{q}}_s\boldsymbol{f}_{sc}\rangle_T \geqslant 0. \tag{68}$$

which mean: the structure and the semi-active controller constitute a negative-feedback loop, the structure is *strictly-velocity passive*, and the semi-active controller is *passive*, respectively.

The substitution of Eq. (66) into (67) leads to the following inequality:

$$\langle(\boldsymbol{f}_{se} - \boldsymbol{f}_{sc}),\dot{\boldsymbol{q}}_s\rangle_T = \langle \boldsymbol{f}_{se},\dot{\boldsymbol{q}}_s\rangle_T - \langle \boldsymbol{f}_{sc},\dot{\boldsymbol{q}}_s\rangle_T \geqslant_s ||\dot{\boldsymbol{q}}_s||_{2,T}^2 \tag{69}$$

in which Eq. (68) can be applied to obtain:

$$\langle \boldsymbol{u}_{se}\boldsymbol{y}_s\rangle_T = \langle \boldsymbol{f}_{se},\dot{\boldsymbol{q}}_s\rangle_T \geqslant_s ||\dot{\boldsymbol{q}}_s||_{2,T}^2 \geqslant 0 \tag{70}$$

Eq. (70) means that, not only the *structure*, but also the whole *SAC system* is *strictly-velocity passive* (see Eq. (14)) when the semi-active controller is *passive*.

Then, the CBS inequality can be used to transform the inner product of Eq. (70) into a product of norms as follows:

$$||\boldsymbol{f}_{se}||_{2,T}||\dot{\boldsymbol{q}}_s||_{2,T} \geqslant_s ||\dot{\boldsymbol{q}}_s||_{2,T}^2 \tag{71}$$

Again, by solving for $||\dot{\boldsymbol{q}}_s||_{2,T}$, it is concluded that:

$$||\dot{\boldsymbol{q}}_s||_{2,T} \leqslant \gamma_{sa}||\boldsymbol{f}_{se}||_{2,T}, \tag{72}$$

where $\gamma_{sa} = \varepsilon_s^{-1}$. Finally, letting $T \to \infty$, it can be stated that the SAC system is *inherently BIBO stable*, in particular it is *inherently bounded-input bounded-velocity stable* with finite gain γ_{sa} if its semi-active controller is *inherently passive* (compare to Eq. (11)). Moreover, since $\gamma_{sa} = \gamma_s = \varepsilon_s^{-1} = \lambda_{\min}^{-1}(\mathbf{C_s})$, it is shown that both, the *structure* and the whole *SAC control system*, are *bounded-input bounded-velocity stable* with the *same* finite gain.

The boundedness of displacements is shown below from Eq. (72). As shown in [26], storage functions are additive in feedback systems as that of Fig. 2. Therefore, the total energy in the whole SAC system is simply $V(t) = V_s(t) + V_c(t)$; and, due to Eq. (12), its first time derivative results in:

$$\dot{V}(t) = \dot{V}_s(t) + \dot{V}_c(t) = \boldsymbol{f}'_s(t)\dot{\boldsymbol{q}}_s(t) - g_s(t) + \dot{\boldsymbol{q}}'_s(t)\boldsymbol{f}_{sc}(t) - g_c \tag{73}$$

which, after inserting Eq. (66), and integrating it in the interval $[0,T]$, yields the total energy:

$$V(T) = \langle \dot{\boldsymbol{q}}_s\boldsymbol{f}_{se}\rangle_T - \int_0^T g_s(t) + g_c(t)dt \leqslant \langle \dot{\boldsymbol{q}}_s\boldsymbol{f}_{se}\rangle_T. \tag{74}$$

Being $V_s(t) \leqslant V(t)$, and appealing to Eq. (39) along with CBS inequality and Eq. (5), Eq. (74) leads to:

$$\frac{1}{2}\lambda_{\min}(\mathbf{K_s})\boldsymbol{q}'_s(T)\boldsymbol{q}_s(T) + \frac{1}{2}\lambda_{\min}(\mathbf{M_s})\dot{\boldsymbol{q}}'_s(T)\dot{\boldsymbol{q}}_s(T) \leqslant ||\boldsymbol{f}_{se}||_{2,T}||\dot{\boldsymbol{q}}_s||_{2,T}. \tag{75}$$

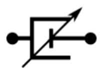
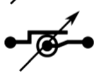
Similar to the case of the structure, Eq. (72) is substituted into Eq. (75), leading to the following two important results:

$$||\boldsymbol{q}_s||_{\infty,T} \leqslant \sqrt{\frac{2\gamma_{sa}}{\lambda_{\min}(\mathbf{K_s})}}||\boldsymbol{f}_{se}||_{2,T}, \tag{76}$$

$$||\dot{\boldsymbol{q}}_s||_{\infty,T} \leqslant \sqrt{\frac{2\gamma_{sa}}{\lambda_{\min}(\mathbf{M_s})}}||\boldsymbol{f}_{se}||_{2,T}, \tag{77}$$

When comparing Eqs. (76) and (77) to (52)–(53), it is interesting to note that $\gamma_{sa} = \gamma_s = \varepsilon_s^{-1} = \lambda_{min}^{-1}(C_s)$. In terms of vibration control engineering, this means that peak displacements and peak velocities of a structure with an *inherently passive* semi-active controller can never be greater than the largest ones of the same structure without that semi-active controller (assuming external input forces with the same RMS intensity). This remains to be true even in the worst scenarios, such as failure in non-mechanical subsystems of the semi-active controller or absence of stability analysis. It is worth remembering that, in general, this is false for AC systems and SAC systems in which $g_c(t) \ngeq 0\ \forall\ t$ (see

Table 1
Classes of semi-active controllers.

	Damping parameters		Stiffness parameters		Inertance parameters	
Class	Range	Rate	Range	Rate	Range	Rate
Variable-damping 	$c_{cimin} < 0$ $c_{cimax} > 0$	$c_{rimin} < 0$ $c_{rimax} > 0$	0	0	0	0
Variable-stiffness 	0	0	$k_{cimin} \geqslant 0$	$k_{rimax} > 0$ $k_{rimin} < 0$	0	0
Variable-inertance 	0	0	0	0	$m_{cimin} \geqslant 0$	$m_{rimax} > 0$ $m_{rimin} < 0$

$\forall\ i = 1,\ldots,p$.

for example [48]).

In terms of *BIBO stability*, Eqs. (76) and (77) imply that (by lemma 6.1.24 in [28]): $\boldsymbol{f}_{se} \in L_2 \Rightarrow \boldsymbol{q}_s, \dot{\boldsymbol{q}}_s \in L_\infty$. Summarizing, if a semi-active controller is *inherently passive*, the corresponding SAC system is *inherently stable*.

5. Classification of semi-active controllers

To characterize *SAC systems* according to their *inherent stability*, from the definition of *inherently passive,* a rational classification of *semi-active controllers* is developed below. This classification is used then to review the published literature V_c.

5.1. Classes

From Eq. (64)[3], three dissipation mechanisms are possible, therefore three classes of *semi-active controllers* are defined in Table 1: (1) "variable-damping"; (2) "variable-stiffness"; and (3) "variable-inertance".

Note that dampers with negative *range characteristic parameters* (i.e. $c_{cimin} < 0$) are considered in this work, although they are usually emulated by means not purely mechanical (e.g. electromechanically [49–51]). On the other hand, springs and inerters with negative range characteristic parameters (i.e. $k_{cimin} < 0$ and $m_{cimin} < 0$) are not considered because they violate the lower-boundedness condition of the storage function.

By examining conditions given in Table 1 under Eq. (64), it is found that, in general, none of these three classes of semi-active controller is *inherently passive*. Notably, the *inherent passivity* of "variable-damping" semi-active controllers depends only on their *range characteristic parameter* c_{cimin}. On the contrary, the *inherent passivity* of "variable-stiffness" and "variable-inertance" semi-active controllers depends also on their *rate characteristic parameters* k_{rimax} and m_{rimax}.

Two special *inherently-passive* cases are the following: a "variable-stiffness" semi-active controller with $k_{rimax} \leqslant 0$ and a "variable-inertance" semi-active controller with $m_{rimax} \leqslant 0$. However, although the corresponding SAC systems are *inherently stable*, they are impractical for most vibration control applications. On the other hand, "variable-stiffness" with $k_{rimin} < 0 < k_{rimax}$ and "variable-inertance" with $m_{rimin} < 0 < m_{rimax}$ semi-active controllers that are *inherently passive* (i.e., $g_c(t) \geqslant 0\ \forall\ t$) can be built by using resettable devices.

Based on these aspects, the following three sub-classifications are developed.

[3] Note Eq. (64) is used instead of Eq. (63), for simplicity, without loss of generality. The only consideration is that the term "variable" (stiffness or inertance) refers to variation as a consequence of control parameters, not to variation due intrinsic nonlinearity of the mechanical devices.

5.2. Sub-classes

5.2.1. Non-negative and possibly-negative variable-damping

As inferred from Eq. (64), "variable-damping" semi-active controllers can be sub-classified from the sign in their range characteristic parameter c_{cimin}, irrespective of rate characteristic parameters c_{rimin} and c_{rimax}. In this regard, Table 2 summarizes the two cases along with their passivity and stability results.

"Non-negative variable-damping" is, evidently, the oldest and most common sub-class of semi-active controller. The first example was proposed in 1974 by Karnopp et al. [7], and it can be implemented as, for instance: variable-orifice valve hydraulic-dampers [7,52]; MR dampers [33,53], ER dampers [54,55], variable friction dampers [56–58], SAFTs [11,59], magnetically controllable particle-based dampers [60], and resistively shunted electromechanical machines as synchronous motors [61], DC motors [62], piezoceramic elements [4,63,64] and voice-coil transducers [4,65].

On the other hand, semi-active controllers capable of providing negative damping can also be found in recent literature [49]. Certainly, these controllers potentially have a better performance because they can deliver forces in the four quadrants of the force-velocity diagram. However they are not *inherently passive* and, therefore, the corresponding SAC systems are not *inherently stable* (Section 4.3). A simple implementation of such devices consists in electromechanical machines shunted through possibly-negative resistance resistors; which are obviously *active* electrical devices. Such implementation allows parasitic resistances to be compensated (e.g. copper resistance in DC motors) at the expense of possible *instability*.

5.2.2. Resettable-stiffness and independently-variable-stiffness

As summarized in Table 3, "variable-stiffness" semi-active controllers can be sub-classified into: "resettable-stiffness", which are *inherently passive*; and "independently-variable-stiffness" which are not.

By inspecting Eq. (64), it can be inferred that "variable-stiffness" semi-active controllers in which k_{ci} can increase only at instants of time t such that $q_{ci}(t) = 0$ and can decrease at any instant of time t are *inherently passive* and usually denominated "resettable-stiffness" springs. A detailed *passivity* analysis of resettable-stiffness springs is addressed in Appendix A. Devices with these properties were proposed in 1993 by Kobori et al. [66], though previously suggested by Klein and Healey [67]. Since then, several implementations have been studied [19,36,68–70].

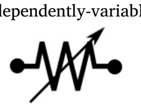
In general, these devices are modelled by a constant-stiffness spring connected in series with a resetting device functioning as an on–off switch. The spring and resetting device can be implemented by using pressurized gas and valves [19]. From an energetic point of view, a *resettable-stiffness spring* is composed of a *spring* that can suddenly be disconnected from the structure and *released* to eliminate its *potential* energy by nullifying its *displacement.*

On the other hand, "variable-stiffness" semi-active devices that are

Table 2
Sub-classes of "variable-damping" semi-active controllers.

Sub-class	Characteristic parameters	Dissipated power	Passivity of the Semi-active controller	Stability of the SAC system
Non-negative variable-damping	$c_{cimin} \geqslant 0 \; \forall \; i = 1,\dots,p$	$g_c(t) \geqslant 0 \; \forall \; t$	Inherently passive $\langle \boldsymbol{y_s},\boldsymbol{u_{sc}} \rangle_T \geqslant 0$	Inherently stable $\exists \; \gamma_d,\gamma_v > 0$: $\lVert \boldsymbol{q_s} \rVert_\infty \leqslant \gamma_d \lVert \boldsymbol{f}_{\boldsymbol{se}} \rVert_2 \lVert \dot{\boldsymbol{q}}_{\boldsymbol{s}} \rVert_\infty \leqslant \gamma_v \lVert \boldsymbol{f}_{\boldsymbol{se}} \rVert_2$
Possibly-negative variable-damping	$c_{cimin} < 0$ for some i	$g_c(t) \ngeqslant 0 \; \forall \; t$	Active $\nexists \beta \geqslant 0$: $\langle \boldsymbol{y_s},\boldsymbol{u_{sc}} \rangle_T \geqslant -\beta$	Non-inherently stable $\nexists \gamma_d,\gamma_v > 0$: $\lVert \boldsymbol{q_s} \rVert_\infty \leqslant \gamma_d \lVert \boldsymbol{f}_{\boldsymbol{se}} \rVert_2 \lVert \dot{\boldsymbol{q}}_{\boldsymbol{s}} \rVert_\infty \leqslant \gamma_v \lVert \boldsymbol{f}_{\boldsymbol{se}} \rVert_2$

Table 3
Sub-classes of "variable-stiffness" semi-active controllers.

Sub-class	Characteristic parameters	Dissipated power	Passivity of the Semi-active controller	Stability of the SAC system
Resettable-stiffness 	$k_{rimax} \begin{cases} c \leqslant 0, q_{ci} \neq 0 \\ > 0, q_{ci} = 0 \end{cases}$ $k_{rimin} < 0$ for some i	$g_c(t) \geqslant 0 \; \forall \; t$	Inherently passive $\langle \boldsymbol{y_s},\boldsymbol{u_{sc}} \rangle_T \geqslant 0$	Inherently stable $\exists \; \gamma_d,\gamma_v > 0$: $\lVert \boldsymbol{q_s} \rVert_\infty \leqslant \gamma_d \lVert \boldsymbol{f}_{\boldsymbol{se}} \rVert_2 \lVert \dot{\boldsymbol{q}}_{\boldsymbol{s}} \rVert_\infty \leqslant \gamma_v \lVert \boldsymbol{f}_{\boldsymbol{se}} \rVert_2$
Independently-variable-stiffness 	$k_{rimax} > 0$ $k_{rimin} < 0$ for some i	$g_c(t) \ngeqslant 0 \; \forall \; t$	Active $\nexists \beta \geqslant 0$: $\langle \boldsymbol{y_s},\boldsymbol{u_{sc}} \rangle_T \geqslant -\beta$	Non inherently stable $\nexists \gamma_d,\gamma_v > 0$: $\lVert \boldsymbol{q_s} \rVert_\infty \leqslant \gamma_d \lVert \boldsymbol{f}_{\boldsymbol{se}} \rVert_2 \lVert \dot{\boldsymbol{q}}_{\boldsymbol{s}} \rVert_\infty \leqslant \gamma_v \lVert \boldsymbol{f}_{\boldsymbol{se}} \rVert_2$

not *inherently passive* (i.e. $g_c(t) \ngeqslant 0 \; \forall \; t$) can also be found in the literature [71]. These devices, formerly proposed by Nagarajaiah in 2000 [72], are usually referred to as "independently-variable-stiffness" springs. They can be implemented by means of an arrangement of springs whose geometry is adjustable [72], helicoidal springs whose number of coils can be modified through a moving collar [6], transversally-forced tense cables with variable tension force [23], prestressable leaf-springs [73], variable-pressure granular structures [74], variable-length pendula [47], capacitively shunted piezoceramic elements [71,75], and other smart materials [71].

Typically, "independently-variable-stiffness" springs are used as a means to conveniently tune/detune oscillatory systems. For example, performance of TMDs can be significantly improved by pursuing resonance [35,47,76,77]; whereas performance of isolation systems can be enhanced by avoiding it [20,78]. It is worth noting that none of these investigations formally addresses the stability issue, although Eq. (64) shows that *stability* is not guaranteed and actual cases of *instability* have been shown [48]. In practice, instability is avoided by choosing a sufficiently low value for k_{rimax} which is suitable for vibration control based on tuning/detuning.

5.2.3. Resettable-inertance and independently-variable-inertance

Similarly to the case of "variable-stiffness", when the case of "variable-inertance" is considered in Eq. (64), two possibilities are devised: (1) "independently-variable-inertance", i.e. $m_{ci}(t)$ can increase or decrease at any instant of time, leading to a non-*inherently passive* semi-active controller because $g_c(t) \ngeqslant 0 \; \forall \; t$; and (2) "resettable-inertance", i.e. $m_{ci}(t)$ can increase only at instants with zero relative velocities (i.e., $\dot{q}_{ci}(t) = 0$) otherwise decrease, leading to an *inherently passive* semi-active controller ($g_c(t) \geqslant 0 \; \forall \; t$). These two cases are summarized in Table 4.

The concept of adjustable inertia was first mentioned in 2001 by Jalili et al. [79]. On the other hand, implementations of "independently-variable-inertance" inerters have been proposed very recently. Brzeski et al. patented a device in 2014 consisting of a gear rack linked to a flywheel through a continuously variable transmission [25]. In 2014, Chen et al. [24] numerically studied the benefits of variable inertance interters applied to semi-active car suspension systems. Notably, none of these works address BIBO stability in detail.

So far, there is no mention of "resettable inertance" in the current literature.

Authors of the present paper propose using the principle of duality for devising an implementation of *resettable-inertance inerters* from *resettable-stiffness springs*. Thus, a *resettable-inertance inerter* should be composed of a *flywheel-pinion-rack system* that can suddenly be disconnected from the structure and *blocked* to eliminate its *kinetic* energy by nullifying its *velocity*. A detailed *passivity* analysis of these resettable-inertance inerters is addressed in Appendix B.

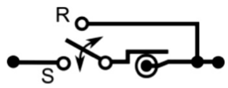
5.3. Hybrid combinations

In this subsection, two possible hybrid combinations that appear in current literature and are interesting from the perspective of *inherent stability* are assessed: (1) non-negative variable damping dampers in parallel with independently-variable inertance inerters, and (2) non-negative variable damping dampers in parallel with independently-variable stiffness springs. The aim of these combinations is to transform an *active* device into an *inherent passive* device by the addition of other device that is already *inherently passive*.

When inspecting the first and last term of Eq. (64), it is evident that negative inertance rate plays the same role as positive damping coefficient (a similar result is obtain in [44]). Moreover, it is easy to show that the hybrid combination of "non-negative variable damping" dampers in parallel with "independently-variable inertance" inerters, such that $\frac{\partial m_{ci}}{\partial \alpha_{mi}} m_{rimax} \leqslant 2c_{cimin}$, is *inherently passive* since $g_c(t) \geqslant 0$. Interestingly, this hybrid combination is considered in [24] when an inerter is added to a car suspension system; although stability issues are not mentioned.

Finally, the combination of "non-negative variable damping dampers" in parallel with "independently-variable stiffness springs" does not lead to the rigorous form of *inherent stability* (Eq. (10)), since one device is velocity-driven while the other is displacement-driven. However, under the approximate assumption that $\lVert q_{si} \rVert_2 \approx \frac{1}{\omega} \lVert \dot{q}_{si} \rVert_2$ (where ω is the fundamental natural frequency of the system), a design that meets $\frac{\partial k_{ci}}{\partial \alpha_{ki}} k_{rimax} \leqslant 2\omega c_{cimin}$ allows dissipating, in the damper, approximately the same energy that is supplied by the spring and therefore could be considered as inherently stable. A formal study of this case should rely on the dissipativity approach rather than in the passivity formalism, and therefore is out of the scope of the present paper.

Table 4
Sub-classes of "variable-inertance" semi-active controllers.

Sub-class	Characteristic parameters	Dissipated power	Passivity of the Semi-active controller	Stability of theSAC system
Resettable-inertance 	$m_{r i\max} \begin{cases} c \leqslant 0, \dot{q}_{ci} \neq 0 \\ >0, \dot{q}_{ci} = 0 \end{cases}$ $m_{r i\min} < 0$ for some i	$g_c(t) \geqslant 0\ \forall\ t$	Inherently passive $\langle \boldsymbol{y_s}, \boldsymbol{u_{sc}} \rangle_T \geqslant 0$	Inherently stable $\exists\ \gamma_d, \gamma_v > 0$: $\|\|\boldsymbol{q_s}\|\|_\infty \leqslant \gamma_d \|\|\boldsymbol{f}_{se}\|\|_2 \|\|\dot{\boldsymbol{q}}_s\|\|_\infty \leqslant \gamma_v \|\|\boldsymbol{f}_{se}\|\|_2$
Independently-variable-inertance 	$m_{r i\max} > 0$ $m_{r i\min} < 0$ for some i	$g_c(t) \ngeqslant 0\ \forall\ t$	Active $\nexists\, \beta \geqslant 0$: $\langle \boldsymbol{y_s}, \boldsymbol{u_{sc}} \rangle_T \geqslant -\beta$	Non inherently stable $\nexists\ \gamma_d, \gamma_v > 0$: $\|\|\boldsymbol{q_s}\|\|_\infty \leqslant \gamma_d \|\|\boldsymbol{f}_{se}\|\|_2 \|\|\dot{\boldsymbol{q}}_s\|\|_\infty \leqslant \gamma_v \|\|\boldsymbol{f}_{se}\|\|_2$

6. Use of the presented framework in a practical example

In this section, a novel control device which enables changes of inertance, as an alternative to other recently studied devices [25,80], is first depicted and modelled. Then, the framework developed in the present paper is used to classify it (TableS 1 and 4) and assess whether a semi-active control system based on it is inherently stable. Finally, main results are demonstrated through numerical simulations of a frame structure provided with the control device. Instability caused by variable stiffness is not illustrated in the present paper since examples can be found in the literature [48].

6.1. Model of the novel semi-active control device

Fig. 3 shows the control device, which consists of: a pinion-rack mechanism (with constant radius r) that links the rotational and translational degrees of freedom $\theta(t)$ and $q_c(t)$, two arms with constant masses $\frac{M}{2}$ at their ends, and actuators exerting variable forces $f_a(t)$ capable of changing the length $r_M(t)$ of the arms. For simplicity of modelling, one of the device ends is considered grounded.

Through inspection of Fig. 3, it can be seen that the inertance experienced when varying $q_c(t)$ is $m_c(t) = \frac{M r_M(t)^2}{r^2}$, where $r_M(t)$ corresponds to the control parameter $\alpha_m(t)$ (see Eq. (31)) of the semi-active control device. However, detailed modelling shows that $f_c(t) \neq m_c(t)\ddot{q}_c(t)$ when $\alpha_m(t)$ changes in real-time, as expected in actual implementations of semi-active control systems.

First, note that the Lagrangian of the system is simply:

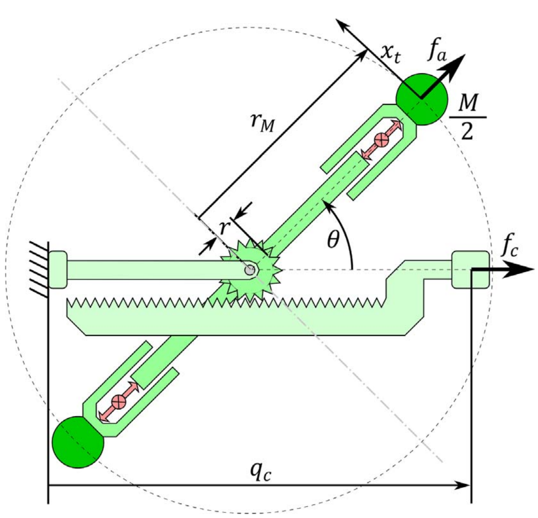


Fig. 3. Inerter which enables changes of inertance.

$$L = T(\dot{r}_M(t), \dot{x}_t(t)) = \frac{M}{2}(\dot{r}_M(t)^2 + \dot{x}_t(t)^2) = \frac{M}{2}\left(\dot{r}_M(t)^2 + r_M(t)^2 \frac{\dot{q}_c(t)^2}{r^2}\right) \tag{78}$$

where $q_c(t)$ and $r_M(t)$ are chosen generalized coordinates, and $f_c(t)$ and $f_a(t)$ are their corresponding generalized forces. Hence, by using the Lagrange equations of the second kind [81], for the first generalized coordinate, it is found that:

$$f_c(t) = \frac{M r_M(t)^2}{r^2}\ddot{q}_c(t) + \frac{2M r_M(t)\dot{r}_M(t)}{r^2}\dot{q}_c \tag{79}$$

6.2. Classification of the control device and inherent stability assessment

To classify the control device of Fig. 3, in the sense of the inherent-stability criterion developed in Sections 3–5, it is noted in Eq. (79) that:

$$m_c(t) = \frac{M r_M(t)^2}{r^2}, c_c(t) = \frac{2M r_M(t)\dot{r}_M(t)}{r^2}. \tag{80}$$

Eqs. (79) and (80) evidence that the control device must be classified as an hybrid combination of an *independently-variable-inertance inerter*, since it is not necessary that $\dot{q}_c(t) = 0$ to increase $r_M(t)$ (see Section 5.2.3), in parallel with a *possibly-negative variable-damping damper*, since $\dot{r}_M(t)$ can be negative (see Section 5.2.1).

This classification is enough to *question* the inherent passivity of the device. However, as demonstrated in Section 5.3, hybrid combinations of individually non-inherently-passive devices can be inherently passive. In this particular case, $c_c(t)$ and $m_c(t)$ are related by $r_M(t)$, so mechanisms that supply and dissipate energy could exactly oppose.

This question can be easily answered by substituting $m_c(t)$ and $c_c(t)$ into Eq. (63) (or into Eq. (64) since $m_c(t)$ is independent of $\dot{q}_c(t)$). Hence, the following dissipation function is found:

$$g_c(t) = \dot{q}_c^2(t)\left(\frac{2M r_M(t)\dot{r}_M(t)}{r^2} - \frac{1}{2}\frac{M 2 r_M(t)\dot{r}_M(t)}{r^2}\right) = \dot{q}_c^2(t)\dot{m}_c(t), \tag{81}$$

from which it can be *concluded* that a semi-active control system using this device is not inherently stable since $g_c(t) \ngeqslant 0\ \forall\ t$, i.e. the device is not inherently passive. Evidently, the net added energy is due to the work of the actuator force; which opposes to the centrifugal force.

It is worthy of note that, in this particular *hybrid combination*, the critical design parameter for stability is $m_{r\min}$; unlike the case of an *independently-variable inertance inerter* alone, in which it is $m_{r\max}$. This interesting aspect is demonstrated through numerical simulations in Section 6.3.3.

6.3. Numerical simulations

In order to show the capability of the control device to reduce vibrations, in contrast to its risk of instability and a possible countermeasure, numerical simulations were performed using a 5th-order Runge-Kutta variable-step explicit solver with a 4th-order estimate of the error. Relative tolerance was set to 0.1% for states whereas energybalance was checked to have less than 4% error. Thus, it is assured that found instability is not numerical.

6.3.1. Control law

Variable inertance inerters, as variable stiffness springs, are often used to tune dynamic vibration absorbers. However, they can be used as primary dissipating devices too and, moreover, their control law can be developed from the dissipation function obtained in Section 4.2 of this paper.

By inspection of Eq. (81), it can be seen that dissipation function $g_c(t)$ is positive when $m_c(t)$ increases. On the other hand, when velocity $\dot{q}_c(t)$ is small, $m_c(t)$ can be decreased without large energy supply. On this basis, the following control law is proposed: *increase* $m_c(t)$ *when* $q_c(t) = 0$; *decrease* $m_c(t)$ *when* $\dot{q}_c(t) = 0$; *keep* $m_c(t)$ *constant otherwise*. Some experiences have shown that this kind of control laws have much less chattering than those such as ground-hook or sky-hook, especially when control forces are proportional to relative accelerations.

6.3.2. Structure model and external excitation

The structure to be controlled and the placement of the semi-active control device were taken from the work of Jansen and Dyke [53]. It is a 6-story steel plane frame structure with a fundamental period of 0.72 s and 0.5% of critical damping. The magnetorheological dampers of that work were replaced with the control device of Fig. 3, installed on a diagonal brace in the first story. The maximal inertance m_{cmax} was set to 5 times the floor mass, which can be achieved with a rotating mass M (at the ground) equal to the 5% of the floor mass and $\frac{r_M}{r} = 10$. The minimal inertance m_{cmin} was set to a fourth of m_{cmax}, i.e. $\frac{r_M}{r} = 5$. The rate parameters m_{rmax} and m_{rmin} were set to allow a complete sweep of inertance range in 0.025 s. The excitation was defined as a resonant harmonic base acceleration with amplitude of 0.01 g and 5 s long.

6.3.3. Definition scenarios and discussion of results

Four scenarios, consisting of three cases each, were simulated. The three cases are: (1) a non-controlled case (baseline); (2) the case of using an ideal controller (without delay); and (3) an accidental delay of 0.2 s in the controller (to assess the risk of instability). These are referred to as 'without controller', 'in-phase controller', and 'delayed controller', respectively, in Fig. 4 which shows the top displacement of the frame structure.

In the first scenario (Fig. 4a), it can be seen that, with the parameters defined above, the in-phase controller improves the performance of the structure without controller, but the accidental delay leads to instability as warned in Section 6.2. In the second scenario (Fig. 4b), the critical parameter $m_{r\min}$ has been reduced to a half of the value used in the first scenario. This countermeasure reduces the instability of the delayed controller, without losing much performance in the case of the in-phase controller, though the response is still worse than that of the structure without controller. In the third scenario (Fig. 4c), the parameter $m_{r\max}$ has been reduced to a half of the value used in the first scenario. These results clearly show that, for the proposed device, $m_{r\max}$ is not a critical parameter for stability; which is in agreement with the inference made at the end of Section 6.2.

Finally, in the fourth scenario (Fig. 4d), $m_{c\min}$ has been set equal to $m_{c\max}$ to show that a *passive* inerter, with the same mechanical characteristics, would only produce a small detuning of the structure without significantly reducing its response.

7. Conclusions

This paper formally addresses the issue of the assumed *inherent stability of semi-active control systems*. Thus, this investigation represents a guiding framework for stability assessment and for the classification of existing and novel semi-active control systems. In addition it acts as a reference for the practising engineer that is both: rigorous and easy to use. The application of the developed framework is demonstrated with a practical example where it is useful in the task of proposing control-laws and instability-countermeasures.

A direct relationship between the *inherent passivity* of the semi-active controller and the *inherent stability* of the semi-active control system is found by means of the passivity theorem. Both properties, *passivity* and *stability*, are considered *inherent* in the sense that they only rely on the mechanical subsystems.

In particular, it is misguided to assume that any semi-active control system is inherently stable. However, when these are classified into (1) variable damping, (2) variable stiffness and (3) variable inertance; and,

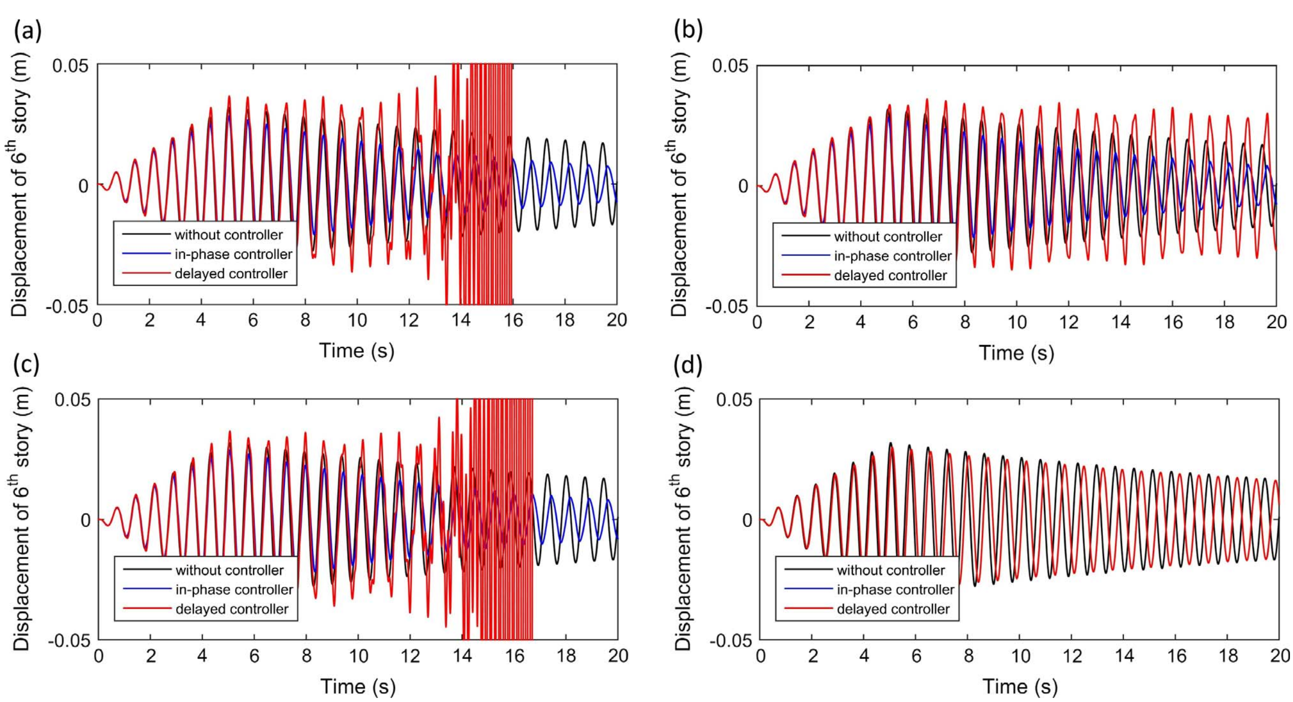


Fig. 4. Simulations results: (a) initial parameters; (b) reducing m_{rmin} to a fifth; (c) reducing m_{rmax} to a fifth; (d) passive case as reference ($m_{cmin} = m_{cmax}$).

furthermore,
variable damping is sub-classified into

(1.1) non-negative variable damping and
(1.1) possibly-negative variable damping;

variable stiffness into

(2.1) independently-variable-stiffness and
(2.2) resettable-stiffness;

and variable inertance into

(3.1) independently-variable-inertance and
(3.2) resettable-inertance:

it is found that sub-classes (1.1), (2.2) and (3.2) are the only ones *inherently stable*.

Interestingly, sub-class (3.2) is absent in the current literature. In addition, the hybrid combination of (1.1) and (3.1) can be designed to be *inherently stable*.

The generality of the developed framework was demonstrated through the systematic classification of a large number of literature examples.

The stability results found in the present work are of practical significance since they not only guarantee output boundedness but also show that the output bounds are the same with and without the controller.

The authors hope that the present paper serves as a rigorous and clear link between two disciplines: Vibration-Control Design and Nonlinear-Systems Analysis.

Acknowledgements

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Appendix A. Resettable stiffness springs with offset

This appendix addresses the general case of resettable-stiffness springs, shown in Fig. 5, whose stiffness $k_{ci}(t)$ can suddenly decrease from a constant value k_{0i} to 0 at any instant of time t; but can increase only at instants of time t such that $q_{ci}(t) = 0$, although the displacement across the corresponding DOFs of the structure is not necessarily 0.

The matrix containing the direction cosines that links the control devices to the structure is defined as $B_{sc} = [b_{sc1}...b_{sci}...b_{scp}]$; so the displacement across the i-th resettable-stiffness spring is $b'_{sci}q_s(t)$, as labelled in Fig. 5. When the resetting device of Fig. 5 turns to position R (Reset), both stiffness $k_{ci}(t)$ and deformation $q_{ci}(t)$ drop suddenly to zero. On the other hand, assuming $q_{ci}(t) = 0$, when the resetting device turns to position S (Set), stiffness $k_{ci}(t)$ increases from zero to k_{0i}; however, a constant displacement $\Delta_{ci}(t)$ (offset) can appear between the spring and the structure, i.e.:

$$B'_{sc}q_s(t) = q_c(t) + \Delta_c(t), \tag{A. 1}$$

where $\Delta_c(t) = [\Delta_{c1}(t)...\Delta_{ci}(t)...\Delta_{cp}(t)]'$. Note that $\Delta_{ci}(t)$ is constant when the corresponding stiffness is set (position S in Fig. 5) and changes when stiffness is reset (position R in Fig. 5). Nevertheless, in general, velocities follow the form:

$$B'_{sc}\dot{q}_s(t) = \dot{q}_c(t) + \dot{\Delta}_c(t). \tag{A. 2}$$

For its part, the forces that the resettable-stiffness springs exert on the structure are:

$$f_{sc}(t) = B_{sc}K_c(t)q_c(t). \tag{A. 3}$$

Considering the linear case, without loss of generality, the stored energy in the semi-active controller is computed as:

$$V_c(t) = \frac{1}{2}q_c'(t)K_c(t)q_c(t) \tag{A. 4}$$

so $\dot{V}_c(t) = \frac{1}{2}q_c'(t)\dot{K}_c(t)q_c(t) + \dot{q}_c'(t)K_c(t)q_c(t)$, by the product rule. Besides, Eq. (12) allows stating that the dissipated power is $g_c(t) = \dot{q}_s'(t)f_{sc} - \dot{V}_c(t)$. Then, combining these two expressions with Eq. (A. 3) yields the following expression of the dissipated power:

$$g_c(t) = \dot{q}_s^T(t)B_{sc}K_c(t)q_c(t) - \frac{1}{2}q_c'(t)\dot{K}_c(t)q_c(t) - \dot{q}_c'(t)K_c(t)q_c(t) \tag{A. 5}$$

in which Eq. (A. 2) is inserted transposed to obtain:

$$g_c(t) = \dot{\Delta}_c'(t)K_c(t)q_c(t) - \frac{1}{2}q_c'(t)\dot{K}_c(t)q_c(t). \tag{A. 6}$$

Noteworthy is that the first term of Eq. (A. 6) is always equal to zero since the offset $\Delta_{ci}(t)$ only changes when $q_{ci}(t) = 0$ (position R in Fig. 5). Besides, $\dot{k}_{ci}(t) > 0$ only when $q_{ci}(t) = 0$. Therefore $g_c(t) \geqslant 0 \ \forall \ t$, i.e. resettable-stiffness semi-active controllers are *inherently passive*.

From a thermodynamically point of view, the action of the resetting device cannot be instantaneous. Instead, it occurs in a fraction of time in which, it can be shown, the combination of signs of factors in the first term of Eq. (A. 6) gives always a positive product. Indeed the actual dissipation occurs in such a fraction of time. Nevertheless, energy can be considered to “instantaneously disappear” in the resetting action for the purposes of the present discussion.

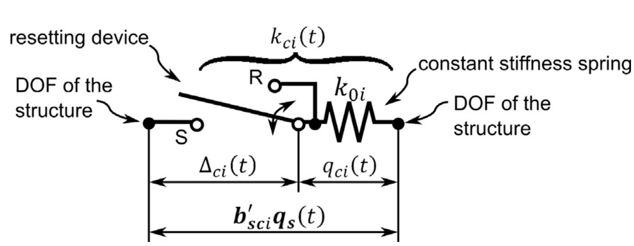

Fig. 5. Resettable-stiffness spring with offset.

Appendix B. Resettable-inertance inerters with offset

This appendix addresses the general case of resettable-inertance inerters, as that shown in Fig. 6, whose inertance $m_{ci}(t)$ can suddenly drop from a constant value m_{0i} to 0 at any instant of time t; but can increase only at instants of time t such that $\dot{q}_{ci}(t) = 0$, although the velocity across the corresponding DOFs of the structure is not necessarily 0.

When the resetting device of Fig. 6 turns to position R (Reset), both inertance $m_{ci}(t)$ and velocity $\dot{q}_{ci}(t)$ drop suddenly to zero. On the other hand, assuming $\dot{q}_{ci}(t) = 0$, when the resetting device turns to position S (Set), inertance $m_{ci}(t)$ increases up to m_{0i}. However, a constant displacement $\Delta_{ci}(t)$ (offset) can appear between the inerter and the structure; i.e. Eqs. (A. 1) and (A. 2) are also valid for this case. Note that $\dot{\Delta}_{ci}(t) = 0$ when the corresponding inertance is set (position S in Fig. 6), while $\dot{q}_{ci}(t) = 0$ when inertance is reset (position R in Fig. 6).

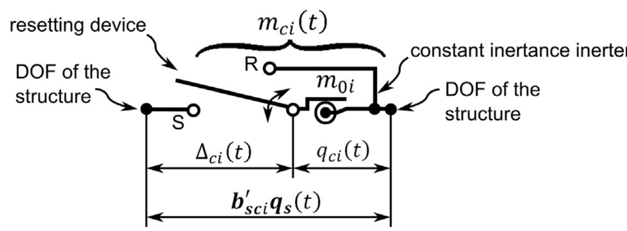

Fig. 6. Resettable-inertance inerter with offset.

The forces that the resettable-inertance inerters exert on the structure are:

$$f_{sc}(t) = B_{sc}M_c(t)\ddot{q}_c \tag{B. 1}$$

Considering the linear case, without loss of generality, the stored energy in the controller can be computed as:

$$V_c(t) = \frac{1}{2}\dot{q}_c'(t)M_c(t)\dot{q}_c(t) \tag{B. 2}$$

so $\dot{V}_c(t) = \frac{1}{2}\dot{q}_c'(t)\dot{M}_c(t)\dot{q}_c(t) + \dot{q}_c'(t)M_c(t)\ddot{q}_c(t)$, by the product rule. Besides, Eq. (12) allows stating that the dissipated power is $g_c(t) = \dot{q}_s'(t)f_{sc} - \dot{V}_c(t)$. Then, combining these two expressions with Eq. (B. 1) yields the following expression for the dissipated power:

$$g_c(t) = \dot{q}_s'(t)B_{sc}M_c(t)\ddot{q}_c(t) - \frac{1}{2}\dot{q}_c'(t)\dot{M}_c(t)\dot{q}_c(t) - \dot{q}_c'(t)M_c(t)\ddot{q}_c \tag{B. 3}$$

in which Eq. (A. 2) is substituted transposed, thus obtaining:

$$g_c(t) = \dot{\Delta}_c'(t)M_c(t)\ddot{q}_c(t) - \frac{1}{2}\dot{q}_c'(t)\dot{M}_c(t)\dot{q}_c \tag{B. 4}$$

Noteworthy is that the first term of Eq. (B. 4) is always equal to zero since the offset $\Delta_{ci}(t)$ only changes when $\dot{q}_{ci}(t) = 0$ (position R in Fig. 6), which implies that $\ddot{q}_{ci}(t) = 0$. Besides, $\dot{m}_{ci}(t) > 0$ only when $\dot{q}_{ci}(t) = 0$. Therefore $g_c(t) \geqslant 0\ \forall\ t$ i.e. resettable-inertance semi-active controllers are *inherently passive*. Similar thermodynamic considerations to those of the case of resettable-stiffness also apply to this case.

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