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xxx 00 (2015) 1–14

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A nodal inverse problem for second order Sturm-Liouville operators with indefinite weights

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Abstract

In this paper we study an inverse problem for weighted second order Sturm-Liouville equations. We show that the zeros of any subsequence of eigenfunctions, or a dense set of nodes, are enough to determine the weight. We impose weaker hypotheses for positive weights, and we generalize the proof to include indefinite weights. Moreover, the parameters in the boundary conditions can be determined numerically by using a shooting method.

Keywords: Inverse problems, eigenvalues, nodal points, indefinite weights.
2000 MSC: 34A55, 34C10

1. Introduction

In this work we will study an inverse problem for the following weighted Sturm-Liouville equation,

$$-u'' = \lambda\rho(x)\,u, \qquad x \in (0,1) \tag{1}$$

with homogeneous boundary conditions

$$\begin{aligned} \cos(\alpha)u(0) - \sin(\alpha)u'(0) = 0 \\ \cos(\beta)u(1) - \sin(\beta)u'(1) = 0 \end{aligned} \tag{2}$$

Here, $\rho \in BV[0,1]$ is the unknown weight function, λ is a real parameter, and $0 \le \alpha < \pi$, $0 < \beta \le \pi$. We assume also that $\sqrt{\rho} \in BV[0,1]$.

For positive weights it is well known that problem (1)-(2) has a sequence of simple eigenvalues $\{\lambda_n\}_{n\ge1}$, the n-th eigenvalue λ_n behaves asymptotically as

$$\lambda_n = \left(\frac{\pi n}{\int_0^1 \sqrt{\rho}\, dx} \right)^2 + o(n^2), \tag{3}$$

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and the associated eigenfunction u_n has exactly n nodal domains in $[0, 1]$. By a nodal domain we understand a maximal connected set in $(0, 1)$ where the solution does not change signs.

A classical inverse problem is to characterize the weight function ρ and the parameters α and β in the boundary conditions by using some spectral data of the problem. Usually, this function describes the tension or the mass distribution of a string or a rod, or some physical properties such as diffusivity or conductivity of the material, since a simple change of variables enable us to consider the related equation

$$-(\sigma(x)v')' = \lambda v.$$

The first result for this kind of problems was obtained by Goran Borg [2], who proved that ρ can be determined from two spectra, for example the sequences of eigenvalues of equation (1) with Dirichlet and Neumann boundary condition. Later, Gelfand and Levitan showed in [11] that one sequence of eigenvalues is enough, together with a sequence of constants $\{a_n\}_n$, the values of $u_n'(0)$ where u_n is a normalized eigenfunction corresponding to λ_n.

Of course, in these cases we need a deep knowledge of the eigenvalue problem. From a practical point of view, when we wish to use two different spectra, we need to change the boundary conditions, and this is not always possible (think of a tensor of a bridge, or a fixed rod which cannot be removed from some structure). Also, the constants of normalized eigenfunctions cannot be easily obtained.

However, in several cases, it is possible to observe the zeros of the eigenfunctions. This procedure goes back to Chladni (see [5]) in the eighteen century, who determined the nodal lines of surfaces by covering them with sand and then the surfaces were forced to vibrate by stroking them with a violin bow. Also, we can force a string into resonance by varying a source of sound, and a velocity scanner (as the ones used in traffic control) is enough to determine its nodal points, see chapter 12 in Gladwell's book [13].

On the other hand, the full set of nodes has redundant information, and only a part of the nodes is enough to determine the weight ρ. Usually, two sets of nodes are used:

- The nodes of a subsequence of eigenfunctions.
- A dense subset of pairs of consecutive nodal points x_j^n, x_{j+1}^n.

For zero Dirichlet boundary conditions and positive weights, the characterization of the weight was proved independently in different ways.

Given the zeros of a subsequence of eigenfunctions, for $\rho \in C^2([0, 1])$, Shen in [20] studied the lengths of nodal domains, and show that the following formula holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} f(x_j) = \frac{\int_0^1 f(x)\sqrt{\rho(x)}\, dx}{\int_0^1 \sqrt{\rho(x)}\, dx}$$

Now, the weight can be recovered by using the theory of moments, since we can compute the moment c_n with $f(x) = x^n$. Later, Shen and Tsai in [21] obtained $\rho \in C^2([0, 1])$ by using both the zeros and a subsequence of eigenvalues. The regularity condition $\rho \in C^2([0, 1])$ cannot be avoided since they use the WKB method in order to estimate the lengths of the nodal domains. This is a common point with the proof of Martínez-Finkelshtein et al. in [17], since they have used that a solution has the asymptotic formula

$$u \approx C_1[\lambda\rho]^{-1/4} \sin\left(C_2 + \int_0^x \sqrt{\lambda\rho(t)}dt\right),$$

under the extra assumption that

$$-\frac{1}{4}\frac{\rho''}{\lambda\rho^2} + \frac{5}{16}\frac{\rho'}{\lambda^2\rho^3} \ll 1,$$

which holds for λ big enough.

By using a dense set of pairs of consecutive nodes, Hald and McLaughlin in [10] obtained bounds for the lengths of nodal domains, and gave an algorithm to construct piecewise functions ρ_n converging to the weight ρ. They considered positive functions $\rho \in BV([0,1])$ bounded away from zero. This last hypothesis is crucial in their proof, since $V[\ln(\rho)]$, the total variation of $\ln(\rho)$, is needed in a better asymptotic estimate of eigenvalues than equation (3),

$$\lambda_n^{1/2} = \frac{\pi n}{\int_0^1 \sqrt{\rho}\, dx} + O(V[\ln(\rho)]). \tag{4}$$

Also, a very precise estimate on the length of nodal domains was needed; and this estimate depend on a positive lower bound of the weight, see Lemma 2 in [10].

Let us recall that, for rough coefficients, an arbitrary dense set of nodes is not enough to determine ρ, and in [10] is required that if x_j^n is in the dense set, then x_{j-1}^n or x_{j+1}^n is also in the dense set. In [9], Hald and McLaughlin proved that a dense set of zeros characterizes a smooth positive weight ρ, with a second derivative $\rho'' \in L^1$.

The aim of this work is to show that it is possible to determine the weight function ρ under milder hypotheses, given the zeros of a subsequence of eigenfunctions or just a dense set of pairs of consecutive nodes. We assume only that $\sqrt{|\rho|} \in BV([0,1])$, allowing ρ to change sings. Moreover, we show that it is possible to determine numerically the parameters α, β in the boundary condition. Essentially, the main contributions are the following, and precise statements of the Theorems can be found in Section 2:

- In previous works, the boundary condition was given. We show that we can compute α and β *a posteriori*.
- The allowed regularity of the weight is improved. Indeed, it is enough that $\sqrt{\rho}$ be Riemann integrable, and a positive lower bound is not needed.
- We consider sign changing weights, and there are no previous results for them. By allowing ρ to change signs, we can study problems where the weight is related to the index of refraction of the media, and problems of population dynamics where the weight represents the intrinsic growth rate of species, and it is positive (resp., negative) in the favorable (resp., unfavorable) zone of habitat, see [3, 16]. Recently, Eckhardt and Kostenko studied the spectral inverse problem for indefinite strings in terms of the Weyl-Titchmarsh function, motivated by the study of Camassa-Holm and Hunter-Saxton equations, see [7] for details.
- We give a different proof of Theorem 7 in [10], using only the weaker estimate (3) instead of (4), without using any information of the lengths of nodal domains, and we extend it to indefinite weights.

Let us remark that the method can be used for more general classes of operators, like Krein-Feller operators arising from general diffusion processes. The inverse problem based on two spectra for these operator can be found in the book of Dym and McKean [6]. Also, quasilinear problems of $p-$Laplacian type can be handled in this way.

Our proof was motivated by the theory of orthogonal polynomials. We define a sequence of probability measures associated to the zeros, and by using the machinery of weak convergence of tight sequences of measures, we obtain a (weak) limit which is enough to characterize ρ. The existence of a weakly convergent subsequence follows from Helly's Selection Theorem (or can be proved by using Prokhorov's Theorem), and Sturmian type arguments show that the full sequence converges. As a by product, we give a different proof of Theorem 7 in [10].

The paper is organized as follows: in Section §2 we present our main theorems and we collect some previous results about the eigenvalue problem that we will need in the proofs. Section §3 is devoted to the proof of the main Theorems. In Section §4 we show that only a dense set of pairs of nodes is enough to determine the weight. We close the paper in Section §5 with several remarks.

2. Notations, eigenvalues, and statement of main results

2.1. Notation

Given ρ such that $\sqrt{\rho} \in BV[0,1]$, we introduce the positive and negative parts of ρ,

$$\rho^+(x) = \max\{\rho(x), 0\},$$

$$\rho^-(x) = \max\{-\rho(x), 0\}.$$

Clearly, $\rho = \rho^+ - \rho^-$. Also we denote

$$\begin{aligned}\Omega^+ &= \{x \in (0,1) : \rho(x) > 0\} = supp(\rho^+),\\ \Omega^- &= \{x \in (0,1) : \rho(x) < 0\} = supp(\rho^-),\\ \Omega^0 &= \{x \in (0,1) : \rho(x) \equiv 0\}.\end{aligned}$$

When we consider indefinite weights, we assume that Ω^+ and Ω^- are open sets with positive measure.

We call δ_{x_0} the Dirac delta function in x_0, i.e. the measure such that

$$\delta_{x_0}(a,b) = \begin{cases} 1 & x_0 \in (a,b) \\ 0 & x_0 \notin (a,b). \end{cases}$$

By a nodal domain of an eigenfunction we understand a connected set in $[0,1]$ between two consecutive zeros, or between one boundary point and the closest zero.

2.2. Main results

Let us state our main result:

Theorem 2.1 (Definite case). *Let ρ be a nonnegative function such that $\sqrt{\rho} \in BV[0,1]$. Let $\{x_0^n, \dots, x_n^n\}_{n\geq 1}$ be the zeros of the $n-$th. eigenfunction of the following Sturm-Liouville problem:*

$$\begin{cases} -u'' = \lambda\rho(x)\, u, & x \in [0,1] \\ \cos(\alpha)u(0) - \sin(\alpha)u'(0) = 0 \\ \cos(\beta)u(1) - \sin(\beta)u'(1) = 0 \end{cases} \tag{5}$$

where $x_0^n = 0$ and $x_n^n = 1$, being zeros or not. Let Z_n be the function

$$Z_n(x) = \frac{\#\{j \geq 0 \colon x_j^n \leq x\}}{n+1}.$$

Then, there exists $Z(x) = \lim_{n\to\infty} Z_n(x)$ and

$$Z(x) = \frac{1}{\int_0^1 \sqrt{\rho}\, dt} \int_0^x \sqrt{\rho(t)}\, dt. \tag{6}$$

This theorem implies that the weight ρ can be obtained from the zeros of eigenfunctions:

Corollary 2.2. *Let ρ be a nonnegative function such that $\sqrt{\rho} \in BV[0,1]$ and $\int_0^1 \sqrt{\rho}\, dt = 1$. Given the zeros of any sequence of eigenfunctions of problem (5), we can recover the weight ρ.*

With slight modifications in the method of proof we obtain the following piecewise approximation of ρ:

Algorithm for ρ:

$$\rho^n(x) = \frac{1}{n^2(x_j^n - x_{j-1}^n)^2} \qquad x_{j-1}^n \leq x < x_j^n. \tag{7}$$

Observe that this algorithm requires only a dense subset of pairs of nodes converging to x. This approximation can be compared with Algorithm A in [10], namely

$$\rho^n(x) = \frac{\pi^2}{\lambda_n^2(x_j^n - x_{j-1}^n)^2} \qquad x_{j-1}^n \leq x < x_j^n,$$

where λ_n^2 is the n−th. eigenvalue, which must be known in advance. The convergence of the algorithm to the weight ρ will be proved below, although we can deduce it from the previous algorithm and the asymptotic behavior of eigenvalues given by equation (3), since $\int_0^1 \sqrt{\rho} = 1$.

A different approximation is given by Shen and Tsai, see Theorem 2.1 in [21], namely

$$\rho^n(x) = \frac{\pi^2}{\lambda_n^2}\left[\left(\frac{x_{j+1}^n - x_j^n}{x_j^n - x_{j-1}^n} - 1\right)(x - x_{j-1}^n) + (x_j^n - x_{j-1}^n)\right]^{-2}, \tag{8}$$

for $x \in [x_{j-1}^n, x_j^n)$ under the assumptions that λ_n is known, and $\rho \in C^2([0,1])$. A similar formula can be obtained replacing π^2/λ_n^2 by n^2.

Now, a shooting argument enable us to recover the parameters in the boundary conditions:

Theorem 2.3. *Let ρ be a nonnegative function such that $\sqrt{\rho} \in BV[0,1]$ and $\int_0^1 \sqrt{\rho}\, dt = 1$. Given the zeros of an eigenfunction which has at least two interior zeros, we can recover the parameters α and β in the boundary condition.*

Remark 2.4. *As usual, the normalization condition $\int_0^1 \sqrt{\rho}\, dt = 1$ cannot be avoided. Observe that the eigenfunctions of problem (5) with the weight $\hat{\rho} = c\rho$ are the same, and the eigenvalues change to λ/c.*

When the weight ρ changes signs, we have the following result:

Theorem 2.5 (Indefinite case). *Let ρ such that $\sqrt{|\rho|} \in BV[0,1]$ and Ω^+, Ω^- are open sets with positive measure. Let $\{x_0^{n,\pm} \ldots, x_n^{n,\pm}\}_{n\geq 1}$ be the zeros of u_n^+ and u_n^-, the eigenfunctions of problem (5) corresponding to λ_n^+ and λ_n^-, where $x_0^{n,\pm} = 0$ and $x_n^{n,\pm} = 1$, being zeros or not. Let $Z_n^\pm$ be the functions*

$$Z_n^+(x) = \frac{\#\left\{j \geq 0 \colon x_j^{n,+} \leq x\right\}}{n+1}, \qquad Z_n^-(x) = \frac{\#\left\{j \geq 0 \colon x_j^{n,-} \leq x\right\}}{n+1}.$$

Then, there exist $Z^\pm(x) = \lim_{n\to\infty} Z_n^\pm(x)$, and

$$Z(x)^\pm = \frac{1}{\int_0^1 \sqrt{\rho(t)^\pm}\, dt} \int_0^x \sqrt{\rho(t)^\pm}\, dt. \tag{9}$$

As before, from Theorem 2.5 it is easy to prove now that ρ and α, β can be recovered.

Finally, it is well known that a dense set of pairs of consecutive nodes are enough to determine $\rho \in BV$ when it is bounded away from zero, see Theorem 7 in [10]. We can extend this result to the class of weights considered here, and we prove the following theorem:

Theorem 2.6. *Let ρ be a function continuous from the right and at $x = 1$, such that $\sqrt{\rho^\pm} \in BV[0,1]$. Then ρ^+ and ρ^- are uniquely determined up to multiplicative constants by a dense subset of pairs of consecutive nodes x_j^n, x_{j+1}^n from eigenfunctions of the following Sturm-Liouville problem:*

$$\begin{cases} -u'' = \lambda\rho(x)\, u, & x \in [0,1] \\ \cos(\alpha)u(0) - \sin(\alpha)u'(0) = 0 \\ \cos(\beta)u(1) - \sin(\beta)u'(1) = 0. \end{cases} \tag{10}$$

2.3. Eigenvalues

The eigenvalue problem (1) with positive weights has a sequence of nonnegative eigenvalues,

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \nearrow +\infty,$$

which are simple, so we can choose u_n and any other eigenfunction corresponding to λ_n is a multiple of u_n.

The eigenfunction u_n has n nodal domains, although the number of zeros of solutions depends on the boundary conditions. Two particular cases are the following ones:

- Dirichlet boundary conditions, $u(0) = 0 = u(1)$: the $n-$th eigenfunction has $n + 1$ zeros, counting both 0 and 1. In this case, the first eigenvalue is strictly positive.
- Neumann boundary conditions, $u'(0) = 0 = u'(1)$: the $n-$th eigenfunction has $n - 1$ zeros. In this case, the first eigenvalue is zero and the associated eigenfunction is constant.

For general homogeneous boundary conditions, the n-th eigenfunction has $n - 1$ or n zeros.

Let us note that 0 is an eigenvalue only for the Neumann boundary condition.

The classical Sturm-Liouville theory states several important facts, we will need the following quantitative estimate on the number of zeros of solutions, where $\lfloor x \rfloor$ denotes the largest integer not greater than x:

Lemma 2.7. *Let $r, R \in \mathbb{R}$ such that $r \leq \rho(x) \leq R$ for $x \in [a, b]$. Given $\lambda > 0$ and any solution u of*

$$-u'' = \lambda \rho u,$$

then u has at least $\lfloor \sqrt{\lambda r}(b-a)/\pi \rfloor$ zeros, and at most $\lfloor \sqrt{\lambda R}(b-a)/\pi \rfloor + 1$ zeros in $[a, b]$.

Proof. The proof follows directly by comparing with the number of zeros of $v = \sin(\sqrt{\lambda r}(x-a))$ and $w = \sin(\sqrt{\lambda R}(x-a))$, two solutions of

$$-v'' = \lambda r v, \qquad -w'' = \lambda R w$$

in $[a, b]$.

The eigenvalues can be obtained by using the Rayleigh quotient and one of the variants of the Courant-Fischer minimax method in some subspace $H \subset H^1(0, 1)$ where the boundary condition are imposed if necessary. Alternatively, we can obtain the sequence of eigenvalues by using the shooting method, based on the Prufer's transform, which represents the solution of problem (1)-(2) in polar coordinates:

$$\begin{aligned} u(x) &= r(x)\sin(\theta(x)), \\ u'(x) &= r(x)\cos(\theta(x)), \end{aligned}$$

where θ and r are solutions of the system

$$\begin{cases} \theta' = \cos^2(\theta) + \lambda\rho\sin^2(\theta), \\ r' = r(1-\lambda\rho)\sin(\theta)\cos(\theta) \end{cases} \tag{11}$$

with $\theta(0) = \alpha\, mod(\pi)$, and $\theta(1) = \beta\, mod(\pi)$.

Now, the eigenvalues can be found solving for λ for each $n \in \mathbb{N}$, the problem

$$\theta' = \cos^2(\theta) + \lambda\rho\sin^2(\theta),$$

with $\theta(0) = \alpha$ and $\theta(1) = \beta + (n-1)\pi$.

2.4. Indefinite weights

For indefinite weights $\rho = \rho^+ - \rho^-$ the eigenvalue problem (1) is similar, see the book of Ince [15]. However, certain differences arise: now there are two sequences of eigenvalues,

$$\lambda_1^+ < \lambda_2^+ < \cdots < \lambda_n^+ < \cdots \nearrow +\infty,$$

$$\lambda_1^- > \lambda_2^- > \cdots > \lambda_n^- > \cdots \searrow -\infty,$$

and the variational characterization of eigenvalues includes an additional condition:

$$u \in H^1(0,1) : \int_0^1 \rho(x)u^2(x)dx > 0$$

for positive eigenvalues, and

$$u \in H^1(0,1) : \int_0^1 \rho(x)u^2(x)dx < 0$$

for the negative ones. We have now the following asymptotic formula for the eigenvalues

$$\lambda_n^\pm = \left(\frac{\pi n}{\int_0^1 \sqrt{\rho^\pm}\, dx} \right)^2 + o(n^2), \tag{12}$$

see for instance [8].

2.5. Prokhorov and Helly's theorems

The key point in our proofs are the following results due to Prokhorov and Helly. We state it only for probability measures in the real line, although they hold for separable metric spaces and also for finite signed measures.

Definition 2.8. *A family $\{\mu_i\}_{i\in I}$ of Borel probability measures on $\mathbb{R}$ is called tight if for every $\varepsilon > 0$ there exists a compact set K_ε such that*

$$\mu_i(K_\varepsilon) \geq 1 - \varepsilon$$

for all $i \in I$.

Definition 2.9. *A sequence $\{\mu_n\}_{n\geq 1}$ of Borel probability measures on $\mathbb{R}$ converges weakly to a measure μ if*

$$\int_{\mathbb{R}} f d\mu_n \to \int_{\mathbb{R}} f d\mu$$

for all $f \in C_b(\mathbb{R})$.

Theorem 2.10 (Prokhorov). *A family $\{\mu_i\}_{i\in I}$ of Borel probability measures on $\mathbb{R}$ is tight if and only if its closure is weakly compact.*

This theorem can be stated in terms of the distribution functions $F_{\mu_i}(x) = \mu_i(-\infty, x]$, and it was proved by Helly:

Theorem 2.11 (Helly's selection theorem). *Let $\{Z_n\}_{n\geq 1}$ be a sequence of non-decreasing real functions on $\mathbb{R}$ satisfying $0 \leq Z_n(x) \leq 1$ for all $x \in \mathbb{R}$ and $n \geq 1$. Then, there exists a subsequence $\{Z_{n_j}\}_{j\geq 1}$ converging to a real function Z.*

Moreover, for any $f \in C_b(\mathbb{R})$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)dZ_n(x) \to \int_{\mathbb{R}} f(x)dZ(x).$$

We refer the interested reader to [19] or the classical book of Billingsley [1] for more details.

3. Proof of the main results

3.1. The nodal inverse problem

Proof of Theorem 2.1. The existence of a limit Z follows directly from Helly's Theorem 2.11, or by defining the family of probability measures

$$dZ_n = \frac{1}{n+1}\sum_{j=0}^{n}\delta_{x_j^n}.$$

This family of probability measures $\{dZ_n\}_{n\geq 1}$ is tight, being supported in $[0,1]$, and Prokhorov's Theorem 2.10 implies that there exists a measure dZ such that some subsequence $\{dZ_{n_k}\}_{k\geq 1}$ converges weakly to dZ.

Now we will prove that

$$dZ = \sqrt{\rho}dx.$$

Indeed, we will show that the limit does not depend on a particular convergent subsequence $\{dZ_{n_k}\}_{k\geq 1}$. So, the full sequence converges weakly to dZ.

Let us assume first that ρ is a continuous function.

We fix $\varepsilon > 0$, and taking M big enough, we can subdivide the interval $[0,1]$ in M subintervals of length $h = M^{-1}$, such that

$$h\cdot\max\sqrt{\rho} < \frac{\varepsilon}{2},$$
$$0 \leq \sqrt{R_i} - \sqrt{r_i} < \frac{\varepsilon}{2}$$

for $1\leq i\leq M$, where

$$r_i = \inf\{\rho(x) : x\in[(i-1)h, ih)\},$$
$$R_i = \sup\{\rho(x) : x\in[(i-1)h, ih)\}.$$

Let us call $y_{n,i}$ the number of zeros of u_n in the interval $[(i-1)h, ih)$ for $1\leq i < M$, and let $y_{n,M}$ be the number of zeros of u_n in the interval $[1-h, 1]$. We have $n+1 = \sum_i y_{n,i}$.

Let us fix $x\in(0,1)$, and let us estimate $Z_n(x)$ for n big enough. There exists some I_0 such that $x\in[(I_0-1)h, I_0h)$, so

$$\sum_{i=1}^{I_0-1} y_{n,i} \leq (n+1)Z_n(x) \leq \sum_{i=1}^{I_0} y_{n,i}.$$

From Lemma 2.7 we can bound each $y_{n,i}$,

$$\sum_{i=1}^{I_0-1}\frac{\sqrt{\lambda_n r_i}h}{\pi} - 1 \leq (n+1)Z_n(x) \leq \sum_{i=1}^{I_0}\frac{\sqrt{\lambda_n R_i}h}{\pi} + 1,$$

and we get

$$-M + \frac{\sqrt{\lambda_n}}{\pi}\sum_{i=1}^{I_0-1}\sqrt{r_i}h \leq (n+1)Z_n(x) \leq M + \frac{\sqrt{\lambda_n}}{\pi}\sum_{j=1}^{I_0}\sqrt{R_i}h.$$

Let us observe that

$$\sum_{i=1}^{I_0-1}\sqrt{r_i}h \leq \int_0^x\sqrt{\rho(t)}dt \leq \sum_{I=1}^{I_0}\sqrt{R_i}h,$$

and the hypothesis on h implies

$$\sum_{I=1}^{I_0}\sqrt{R_i}h - \sum_{i=1}^{I_0-1}\sqrt{r_i}h \leq h\cdot\max\rho + \sum_{i=1}^{I_0-1}(\sqrt{R_i} - \sqrt{r_i})h < \varepsilon.$$

Finally, we use now the asymptotic formula for the eigenvalues given by equation (3) obtaining

$$\lim_{n\to\infty} \pm\frac{M}{n+1} + \frac{\sqrt{\lambda_n}}{\pi(n+1)} = \frac{1}{\int_0^1 \sqrt{\rho}dx},$$

which implies the desired result,

$$Z_n(x) \to \frac{1}{\int_0^1 \sqrt{\rho}dx} \int_0^x \sqrt{\rho}dx.$$

Let us consider now $\sqrt{\rho} \in BV[0,1]$. We fix $\varepsilon > 0$, and there exist finitely many points of discontinuity $z_1, \cdots, z_{k(\varepsilon)}$ in $[0,x]$, if any, such that

$$\left| \sqrt{\rho(z_i^+)} - \sqrt{\rho(z_i^-)} \right| > \varepsilon.$$

We can split $[0,x]$ as

$$[0,x] = [0,z_1] \cup \bigcup_{s=2}^{k(\varepsilon)} [z_{s-1}, z_s] \cup [z_{k(\varepsilon)}, x].$$

Now, by subdividing each interval in M intervals $\{I_i\}_{i=1}^M$ such that

$$\sup\{\sqrt{\rho(x)} : x \in I_i\} - \inf\{\sqrt{\rho(x)} : x \in I_i\} < \frac{\varepsilon}{2}$$

for $1 \le i \le M$, and reducing its lengths if necessary, such that

$$|I_i| \cdot \max \rho < \frac{\varepsilon}{2},$$

we can repeat now the previous arguments, and the proof is finished.

Proof of Corollary 2.2. Let us observe that, being $\sqrt{\rho} \in BV[0,1]$, we have

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{x-\varepsilon}^x \sqrt{\rho(y)}dy = \sqrt{\rho(x)}$$

almost everywhere.

3.2. An algorithm for ρ

In the previous proof we have constructed a sequence of measures dZ_n by using the familiy $\{\delta_{x_j^n}\}_{0\le j\le n}$. However, since $\int_0^x dZ_n$ is an step function, its derivative is just the original sum of deltas, which is not a convenient way to approximate $\sqrt{\rho}$.

So, we can introduce f_n defined as follows:

$$f_n(x) = \frac{1}{n(x_j^n - x_{j-1}^n)} \qquad x_{j-1}^n \le x < x_j^n,$$

and we have a new sequence of measures $\mu_n = f_n(x)dx$.

Let us call $F_n(x) = \int_0^x f_n(t)dt$. Clearly,

$$\int_0^x dZ_n - \frac{1}{n} \le \int_0^x d\mu_n \le \int_0^x dZ_n,$$

and therefore they converge to $\int_0^x \sqrt{\rho}dt$.

Now, the sequence $\{f_n\}_{n\ge1}$ converges in L^1 to some $f \in L^1$, and a subsequence converge to f almost everywhere; and, from Theorem 2.1, the full sequence converges as $n \to \infty$.

Since $F_n(x) \to \int_0^x f(t)dt$, it follows that

$$f(x) = \sqrt{\rho(x)},$$

and $f_n \to \sqrt{\rho}$ as $n \to \infty$ almost everywhere.

3.3. Boundary conditions

The parameters α and β in the boundary conditions can be obtained numerically by using a shooting argument, by solving the Prufer's system (11).

Proof of Theorem 2.3. Given the zeros of u_n and ρ, we can compute with arbitrary accuracy the first eigenvalue of

$$-u'' = \mu_1 \rho u, \qquad u(x_j^n) = u(x_{j+1}^n) = 0$$

where x_j^n and x_{j+1}^n are two consecutive zeros of u_n.

Since $\mu_1 = \lambda_n$, and we know ρ, the zeros of u_n and λ_n, we can solve the initial boundary value problem backwards

$$\begin{cases} -u'' = \lambda_n \rho(x)\, u, & x \in [0, x_1^n] \\ u(x_1^n) = 0 \\ u'(x_1^n) = 1, \end{cases}$$

starting the shooting at x_1^n, and compute the values $u(0)$, $u'(0)$. Therefore,

$$\alpha = \tan^{-1}\left(\frac{u(0)}{u'(0)}\right),$$

or $\alpha = \pi/2$ if $u'(0) = 0$.

In much the same way, we solve the initial boundary value problem forward from x_{n-1}^n

$$\begin{cases} -u'' = \lambda_n \rho(x)\, u, & x \in [x_{n-1}^n, 1] \\ u(x_{n-1}^n) = 0 \\ u'(x_{n-1}^n) = 1, \end{cases}$$

and we obtain

$$\beta = \tan^{-1}\left(\frac{u(1)}{u'(1)}\right),$$

or $\beta = \pi/2$ if $u'(1) = 0$, and the proof is finished.

3.4. Indefinite weights

Let us prove now Theorem 2.5. Although it is very similar to the one of Theorem 2.1, it includes some technical points. We prove it only for ρ^+, the other case being identical.

Proof of Theorem 2.5. Let us recall that $\Omega^\pm$, the positivity and negativity regions of ρ, are open sets. So, we have a denumerable partition of Ω^-, say $\{J_k^-\}_{k\geq 1}$, where J_k^- are open intervals, and $\rho < 0$ in J_k^- for any $k \geq 1$.

In the closure of any of these intervals we cannot have two zeros of an eigenfunction u_n^+ corresponding to a positive eigenvalue. If not, suppose that x_a and x_b are two consecutive zeros of u_n^+ in $\bar{J}_k^-$ for some $k \geq 1$. Hence,

$$-u_n^{+\,\prime\prime} = \lambda_n^+ \rho(x) u_n^+ \qquad x \in (x_a, x_b),$$

and integrating by parts after multiplying by u_n^+ we get

$$0 \leq \int_{x_a}^{x_b} (u_n^{+\,\prime})^2 dx = \lambda_n^+ \int_{x_a}^{x_b} \rho(x)(u_n^+)^2 dx < 0,$$

a contradiction, since $\lambda_n^+ > 0$ and $\rho < 0$ in (x_a, x_b). Therefore, for any $J_k^- \subset \Omega^-$ we have

$$\int_{J_k^-} dZ_n^+ \leq \frac{1}{n+1}, \tag{13}$$

and given any subsequence $dZ^+_{n_j}$ which converges weakly to some dZ^+, we have

$$\int_{J_k^-} dZ^+ = 0,$$

that is, Z^+ is constant in any connected component of Ω^-.

Let us prove now that $dZ^+ = \sqrt{\rho^+}dx$. Given $x \in [0,1]$, we fix $\varepsilon > 0$ and there exist at most finitely many points of discontinuity $z_1, \cdots, z_{k(\varepsilon)} \in [0,x]$ such that

$$\left| \sqrt{\rho^+(z_i^+)} - \sqrt{\rho^+(z_i^-)} \right| > \varepsilon.$$

Again, we split $[0,x]$ as

$$[0,x] = [0,z_1] \cup \bigcup_{s=2}^{k(\varepsilon)} [z_{s-1}, z_s] \cup [z_{k(\varepsilon)}, x],$$

and we can refine the partition obtaining M intervals $\{I_i\}_{i=1}^M$ such that

$$|I_i| \cdot \max \rho < \frac{\varepsilon}{2}, \quad \text{and} \quad \sup\left\{ \sqrt{\rho(x)} : x \in I_i \right\} - \inf\left\{ \sqrt{\rho(x)} : x \in I_i \right\} < \frac{\varepsilon}{2}$$

for $1 \le i \le M$. Repeating the arguments on the proof of Theorem 2.1, we call $y_{n,i}$ the number of zeros of u_n^+ in I_i, and we get the bound

$$\frac{\sqrt{\lambda_n^+ r_i^+}|I_i|}{\pi} - 1 \le y_{n,i} \le \frac{\sqrt{\lambda_n^+ R_i^+}|I_i|}{\pi} + 1,$$

where

$$r_i^+ = \inf\{\rho^+(x) : x \in I_i\},$$
$$R_i^+ = \sup\{\rho^+(x) : x \in I_i\},$$

Observe that, when $I_i \subset \Omega^-$, the bound is trivially satisfied due to (13).

Now,

$$\frac{1}{n+1}\sum_{i=1}^M y_{n,i} - \frac{1}{\int_0^1 \sqrt{\rho^+}dt}\int_0^x \sqrt{\rho+(t)}dt \le \frac{\sqrt{\lambda_n^+}}{\pi(n+1)}\sum_{i=1}^M \sqrt{R_i^+}|I_i| + \frac{M}{n+1} - \frac{1}{\int_0^1 \sqrt{\rho^+}dt}\sum_{i=1}^M r_i^+|I_i|,$$

$$\frac{1}{n+1}\sum_{i=1}^M y_{n,i} - \frac{1}{\int_0^1 \sqrt{\rho^+}dt}\int_0^x \sqrt{\rho+(t)}dt \ge \frac{\sqrt{\lambda_n^+}}{\pi(n+1)}\sum_{i=1}^M \sqrt{r_i^+}|I_i| - \frac{M}{n+1} - \frac{1}{\int_0^1 \sqrt{\rho^+}dt}\sum_{i=1}^M R_i^+|I_i|,$$

and the result follows, since

$$\frac{\sqrt{\lambda_n^+}}{\pi(n+1)} \to \frac{1}{\int_0^1 \sqrt{\rho^+}dt},$$
$$\frac{M}{n+1} \to 0$$

as $n \to \infty$, and

$$\sum_{i=1}^M (R_i^+ - r_i^+)|I_i| < \frac{\varepsilon}{2}.$$

The theorem is proved.

4. A dense set of pairs of nodes

Let us prove now that a dense set of pairs of consecutive zeros is enough to characterize the weight ρ.

Proof of Theorem 2.6. We consider only the positive part of ρ, the other one being similar.

Suppose that there exist two weights ρ_1 and ρ_2 with the same dense set of pairs of nodal points, satisfying

$$\int_0^1 \sqrt{\rho_1^+}dt = \int_0^1 \sqrt{\rho_2^+}dt.$$

Let x be a continuity point of both ρ_1^+ and ρ_2^+. Let us suppose that

$$\rho_1(x) - \rho_2(x) = a > 0.$$

Hence, there exists $\delta > 0$ and some constant $c > 1$ such that $c\rho_2 < \rho_1$ in $[x-\delta, x+\delta]$.

Let us take now a sequence of nodal domains $I_n = [x^n_{j(n)}, x^n_{j(n)+1}]$ with extremes in the given dense set, such that

$$\lim_{n\to\infty} x^n_{j(n)} = \lim_{n\to\infty} x^n_{j(n)+1} = x.$$

Clearly, for n big enough, $I_n \subset [x-\delta, x+\delta]$.

Let us consider now the following eigenvalue problems,

$$\begin{cases} -u'' = \mu^{(1)}\rho_1\, u, \\ u(x^n_{j(n)}) = u(x^n_{j(n)+1}) = 0 \end{cases} \tag{14}$$

$$\begin{cases} -v'' = \mu^{(2)}\rho_2\, v, \\ v(x^n_{j(n)}) = v(x^n_{j(n)+1}) = 0 \end{cases} \tag{15}$$

The first eigenvalues $\mu_1^{(1)}$ and $\mu_1^{(2)}$ coincide with $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$, the $n-$th. eigenvalues of problem (10) with the weights ρ_1 and ρ_2 respectively. The Sturmian comparison theorem implies

$$\mu_1^{(2)} > c\mu_1^{(1)},$$

and then we get

$$1 < c < \frac{\mu_1^{(2)}}{\mu_1^{(1)}} = \frac{\lambda_n^{(2)}}{\lambda_n^{(1)}} \to 1$$

as $n \to \infty$, since the estimate (12) implies

$$\lambda_n^{(1)} = \frac{\pi^2 n^2}{\left(\int_0^1 \sqrt{\rho_1^+}dt\right)^2} + o(n^2),$$

$$\lambda_n^{(2)} = \frac{\pi^2 n^2}{\left(\int_0^1 \sqrt{\rho_2^+}dt\right)^2} + o(n^2),$$

and both integrals are equal, a contradiction.

Hence, ρ_1 and ρ_2 coincide in those points where both functions are continuous. Since they have at most a countable number of discontinuities, and are both continuous from the right, $\rho_1 = \rho_2$ in $[0,1]$ and the proof is finished.

5. Concluding remarks

Several comments are in order.

Remark 5.1. *The proof can be easily modified to handle the eigenvalue problem*

$$-(\sigma(x)v')' = \lambda v.$$

Moreover, for

$$-(\sigma(x)u')' = \lambda\rho(x)u,$$

if σ is known, the nodal domains of a sequence of eigenfunctions or a dense set of zeros determine ρ. Also, we can determine σ given ρ and the zeros of a sequence of eigenfunctions or a dense set of zeros.

Remark 5.2. *Given ρ, the determination of a potential q for*

$$-u'' + q(x)u = \lambda\rho(x)u, \qquad x \in [a,b]$$

in terms of nodal data has attracted a great deal of interest, see for example [4, 9, 12, 18, 14]. Usually, ρ is assumed constant, although an indefinite problem was studied in [22], with $\rho > 0$ in $[a,c)$ and $\rho > 0$ in $(c,b]$. We are not sure that the ideas behind the proof of Theorem 2.1 can be applied to this problem. However, it is known that this problem needs less nodal data, just a dense set of nodes in part of the interval.

Remark 5.3. *An additional problem arises for indefinite weights if the zeros of u_n^+ and u_n^- are given as two sequences of sets $\{x_j^n\}$, $\{\hat{x}_j^n\}$, and we need to know which ones correspond to the positive and the negative eigenvalues.*

To this end, if $\rho \geq c > 0$ in some neighborhood of x, say $(x-\varepsilon, x+\varepsilon)$, the number of zeros of the positive eigenfunctions u_n^+ goes to infinity as $n \to \infty$, as a consequence of Lemma 2.7. Moreover, given a negative eigenvalue λ_n^-, the corresponding eigenfunction u_n^- has at most a single zero in $(x-\varepsilon, x+\varepsilon)$.

So, for n big enough, we take an arbitrary $x \in (0,1)$, $\varepsilon > 0$, and just one of the two sets $(x-\varepsilon, x+\varepsilon) \cap \{x_j^n\}$ and $(x-\varepsilon, x+\varepsilon) \cap \{\hat{x}_j^n\}$ can contain two or more zeros. If both sets contains two or more elements, we need to take a smaller value of ε. Nevertheless, the zeros of positive and negative eigenvalues will concentrate on different subsets of $(x-\varepsilon, x+\varepsilon)$, and it is possible to classify the sequences.

Let us recall that for positive weights we can recover ρ by imposing the extra condition $\int_0^1 \sqrt{\rho} = 1$. Needless to say, the same problem is present for indefinite weights, and worse, since we do not know which sequence correspond to the positive and the negative eigenvalues. As before, since $\lambda\rho = c\lambda\rho/c$, the negative eigenvalues of ρ are the positive ones of $-\rho$. In this case, we are able to determine two functions corresponding to ρ^+ and ρ^-, although we cannot decide which one is the positive part.

Remark 5.4. *The proof of Theorem 2.1 suggest a possible way to deal with higher dimensional inverse problems. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let us consider the following eigenvalue problem:*

$$\begin{cases} -\Delta u &= \lambda\rho u & \Omega \\ u &= 0 & \partial\Omega \end{cases}$$

Given an eigenfunction λ_n with k nodal domains, we can define a probability measure μ_n on Ω,

$$\mu_n = \sum_{j=1}^{k} \frac{\mu_j}{k \cdot |\Omega_j|}$$

where μ_j is the Lebesgue measure restricted to the nodal domain Ω_j, and $|A|$ denotes the Lebesgue measure of a set A.

Now, Prokhorov Theorem implies that there exists a weak limit μ, and the Weyl formula for the eigenvalue distribution suggest that

$$\int_A d\mu = \int_A \rho^{N/2}(x)dx.$$

However, given the complex patterns of nodal domains corresponding to higher eigenvalues, we believe it would be very difficult to prove it.

Acknowledgments

This work was partially supported by ANPCyT PICT2012 0153, CONICET (Argentina) PIP 5478/1438 and Universidad de Buenos Aires under grants 20020100100400 and 20020130100283BA. J.P. Pinasco is member of CONICET. C. Scarola is a Fellow of CONICET. We want to thanks the reviewers for their suggestions and comments, which significantly contributed to improving the quality of this publication.

References

[1] P. Billingsley, Convergence of probability measures. Wiley, New York, 1968.
[2] G. Borg, *Eine umkehrung der Sturm-Liouvilleschen eigenwertaufgabe*, Acta Mathematica 78 (1946) 1-96.
[3] R. S. Cantrell, C. Cosner, *Diffusive logistic equations with indefinite weights: population models in disrupted environments. II.* SIAM J. Math. Anal. 22 (1991) 1043-1064.
[4] Y.H. Cheng, C.K. Law, *The inverse nodal problem for Hill's equation*, Inverse Problems 22 (2006) 891-901.
[5] E. Chladni, *Entdeckungen uber die Theorie des Klanges*, 1787.
[6] H. Dym, H.P. McKean, Gaussian processes, function theory, and the inverse spectral problem. Academic Press, New York (1976).
[7] J. Eckhardt, A. Kostenko, *The inverse spectral problem for indefinite strings*. Preprint 2014
[8] J. Fernández Bonder, J. P. Pinasco, Asymptotic behavior of the eigenvalues of the one-dimensional wheighted p-Laplace operator, Arkiv fur Mathematik 41 (2003) 267-280.
[9] O. H. Hald, J. R. McLaughlin, *Solution of inverse nodal problems*. Inverse Problems 5 (1989) 307-347.
[10] O. H. Hald and J. R. McLaughlin, *Inverse problems: recovery of BV coefficients from nodes*, Inverse Problems 14 (1998) 245-273.
[11] I. M. Gelfand, B. M Levitan, *On the determination of a differential equation from its spectral function*, Trans. Amer. Math. Soc. (2) 1 (1955) 253-304. Translated from Izv. Akad. Nauk SSSR 15 (1951).
[12] F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential, II. The case of discrete spectrum*, Trans. Amer. Math. Soc. 352 (2000), 2765-2787
[13] G. M. L. Gladwell, *Inverse problems in vibration*. Vol. 119. Springer, 2004.
[14] Yongxia Guo, Guangsheng Wei, *Inverse problems: Dense nodal subset on an interior subinterval*, J. Differential Equations 255 (2013) 2002-2017.
[15] E. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.
[16] Y. Lou, E. Yanagida, *Minimization of the principal eigenvalue for an elliptic boundary value problem with indefinite weight, and applications to population dynamics*, Japan J. Indust. Appl. Math. 23 (2006) 275-292.
[17] A. Martínez-Finkelshtein, P. Martínez-González, A. Zarzo, *WKB approach to zero distribution of solutions of linear second order differential equations*, Journal of computational and applied mathematics 145 (2002) 167-182.
[18] E. S. Panakhov, M. Sat, *Reconstruction of potential function for Sturm-Liouville operator with Coulomb potential*, Boundary Value Problems 2013 (2013) 1-9.
[19] Yu.V. Prokhorov, *Convergence of random processes and limit theorems in probability theory*, Theor. Prob. Appl. 1 (1956) 157-214.
[20] C-L. Shen, *On the nodal sets of the eigenfunctions of the string equation*, SIAM journal on mathematical analysis 19 (1988) 1419-1424.
[21] C.-L. Shen, T.-M. Tsai, *On a uniform approximation of the density function of a string equation using eigenvalues and nodal points and some related inverse nodal problems* Inverse Problems, 11 (1995) 1113-1123.
[22] Y.-P. Wang, *A uniqueness theorem for indefinite Sturm-Liouville Operators*, Appl. Math. J. Chinese Univ. 27 (2012) 345-352.