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Letter to the Editor

Some observations in the dynamics of beams with intermediate supports

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1. Introduction

Several investigators have studied the influence of rotational and/or translational restraints at the ends of vibrating beams [1–17]. Kameswara Rao and Mirza [18] have derived exact frequency and normal mode shape expressions for uniform beams with ends elastically restrained against rotation and translation. Nallim and Grossi [19] studied the dynamical behaviour of beams with complicating effects, such as non-uniform cross sections, presence of an arbitrarily placed concentrated mass and an axial force and ends elastically restrained against rotation and translation.

In contrast to the body of information described, there is only a limited amount of information for beams elastically restrained at intermediate points. Rutemberg [20] presented eigenfrequencies for a uniform cantilever beam with a rotational restraint at some position. Lau [21] extended Rutemberg's results including an additional translational restraint. Arenas and Grossi [22] presented exact and approximate frequencies of a uniform beam, with one end spring-hinged and a rotational restraint in a variable position.

This paper deals with a particular case, of the problem of free vibrations of a uniform beam with intermediate constraints and ends elastically restrained against rotation and translation. A rather curious situation is shown to exist in the frequency values and mode shapes, when only an intermediate translational restraint is placed and the beam is simply supported at both ends.

2. Determination of the exact solution

Let us consider the uniform beam of length l, shown in Fig. 1, which has elastically restrained ends, and is constrained at an intermediate point. It has been assumed that the ends are elastically

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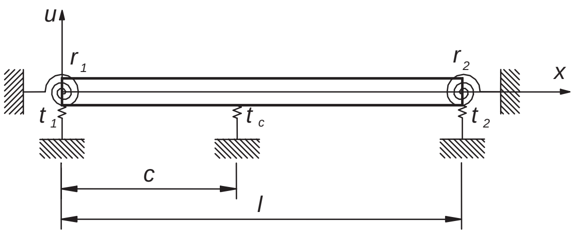

Fig. 1. Vibrating beam with intermediate support.

restrained against rotation and translation. The rotational restraints are characterised by the spring constants r_1, r_2 and the translational restraints by the spring constants t_1, t_2.

Adopting various values of the parameters r_i and $t_i, i = 1, 2$, all the possible combinations of classical end conditions (i.e., clamped, pinned, sliding and free) can be generated. It is also assumed that at the intermediate point, a translational restraint characterised by the spring constant t_c is placed.

In order to analyse the transverse planar displacements of the system under study we suppose that the vertical position of the beam at any time t is described by the function $u = u(x, t),\ x \in [0, l]$. Then u must satisfy the following differential equations:

$$EI \frac{\partial^4 u(x,t)}{\partial x^4} + \rho A \frac{\partial^2 u(x,t)}{\partial t^2} = 0, \quad \forall t, \forall x \in (0, c), \tag{1}$$

$$EI \frac{\partial^4 u(x,t)}{\partial x^4} + \rho A \frac{\partial^2 u(x,t)}{\partial t^2} = 0, \quad \forall t, \forall x \in (c, l). \tag{2}$$

The natural boundary conditions of the problem are given by

$$r_1 \frac{\partial u(0,t)}{\partial x} = EI \frac{\partial^2 u(0,t)}{\partial x^2}, \quad t_1 u(0,t) = -EI \frac{\partial^3 u(0,t)}{\partial x^3}, \tag{3,4}$$

$$\frac{\partial^2 u(c^-,t)}{\partial x^2} = \frac{\partial^2 u(c^+,t)}{\partial x^2}, \quad t_c u(c,t) = EI\left(\frac{\partial^3 u(c^-,t)}{\partial x^3} - \frac{\partial^3 u(c^+,t)}{\partial x^3}\right), \tag{5,6}$$

$$r_2 \frac{\partial u(l,t)}{\partial x} = -EI \frac{\partial^2 u(l,t)}{\partial x^2}, \quad t_2 u(l,t) = EI \frac{\partial^3 u(l,t)}{\partial x^3}. \tag{7,8}$$

Using the well-known method of separation of variables, one assumes as solution of Eq. (1) the expression

$$u^-(x,t) = \sum_{n=1}^{\infty} u_n^-(x) T(t). \tag{9}$$

Similarly for Eq. (2) we write

$$u^+(x,t) = \sum_{n=1}^{\infty} u_n^+(x) T(t). \tag{10}$$

The functions $u_n^-(x), u_n^+(x)$ denote the corresponding nth mode of natural vibration and are respectively given by

$$u_n^-(x) = A_1 \cosh kx + A_2 \sinh kx + A_3 \cos kx + A_4 \sin kx, \tag{11}$$

$$u_n^+(x) = A_5 \cosh kx + A_6 \sinh kx + A_7 \cos kx + A_8 \sin kx, \tag{12}$$

where the parameter k is given by $k = \sqrt{\sqrt{\rho A/(EI)}\omega_n}$.

Substituting Eqs. (11) and (12) into Eqs. (9) and (10) and then in the boundary conditions (3)–(8) and in the continuity conditions which correspond to the point where the intermediate restraint is placed,

$$u(c^-, t) = u(c^+, t) = u(c, t), \tag{13}$$

$$\frac{\partial u(c^-, t)}{\partial x} = \frac{\partial u(c^+, t)}{\partial x} = \frac{\partial u(c, t)}{\partial x}, \tag{14}$$

one obtains a set of eight homogeneous equations in the constants A_i. Since the system is homogeneous, for existence of a non-trivial solution, the determinant of coefficients must be equal to zero. This procedure yields the frequency equation:

$$G(R_i, T_i, T_c, \lambda, c) = 0, \tag{15}$$

where

$$R_i = r_i l/(EI), \;\; T_i = t_i l^3/(EI), \;\; i = 1, 2,$$
$$T_c = t_c l^3/(EI), \;\; \lambda = kl, \;\; \lambda = \sqrt{\sqrt{\rho A/(EI)}\omega l}. \tag{16}$$

3. Analysis of a particular case

An interesting case arises when the intermediate translational restraint with parameter T_c is placed at $x = l/2$ and the beam is simply supported at both ends. Table 1 depicts the values of coefficients λ_i, $i = 1, 2$ for different values of T_c.

When $T_c = 0$ the classical values $\lambda_1 = 3.14159265$ and $\lambda_2 = 6.28318531$ are obtained. The corresponding mode shapes are shown in Figs. 2 and 3. As it can be seen in Table 1, for $T_c = 0.1, 1, 10, 100$ and 700 the respectively values of the first frequency coefficient are: $\lambda_1 =$ 3.14320394, 3.15759089, 3.29131258, 4.13153931 and 5.91342567. The corresponding mode shapes present inflection points as it is illustrated by Fig. 4 in the case $T_c = 700$.

On the other hand, the values and the mode shapes which correspond to the coefficient λ_2 remain the same as T_c is varying. The constant value is $\lambda_2 = 6.28318531$ and the mode shapes coincide with the ones illustrated in Fig. 3.

It is observed that $\lambda_1 \to \lambda_2$ when $T_c \to 995$. A curious situation arises when T_c varies in the interval [995, 996]. For instance, when $T_c = 995.9$ the values of the first two coefficients are $\lambda_1 = 6.28317137$, $\lambda_2 = 6.28318531$ and the corresponding mode shapes are those of Figs. 5 and 3 respectively. But when $T_c = 996$ the values of the first two coefficients are $\lambda_1 = 6.28318531$, $\lambda_2 =$ 6.28327428 and the corresponding mode shapes are illustrated in Figs. 3 and 6.

Table 1
Values of the coefficient λ_1 of simply supported beam with and intermediate point elastically restrained against translation, $T_1 = \infty,\ R_1 = 0,\ T_2 = \infty,\ R_2 = 0,\ c = 0.5$

	T_c									
	0	0.1	1	10	100	700	995	996	1000	∞
λ_1	3.14159265	3.14320394	3.15759089	3.29131258	4.13153931	5.91342567	6.28222446	6.28318531	6.28318531	6.28318531
λ_2	6.28318531	6.28318531	6.28318531	6.28318531	6.28318531	6.28318531	6.28318531	6.28327428	6.28738095	7.85320462

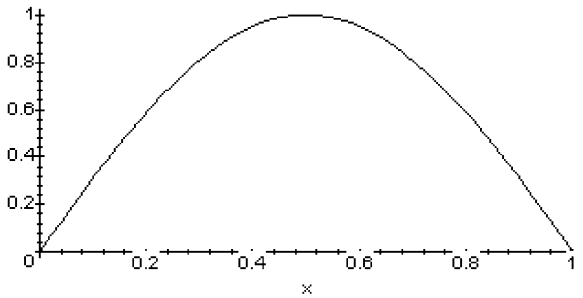

Fig. 2. Mode shape which corresponds to $\lambda_1 = 3.14159265$ and $T_c = 0$.

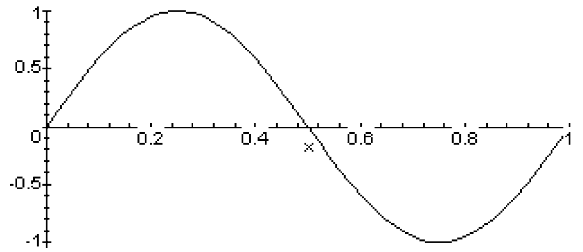

Fig. 3. Mode shape which corresponds to $\lambda_2 = 6.28318531$, $T_c = 0$ and also corresponds to $\lambda_1 = 6.28318531,\ T_c > \bar{T}_c$.

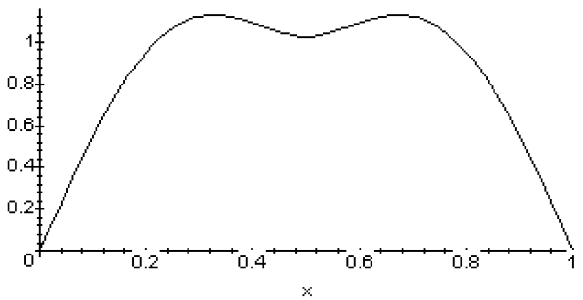

Fig. 4. Mode shape which corresponds to $\lambda_1 = 5.91342567$ and $T_c = 700$.

In this process the first eigenvalue $\lambda_1 \to \lambda_2$ from the left as T_c varies from 0 to 995.9. When the values of T_c increase from 996 to infinity, there is a change. The values of λ_1 remain constant and equal to $\lambda_1 = 6.28318531$ and the values of λ_2 are greater and increase as T_c varies from 996 to

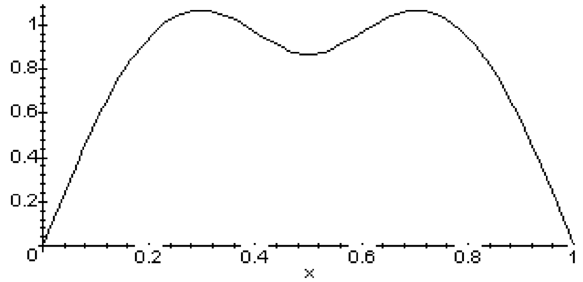

Fig. 5. Mode shape which corresponds to $\lambda_1 = 6.28317137$ and $T_c = 995.9$.

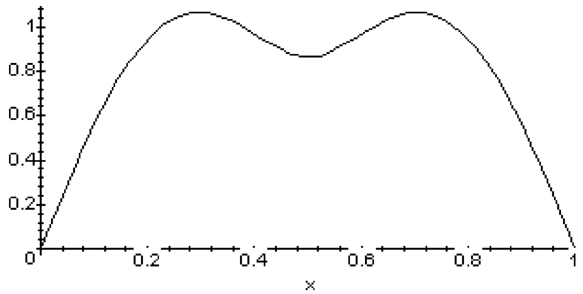

Fig. 6. Mode shape which corresponds to $\lambda_2 = 6.28327428$ and $T_c = 996$.

infinity. The mode shape, analogue to that illustrated in Fig. 6, which corresponded to the second eigenvalue (when $T_c < 996$ in Table 1) now corresponds to the first one. In the transition, there exists a value $\overline{T_c} \in [995.9, 996]$ which corresponds to the case when $\lambda_1 = \lambda_2 = 2\pi$. This situation corresponds to an eigenvalue of multiplicity $m = 2$. Two different eigenfunctions correspond to this eigenvalue and they have a similar aspect as those which are illustrated in Figs. 3 and 6.

No claim of originality is made by the authors since a simple problem has been solved. Nevertheless it is felt that it is important to introduce a discussion on the particular situation analyzed.

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