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The difference between clique graphs and iterated clique graphs

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Abstract

Let $\mathcal{G}$ be the class of all graphs and K the clique operator. The validity of the equality $K(\mathcal{G}) = K^2(\mathcal{G})$ has been an open question for several years. A graph in $K(\mathcal{G})$ but not in $K^2(\mathcal{G})$ is exhibited here.

Keywords: Graph theory, clique operator, clique graph, octahedral graph.

1 Introduction

Dealing with clique graphs is not an easy task and most of the problems about them prove complicated. For example, determining whether a graph is a clique graph is a NP-complete problem [1].

Working on iterated clique graphs is even more difficult and not much is known about techniques to determine whether a graph is an iterated clique graph or not.

Let $\mathcal{G}$ be the class of all graphs, K the clique operator and K^2 the composition of K with itself. In view of the previous paragraph, it is not surprising to discover that it was unknown whether $K(\mathcal{G}) = K^2(\mathcal{G})$. The main goal of

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this paper is to show that the clique graph of the octahedral graph, O_4, is in $K(\mathcal{G})$ but not in $K^2(\mathcal{G})$, thus establishing the falseness of the equality.

For that purpose, after some definitions and properties are given in Section 2, the set $K^{-1}(O_4)$ is characterized in Section 3 helped by the fact that the octahedron is an induced subgraph of any graph in $K^{-1}(O_4)$. A demonstration of the equality $K^{-1}(O_4) \cap K(\mathcal{G}) = \emptyset$, using the terminology of Section 4, follows in Section 5, which concludes the proof.

2 Definitions, basics and goals

For a graph G, $V(G)$ is the set of its vertices and $E(G)$ the set of edges. The subgraph *induced* by $A \subseteq V(G)$, $G[A]$, has A as vertex set and $v, w \in A$ are adjacent in $G[A]$ if and only if they are adjacent in G. The *neighborhood* of $v \in V(G)$, $N[v]$, is the set composed of v and its adjacent vertices. If w is such that $N[v] = N[w]$, we say that v is a *twin* of w, symbolized by $v \sim w$.

Let $\mathcal{F}$ be a family of nonempty sets. $\mathcal{F}$ is *Helly* if the intersection of all the members of any subfamily of pairwise intersecting sets is not empty. The *intersection graph* of $\mathcal{F}$, $L(\mathcal{F})$, has the members of $\mathcal{F}$ as vertices, two of them being adjacent if and only if they are not disjoint.

Let A be a set. $A \ll \mathcal{F}$ means that there exists $F \in \mathcal{F}$ such that $A \subseteq F$. If $A = \{v_1 ... v_n\}$, the notation $v_1 ... v_n \ll \mathcal{F}$ will be used too.

A *complete* of G is a set of pairwise adjacent vertices of G. A *clique* is a maximal complete and the family of all the cliques of G will be denoted by $\mathcal{C}(G)$. The *clique graph* of G is defined as the intersection graph of $\mathcal{C}(G)$. The function $K : \mathcal{G} \to \mathcal{G}$, where $\mathcal{G}$ denotes the class of all the graphs, assigning to each graph its clique graph is called the *clique operator*. The most classical characterization of clique graphs is due to Roberts and Spencer:

Theorem 2.1 *[3] Let G be a graph. Then G is a clique graph if and only if there exists a Helly family $\mathcal{F}$ of completes of G that covers all the edges of G, i.e., for all $vw \in E(G)$, $vw \ll \mathcal{F}$.*

Define the *two section* of a family $\mathcal{F}$, $S(\mathcal{F})$, as a graph whose vertices are the elements of $\mathcal{F}$, two of them being adjacent if and only if there exists a member of $\mathcal{F}$ to which both belong. Then we can say that G is a clique graph if and only if there exists a Helly family $\mathcal{F}$ such that $S(\mathcal{F}) = G$.

The expression $K^{-1}(G)$ will be used instead of $K^{-1}(\{G\})$, that is, the set of all the graphs that have G as a clique graph. Call $\mathcal{F}$ a *separating* family if, for any ordered pair (v, w) of elements, there exists $F \in \mathcal{F}$ such that $v \in F$ and $w \notin F$. The following characterization of $K^{-1}(G)$ can be given:

Theorem 2.2 *[2] $K^{-1}(G)$ is composed of all the graphs of the form $L(\mathcal{F})$, being $\mathcal{F}$ a Helly and separating family such that $S(\mathcal{F}) = G$.*

K^n will indicate the composition of K with itself n times, with K^0 equal to the identity on $\mathcal{G}$. For $i \geq 0$, the *i-th iterated clique graph* of G is defined as $K^i(G)$. The goal of this paper is to determine whether the equality $K(\mathcal{G}) = K^2(\mathcal{G})$ is true or not. If it is true, then it follows that $K^m(\mathcal{G}) = K^n(\mathcal{G})$ for all $m, n \geq 1$. However, it has long been suspected to be false.

Define, for $n \geq 1$, the *n-dimensional octahedron* O_n as a graph such that $V(O_n) = \{1, 2, ..., 2n\}$ and $E(O_n) = \{ij : \; i \neq j \; \wedge \; |i - j| \neq n\}$. If $v \in V(O_n)$, the definition implies that $N[v]$ fails to contain only one vertex of the graph. We name it the *opposite* of v, denoted by v'. As a consequence, it can be inferred that O_n has a total of 2^n cliques, each containing n vertices.

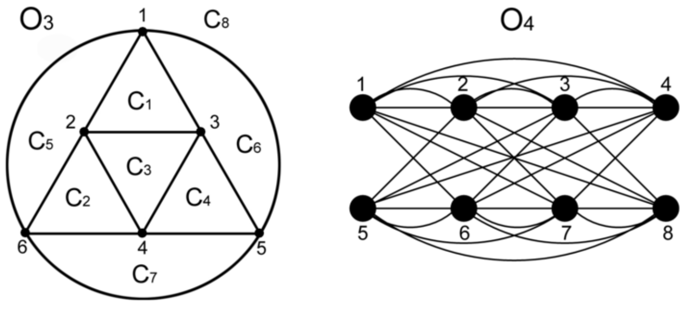

Fig. 1. O_3 and its clique graph O_4. The cliques of O_3 are also labeled.

It is straightforward that $K(O_3) = O_4$. Thus, $O_4 \in K(\mathcal{G})$. The inequality $K(\mathcal{G}) \neq K^2(\mathcal{G})$ will be proved by showing that $O_4 \notin K^2(\mathcal{G})$. We can infer from the definitions and basic set theory that a graph G is in $K^2(\mathcal{G})$ if and only if $K^{-1}(G) \cap K(\mathcal{G}) \neq \emptyset$. Consequently, the following theorem will suffice:

Theorem 2.3 $K^{-1}(O_4) \cap K(\mathcal{G}) = \emptyset$.

3 Finding $K^{-1}(O_4)$

In order to describe $K^{-1}(O_4)$, it will be necessary to prove that any graph in it has O_3 as an induced subgraph. In view of Theorem 2.2, it will be sufficient to verify that, for every Helly family $\mathcal{F}$ such that $S(\mathcal{F}) = O_4$, O_3 is an induced subgraph of $L(\mathcal{F})$. We will consider first those $\mathcal{F}$ that are minimal in the sense that no proper subfamily of $\mathcal{F}$ has O_4 as its two section. From Lemma 3.1 to Lemma 3.3, the families considered will be assumed to satisfy this condition.

Lemma 3.1 *$|F| = 4$ for all $F \in \mathcal{F}$.*

Lemma 3.2 *Let F and F' be two members of $\mathcal{F}$. Then $|F \cap F'|$ is even.*

Lemma 3.3 *Let $\{a, b, c, d\}$ be a member of $\mathcal{F}$. Then $\{a', b', c', d'\} \in \mathcal{F}$.*

Proof. Suppose on the contrary that $\{a', b', c', d'\} \notin \mathcal{F}$. Since $a'b' \ll \mathcal{F}$, $a'c' \ll \mathcal{F}$ and $a'd' \ll \mathcal{F}$, the previous lemma implies that $\{a', b', c, d\}, \{a', c', b, d\}, \{a', d', b, c\} \in \mathcal{F}$. These three sets and $\{a, b, c, d\}$ form an intersecting subfamily of $\mathcal{F}$ but they have no common element, contradicting that $\mathcal{F}$ is Helly. Therefore, $\{a', b', c', d'\} \in \mathcal{F}$. □

Theorem 3.4 *Let $G \in K^{-1}(O_4)$. Then O_3 is an induced subgraph of G.*

Sketch of proof. Let $\mathcal{F}$ be a Helly family such that $L(\mathcal{F}) = G$ and $S(\mathcal{F}) = O_4$. Among all the subfamilies of $\mathcal{F}$ with two section equal to O_4, take $\mathcal{F}'$ minimal with respect to inclusion. Use the previous lemmas to find three members F_1, F_2, F_3 of $\mathcal{F}'$ that are pairwise intersecting. By Lemma 3.3, the sets F_1', F_2', F_3' composed of the opposites of the vertices in F_1, F_2, F_3, respectively, are also members of $\mathcal{F}'$. Then F_1, F_2, F_3, F_1', F_2' and F_3' induce O_3 in $L(\mathcal{F})$. Therefore, O_3 is an induced subgraph of G. □

Let G be any graph such that $K(G) = O_4$. If $G \neq O_3$, then G has an induced subgraph with fewer vertices, namely O_3, whose clique graph also equals O_4. This fact gives several special characteristics to the graphs in $K^{-1}(O_4)$:

Proposition 3.5 *Let G be a graph with $K(G) = O_4$, $V' \subseteq V(G)$ such that $G[V'] = O_3$ and v, w two vertices of G. Then:*

(a) *$N[v] \cap V' \neq \emptyset$.*

(b) *If $N[v] \cap V' \subseteq N[w] \cap V'$ then $N[v] \subseteq N[w]$.*

(c) *If $N[v] \cap V' = N[w] \cap V'$ then $v \sim w$.*

(d) *$N[v] \cap V' \neq V'$.*

4 Classifying sets of cliques of O_3

The information obtained from Section 3 leaves us in a good position to give a proof that $O_4 \notin K^2(G)$. Here we define structures of O_3 that are necessary for that purpose. A *castle* is defined as a subset A of $\mathcal{C}(O_3)$ such that $|A| = 3$ and $|C \cap C'| = 1$ for any pair C, C' of distinct elements of A. Castles can also be characterized as in the next proposition:

Proposition 4.1 *Let A be a castle of O_3. Then there exists $C' \in \mathcal{C}(O_3)$ such that $A = \{C \in \mathcal{C}(O_3) : |C \cap C'| = 2\}$.*

A will be said to be *castled* if it contains a castle. We have the following result regarding castled sets:

Proposition 4.2 *Let $A \subseteq \mathcal{C}(O_3)$ such that $|A| \geq 5$. Then A is castled.*

Proposition 4.2 is equivalent to stating that any nonempty non-castled subset of $\mathcal{C}(O_3)$ has at most four elements. The following classification for non-castled sets is proposed:

A is a *triangle* if $|A| = 1$. In case that $|A| = 2$, $A = \{C, C'\}$, A is a *rhombus* if $|C \cap C'| = 2$, is a *bow* if $|C \cap C'| = 1$ or is an *opposite pair* if $C \cap C' = \emptyset$. In case that $|A| = 3$, A is an *umbrella* if it contains an opposite pair or is a *fan* if the intersection of all its elements is nonempty. In case that $|A| = 4$, A is a *round* if the intersection of all its elements is nonempty; A is a *worm* if its elements can be listed in such a way that two of them are consecutive if and only if they share two vertices; and A is a *rhombic circle* if it contains two distinct opposite pairs.

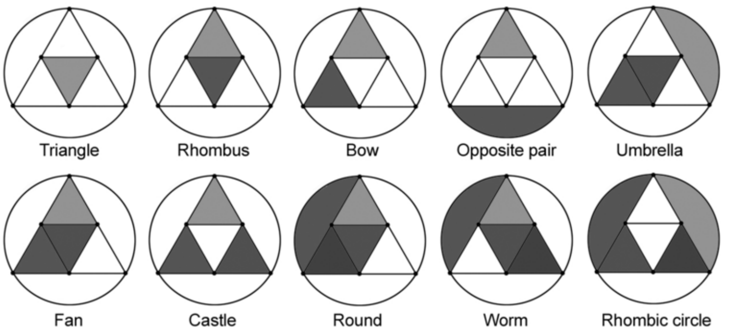

Fig. 2. Graphical representation of the sets defined in this section.

5 Extract of the proof of Theorem 2.3

Suppose that there exists a graph G in $K(\mathcal{G}) \cap K^{-1}(O_4)$. Let $\mathcal{F}$ be a Helly family such that $S(\mathcal{F}) = G$. For any set $V' = \{a, a', b, b', c, c'\}$ such that $G[V'] = O_3$ define $C(\mathcal{F}, V') := \{C \in \mathcal{C}(G[V']) : C \ll \mathcal{F}\}$. We also choose, for every clique $\{x, y, z\}$ of $G[V']$, a vertex v_{xyz} which is in each member of $\mathcal{F}$ that contains $\{x, y\}$, $\{x, z\}$ or $\{y, z\}$. Note that two of these vertices are adjacent (or equal) if their subscripts share two vertices. Now we study what kind of set $C(\mathcal{F}, V')$ is.

$C(\mathcal{F}, V')$ is not castled:

Suppose that C_1, C_2, C_3 are elements of $C(\mathcal{F}, V')$ forming a castle, and let F_1, F_2, F_3 be members of $\mathcal{F}$ such that $C_i \subseteq F_i$, $1 \leq i \leq 3$. Then F_1, F_2, F_3 are pairwise intersecting and we can take $v \in V(G)$ such that $v \in F_1 \cap F_2 \cap F_3$. We can deduce from Proposition 4.1 that $V' \subseteq N[v]$, contradicting part (d) of Proposition 3.5. Therefore, $C(\mathcal{F}, V')$ is not castled.

$C(\mathcal{F}, V')$ does not contain an umbrella:

Suppose that $C(\mathcal{F}, V')$ contains an umbrella with elements $\{a, b, c\}, \{a', b', c\}$, $\{a', b', c'\}$. Then $\{a, a', b, b', c\} \subseteq N[v_{ab'c}]$ and we deduce from part (c) of Proposition 3.5 that $v_{ab'c} \sim c$. Furthermore, $\{a, a', b', c'\} \subseteq N[v_{ab'c'}]$ and $v_{ab'c'} \in N[v_{ab'c}]$. Thus $v_{ab'c'} \sim b'$. Similarly, $v_{a'bc} \sim c$, $\{a', b, b', c'\} \subseteq N[v_{a'bc'}]$, $v_{a'bc} \in N[v_{a'bc'}]$ and $v_{a'bc'} \sim a'$. Note that $\{a, b, v_{ab'c'}, c, c'\} \subseteq N[v_{abc'}]$, so $v_{abc'} \sim a$. This contradicts that $v_{a'bc'} \in N[v_{abc'}]$. Therefore, $C(\mathcal{F}, V')$ does not contain an umbrella.

$C(\mathcal{F}, V')$ is not a round:

Suppose that $C(\mathcal{F}, V')$ is a round and that $\{a, b, c\}, \{a, b, c'\}$, $\{a, b', c\}$, $\{a, b', c'\}$ are its elements. Then $\{a, a', b', c\} \subseteq N[v_{a'b'c}]$ and $\{a, a', b, c\} \subseteq N[v_{a'bc}]$. Let $C \in \mathcal{C}(G)$ such that $\{v_{a'bc}, v_{a'b'c}\} \subseteq C$. Since $b \in C$ or $b' \in C$, $v_{a'b'c} \sim c$ or $v_{a'bc} \sim c$. If $v_{a'b'c} \sim c$, let $V'' = \{a, a', b, b', v_{a'b'c}, c'\}$. Then $G[V''] = O_3$ and use the definition of $v_{a'b'c}$ to conclude that $\{a, b, c'\}$, $\{a, b', c'\}$ and $\{a', b', v_{a'b'c}\}$ are elements of $C(\mathcal{F}, V'')$. Thus, $C(\mathcal{F}, V'')$ contains an umbrella, which is a contradiction. If $v_{a'bc} \sim c$ we proceed similarly, also getting a contradiction. Therefore, $C(\mathcal{F}, V')$ is not a round.

The reasonings displayed so far have to continue being applied to prove that there is not a set that $C(\mathcal{F}, V')$ can equal. Each possibility can be discarded by following a decreasing order of $|C(\mathcal{F}, V')|$. This is the contradiction that allows to conclude that $K(\mathcal{G}) \cap K^{-1}(O_4) = \emptyset$.

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