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Invariant almost complex structures on real flag manifolds

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Abstract In this work we study the existence of invariant almost complex structures on real flag manifolds associated to split real forms of complex simple Lie algebras. We show that, contrary to the complex case where the invariant almost complex structures are well known, some real flag manifolds do not admit such structures. We check which invariant almost complex structures are integrable and prove that only some flag manifolds of the Lie algebra C_l admit complex structures.

Keywords Homogeneous manifold · Real flag manifold · Isotropy representation · Invariant almost complex structure

Mathematics Subject Classification (2000) MSC 22F30 · 32Q60 · 53C55

1 Introduction

A flag manifold of a non compact semisimple Lie algebra $\mathfrak{g}$, is a quotient space $\mathbb{F}_\Theta = G/P_\Theta$, where G is a connected group with Lie algebra $\mathfrak{g}$ and

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P_Θ is a parabolic subgroup. If $K \subset G$ is a maximal compact subgroup and $K_\Theta = K \cap P_\Theta$, then the flag $\mathbb{F}_\Theta$ can be written in the form $\mathbb{F}_\Theta = K/K_\Theta$.

In this work, we study the existence and integrability of invariant almost complex structures on real flag manifolds $\mathbb{F}_\Theta$ in the case that $\mathfrak{g}$ is a split real form of a complex simple Lie algebra. Our goal is to make an exhaustive investigation of the real flag manifolds $\mathbb{F}_\Theta$ that admit K-invariant almost complex structures and to verify their integrability, that is, when they are indeed complex structures.

The invariant geometry of complex flag manifolds has been extensively studied. Regarding invariant geometry of complex flag manifolds, the literature is exhaustive and goes back to Borel [2] and Wolf-Gray [22], [21]. Recent works are [1], [13], [3], [4], [14], [19], [20], [7], [9], [5] and [1].

For real flag manifolds the literature is much more sparse. There is no systematic treatment of the invariant geometric structures on these flag manifolds. An attempt to fill this gap was made recently by Patrão and San Martin [15] who provide a detailed analysis of the isotropy representations for the flag manifolds of the split real forms of the complex simple Lie algebras.

In this paper we rely on the results of [15] to build (or to prove the non-existence of) K-invariant almost complex structures on the real flag manifolds. The conclusion is that only a few flag manifolds (associated to split real forms) admit K-invariant almost complex structures. In this sense we obtain the following result.

Theorem 1 *A real flag manifold $\mathbb{F}_\Theta = K/K_\Theta$ admits a K-invariant almost complex structure structure if and only if it is a maximal flag of type A_3, B_2, G_2, C_l for l even or D_l for $l \geq 4$, or if it is one of the following intermediate flags:*

- *of type B_3 and $\Theta = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3\}$;*
- *of type C_l with $\Theta = \{\lambda_d - \lambda_{d+1}, \ldots, \lambda_{l-1} - \lambda_l, 2\lambda_l\}$ for $d > 1$, d odd.*
- *of type D_l with $l = 4$ and Θ being one of: $\{\lambda_1 - \lambda_2, \lambda_3 - \lambda_4\}$, $\{\lambda_1 - \lambda_2, \lambda_3 + \lambda_4\}$, $\{\lambda_3 - \lambda_4, \lambda_3 + \lambda_4\}$.*

The next step is to check which of the existing almost complex structures are integrable. By making computations with the Nijenhuis tensor we arrive at the following result.

Theorem 2 *A real flag manifold $\mathbb{F}_\Theta = K/K_\Theta$ admits K-invariant complex structures if and only if it is of type C_l and $\Theta = \{\lambda_d - \lambda_{d+1}, \ldots, \lambda_{l-1} - \lambda_l, 2\lambda_l\}$ with $d > 1$, d odd.*

These complex flag manifolds are realized as manifolds of flags $(V_1 \subset \cdots \subset V_k)$ of subspaces of $\mathbb{R}^{2l}$ that are isotropic with respect to the standard symplectic form of $\mathbb{R}^{2l}$. Moreover $\mathbb{F}_\Theta$ is finitely covered by $U(l)/U(l-d)$ and the complex structures on $\mathbb{F}_\Theta$ can be lifted to this covering space.

To prove the results above we mainly use the isotropy decomposition of $T_{b_\Theta}\mathbb{F}_\Theta$, the tangent space of the flag a the origin b_Θ. In [15] there are described the K_Θ-invariant and irreducible components of this representation obtaining a decomposition

$$T_{b_\Theta}\mathbb{F}_\Theta = V_1 \oplus \ldots \oplus V_k.$$

This decomposition is essential to find K-invariant geometries on $\mathbb{F}_\Theta$. It is well known that the compact isotropy group is a product $K_\Theta = M(K_\Theta)_0$ where M is the isotropy of the maximal flag and $(K_\Theta)_0$ the connected component of the identity. An almost complex structure commutes with the isotropy representation of K_Θ if and only if it commutes with the M and $(K_\Theta)_0$ representations on the tangent space. This allows us to split the proofs in two stages: study M-invariance on the one hand, and the condition of commutativity with ad_X for all $X \in \mathfrak{k}_\Theta = \mathrm{Lie}(K_\Theta)$, on the other hand.

A necessary and sufficient condition for a real flag to admit M-invariant almost complex structures is that every M-equivalence class on $\Pi^+ \setminus \langle\Theta\rangle^+$ has an even amount of elements. Two roots α and β lie in the same M-equivalence class if the representations of M on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are equivalent. This condition is necessary for $\mathbb{F}_\Theta$ to admit K_Θ invariant almost complex structures, so by inspection of these equivalence classes we discard many flags manifolds. For the remaining cases we focus on the $\mathfrak{k}_\Theta$ representation on $T_{b_\Theta}\mathbb{F}_\Theta$. We should remark that in all cases we give the almost complex structures explicitly, in a constructive way. Integrability is proved by computing the Nijenhuis tensor.

It is worth stressing a main difference in the isotropy representation of K_Θ between the real case and the complex case. In the real flag, there are cases where two K_Θ-invariant and irreducible components are equivalent. In the complex case this fact does not occur. Consequently, on the complex case, the K_Θ-invariant and irreducible components, in the isotropy representation of $\mathbb{F}_\Theta$, are invariant by almost complex structures. On the real flag, there are cases where $JV_i = V_j$, for V_i and V_j equivalent K_Θ-invariant and irreducible components.

This work is organized in the following manner. In Section 2 we fix notations and present the first results on existence of M-invariant complex structures. We give necessary and sufficient conditions for a flag manifold to admit such structure. In the case of a maximal flag, that is $\Theta = \emptyset$, this is all we need to pursue our study since $K_\Theta = M$. Section 3 focuses in this case. Section 4 deals with intermediate flags, that is $\Theta \neq \emptyset$. We only consider those intermediate flags verifying the necessary condition of Section 2. The full comprehension of the isotropy representation of K_Θ is needed, so we fully describe it for the cases under study. The propositions in Sections 3 and 4 account to Theorems 1 and 2 above.

2 Notation and preliminary results

We refer to [17, 11] for further developments of the concepts in this section. We assume throughout the paper that $\mathfrak{g}$ is the split real form of a complex simple Lie algebra $\mathfrak{g}_\mathbb{C}$. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition then $\mathfrak{a}$ is a Cartan subalgebra. Denote Π the set of roots of $\mathfrak{g}$ associated to $\mathfrak{a}$. If $\alpha \in \mathfrak{a}^*$ is a root then we write

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \mathrm{ad}\,(H)\,X = \alpha\,(H)\,X,\ H \in \mathfrak{a}\}$$

for the corresponding root space, which is one-dimensional since $\mathfrak{g}$ is split. Let Π^+ be a set of positive roots and Σ the corresponding positive simple roots.

The set of parabolic Lie subalgebras of $\mathfrak{g}$ is parametrized by the subsets of simple roots Σ. Given $\Theta \subset \Sigma$, the corresponding parabolic subalgebra is given by

$$\mathfrak{p}_\Theta = \mathfrak{a} \oplus \sum_{\alpha\in\Pi^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\langle\Theta\rangle^-} \mathfrak{g}_\alpha = \mathfrak{a} \oplus \sum_{\alpha\in\langle\Theta\rangle^+\cup\langle\Theta\rangle^-} \mathfrak{g}_\alpha \oplus \sum_{\alpha\in\Pi^+\setminus\langle\Theta\rangle^+} \mathfrak{g}_\alpha$$

where $\langle\Theta\rangle^\pm$ is the set of positive/negative roots generated by Θ.

Denote by G the group of inner automorphisms of $\mathfrak{g}$, which is connected and generated by $\exp\mathrm{ad}(\mathfrak{g})$ inside $\mathrm{GL}(\mathfrak{g})$. Let K be the maximal compact subgroup of G, then K is generated by $\mathrm{ad}(\mathfrak{k})$. The standard parabolic subgroup P_Θ of G is the normalizer of $\mathfrak{p}_\Theta$ in G. The associated flag manifold is defined by $\mathbb{F}_\Theta = G/P_\Theta$. The compact subgroup K acts transitively on $\mathbb{F}_\Theta$ so we obtain $\mathbb{F}_\Theta = K/K_\Theta$ where $K_\Theta = K \cap P_\Theta$. Fixing an origin b_Θ in $\mathbb{F}_\Theta$ we identify the tangent space $T_{b_\Theta}\mathbb{F}_\Theta$ with the nilpotent Lie algebra

$$\mathfrak{n}_\Theta^- = \sum_{\alpha\in\Pi^-\setminus\langle\Theta\rangle^-} \mathfrak{g}_\alpha.$$

In $\mathfrak{n}^-$, the isotropy representation of K_Θ on $T_{b_\Theta}\mathbb{F}_\Theta$ is just the adjoint representation, since $\mathfrak{n}_\Theta^-$ is normalized by K_Θ. The Lie algebra $\mathfrak{k}_\Theta$ of K_Θ is

$$\mathfrak{k}_\Theta = \sum_{\alpha\in\langle\Theta\rangle^+\cup\langle\Theta\rangle^-} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}.$$

Compactness of K implies that $\mathfrak{k}_\Theta$ admits a reductive complement $\mathfrak{m}_\Theta$ so that $\mathfrak{k} = \mathfrak{k}_\Theta \oplus \mathfrak{m}_\Theta$ and $T_{b_\Theta}\mathbb{F}_\Theta$ is identified also with $\mathfrak{m}_\Theta$. The map $X_\alpha \longrightarrow X_\alpha - X_{-\alpha}$ for $\alpha \in \Pi^-\setminus\langle\Theta\rangle^-$ is a K_Θ invariant map from $\mathfrak{n}_\Theta^-$ to $\mathfrak{m}_\Theta$. Along the paper we will call isotropy representation either the representation of K_Θ on $\mathfrak{n}_\Theta^-$ or on $\mathfrak{m}_\Theta$, without making any difference or special mention. In some cases we will even use $\mathfrak{n}_\Theta^+$ instead of $\mathfrak{n}_\Theta^-$.

Let M be the centralizer of $\mathfrak{a}$ in K. Then $K_\Theta = M\cdot(K_\Theta)_0$ where $(K_\Theta)_0$ is the connected component of the identity of K_Θ. Thus M acts on $T_{b_\Theta}\mathbb{F}_\Theta$ by restricting the isotropy representation of K_Θ. The group M is finite and acts on $\mathfrak{n}_\Theta^-$ leaving each root space $\mathfrak{g}_\alpha$ invariant. Moreover if $m \in M$ and $X \in \mathfrak{g}_\alpha$ then $\mathrm{Ad}(m)X = \pm X$. Two roots α and β are called M-equivalent, which we will denote by $\alpha \sim_M \beta$, if the representations of M on the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are equivalent. The M-equivalence classes were described in [15].

When $\Theta = \emptyset$, we drop all the sub indexes Θ. The associated flag manifold is the maximal flag $\mathbb{F} = K/M$ and the tangent space at the origin b will be identified with $\mathfrak{n}^-$.

Let U be a group of linear maps of the vector space V. A subspace $W \subset U$ is U-invariant if $ux \in W$ for all $x \in W$ and for all $u \in U$. A complex structure on V is endomorphism $J : V \longrightarrow V$ such that $J^2 = -1$ and it is said to be U-invariant if $uJ = Ju$ for all $u \in U$. We shall prove two technical results.

Lemma 1 *Let $W \subset V$ be a U-invariant space. Then the following statements are true:*

1. *JW is U-invariant as well.*
2. *W is irreducible if and only if JW is irreducible.*
3. *The representations of U on W and JW are equivalent.*
4. *If W is irreducible then either $W \cap JW = \{0\}$ or $JW = W$.*
5. *If $\dim W = 1$ then $W \cap JW = \{0\}$.*

Proof Take $u \in U$ and $x \in W$. Then, $uJx = Jux \in JW$ showing that JW is U-invariant.

Suppose that W is irreducible and let $A \subset JW$ be a U-invariant subspace. Then $J^{-1}A = JA \subset W$ is also U-invariant. Hence, $JA = W$ or $JA = \{0\}$, which implies that $A = JW$ or $A = \{0\}$. Thus JW is irreducible.

As J commutes with the elements of U, the map $J : W \to JW$ intertwines the representations on W and JW so that they are equivalent. Since $W \cap JW \subset W$ is U-invariant and W is irreducible we get item 4. Finally $W \cap JW = \{0\}$ if $\dim W = 1$ because the eigenvalues of J are $\pm i$ hence W is not invariant by J.

Lemma 2 *Let W_i, $i = 1, 2$ be U-invariant and irreducible subspaces of V such that $W_1 \cap W_2 = 0$ and the representation of U on W_1 is not equivalent to that on W_2. If $V = W_1 \oplus W_2 \oplus W$ for some complementary subspace W and J is a U-invariant complex structure, then $Jw_1 \in W_1 \oplus W$ for all $w_1 \in W_1$.*

Proof Consider $P : V \longrightarrow W_2$ the projection map with respect to the decomposition above. The map $P \circ J : W_1 \longrightarrow W_2$ is U-invariant and bijective if non-zero, since its domain and target spaces are irreducible. Thus it is an equivalence between the representations of U, if non-zero. Therefore, $P \circ J \equiv 0$ and the result follows.

Under the hypothesis of the lemma above, in the particular case of $V = W_1 \oplus W_2$ we have $JW_i = W_i$, $i = 1, 2$.

From the general theory of invariant tensors on homogeneous manifolds we know that K-invariant almost complex structures on the flag manifold $\mathbb{F}_\Theta = K/K_\Theta$ are in one to one correspondence with K_Θ-invariant complex structures $J : T_{b_\Theta}\mathbb{F}_\Theta \to T_{b_\Theta}\mathbb{F}_\Theta$. Recall that $T_{b_\Theta}\mathbb{F}_\Theta$ identifies with $\mathfrak{n}_\Theta^-$ (or $\mathfrak{m}_\Theta$) and this identification preserves the K_Θ representation. So K-invariant almost complex structures on $\mathbb{F}_\Theta$ also correspond to K_Θ-invariant complex structures on $\mathfrak{n}_\Theta^-$.

Let $J : \mathfrak{n}_\Theta^- \longrightarrow \mathfrak{n}_\Theta^-$ be a complex structure and assume it is only M-invariant. Since $K_\Theta = M(K_\Theta)_0$ we have that J is also K_Θ-invariant if and only if J commutes with the elements in $(K_\Theta)_0$, or equivalently, $\mathrm{ad}_X J = J \,\mathrm{ad}_X$ for all $X \in \mathfrak{k}_\Theta$ (because of connectedness).

Proposition 1 *Let $\mathbb{F}_\Theta$ be a real flag manifold associated to a split real form. Then a necessary and sufficient condition for the existence of a M-invariant complex structure $J : T_{b_\Theta}\mathbb{F}_\Theta \to T_{b_\Theta}\mathbb{F}_\Theta$ is that the amount of elements in each M-equivalence class $[\alpha]$ in $\Pi^- \setminus \langle\Theta\rangle^-$ is even.*

In this case the M-invariant complex structures are given by direct sums of invariant structures on the subspaces $V_{[\alpha]} = \sum_{\beta\sim_M\alpha} \mathfrak{g}_\beta \subset \mathfrak{n}_\Theta^-$. In a subspace $V_{[\alpha]}$ the set of M-invariant structures is parametrized by $\mathrm{Gl}(d,\mathbb{R})/\mathrm{Gl}(d/2,\mathbb{C})$, where $d = \dim V_{[\alpha]}$.

Proof If $\alpha \in \Pi^-\setminus\langle\Theta\rangle^-$ then $\mathfrak{g}_\alpha \in \mathfrak{n}_\Theta^-$ and $\dim \mathfrak{g}_\alpha = 1$ (because $\mathfrak{g}$ is a split real form). The subspace $J\mathfrak{g}_\alpha \subset \mathfrak{n}_\Theta^-$ is different of $\mathfrak{g}_\alpha$ by 5. in Lemma 1 and the representation of M in $J\mathfrak{g}_\alpha$ is equivalent to the representation on $\mathfrak{g}_\alpha$. Lemma 2 implies that $J\mathfrak{g}_\alpha$ is contained in the subspace $V_{[\alpha]} = \sum_{\beta\sim_M\alpha} \mathfrak{g}_\beta$. Applying the same argument to the roots β that are M-equivalent to α, we obtain $JV_\alpha = V_\alpha$. As $J^2 = -1$, it follows that $\dim V_\alpha$ is even and, hence, the amount of roots M-equivalent to α is even. This proves that the condition is necessary.

To see the sufficiency take a M-equivalent class $[\alpha]$ so that by assumption the subspace $V_{[\alpha]} = \sum_{\beta\sim_M\alpha} \mathfrak{g}_\beta$ is even dimensional. Given $m \in M$ we have $\mathrm{Ad}\,(m)\,X = \pm X$ if X belongs to a root space $X \in \mathfrak{g}_\beta$. In this equality the sign does not change when β runs through a M-equivalence class. It follows that $\mathrm{Ad}\,(m) = \pm 1$ on $V_{[\alpha]}$. Hence any complex structure on $V_{[\alpha]}$ is M-invariant. Taking direct sums of complex structures on the several $V_{[\alpha]}$ we get M-invariant complex structures on $T_{b_\Theta}\mathbb{F}_\Theta \simeq \mathfrak{n}_\Theta^-$.

Finally the set of complex structures in a d-dimensional real space (d even) is parametrized by $\mathrm{Gl}(d,\mathbb{R})/\mathrm{Gl}(d/2,\mathbb{C})$.

We use the results in [15] to present in Table 1 all possible subsets $\Theta \subset \Sigma$ for which the M-equivalence classes in $\Pi^- \setminus \langle\Theta\rangle^-$ have an even amount of elements. Even though we do not give the explicit computations to construct this table, we present the M-equivalence classes for some cases in the followings sections.

Complex structures on $\mathbb{F}_\Theta$ which are invariant under K are induced by K_Θ-invariant complex structures on the tangent space and, in particular, are

Type	Θ
A_3	$\emptyset$
B_2	$\emptyset$
B_3	$\{\lambda_1-\lambda_2, \lambda_2-\lambda_3\}$
C_4	$\emptyset$,$\{\lambda_1-\lambda_2, \lambda_3-\lambda_4\}$,$\{\lambda_3-\lambda_4, 2\lambda_4\}$
C_l, $l \neq 4$	$\emptyset$ only for l even, $\{\lambda_d-\lambda_{d+1}, \cdots, \lambda_{l-1}-\lambda_l, 2\lambda_l\}$, $1 < d \leq l-1$, d odd, for all l
D_4	$\emptyset$, $\{\lambda_1-\lambda_2, \lambda_3-\lambda_4\}$, $\{\lambda_1-\lambda_2, \lambda_3+\lambda_4\}$, $\{\lambda_3-\lambda_4, \lambda_3+\lambda_4\}$, $\{\lambda_1-\lambda_2, \lambda_2-\lambda_3, \lambda_3-\lambda_4\}$ $\{\lambda_1-\lambda_2, \lambda_2-\lambda_3, \lambda_3+\lambda_4\}$, $\{\lambda_2-\lambda_3, \lambda_3-\lambda_4, \lambda_3+\lambda_4\}$
D_l, $l \geq 5$	$\emptyset$, $\{\lambda_d-\lambda_{d+1}, \cdots, \lambda_{l-1}-\lambda_l, \lambda_{l-1}+\lambda_l\}$, $1 < d \leq l-1$.
G_2	$\emptyset$

Table 1: M-equivalence classes in $\Pi^- \setminus \langle\Theta\rangle^-$ with even elements

M-invariant. Hence Proposition 1 and a simple inspection of Table 1 give the following result.

Proposition 2 *Let $\mathbb{F}_\Theta$ be a real flag manifold associated to a split real form. If $\mathbb{F}_\Theta$ admits a K-invariant almost complex structure, then Θ is in Table 1.*

An invariant complex structure $J : \mathfrak{n}_\Theta^- \longrightarrow \mathfrak{n}_\Theta^-$ induced is integrable if the Nijenhuis tensor vanishes, that is if

$$N_J(X,Y) := [JX,JY]-[X,Y]-J[JX,Y]-J[X,JY] = 0, \quad \text{for all } X,Y \in \mathfrak{n}_\Theta^-.$$

3 K-invariant complex structures on maximal flags

For a maximal flag manifold the isotropy subgroup $\mathbb{K}_\Theta$ is the centralizer of $\mathfrak{a}$ inside K, that is, $\mathbb{K}_\Theta = M$. Hence Proposition 1 solves the question of existence of almost complex structures, remaining only integrability to be solved. The main result of this section is the following.

Proposition 3 *The maximal real flag $\mathbb{F}$ associated to a split real form admits a K-invariant almost complex structure if and only if $\mathbb{F}$ is of type A_3, B_2, G_2, C_l for even l and D_l for $l \geq 4$. None of these structures is integrable.*

Proof By Proposition 1, a maximal flag $\mathbb{F}$ admits an M-invariant almost complex structure if and only if it appears in Table 1.

Recall that a M-invariant almost complex structure in F is given by an endomorphism $J : \mathfrak{n}^- \longrightarrow \mathfrak{n}^-$ which is a sum of almost complex structures $J_{[\alpha]} : V_{[\alpha]} \longrightarrow V_{[\alpha]}$, for $\alpha \in \Pi^-$. We address integrability of these structures by fixing one of these $J : \mathfrak{n}^- \longrightarrow \mathfrak{n}^-$ and we study case by case.

Notice that if $V_{[\alpha]}$ is two dimensional with basis $\mathcal{B}$, then the matrix of $J_{[\alpha]}$ in $\mathcal{B}$ is

$$\begin{pmatrix} a & \frac{-(1+a^2)}{c} \\ c & -a \end{pmatrix}, \text{ with } a, c \in \mathbb{R},\ c \neq 0. \tag{1}$$

– Case A_3. The M-equivalence classes of negative roots are:

$$\{\lambda_2-\lambda_1, \lambda_4-\lambda_3\},\ \{\lambda_3-\lambda_1, \lambda_4-\lambda_2\} \text{ e } \{\lambda_4-\lambda_1, \lambda_3-\lambda_2\}.$$

Thus for $i = 2,3,4$, $\dim V_{[\lambda_i-\lambda_1]} = 2$ and it is spanned by $\{E_{i1}, E_{st}\}$ with $s > t$, $\{s,t\} \cap \{i,1\} = \emptyset$ and $\{s,t\} \cup \{i,1\} = \{1,\dots,4\}$; here E_{jk} is the 4×4 matrix with 1 in the jk entry and zero elsewhere. For $i = 2,3,4$, let $a_i, c_i \in \mathbb{R}$ such that $J|_{V_{[\lambda_i-\lambda_1]}}$ in this basis has the following form

$$\begin{pmatrix} a_i & \frac{-(1+a_i^2)}{c_i} \\ c_i & -a_i \end{pmatrix}, \quad c_i \neq 0.$$

Explicit computations give

$$N_J(E_{21}, E_{31}) = (c_3-c_2)c_4E_{32} + (c_2a_3 - a_2c_3 + a_4(c_3-c_2))E_{41},$$
$$N_J(E_{21}, E_{41}) = c_4(a_3-a_2)E_{31} + c_4(c_2+c_3)E_{42}.$$

These two equations cannot be zero simultaneously since $c_i \neq 0$. Thus the Nijenhuis tensor does not vanish and J is not integrable.

– Case B_2. The M-equivalence classes of negative roots are

$$\{\lambda_2 - \lambda_1, -\lambda_2 - \lambda_1\} \text{ e } \{-\lambda_1, -\lambda_2\}.$$

Let X_{21}, Y_{21}, X_1 and X_2 be elements of a Weyl basis generating $\mathfrak{g}_{\lambda_2-\lambda_1}$, $\mathfrak{g}_{-\lambda_2-\lambda_1}$, $\mathfrak{g}_{-\lambda_1}$ and $\mathfrak{g}_{-\lambda_2}$, respectively. Thus J verifies

$$\begin{array}{ll} JX_{21} = a_{21}X_{21} + c_{21}Y_{21}, & JX_1 = a_1X_1 + c_1X_2, \\ JY_{21} = -(1+a_{21}^2)X_{21}/c_{21} - a_{21}Y_{21}, & JX_2 = -(1+a_1^2)X_1/c_1 - a_1X_2, \end{array}$$

with $c_1, c_{21} \neq 0$.

Let $m = m_{\lambda_2-\lambda_1,-\lambda_2} \neq 0$ be the corresponding coefficient in the Weyl basis, that is, $[X_{21}, X_2] = mX_1$. Then

$$\begin{aligned} N_J(X_{21}, X_1) &= [JX_{21}, JX_1] - [X_{21}, X_1] - J[X_{21}, JX_1] - J[JX_{21}, X_1] \\ &= -mc_1^2X_2 + mc_1(a_{21} - a_1)X_1 \end{aligned}$$

which is never zero since $mc_1^2 \neq 0$. Therefore J is not integrable.

– Case C_4. The M-equivalence classes are:

$$\{\pm\lambda_2 - \lambda_1, \pm\lambda_4 - \lambda_3\}, \{\pm\lambda_3 - \lambda_1, \pm\lambda_4 - \lambda_2\}, \{\pm\lambda_4 - \lambda_1, \pm\lambda_3 - \lambda_2\},$$
$$\{-2\lambda_i : \ i = 1, \dots, 4\}.$$

Notice that $\dim V_{[2\lambda_1]} = \dim V_{[\lambda_i-\lambda_1]} = 4$ for $i = 2, 3, 4$. Let $(a_{ij})_{ij}$, $(b_{ij})_{ij}$, $(c_{ij})_{ij}$ be the matrices corresponding to $J|_{V_{[\lambda_2-\lambda_1]}}$, $J|_{V_{[\lambda_3-\lambda_1]}}$, $J|_{V_{[\lambda_4-\lambda_1]}}$ respectively in a Weyl basis of $\mathfrak{n}^-$.

Then $N_J(X_{-\lambda_2-\lambda_1}, X_{-2\lambda_2}) = 0$ and $N_J(X_{-\lambda_4-\lambda_3}, X_{-2\lambda_4}) = 0$ imply $a_{12} = a_{34} = 0$ and moreover $a_{14}^2 + a_{24}^2 \neq 0$ because otherwise $X_{-\lambda_4-\lambda_3}$ would be an eigenvector of J. Analogously we obtain $b_{12} = b_{34} = c_{12} = c_{34} = 0$ and $b_{14}^2 + b_{24}^2 \neq 0$, $c_{14}^2 + c_{24}^2 \neq 0$.

With these conditions, $N_J(X_{-\lambda_2-\lambda_1}, X_{-2\lambda_4}) = 0$ imply $a_{32} = 0$ and $a_{42} \neq 0$. Similar computations give $b_{32} = c_{32} = 0$ and $b_{42} \neq 0$, $c_{42} \neq 0$. Now $J^2 = -1$ imply $a_{14} = b_{14} = c_{14} = 0$.

All this account to $N_J(X_{\lambda_2-\lambda_1}, X_{-\lambda_3-\lambda_1}) = 0$ and $N_J(X_{\lambda_2-\lambda_1}, X_{-\lambda_4-\lambda_1}) = 0$ only if, respectively, $a_{31} = c_{42}$ and $a_{31} = -c_{42}$. This clearly cannot hold since $c_{42} \neq 0$.

– Case C_l, l even and $l \geq 6$. The M-equivalence classes are

$$\{\pm\lambda_s - \lambda_i\}, \ 1 \leq i < s \leq l, \quad \text{and} \quad \{2\lambda_1, \dots, 2\lambda_l\}.$$

Let X_{si}, Y_{si} and X_j be the generators of the roots spaces $\mathfrak{g}_{\lambda_s-\lambda_i}$, $\mathfrak{g}_{-\lambda_s-\lambda_i}$ and $\mathfrak{g}_{-2\lambda_j}$, respectively, corresponding to a Weyl basis. In this case we have $\dim V_{[\lambda_s-\lambda_i]} = 2$ while $\dim V_{[2\lambda_1]} = l$, even. Thus $JX_1 = \sum_{j=1}^{l} b_jX_j$ and for $s = 1, \dots, l$ we have

$$JX_{s1} = a_{s1}X_{s1} + c_{s1}Y_{s1}, \quad JY_{s1} = -\frac{(1+a_{s1}^2)}{c_{s1}}X_{s1} - a_{s1}Y_{s1}, \quad c_{s1} \neq 0.$$

We compute the Nijenhuis tensor on the vectors X_1 and X_{s1}, for $s = 2, \dots, l$. Denote $m = m_{\lambda_s - \lambda_1, -2\lambda_s} \neq 0$, then we get

$$\begin{aligned} N_J(X_{s1}, X_1) &= [JX_{s1}, JX_1] - [X_{s1}, X_1] - J[X_{s1}, JX_1] - J[JX_{s1}, X_1] \\ &= [a_{s1}X_{s1} + c_{s1}Y_{s1}, \sum_{j=1}^{l} b_j X_j] - b_s m JY_{s1} \\ &= a_{s1} b_s m Y_{s1} - b_s m(-\frac{(1+a_{s1}^2)}{c_{s1}} X_{s1} - a_{s1} Y_{s1}) \\ &= b_s m \frac{(1+a_{s1}^2)}{c_{s1}} X_{s1} + a_{s1}(b_s m + 1) Y_{s1}. \end{aligned}$$

Hence $N_J(X_{s1}, X_1) = 0$ if and only if $b_s m = 0$. Thus J integrable implies $b_s = 0$ for $s = 2, \dots, l$. and therefore $JX_1 = b_1 X_1$, which contradicts the fact that $J^2 = -1$. Thus J is not integrable.

– Case D_4. The M-equivalence classes are

$$\{\pm\lambda_2 - \lambda_1, \pm\lambda_4 - \lambda_3\}, \{\pm\lambda_3 - \lambda_1, \pm\lambda_4 - \lambda_1\}, \{\pm\lambda_4 - \lambda_1, \pm\lambda_3 - \lambda_2\}.$$

Clearly, $\dim V_{[\lambda_i - \lambda_1]} = 4$ for $i = 2, 3, 4$. We proceed as in the C_4 case. Let $(a_{ij})_{ij}$, $(b_{ij})_{ij}$, $(c_{ij})_{ij}$ be the matrices corresponding to $J|_{V_{[\lambda_2-\lambda_1]}}$, $J|_{V_{[\lambda_3-\lambda_1]}}$, $J|_{V_{[\lambda_4-\lambda_1]}}$, respectively, in a Weyl basis of $\mathfrak{n}^-$.
By imposing $N_J(X_\gamma, X_\delta) = 0$ for $\gamma \in [\lambda_3 - \lambda_1]$ and $\delta \in [\lambda_4 - \lambda_1]$ we obtain that the matrix of $J|_{V_{[\lambda_4-\lambda_1]}}$ in the Weyl basis is

$$\begin{pmatrix} -b_{44} & -b_{34} & b_{24} & b_{14} \\ -b_{43} & -b_{33} & b_{23} & b_{13} \\ b_{42} & b_{32} & -b_{22} & -b_{12} \\ b_{41} & b_{31} & -b_{21} & -b_{11} \end{pmatrix}.$$

With this, $N_J(X_{\lambda_2-\lambda_1}, X_{-\lambda_4-\lambda_1}) = 0$, $N_J(X_{-\lambda_4-\lambda_3}, X_{-\lambda_3-\lambda_1}) = 0$ and $N_J(X_{\lambda_4-\lambda_3}, X_{-\lambda_3-\lambda_1}) = 0$ imply $b_{12}b_{32} = 0$, $b_{12}b_{42} = 0$ and $b_{32}b_{42} = 0$. But we know that $a_{12}^2 + a_{32}^2 + a_{42}^2 \neq 0$ since $X_{-\lambda_2-\lambda_1}$ is not an eigenvector. So we conclude that only one of b_{12}, b_{32}, b_{42} is not zero. In each of the three cases we obtain $a_{12} = a_{32} = a_{42} = 0$ if N_J vanishes, which cannot happen since $X_{-\lambda_2-\lambda_1}$ is not an eigenvector of J.

– Case D_l, $l \geq 5$. The M-equivalence classes are:

$$\{\pm\lambda_j - \lambda_i\}, \quad 1 \leq i < j \leq l.$$

For $1 \leq i < j \leq l$, we have $\dim V_{[\lambda_j - \lambda_i]} = 2$; let X_{ij} be a generator of $\mathfrak{g}_{\lambda_i - \lambda_j}$ and let Y_{ij} be a generator of $\mathfrak{g}_{\lambda_i + \lambda_j}$. Thus $V_{[\lambda_j - \lambda_i]}$ is spanned by $\{X_{ij}, Y_{ij}\}$ and J in this basis has a matrix of the form

$$\begin{pmatrix} a_{ij} & \frac{-(1+a_{ij}^2)}{c_{ij}} \\ c_{ij} & -a_{ij} \end{pmatrix}, \text{ where } c_{ij} \neq 0.$$

Conditions $N_J(X_{13}, X_{23}) = 0$ and $N_J(X_{12}, X_{23}) = 0$ imply

$$\frac{m_{\lambda_1-\lambda_2,\lambda_2+\lambda_3}}{m_{\lambda_1-\lambda_2,\lambda_2-\lambda_3}} = \frac{c_{13}}{c_{23}} = -\frac{m_{\lambda_1-\lambda_3,\lambda_2+\lambda_3}}{m_{\lambda_1+\lambda_3,\lambda_2-\lambda_3}}. \tag{2}$$

Now using Jacobi identity, we have

$$\begin{aligned}0 &= [Y_{23}, [X_{12}, X_{23}]] - [[Y_{23}, X_{12}], X_{23}] - [X_{12}, [Y_{23}, X_{23}]] \\ &= m_{\lambda_1-\lambda_2,\lambda_2-\lambda_3}[Y_{23}, X_{13}] + m_{\lambda_2+\lambda_3,\lambda_1-\lambda_2}[X_{23}, Y_{13}] \\ &= (m_{\lambda_1-\lambda_2,\lambda_2-\lambda_3} m_{\lambda_2+\lambda_3,\lambda_1-\lambda_3} + m_{\lambda_2+\lambda_3,\lambda_1-\lambda_2} m_{\lambda_2-\lambda_3,\lambda_1+\lambda_3}) Y_{12}.\end{aligned}$$

Thus

$$\begin{aligned}m_{\lambda_1-\lambda_2,\lambda_2-\lambda_3} m_{\lambda_2+\lambda_3,\lambda_1-\lambda_3} &= -m_{\lambda_2+\lambda_3,\lambda_1-\lambda_2} m_{\lambda_2-\lambda_3,\lambda_1+\lambda_3} \\ &= -m_{\lambda_1-\lambda_2,\lambda_2+\lambda_3} m_{\lambda_1+\lambda_3,\lambda_2-\lambda_3},\end{aligned}$$

and therefore

$$\frac{m_{\lambda_1-\lambda_2,\lambda_2+\lambda_3}}{m_{\lambda_1-\lambda_2,\lambda_2-\lambda_3}} = \frac{m_{\lambda_1-\lambda_3,\lambda_2+\lambda_3}}{m_{\lambda_1+\lambda_3,\lambda_2-\lambda_3}}. \tag{3}$$

This equation clearly contradicts (2) and hence J is not integrable.

– Case G_2. The M-equivalence classes are

$$\{-\lambda_1, -2\lambda_2 - \lambda_1\},\ \{-\lambda_2 - \lambda_1, -3\lambda_2 - \lambda_1\},\ \{-\lambda_2, -3\lambda_2 - 2\lambda_1\}.$$

For $(i, j) \in \{(1, 0), (0, 1), (1, 1)\}$, $\dim V_{[-i\lambda_1 - j\lambda_2]} = 2$. In a Weyl basis of $\mathfrak{n}^-$ we have that the matrix of $J|_{V_{[-i\lambda_1-j\lambda_2]}}$ has the form

$$\begin{pmatrix} a_{ij} & \frac{-(1+a_{ij}^2)}{c_{ij}} \\ c_{ij} & -a_{ij} \end{pmatrix},\ \text{where } c_{ij} \neq 0.$$

Denote $m = m_{-(\lambda_1+\lambda_2),-\lambda_2}$ then

$$\begin{aligned}N_J(X_{-\lambda_1-\lambda_2}, X_{-\lambda_2}) &= m(a_{11}a_{01} - 1)X_{-\lambda_1-2\lambda_2} - m(a_{11} + a_{01})JX_{-\lambda_1-2\lambda_2} \\ &= m\left((a_{11}a_{01} - 1) + a_{10}(a_{11} + a_{01})\right) X_{-\lambda_1-2\lambda_2} \\ &\quad + m(a_{11} + a_{01})\frac{1 + a_{10}^2}{c_{10}} X_{-\lambda_1}.\end{aligned}$$

Thus

$$N_J(X_{-\lambda_1-\lambda_2}, X_{-\lambda_2}) = 0 \ \Leftrightarrow\ a_{01} = -a_{11} \ \text{and}\ a_{11}a_{01} = 1,$$

and J is not integrable.

4 *K*-Invariant complex structures on intermediate flags

In this section we study existence of invariant almost complex structures on intermediate flags $\mathbb{F}_\Theta$, and their integrability. We obtain the classification of the flags admitting K-invariant complex structures, only some of type C_l do, and also we describe the complex structures explicitly.

Proposition 2 states that if $\mathbb{F}_\Theta = K/K_\Theta$ with $\Theta \neq \emptyset$ admits a K-invariant almost complex structure, then $\mathbb{F}_\Theta$ is one of the following:

- of type B_3 and $\Theta = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3\}$;
- of type C_l with $l = 4$ and $\Theta = \{\lambda_1 - \lambda_2, \lambda_3 - \lambda_4\}$ or $\Theta = \{\lambda_3 - \lambda_4, 2\lambda_4\}$; or $l \neq 4$ and $\Theta = \{\lambda_d - \lambda_{d+1}, \dots, \lambda_{l-1} - \lambda_l, 2\lambda_l\}$ for $d > 1$, d odd.
- of type D_l with $l = 4$ and Θ being one of: $\{\lambda_1 - \lambda_2, \lambda_3 - \lambda_4\}$, $\{\lambda_1 - \lambda_2, \lambda_3 + \lambda_4\}$, $\{\lambda_3 - \lambda_4, \lambda_3 + \lambda_4\}$, $\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - \lambda_4\}$, $\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 + \lambda_4\}$, $\{\lambda_2 - \lambda_3, \lambda_3 - \lambda_4, \lambda_3 + \lambda_4\}$; or $l \geq 5$ and $\Theta = \{\lambda_d - \lambda_{d+1}, \cdots, \lambda_{l-1} - \lambda_l, \lambda_{l-1} + \lambda_l\}$ for $1 < d \leq l - 1$.

We analyse the cases B, C and D separately in the next subsections. We need to treat them separately since the isotropy representations differ significantly. In the three cases we start by imposing necessary conditions for the flag to admit an invariant complex structure, which we shall describe in the next paragraph. We obtain that only in few cases one can obtain that type of structure.

Recall that K-invariant almost complex structures on $\mathbb{F}_\Theta$ are in one to one correspondence with K_Θ-invariant maps $J : \mathfrak{n}_\Theta^- \longrightarrow \mathfrak{n}_\Theta^-$ such that $J^2 = -1$.

Assume $J : \mathfrak{n}_\Theta^- \longrightarrow \mathfrak{n}_\Theta^-$ is K_Θ-invariant and $J^2 = -1$. Then J is necessarily M-invariant since $M \subset K_\Theta = M(K_\Theta)_0$, hence by Proposition 1 we have

$$JV_{[\alpha]} = V_{[\alpha]} \text{ for each } \alpha \in \Pi^- \setminus \langle \Theta \rangle^- . \tag{4}$$

In addition, J is also $(K_\Theta)_0$ invariant and therefore

$$\operatorname{ad}_X J = J \operatorname{ad}_X \text{ for all } X \in \mathfrak{k}_\Theta. \tag{5}$$

Assume $\mathfrak{n}_\Theta^- = W_1 \oplus \cdots \oplus W_s$ is a decomposition on K_Θ-invariant and irreducible subspaces. If the representation on W_i is not equivalent to the representation on any other W_j, $j \neq i$ then $JW_i = W_i$ because of Lemma 2. Notice that if this is the case W_i is even dimensional. To the contrary, if $JW_i = W_j$ for some $i \neq j$, then the K_Θ representation on these subspaces are equivalent, and J gives such an equivalence.

To address the non-existence of almost complex structures, we prove that some of the necessary conditions above cannot hold simultaneously. For the cases where an invariant almost complex structure does exist, we use these necessary conditions to build them explicitly. Notice that, for instance, if $J : \mathfrak{n}_\Theta^- \longrightarrow \mathfrak{n}_\Theta^-$ with $J^2 = -1$ satisfying (4) and (5) is K_Θ invariant.

We remark that the conditions related to the K_Θ and $\mathfrak{k}_\Theta$ representation on $\mathfrak{n}_\Theta^-$ are dealt with through a description of $\mathfrak{g}$ as a matrix Lie algebra. Integrability of the almost complex structure is established by computing the Nijenhuis tensor, as in the maximal flag case.

4.1 Flags of $B_3 = \mathfrak{so}(3,4)$.

The set of simple roots is $\Sigma = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3\}$, and we take $\Theta = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3\}$ obtaining $\langle\Theta\rangle = \pm\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_1 - \lambda_3\}$. Notice that the flag is a six dimensional manifold. The M-equivalence classes outside of Θ are: $\{\lambda_1 + \lambda_2, \lambda_3\}$, $\{\lambda_1 + \lambda_3, \lambda_2\}$ and $\{\lambda_2 + \lambda_3, \lambda_1\}$. The compact subgroup $(K_\Theta)_0$ is isomorphic to SO(3).

We consider the realization of $B_3 = \mathfrak{so}(3,4)$ in real matrices of the type

$$\begin{pmatrix} 0 & \beta & \gamma \\ -\gamma^T & A & B \\ -\beta^T & C & -A^T \end{pmatrix},$$

with A, B, C are 3×3 matrices, β, γ 1×3 matrices and $B + B^T = C + C^T = 0$. Then, $(K_\Theta)_0$ (respectively M) is given by matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{pmatrix},$$

with $g \in \mathrm{SO}(3)$ (respectively g diagonal with entries ± 1 and an even amount of -1 entries). The root space corresponding to the short root λ_1 is given by matrices where the components A, B, C and β vanish and γ is a multiple of $e_1 = (1,0,0)$. The same holds for the roots λ_2 and λ_3 with $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$, respectively. The root spaces corresponding to $\lambda_i + \lambda_j$ have B as unique non-vanishing component and it has the following form, depending on the long root:

$$\lambda_1 + \lambda_2 : B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_1 + \lambda_3 : B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\lambda_2 + \lambda_3 : B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The subspaces $V_c = \sum_i \mathfrak{g}_{\lambda_i}$ and $V_l = \sum_{i,j} \mathfrak{g}_{\lambda_i + \lambda_j}$ are both invariant subspaces under the adjoint representation of $K_\Theta = M \cdot \mathrm{SO}(3)$. The representation of the SO (3) on V_c is isomorphic to canonical representation on $\mathbb{R}^3$, while the representation on V_l is the adjoint representation. These two representations of SO(3) are isomorphic. In fact, an isomorphism is constructed via the identification of $\mathbb{R}^3$ with the imaginary quaternions $\mathbb{H}$: if $p, q \in \mathbb{H}$ then $\mathrm{ad}(q)p = [q,p] \in \mathrm{Im}\ \mathbb{H}$ and $\mathrm{ad}(q) \in \mathfrak{so}(3)$ that commutes with the representations of the SO(3). This isomorphism also commutes with the representations of M. Indeed, considering the basis $\{e_1, e_2, e_3\} = \{i, j, k\} \in \mathbb{R}^3 = \mathrm{Im}\ \mathbb{H}$, we have

$$\operatorname{ad}(i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \ \operatorname{ad}(j) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$
$$\text{and } \operatorname{ad}(k) = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The isomorphism $P : V_c \to V_l$ takes the root spaces $\mathfrak{g}_{\lambda_1}$, $\mathfrak{g}_{\lambda_2}$ and $\mathfrak{g}_{\lambda_3}$ to the root spaces $\mathfrak{g}_{\lambda_2+\lambda_3}$, $\mathfrak{g}_{\lambda_1+\lambda_3}$ and $\mathfrak{g}_{\lambda_1+\lambda_2}$, respectively. In addition, it commutes with the representation of $(K_\Theta)_0$ and with the representations of M. Therefore, $P : V_c \to V_l$ commutes with the representation of K_Θ.

Proposition 4 *The flag manifold $\mathbb{F}_\Theta$ of B_3 with $\Theta = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3\}$ admits K-invariant almost complex structures and each of them is given by J_a for some $a \neq 0$ where $J_a : \mathfrak{n}_\Theta^+ \longrightarrow \mathfrak{n}_\Theta^+$ is defined by*

$$J_a(X) = aP(X) \text{ if } X \in V_c, \quad J_a(X) = -a^{-1}P^{-1}(X) \text{ if } X \in V_l.$$

These structures are not integrable.

Proof We have $\mathfrak{n}_\Theta^+ = V_c \oplus V_l$ as K_Θ-invariant irreducible subspaces and because of the reasoning above, J_a is indeed invariant by K_Θ. Thus, there is a one-parameter family of invariant almost complex structures on $\mathbb{F}_\Theta$.

Furthermore, a K_Θ-invariant complex structure J on $\mathfrak{n}_\Theta^+$ is of this form. In fact, any K_Θ-invariant complex structure $J : \mathfrak{n}_\Theta^+ \longrightarrow \mathfrak{n}_\Theta^+$ interchanges V_c with V_l by 4. in Lemma 1, since these are irreducible odd dimensional subspaces. Moreover the subspaces $\mathfrak{g}_{\lambda_1+\lambda_2} \oplus \mathfrak{g}_{\lambda_3}$, $\mathfrak{g}_{\lambda_1+\lambda_3} \oplus \mathfrak{g}_{\lambda_2}$, $\mathfrak{g}_{\lambda_2+\lambda_3} \oplus \mathfrak{g}_{\lambda_1}$ are J-invariant because of (4). The fact that $\operatorname{ad}_X J = J \operatorname{ad}_X$ for all $X \in \mathfrak{k}_\Theta$ implies that J is actually a multiple of P.

These structures are never integrable. In fact, $[V_c, V_c] = V_l$ and $[V_l, \mathfrak{n}_\Theta^+] = 0$. Thus, for $X, Y \in V_c$ we have $J_a X, J_a Y \in V_l$ and therefore $N_{J_a}(X, Y) = -[X, Y]$. Hence N_J never vanishes.

Remark 1 This flag $\mathbb{F}_\Theta$ of type B_3 and $\Theta = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3\}$ is the Grassmannian of three dimensional isotropic subspaces of $\mathbb{R}^7$, that is, three dimension subspaces in which the quadratic form matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_{3\times 3} \\ 0 & 1_{3\times 3} & 0 \end{pmatrix}$$

vanishes. The proposition above gives a family of K-invariant almost complex structures on this flag which is parametrized by $\mathbb{R} \backslash \{0\}$.

4.2 Flags of $C_l = \mathfrak{sp}(l, \mathbb{R})$

The set of simple roots is $\Sigma = \{\lambda_1 - \lambda_2, \ldots, \lambda_{l-1} - \lambda_l, 2\lambda_l\}$. For the analysis of these flags, we separate the case $l = 4$ where the M-equivalence classes are different from the general case.

4.2.1 Case C_l, $l \neq 4$

Assume $l \neq 4$ and let $\Theta = \{\lambda_{d+1} - \lambda_{d+2}, \ldots, \lambda_{l-1} - \lambda_l, 2\lambda_l\}$ with $d \in \{0, \cdots, l\}$ and d even. Notice that Θ gives a Dynkin sub diagram C_p of C_l with $p = l - d$, thus $\mathfrak{k}_\Theta$ is the maximal compact subalgebra of $\mathfrak{sp}(p, \mathbb{R})$, that is, $\mathfrak{k}_\Theta \simeq \mathfrak{u}(p)$.

The M-equivalence classes in $\Pi^+ \setminus \langle\Theta\rangle^+$ are

$$\{\lambda_i - \lambda_j, \lambda_i + \lambda_j\},\ 1 \leq i \leq d,\ i < j \leq l, \quad \text{and} \quad \{2\lambda_1, \ldots, 2\lambda_d\}.$$

For each positive root α denote $\mathfrak{t}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k}$. Then $\mathfrak{k} = \mathfrak{k}_\Theta \oplus \mathfrak{m}_\Theta$ where $\mathfrak{k}_\Theta$ is the vector space sum of $\mathfrak{t}_\alpha$ where α runs in $\langle\Theta\rangle^+$ and

$$\mathfrak{m}_\Theta = \sum_{1 \leq i \leq d, i < j \leq l} \mathfrak{t}_{\lambda_i - \lambda_j} \oplus \mathfrak{t}_{2\lambda_1} \oplus \cdots \oplus \mathfrak{t}_{2\lambda_d}$$

is a reductive complement of $\mathfrak{k}_\Theta$.

The invariant and irreducible subspaces of $\mathfrak{m}_\Theta$ by the K_Θ action were described in [15, Section 5.3] and we present them below. Define

$$R = \{\lambda_i \pm \lambda_j : 1 \leq i < j \leq d\} \cup \{2\lambda_i : 1 \leq i \leq d\}.$$
$$\Pi_i = \{\lambda_i \pm \lambda_j : d + 1 \leq j \leq l\}, \quad i = 1 \ldots, d,$$

and let $W_R = \sum_{\alpha \in R} \mathfrak{k}_\alpha$ and $W_i = \sum_{\alpha \in \Pi_i} \mathfrak{k}_\alpha$, $i = 1, \ldots, d$. We have

$$\mathfrak{m}_\Theta = W_R \oplus \sum_{i=1}^{d} W_i \tag{6}$$

and the subspaces above are M-invariant.

If $\alpha \in R$ and $\beta \in \Theta$, then $\pm\alpha \pm \beta$ is never a root so $[Y, X] = 0$ for any $Y \in \mathfrak{k}_\Theta$ and $X \in W_R$. Thus $\mathrm{Ad}(g)X = X$ for any $g \in (K_\Theta)_0$, since $(K_\Theta)_0$ is connected, and therefore W_R is invariant by $\mathrm{Ad}(K_\Theta)$.

Each subspace W_i is K_Θ invariant and is irreducible subspace and the respective representations are not equivalent if $i \neq j$ (see [15, Lemma 5.11]). We make use of the following isomorphism between the compact algebra $\mathfrak{k}$ and $\mathfrak{u}(l)$ given by

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \longmapsto A + iB, \quad A + A^T = B - B^T = 0.$$

The isomorphism takes $\mathfrak{k}_\Theta$ in the algebra of anti-hermitian matrices of the form

$$\mathfrak{k}_\Theta : \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, \tag{7}$$

being X a $p \times p$ matrix. Moreover W_R corresponds to the matrices of the form

$$W_R : \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},$$

with $d \times d$ upper left block, while the subspace $W = \sum_{i=1}^{d} W_i$ corresponds to

$$W : \begin{pmatrix} 0 & -\overline{C}^T \\ C & 0 \end{pmatrix}, \tag{8}$$

where C is $d \times p$. A subspace W_j is given by those matrices C having non vanishing entries in column j. The representation of $\mathfrak{k}_\Theta$ in W is given by the adjoint action:

$$\left[\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, \begin{pmatrix} 0 & -\overline{C}^T \\ C & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & \overline{C}^T X \\ XC & 0 \end{pmatrix}.$$

Thus C having non-vanishing entries on column j implies that the same occurs for XC. So the subspaces W_j are, in fact, invariant.

The image of $\mathfrak{k}_{\lambda_j - \lambda_k}$ in $\mathfrak{u}(l)$ through the isomorphism is generated by the real anti-symmetric matrix $A_{jk} = E_{jk} - E_{kj}$, while the image of $\mathfrak{k}_{\lambda_j + \lambda_k}$ is generated by the imaginary symmetric matrix $S_{jk} = i(E_{jk} + E_{kj})$.

Lemma 3 *1. An almost complex structure $J : \mathfrak{m}_\Theta \longrightarrow \mathfrak{m}_\Theta$ is M-invariant if and only if J leaves invariant each subspace $\mathfrak{k}_{\lambda_i - \lambda_j} \oplus \mathfrak{k}_{\lambda_i + \lambda_j}$ and $\mathfrak{k}_{2\lambda_1} \oplus \cdots \oplus \mathfrak{k}_{2\lambda_d}$.*
2. An M-invariant almost complex structure J is K_Θ-invariant if and only if for each $j = 1, \ldots, d$ there is some $\varepsilon_j = \pm 1$ such that $JA_{kj} = \varepsilon_j S_{kj}$ and $JS_{kj} = -\varepsilon_j A_{kj}$ for all $d < k \leq l$.

Proof Let $J : \mathfrak{m}_\Theta \longrightarrow \mathfrak{m}_\Theta$ be an isomorphism such that $J^2 = -1$. From Proposition 1 and taking into account the M-equivalence classes given above we have that J is M-invariant if and only if it preserves each $\mathfrak{k}_{\lambda_i - \lambda_j} \oplus \mathfrak{k}_{\lambda_i + \lambda_j}$ and $\mathfrak{k}_{2\lambda_1} \oplus \cdots \oplus \mathfrak{k}_{2\lambda_d}$.

Now assume J is M-invariant, then J is K_Θ-invariant if and only if $\mathrm{ad}_Y J = J \,\mathrm{ad}_Y$ for all $Y \in \mathfrak{k}_\Theta$.

Notice that J preserves each W_i and W_R in (6). Since $[X, Y] = 0$ for all $Y \in \mathfrak{k}_\Theta$, $X \in W_R$ we see that $J|_{W_R}$ is K_Θ-invariant. Recall that W_i is spanned by A_{ji}, S_{ji} with $d + 1 \leq j \leq l$.

Let $Y \in \mathfrak{k}_\Theta$ be as in (7) with X imaginary diagonal matrix, i.e., $X = \mathrm{diag}(ia_1, \ldots, ia_m)$. We have $\mathrm{ad}(Y)A_{kj} = a_j S_{kj}$ and $\mathrm{ad}(Y)S_{kj} = -a_j A_{kj}$ for some $a_j \in \mathbb{R}$. That is, $\mathfrak{k}_{\lambda_j - \lambda_k} \oplus \mathfrak{k}_{\lambda_j + \lambda_k}$ is invariant by $\mathrm{ad}(Y)$ and the matrix of $\mathrm{ad}(Y)$ in the basis $\{A_{kj}, S_{kj}\}$ is

$$\begin{pmatrix} 0 & -a_j \\ a_j & 0 \end{pmatrix}. \tag{9}$$

If we denote J_{kj} the restriction of J to $\mathfrak{k}_{\lambda_j-\lambda_k} \oplus \mathfrak{k}_{\lambda_j+\lambda_k}$, for $k > j$ we see that J_{kj} commutes with $\mathrm{ad}(Y)$ only when its matrix in the basis $\{A_{kj}, S_{kj}\}$ is

$$J_{kj} = \varepsilon_{kj} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{with } \varepsilon_{kj} = \pm 1. \tag{10}$$

Fix $j \in \{1, \ldots, d\}$ and let $l \geq s, t \geq d+1$, consider Z be as in Eq. (7) with $X = E_{ts} - E_{st}$ and let D be as in Eq. (8) with $C = E_{sj}$. Then

$$\mathrm{ad}(Z)D = \begin{pmatrix} 0 & -\overline{XC}^T \\ XC & 0 \end{pmatrix}, \qquad \text{with } XC = E_{tj}.$$

This implies that that $\mathrm{ad}(Z)A_{sj} = A_{tj}$ and $\mathrm{ad}(Z)S_{sj} = S_{tj}$. Recall that J in the basis restricted to $\mathfrak{k}_{\lambda_j-\lambda_k} \oplus \mathfrak{k}_{\lambda_j+\lambda_k}$ has a matrix of the form in Eq. (10) in the appropriate basis. In order J to commute with $\mathrm{ad}(Z)$ above, we need

$$\varepsilon_{tj} S_{tj} = JA_{tj} = J\,\mathrm{ad}(Z)A_{sj} = \mathrm{ad}(Z)JA_{sj} = \mathrm{ad}(Z)\varepsilon_{sj}S_{sj} = \varepsilon_{sj}S_{tj}.$$

Thus $\varepsilon_{sj} = \varepsilon_{tj}$ for all $l \geq s, t \geq d+1$, and we define ε_j this value. We have then $JA_{kj} = \varepsilon_j S_{kj}$ and $JS_{kj} = -\varepsilon_j A_{kj}$ for all $d < k \leq l$.

Next we prove that this condition is sufficient for J to commute with the adjoint of elements in $\mathfrak{k}_\Theta$. Indeed, for j, s, t as above, we only have left to verify that J commutes with matrices Z as in Eq. (7) with with $X = i(E_{ts} + E_{st})$. We consider D as in Eq. (8) with $C = E_{sj}$, then $XC = iE_{tj}$ and we obtain $\mathrm{ad}(Z)A_{sj} = S_{tj}$. Likewise, if $C = iE_{sj}$, then $XC = -E_{tj}$ and thus $\mathrm{ad}(Z)S_{sj} = -A_{tj}$. Therefore

$$\mathrm{ad}(Z)JA_{sj} = \varepsilon_j \mathrm{ad}(Z)S_{sj} = -\varepsilon_j A_{tj} = JS_{tj} = J\mathrm{ad}(Z)A_{sj}$$

and

$$\mathrm{ad}(Z)JS_{sj} = -\varepsilon_j \mathrm{ad}(Z)A_{sj} = -\varepsilon_j S_{tj} = -JA_{tj} = J\mathrm{ad}(Z)S_{sj}.$$

Remark 2 The set of K invariant almost complex structures on the flags $\mathbb{F}_\Theta$ in Lemma 3 is parametrized by $\mathrm{Gl}(d-1,\mathbb{R})/\mathrm{Gl}(d-1/2,\mathbb{C}) \times (\mathbb{R}^2 \cup \mathbb{R}^2)^{d(d-1)} \times \mathbb{Z}_2^d$.

The component $\mathrm{Gl}(d-1,\mathbb{R})/\mathrm{Gl}(d-1/2,\mathbb{C})$ corresponds to the complex structures on the space generated by long roots outside $\langle\Theta\rangle^+$. The component $(\mathbb{R}^2 \cup \mathbb{R}^2)^{d(d-1)}$ corresponds to the structures on the spaces generated by the roots $\{\lambda_j - \lambda_k, \lambda_j + \lambda_k\}$. The set $\mathbb{R}^2 \cup \mathbb{R}^2$ is the disjoint union of the two copies of $\mathbb{R}^2$, that is $\mathrm{Gl}(2,\mathbb{R})/\mathrm{Gl}(1,\mathbb{C})$. Finally, $\mathbb{Z}_2^{(d-1)}$ parametrizes the signs ε_j.

We introduce two technical lemmas which will lead to the determination of the integrable structures.

Lemma 4 *Let J be a K_Θ-invariant almost complex structure. If J is integrable then for each $i, j \in \{1, \ldots, d\}$, $j > i$, we have $JA_{ji} = c_{ji}S_{ji}$ and $JS_{ji} = -c_{ji}A_{ji}$, with $c_{ji} = \pm 1$.*

Proof Take $1 \leq i < j \leq d$ then by M-invariance $JS_{ii} = \sum_k b_{ki} S_{kk}$ and

$$J|_{\{A_{ji},S_{ji}\}} = \begin{pmatrix} a_{ji} & -\frac{1+a_{ji}^2}{c_{ji}} \\ c_{ji} & -a_{ji} \end{pmatrix} \text{ where } c_{ji} \neq 0.$$

We have

$$\begin{aligned} N_J(S_{ii}, A_{ji}) = A_{ji} & \Big(2c_{ji}(b_{ii} - b_{ji}) + 2(b_{ji} - b_{ii}) \tfrac{(1+a_{ji}^2)}{c_{ji}} - 2c_{ji}a_{ji} - 2a_{ji} \tfrac{(1+a_{ji}^2)}{c_{ji}} \Big) \\ & + S_{ji} \left(2a_{ji}(b_{ji} - b_{ii}) + 2 + 2a_{ji}(b_{ji} - b_{ii}) - 2(a_{ji}^2 + c_{ji}^2) \right) \end{aligned}$$

Therefore

$$\begin{aligned} N_J(S_{ii}, A_{ji}) = 0 &\Leftrightarrow \begin{cases} (b_{ii} - b_{ji})(c_{ji}^2 - 1 - a_{ji}^2) - a_{ji}(c_{ji}^2 + 1 + a_{ji}^2) = 0 \\ c_{ji}^2 = 2a_{ji}(b_{ji} - b_{ii}) + 1 - a_{ji}^2 \end{cases} \\ &\Leftrightarrow \begin{cases} c_{ji}^2 = 2a_{ji}(b_{ji} - b_{ii}) + 1 - a_{ji}^2 \\ ((b_{ii} - b_{ji})^2 + 1)a_{ji} = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} a_{ji} = 0 \\ c_{ji} = \pm 1 \end{cases} . \end{aligned}$$

Up to this moment we have proved that if J is K_Θ-invariant and integrable then for each $j = 1, \ldots, d$:

- $JA_{kj} = c_{kj}S_{kj}$ and $JS_{kj} = -c_{kj}A_{kj}$ for $k = 1, \ldots, d$, $k \neq j$ and
- $JA_{kj} = \varepsilon_j S_{kj}$ and $JS_{kj} = -\varepsilon_j A_{kj}$ for all $k = d+1, \ldots, l$.

where $\varepsilon_j, c_{kj} \in \{\pm 1\}$. To simplify notation in the following lemma we write

$$JA_{kj} = \mu_{kj} S_{kj}, \quad JS_{kj} = -\mu_{kj} A_{kj} \text{for all } j = 1, \ldots, d,\ j < k \neq l. \qquad (11)$$

Lemma 5 *Let J be a K_Θ-invariant (integrable) complex structure. Then for any triple $k > j > s$ such that $j, s \in \{1, \ldots, d\}$ the possible values for $(\mu_{ks}, \mu_{kj}, \mu_{js})$ are:*

$$(\mu_{ks}, \mu_{ks}, \mu_{ks}),\ (\mu_{ks}, -\mu_{ks}, \mu_{ks}) \text{ and } (\mu_{ks}, \mu_{ks}, -\mu_{ks}).,\quad \mu_{ks} = \pm 1.$$

In particular, if $\varepsilon_j = -\varepsilon_s$ then $c_{js} = \varepsilon_s$.

Proof By equation (11) we obtain

$$\begin{aligned} 0 \ = \ N_J(A_{kj}, A_{ks}) &= (1 + \mu_{kj}\mu_{js} - \mu_{kj}\mu_{ks} - \mu_{ks}\mu_{js})\, A_{js} \\ &= ((\mu_{kj} - \mu_{ks})\mu_{js} + (\mu_{ks} - \mu_{kj})\mu_{ks})\, A_{js} \\ &= ((\mu_{js} - \mu_{ks})\mu_{kj} + (\mu_{js} - \mu_{ks})\mu_{js})\, A_{js}. \end{aligned}$$

From the second row of this equation we see that $\mu_{kj} = -\mu_{ks}$ implies $\mu_{js} = \mu_{ks}$; while the third row implies $\mu_{kj} = -\mu_{js} = \mu_{ks}$ if $\mu_{js} = -\mu_{ks}$. We conclude then that the possible values for the triple $(\mu_{ks}, \mu_{kj}, \mu_{js})$ are: $(\mu_{ks}, \mu_{ks}, \mu_{ks})$, $(\mu_{ks}, -\mu_{ks}, \mu_{ks})$ and $(\mu_{ks}, \mu_{ks}, -\mu_{ks})$.

Proposition 5 *Let $J : \mathfrak{m}_\Theta \longrightarrow \mathfrak{m}_\Theta$ be such that $J^2 = -1$ and moreover it preserves $\mathfrak{k}_{2\lambda_1} \oplus \cdots \oplus \mathfrak{k}_{2l_d}$ and $JA_{kj} = \mu_{kj} S_{kj}$, $JS_{kj} = -\mu_{kj} A_{kj}$ for all $j = 1, \ldots, d$, $j < k \leq d$, with $\mu_{kj} = \pm 1$.*

Then J is K_Θ-invariant and integrable if and only if the following hold:

- *for each $j = 1, \ldots, d$, $\mu_{kj} = \varepsilon_j$ for all $k = d+1, \ldots, l$.*
- *for each triple $k > j > s$ such that $j, s \in \{1, \ldots, d\}$ the coefficients $(\mu_{ks}, \mu_{kj}, \mu_{js})$ are one of the following:*

$$(\mu_{ks}, \mu_{ks}, \mu_{ks}),\ (\mu_{ks}, -\mu_{ks}, \mu_{ks}) \text{ and } (\mu_{ks}, \mu_{ks}, -\mu_{ks}).$$

Conversely, any K-invariant complex structure on $\mathbb{F}_\Theta$ is induced by J as above.

Proof It is necessary for J to be M-invariant to preserve $\mathfrak{k}_{2\lambda_1} \oplus \cdots \oplus \mathfrak{k}_{2l_d}$ and $\mathfrak{k}_{\lambda_j - \lambda_k} \oplus \mathfrak{k}_{\lambda_j + \lambda_k}$. The conditions above are necessary as proved in Lemma 3 in order J to be K_Θ-invariant and Lemma 4 and Lemma 5 to be integrable. As seen there, such J verifies $N_J(S_{kk}, A_{kj}) = 0$ $j = 1, \ldots, d$, $j < k \leq l$ and $N_J(A_{kj}, A_{ks}) = 0$ for each triple in the second item. To show that these conditions are sufficient we have to show that i) $N_J(S_{kk}, S_{kj}) = 0$, ii) $N_J(S_{kj}, S_{ks}) = 0$, iii) $N_J(S_{kj}, A_{ks}) = 0$ and iv) $N_J(S_{jj}, S_{ss}) = 0$ for all $j > s \in \{1, \ldots, d\}$ and $k > j > s$.

Clearly iv) holds since these matrices are diagonal. Moreover, $N_J(A_{kj}, A_{ks}) = N_J(S_{kj}, S_{ks})$ so ii) also holds. Similar computations as in the proof of Lemma 4 give i). Finally $N_J(S_{kj}, A_{ks}) = (-1 - \mu_{kj}\mu_{js} + \mu_{kj}\mu_{ks} + \mu_{ks}\mu_{js})\, S_{js}$ so reasoning as in Lemma 5 one obtains iii).

Example 1 We consider the flag $\mathbb{F}_\Theta$ of C_3, with $\Theta = \{2\lambda_3\}$. The component W_R of tangent space at the origin of flag is given by sum of $\mathfrak{k}_\alpha$, $\alpha \in R$, and has the following form: $R = \{\lambda_1 \pm \lambda_2\} \cup \{2\lambda_1, 2\lambda_2\}$. The components W_j are determined by the sets of roots

$$\Pi_1 = \{\lambda_1 \pm \lambda_3\}, \quad \Pi_2 = \{\lambda_2 \pm \lambda_3\}.$$

Fix $\varepsilon_j = \pm 1$ $j = 1, 2$ $\nu = \pm 1$ such that

$$(\varepsilon_1, \varepsilon_2, \nu) \in \{(1,1,1), (-1,-1,-1), (1,-1,1), (-1,1,-1), (1,1,-1), (-1,-1,1)\},$$

and let $a_{11}, c_{11} \in \mathbb{R}$ s.t. $c_{11} \neq 0$. The following table gives all K_Θ-invariant integrable complex structures J in $\mathbb{F}_\Theta$.

Components	K_Θ-invariant complex structures
W_1	$JA_{31} = \varepsilon_1 S_{31}$, $JS_1 = -\varepsilon_1 A_{31}$,
W_2	$JA_{32} = \varepsilon_2 S_{32}$, $JS_{32} = -\varepsilon_2 A_{32}$
W_R	$JA_{21} = \nu S_{21}$, $JS_{21} = -\nu A_{21}$, $JS_{11} = a_{11} S_{11} + c_{11} S_{22}$, $JS_{22} = -\frac{1+a_{11}^2}{c_{11}} S_{11} - a_{11} S_{22}$

4.2.2 Case C_4

The M-equivalence classes of positive roots are

$$\{\lambda_1 \pm \lambda_2, \lambda_3 \pm \lambda_4\},\ \{\lambda_1 \pm \lambda_3, \lambda_2 \pm \lambda_4\}\{\lambda_1 \pm \lambda_4, \lambda_2 \pm \lambda_3\},\ \{2\lambda_1, 2\lambda_2, 2\lambda_3, 2\lambda_4\}.$$

Proposition 6 *The real flag $\mathbb{F}_\Theta$ of C_4 with $\Theta = \{\lambda_1 - \lambda_2, \lambda_3 - \lambda_4\}$ does not admit K-invariant almost complex structures.*

Proof According to [15, Section 5.3] the K_Θ irreducible components of $\mathfrak{n}_\Theta^-$ are given by

$$\begin{array}{ll} V_1 = \langle X_{2\lambda_1} - X_{2\lambda_2}, X_{-\lambda_2-\lambda_1} \rangle & V_4 = \langle X_{2\lambda_3} + X_{2\lambda_4} \rangle, \\ V_2 = \langle X_{2\lambda_1} + X_{2\lambda_2} \rangle, & V_3 = \langle X_{2\lambda_3} - X_{2\lambda_4}, X_{-\lambda_4-\lambda_3} \rangle \end{array}$$
$$\begin{aligned} V_5 &= \langle X_{\lambda_3-\lambda_1} + X_{\lambda_4-\lambda_2}, X_{\lambda_3-\lambda_2} - X_{\lambda_4-\lambda_1} \rangle \\ V_6 &= \langle X_{\lambda_3-\lambda_2} + X_{\lambda_4-\lambda_1}, X_{\lambda_4-\lambda_2} - X_{\lambda_3-\lambda_1} \rangle, \\ V_7 &= \langle X_{-\lambda_3-\lambda_1} + X_{-\lambda_4-\lambda_2}, X_{-\lambda_3-\lambda_2} - X_{-\lambda_4-\lambda_1} \rangle \\ V_8 &= \langle X_{-\lambda_3-\lambda_2} + X_{-\lambda_4-\lambda_1}, X_{-\lambda_4-\lambda_2} - X_{-\lambda_3-\lambda_1} \rangle, \end{aligned}$$

where X_α is a generator of root space $\mathfrak{g}_\alpha$.

The components V_2, V_5 and V_6 are equivalent to the components V_4, V_7 and V_8, respectively. The subspaces V_1 and V_3 are neither equivalent between them nor to any other representation subspace.

Assume J is a K_Θ-invariant complex structure J on $\mathfrak{n}_\Theta^-$. Then $JV_1 = V_1$ since it is irreducible and non-equivalent to any other representation subspace. Moreover, $V_{[-\lambda_2-\lambda_1]} = \mathfrak{g}_{-\lambda_2-\lambda_1} \oplus \mathfrak{g}_{-\lambda_4-\lambda_3}$ and J preserves this subspaces too because of its M-invariance. Therefore $V_1 \cap V_{[-\lambda_2-\lambda_1]} = \langle X_{-\lambda_2-\lambda_1} \rangle$ is an invariant subspace of J, which is a contradiction. So we conclude that no K-invariant complex structure exists in this case.

Fix $\Theta = \{\lambda_3 - \lambda_4, 2\lambda_4\}$ for C_4. The K_Θ-irreducible components of $\mathfrak{m}_\Theta$ are ([15, Section 5.3]):

$$\begin{array}{ll} V_1 = \mathfrak{g}_{-2\lambda_1}, & V_3 = \mathfrak{g}_{\lambda_2-\lambda_1}, \\ V_2 = \mathfrak{g}_{-2\lambda_2}, & V_4 = \mathfrak{g}_{-\lambda_2-\lambda_1}, \end{array}$$
$$\begin{aligned} V_5 &= \mathfrak{g}_{\lambda_3-\lambda_1} \oplus \mathfrak{g}_{-\lambda_3-\lambda_1} \oplus \mathfrak{g}_{\lambda_4-\lambda_1} \oplus \mathfrak{g}_{-\lambda_4-\lambda_1}, \\ V_6 &= \mathfrak{g}_{\lambda_3-\lambda_2} \oplus \mathfrak{g}_{-\lambda_3-\lambda_2} \oplus \mathfrak{g}_{\lambda_4-\lambda_2} \oplus \mathfrak{g}_{-\lambda_4-\lambda_2}. \end{aligned} \tag{12}$$

The components V_1 and V_3 are equivalent to, respectively, the components V_2 and V_4. The components V_5 and V_6 are not equivalent.

As in the previous section, we consider the isomorphism between $\mathfrak{k}$ and $\mathfrak{u}(4)$. Under this map, $\mathfrak{k}_\Theta = \langle \{A_{43}, S_{43}, S_{33}, S_{44}\} \rangle$ and

$$\mathfrak{m}_\Theta = W_R \oplus \bigoplus_{\substack{j=1,2 \\ k=3,4}} W_{kj}$$

where $W_R = W_R^1 \oplus W_{21}$ with $W_R^1 = \langle \{S_{11}, S_{22}\} \rangle$ and $W_{kj} = \langle \{A_{kj}, S_{kj}\} \rangle$.

Proposition 7 *The flag manifold $\mathbb{F}_\Theta$ of C_4 with $\Theta = \{\lambda_3 - \lambda_4, 2\lambda_4\}$ admits K-invariant almost complex structures and each of them is induced by a map $J : \mathfrak{m}_\Theta \longrightarrow \mathfrak{m}_\Theta$ verifying*

$$\begin{array}{ll} JS_{11} = \nu_1 S_{22},\ JS_{22} = -\nu_1^{-1} S_{11} & \text{with } \nu_1 \neq 0, \\ JA_{21} = \nu_2 S_{21},\ JS_{21} = -\nu_2^{-1} A_{21} & \text{with } \nu_2 \neq 0, \\ JA_{kj} = \varepsilon_j S_{kj},\ JS_{kj} = -\varepsilon_j A_{kj} & \text{for } k \in \{3,4\},\ j \in \{1,2\}, \\ & \text{with } \varepsilon_j = \pm 1. \end{array}$$

Such structure is integrable if and only if $\nu_2 = \pm 1$ and $\nu_2 = \varepsilon_1$ if $\varepsilon_2 = -\varepsilon_1$.

Proof We already know that $\mathbb{F}_\Theta$ admits M-invariant almost complex structures and such J is the direct sum of almost complex structures in each $V_{[\alpha]}$, $\alpha \in \Pi^+ \setminus \langle\Theta\rangle^+$. In this case, the M-equivalence classes are

$$\{\lambda_1 - \lambda_2, \lambda_1 + \lambda_2\},\ \{\lambda_1 \pm \lambda_3, \lambda_2 \pm \lambda_4\},\ \{\lambda_1 \pm \lambda_4, \lambda_2 \pm \lambda_3\},\ \{2\lambda_1, 2\lambda_2\}.$$

So, in particular, W_R^1, W_{21}, $W_{31} \oplus W_{42}$ and $W_{32} \oplus W_{41}$ are J-invariant.

Moreover, since $V_5 = W_{31} \oplus W_{41}$ and $V_6 = W_{32} \oplus W_{42}$ in (12) are irreducible and non-equivalent, we have $JV_5 = V_5$ and $JV_6 = V_6$. Therefore each W_{kj}, $k = 3, 4$, $j = 1, 2$ is invariant, since it can be described as an intersection of $V_{[\alpha]}$ and V_t for suitable root and index.

We proceed as in the general case C_l, $l \neq 4$ to show that J has the form given in the statement of the proposition.

For any $Y \in \mathfrak{k}_\Theta$ and $Z \in W_R$, we have $[Y, Z] = 0$ so J restricted to this subspace is also $\mathfrak{k}_\Theta$-invariant. Let $Y = a_3 S_{33} + a_4 S_{44} \in \mathfrak{k}_\Theta$, then $\mathrm{ad}_Y J = J\,\mathrm{ad}_Y$ implies that for $k = 3, 4$, $j = 1, 2$ the matrix of $J|_{W_{kj}}$ in the basis $\{A_{kj}, S_{kj}\}$ is

$$\mu_{kj} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \mu_{kj} = \pm 1.$$

Now let $Y = a_3 A_{43} + a_4 S_{43} \in \mathfrak{k}_\Theta$ and let $Z \in W_{kj}$ with $k = 3, 4$, then $\mathrm{ad}_Y JZ = J\,\mathrm{ad}_Y Z$ holds if and only if $\varepsilon_{4j} = \varepsilon_{3j}$ for $j = 1, 2$. It is not hard to see that these conditions are also sufficient for J to be K_Θ-invariant.

To address integrability, notice that, as in the general case, we have

$$\begin{aligned} N_J(S_{11}, A_{21}) &= -2\left(\nu_1(\nu_2 - \nu_2^{-1})A_{21} + (-1 + \nu_2^2)S_{21}\right) \\ N_J(A_{41}, A_{42}) &= \left(\varepsilon_1\varepsilon_2 - 1 + (\varepsilon_1 - \varepsilon_2)\nu_2^{-1}\right) A_{21} \end{aligned}$$

Therefore J is integrable if $\nu_2 = \pm 1$ and $\nu_2 = \varepsilon_1$ in the case that $\varepsilon_1 = \varepsilon_2$. One can check that these conditions are sufficient for J to be integrable.

4.3 Flags of $D_l = \mathfrak{so}(l, l)$

A root system is given by $\pm\lambda_i \pm \lambda_j$, $i \neq j$, and the corresponding set of simple roots is given by $\Sigma = \{\lambda_1 - \lambda_2, \ldots, \lambda_{l-1} - \lambda_l, \lambda_{l-1} + \lambda_l\}$, $1 \leq i < j \leq l$. The maximal compact subalgebra of $\mathfrak{so}(l, l)$ is $\mathfrak{k} \simeq \mathfrak{so}\,(l) \oplus \mathfrak{so}\,(l)$.

As in the C_l case, we deal first with the case D_l with $l \geq 5$ and later we address the case of $l = 4$ because of the difference between the M-equivalence classes.

4.3.1 Case D_l, $l \geq 5$

We consider $\Theta = \{\lambda_d - \lambda_{d+1}, \ldots, \lambda_{l-1} - \lambda_l, \lambda_{l-1} + \lambda_l\}$, this gives a sub diagram D_p of D_l with $p = l - d + 1$, thus $\mathfrak{k}_\Theta \simeq \mathfrak{so}(p)_1 \oplus \mathfrak{so}(p)_2$. The set $\langle\Theta\rangle$ of roots generated by Θ is given by

$$\langle\Theta\rangle = \{\pm(\lambda_i \pm \lambda_j) : d \leq i < j \leq l\}.$$

The roots in $\Pi^+ \setminus \langle\Theta\rangle^+$ are

$$\lambda_i \pm \lambda_j \text{ with } 1 \leq i < j \leq d, \quad \text{and } \lambda_i \pm \lambda_j \text{ with } i = 1, \ldots, d-1,\ j = d, \ldots l.$$

and the M-equivalence classes are $\{\lambda_i - \lambda_j, \lambda_i + \lambda_j\}$. Consider the subsets of roots in $\Pi^+ \setminus \langle\Theta\rangle^+$:

$$R = \{\lambda_i \pm \lambda_j : 1 \leq i < j \leq d\}$$
$$\Pi_i = \{\lambda_i \pm \lambda_j : d \leq j \leq l\}, \quad i = 1, \ldots, d-1$$

and let $W_R = \sum_{\alpha \in R} \mathfrak{g}_\alpha$ and $W_i = \sum_{\alpha \in \Pi_i} \mathfrak{g}_\alpha$. Clearly we obtain

$$\mathfrak{n}_\Theta^+ = W_R \oplus \sum_{i=1}^{d-1} W_i. \tag{13}$$

The subspace W_R is K_Θ invariant and irreducible. Moreover, each W_i decomposes as $W_i = V_i^1 \oplus V_i^2$, where V_i^j is irreducible K_Θ-invariant and the representations are not equivalent [15]. We present an explicit description of these subspaces.

A split real form of D_l is $\mathfrak{so}(l,l)$ and it is represented by real matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}, \text{ where } B + B^T = C + C^T = 0. \tag{14}$$

The algebra $\mathfrak{g}(\Theta)$ generated by $\mathfrak{g}_\alpha$, $\alpha \in \langle\Theta\rangle$ is given by matrices in Eq. (14) such that A, B and C have the form

$$\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix},$$

where the non-zero part is squared of size $p = l - d + 1$. The Lie algebra $\mathfrak{g}(\Theta)$ is of type D_p, isomorphic to $\mathfrak{so}(p,p)$.

The compact part $\mathfrak{k}$ inside $\mathfrak{so}(l,l)$ is given by the subset matrices in (14) having the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \text{ where } A + A^T = B + B^T = 0.$$

It is well known that $\mathfrak{k}$ decomposes as a sum of two ideals, both isomorphic to $\mathfrak{so}(l)$. The compact Lie algebra $\mathfrak{k}_\Theta$ lies inside $\mathfrak{k}$ and also inside $\mathfrak{g}(\Theta)$ and consists of matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \text{ with } A, B \in \langle\{E_{st} - E_{ts} : d \leq s < t \leq l\}\rangle. \tag{15}$$

The Lie algebra $\mathfrak{k}_\Theta$ also decomposes as a sum of two ideals, both isomorphic to $\mathfrak{so}(p)$, which are

$$\mathfrak{so}(p)_1 = \left\{ \begin{pmatrix} A & A \\ A & A \end{pmatrix} : A \in \langle \{E_{st} - E_{ts} : d \leq s < t \leq l\}\rangle \right\},$$
$$\mathfrak{so}(p)_2 = \left\{ \begin{pmatrix} A & -A \\ -A & A \end{pmatrix} : A \in \langle \{E_{st} - E_{ts} : d \leq s < t \leq l\}\rangle \right\}.$$

Fix $i \in \{1, \ldots, d-1\}$ and denote $S_i = \{X = (a_{st}) \in \mathfrak{gl}(l, \mathbb{R}) : a_{st} = 0 \text{ for all } (st) \notin \{(ij) : j = d, \ldots, l\}\}$. For any $j = d, \ldots, l$ the root space $\mathfrak{g}_{\lambda_i - \lambda_j}$ is represented by matrices (14) where $A = E_{ij}$, $C = B = 0$; meanwhile, $\mathfrak{g}_{\lambda_i + \lambda_j}$ is represented by the matrices of the above form where $B = E_{ij} - E_{ji}$, $A = C = 0$. Thus W_i is given by

$$\begin{pmatrix} X & Y - Y^t \\ 0 & -X^T \end{pmatrix}, \qquad \text{where } X, Y \in S_i. \tag{16}$$

For $Z \in S_i$ denote

$$X_Z = \begin{pmatrix} Z & Z - Z^T \\ 0 & -Z^T \end{pmatrix}, \quad Y_Z = \begin{pmatrix} Z & -Z + Z^T \\ 0 & -Z^T \end{pmatrix} \tag{17}$$

and define $V_i^1 = \{X_Z : Z \in S_i\}$, $V_i^2 = \{Y_Z : Z \in S_i\}$. Clearly, $V_i^1, V_i^2 \subset W_i$. Moreover, a matrix as in (16) can be written as the sum of two matrices in (17) by taking $Z = (X+Y)/2$, $Z' = (X-Y)/2$. Thus we obtain $W_i = V_i^1 \oplus V_i^2$ and $\dim V_i^1 = \dim V_i^2 = l - d + 1 = p = |\Theta|$.

We compute the $\mathfrak{k}_\Theta$ action on V_i^1 and V_i^2: let $N \in \mathfrak{k}_\Theta$ as in (15) and let $X_Z \in V_i^1$, $Y_Z \in V_i^2$ then $AZ = 0 = BZ$, $Z^T A = 0 = Z^T B$ so

$$[N, X_Z] = X_{-Z(A+B)}, \quad \text{and} \quad [N, Y_Z] = Y_{-Z(A-B)}.$$

This implies that $\mathfrak{so}(p)_2$ acts trivially on V_i^1 while for $N \in \mathfrak{so}(p)_1$ the action is $[N, X_Z] = X_{-2ZA}$. Similarly, $\mathfrak{so}(p)_1$ acts trivially on V_i^2 while for $N \in \mathfrak{so}(p)_2$ the action is $[N, X_Z] = X_{-2ZA}$. We conclude that the $\mathfrak{k}_\Theta$ representation on V_i^1 is equivalent to the $\mathfrak{so}(p) \oplus \mathfrak{so}(p)$ representation on $\mathbb{R}^p$ where the action of $\mathfrak{so}(p)_1$ is the canonical and the action of $\mathfrak{so}(p)_2$ is trivial. Similarly, the $\mathfrak{k}_\Theta$ representation on V_i^2 is equivalent to the $\mathfrak{so}(p) \oplus \mathfrak{so}(p)$ representation on $\mathbb{R}^p$ where the action of $\mathfrak{so}(p)_1$ is by zero and the action of $\mathfrak{so}(p)_2$ is the canonical one.

We keep $i = 1, \ldots, d-1$ fixed. Let $s, t \in \{d \ldots l\}$, $s \neq t$ and consider N_{st}^1 being as in (15) with $A = E_{st} - E_{ts}$ and $B = A$ (i.e. $N \in \mathfrak{so}(p)_1$). Then

$$\left[N_{st}^1, X_{E_{is}}\right] = -2X_{E_{it}} \text{ and } [N_{st}^1, X_{E_{it}}] = 2X_{E_{is}}, \text{ while } [N_{st}^1, Y_{E_{ij}}] = 0 \text{ for all } j. \tag{18}$$

Similarly, denote $N_{st}^2 \in \mathfrak{so}(p)_2$ being the matrix in $\mathfrak{k}_\Theta$ associated to $A = E_{st} - E_{ts}$ and $B = -A$, then

$$\left[N_{st}^2, Y_{E_{is}}\right] = -2Y_{E_{it}} \text{ and } [N_{st}^2, Y_{E_{it}}] = 2Y_{E_{is}}, \text{ while } [N_{st}^2, X_{E_{ij}}] = 0 \text{ for all } j. \tag{19}$$

Having described the $\mathfrak{k}_\Theta$ representation on $\mathfrak{n}_\Theta^+$ we can state:

Proposition 8 *The real flags $\mathbb{F}_\Theta$ of D_l with $l \geq 5$ and $\Theta \neq \emptyset$ do not admit K_Θ-invariant complex structures.*

Proof Assume $J : \mathfrak{n}_\Theta^+ \longrightarrow \mathfrak{n}_\Theta^+$ is a K_Θ-invariant almost complex structure. As it is M-invariant and each subspace in (13) is sum of M-equivalence classes, we have that $JW_R = W_R$ and $JW_i = W_i$ for all $i = 1, \ldots, d-1$.

Recall that W_i is not irreducible, for $i = 1, \ldots, d-1$. Instead $W_i = V_i^1 \oplus V_i^2$ where each of these subspace is invariant and irreducible by the K_Θ action, and the induced representations are not equivalent [15]. By Lemma 2, we conclude that V_i^1, V_i^2 are J-invariant. In particular, V_i^1 and V_i^2 are even dimensional and thus p is even.

Fix $i = 1, \ldots, d-1$ and let $j \in \{d, \ldots, l\}$. In the notation (17) one can see that $\mathfrak{g}_{\lambda_i - \lambda_j} \oplus \mathfrak{g}_{\lambda_i + \lambda_j} = \langle \{X_{E_{ij}}, Y_{E_{ij}}\} \rangle$, which is a J-invariant subspace of W_i because of the M-invariance of J. Thus $JX_{E_{ij}} = a_{ij}X_{E_{ij}} + c_{ij}Y_{E_{ij}}$ with $c_{ij} \neq 0$. For any $s \in \{d, \ldots, l\}$, $s \neq j$ we apply (18) and obtain

$$\begin{aligned} \mathrm{ad}_{N_{sj}^1} JX_{E_{ij}} &= \mathrm{ad}_{N_{sj}^1}(a_{ij}X_{E_{ij}} + c_{ij}Y_{E_{ij}}) = -2a_{ij}X_{E_{is}}, \quad \text{while} \\ J\,\mathrm{ad}_{N_{sj}^1} X_{E_{ij}} &= J(-2X_{E_{is}}) = -2(a_{is}X_{E_{is}} + c_{is}Y_{E_{is}}), \end{aligned}$$

but $c_{is} \neq 0$, contradicting the K_Θ-invariance of J.

4.3.2 Case D_4

Now we proceed to the study of flags of D_4 with Θ as in Table 1. The M-equivalence classes of positive roots in D_4 are:

$$\{\lambda_1 - \lambda_2, \lambda_1 + \lambda_2, \lambda_3 - \lambda_4, \lambda_3 + \lambda_4\},\ \{\lambda_1 - \lambda_3, \lambda_1 + \lambda_3, \lambda_2 - \lambda_4, \lambda_2 + \lambda_4\}$$
$$\{\lambda_1 - \lambda_4, \lambda_1 + \lambda_4, \lambda_2 - \lambda_3, \lambda_2 + \lambda_3\}.$$

As in the general case we work with the split form $\mathfrak{so}(4,4)$. In what follows we denote by $X_{ij} = E_{i,j} - E_{l+j,l+i}$ a generator of $\mathfrak{g}_{\lambda_i - \lambda_j}$ and by $Y_{ij} = E_{i,l+j} - E_{j,l+i}$ a generator $\mathfrak{g}_{\lambda_i + \lambda_j}$, where $E_{i,j}$ is the 8×8 matrix with 1 in the position ij and zeroes elsewhere.

The group M consists of 8×8 diagonal matrices $\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ where $\epsilon_i = \pm 1$ and $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$, that is, there is an even amount of -1's in the diagonal of matrices of M.

Proposition 9 *The real flag manifold $\mathbb{F}_\Theta$ of type D_4 with $\Theta = \{\lambda_1 - \lambda_2, \lambda_3 - \lambda_4\}$ admits K_Θ invariant almost complex structures. These structures are not integrable.*

Proof The following is the decomposition of $\mathfrak{n}_\Theta^+$ in K_Θ invariant and irreducible subspaces

$$\mathfrak{n}_\Theta^+ = \mathfrak{g}_{\lambda_1 + \lambda_2} \oplus \mathfrak{g}_{\lambda_3 + \lambda_4} \oplus \sum_{i=1}^{4} V_i,$$

where

$$V_1 = \langle X_{13} - X_{24}, X_{14} + X_{23}\rangle \qquad V_3 = \langle Y_{13} - Y_{24}, Y_{14} + Y_{23}\rangle$$
$$V_2 = \langle X_{13} + X_{24}, X_{14} - X_{23}\rangle \qquad V_4 = \langle Y_{13} + Y_{24}, Y_{14} - Y_{23}\rangle$$

The map $T_{13} : V_1 \longrightarrow V_3$ defined by $T_{13}(X_{13} - X_{24}) = Y_{13} - Y_{24}$ and $T_{13}(X_{14} + X_{23}) = Y_{14} + Y_{23}$ commutes with $\mathrm{ad}_{\mathfrak{k}_\Theta}$. Moreover, the linear map $T_{24} : V_2 \longrightarrow V_4$, verifying $T_{24}(X_{13} + X_{24}) = Y_{23} + Y_{24}$ and $T_{24}(X_{14} - X_{23}) = Y_{14} - Y_{23}$ commutes with the adjoints of $\mathfrak{k}_\Theta$. Therefore the $(K_\Theta)_0$ representations on V_1 and V_3 and the representations on V_2 and V_4, are equivalent. One can see that these two different representations are not equivalent.

Assume $J : \mathfrak{n}^+_\Theta \longrightarrow \mathfrak{n}^+_\Theta$ is a K_Θ-invariant almost complex structure. The M-invariance implies that $V_{[\alpha]}$ is M-invariant. For instance, $V_{[\lambda_1-\lambda_3]} = \langle\{X_{13}, Y_{13}, X_{24}, Y_{24}\}\rangle$ is invariant under J. Because of the $\mathfrak{k}_\Theta$ representations described above, we have that $JV_1 = V_1$ or $JV_1 = V_3$. In the first case, we may have $X_{13} - X_{24}$ as an eigenvalue of J, which is not possible, so we obtain $JV_1 = V_3$ and $J(X_{13} - X_{24}) = c_1(Y_{13} - Y_{24})$ for some $c_1 \neq 0$. By analogous reasoning we obtain that J is as follows:

$$\begin{aligned}
JY_{12} &= aY_{12} + cY_{34},\\
JY_{34} &= (1 + a^2)Y_{12}/c - aY_{34},\\
J(X_{13} - X_{24}) &= c_1(Y_{13} - Y_{24}),\\
J(X_{14} + X_{23}) &= c_2(Y_{14} + Y_{23}),\\
J(X_{13} + X_{24}) &= c_3(Y_{13} + Y_{24}),\\
J(X_{14} - X_{23}) &= c_4(Y_{14} - Y_{23}),
\end{aligned}$$

where $c_i, c \neq 0$. But $J\,\mathrm{ad}_X = \mathrm{ad}_X J$ for $X \in \mathfrak{k}_\Theta$ implies $c_1 = c_4$ and $c_2 = c_3$. Direct computations show that this is M-invariant and $J\,\mathrm{ad}_X = \mathrm{ad}_X J$ for all $X \in \mathfrak{k}_\Theta$, therefore, a K_Θ-invariant almost complex structure.

Regarding integrability, it suffices to remark that, for instance, $N_J(Y_{12}, X_{13}-X_{24})$ is never zero.

Proposition 10 *The real flag manifold $\mathbb{F}_\Theta$ of type D_4 with $\Theta = \{\lambda_1 - \lambda_2, \lambda_3 + \lambda_4\}$ admits K_Θ invariant almost complex structures. These structures are not integrable.*

Proof We proceed as in the previous proof. The following is a decomposition into K_Θ invariant and irreducible subspaces

$$\mathfrak{n}^+_\Theta = \mathfrak{g}_{\lambda_1+\lambda_2} \oplus \mathfrak{g}_{\lambda_3-\lambda_4} \oplus \sum_{i=1}^{4} V_i,$$

where

$$V_1 = \langle X_{13} - Y_{24}, Y_{14} + X_{23}\rangle \qquad V_3 = \langle Y_{13} - X_{24}, X_{14} + Y_{23}\rangle$$
$$V_2 = \langle X_{13} + Y_{24}, Y_{14} - X_{23}\rangle \qquad V_4 = \langle Y_{13} + X_{24}, X_{14} - Y_{23}\rangle$$

The subspace V_1 is $\mathfrak{k}_\Theta$-equivalent to the subspace V_3 and the subspace V_2 is $\mathfrak{k}_\Theta$-equivalent to the subspace V_4 through the following linear transformations $T_{13} : V_1 \longrightarrow V_3$ and $T_{24} : V_2 \longrightarrow V_4$, given by $T_{13}(X_{13} - Y_{24}) = Y_{13} - X_{24}$, $T_{13}(Y_{14} + X_{23}) = X_{14} + Y_{23}$, $T_{24}(X_{13} + Y_{24}) = Y_{13} + X_{24}$ and $T_{24}(Y_{14} - X_{23}) = X_{14} - Y_{23}$. The other representations are not $\mathfrak{k}_\Theta$ equivalent.

Assume J is a K_Θ-invariant almost complex structure. As before, $JV_1 = V_3$ and $JV_2 = V_4$ and J verifies

$$\begin{aligned} JY_{12} &= aX_{34} + cY_{12}, \\ JX_{34} &= (1 + a^2)X_{34}/c - aY_{12}, \\ J(X_{13} - Y_{24}) &= c_1(Y_{13} - X_{24}), \\ J(Y_{14} + X_{23}) &= c_2(X_{14} + Y_{23}), \\ J(X_{13} + Y_{24}) &= c_3(Y_{13} + X_{24}), \\ J(Y_{14} - X_{23}) &= c_4(X_{14} - Y_{23}), \end{aligned}$$

J commuting with ad_X, for $X \in \mathfrak{k}_\Theta$ implies $c_1 = c_2$ and $c_3 = c_4$, and any such J commutes with all $\mathrm{ad}_X \in \mathfrak{k}_\Theta$, so it is $(K_\Theta)_0$ invariant. One can verify that J is also M-invariant.

Again, it is possible to see that $N_J(Y_{12}, X_{13} - Y_{24})$ never vanishes.

Proposition 11 *The real flag manifold $\mathbb{F}_\Theta$ of type D_4 with $\Theta = \{\lambda_3 - \lambda_4, \lambda_3 + \lambda_4\}$ admits K_Θ invariant almost complex structures. These structures are not integrable.*

Proof The following is a decomposition into $(K_\Theta)_0$ invariant and irreducible subspaces

$$\mathfrak{n}_\Theta^+ = \mathfrak{g}_{\lambda_1-\lambda_2} \oplus \mathfrak{g}_{\lambda_1+\lambda_2} \oplus \sum_{i=1}^{4} V_i,$$

where

$$V_1 = \langle X_{13} + Y_{13}, X_{14} + Y_{14} \rangle \qquad V_3 = \langle X_{23} + Y_{23}, X_{24} + Y_{24} \rangle$$
$$V_2 = \langle X_{13} - Y_{13}, X_{14} - Y_{14} \rangle \qquad V_4 = \langle X_{23} - Y_{23}, X_{24} - Y_{24} \rangle$$

The subspace V_1 is $\mathfrak{k}_\Theta$-equivalent to the subspace V_3 and the subspace V_2 is $\mathfrak{k}_\Theta$-equivalent to the subspace V_4. Indeed, we consider the linear transformations $T_{13} : V_1 \longrightarrow V_3$ given by $T_{13}(X_{13}+Y_{13}) = X_{24}+Y_{24}$ and $T_{13}(X_{14}+Y_{14}) = -(X_{23} + Y_{23})$ and $T_{24} : V_2 \longrightarrow V_4$ given by $T_{24}(X_{13} - Y_{13}) = X_{24} - Y_{24}$ and $T_{24}(X_{14} - Y_{14}) = -(X_{23} - Y_{23})$.

Any $(K_\Theta)_0$-invariant complex structure J is of form

$$\begin{aligned} JX_{12} &= aX_{12} + cY_{12}, \\ JY_{12} &= (1 + a^2)X_{12}/c - aY_{12}, \\ J(X_{13} + Y_{13}) &= c_1(X_{24} + Y_{24}), \\ J(X_{14} + Y_{14}) &= -c_1(X_{23} + Y_{23}), \\ J(X_{13} - Y_{13}) &= c_2(X_{24} - Y_{24}), \\ J(X_{14} - Y_{14}) &= -c_2(X_{23} - Y_{23}), \end{aligned}$$

Direct computations show that this is also M-invariant and therefore K_Θ-invariant. For such structure, $N_J(X_{12}, X_{13} + Y_{13})$ never vanishes.

Proposition 12 *The real flag manifolds $\mathbb{F}_\Theta$ of type D_4 where Θ is one of the following sets:*

- $\Theta_1 = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 - \lambda_4\}$,
- $\Theta_2 = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \lambda_3 + \lambda_4\}$,
- $\Theta_3 = \{\lambda_2 - \lambda_3, \lambda_3 - \lambda_4, \lambda_3 + \lambda_4\}$,

do not admit K_Θ-invariant almost complex structures.

Proof Below we give the respective decompositions of $\mathfrak{n}^+_{\Theta_i}$ in K_Θ invariant and irreducible subspaces.

$$\mathfrak{n}^-_{\Theta_1} = \langle Y_{12} + Y_{34}, Y_{13} - Y_{24}, Y_{14} + Y_{23}\rangle \oplus \langle Y_{12} - Y_{34}, Y_{13} + Y_{24}, Y_{14} - Y_{23}\rangle.$$
$$\mathfrak{n}^-_{\Theta_2} = \langle Y_{12} + X_{34}, Y_{13} - X_{24}, X_{14} + Y_{23}\rangle \oplus \langle Y_{12} - X_{34}, Y_{13} + X_{24}, X_{14} - Y_{23}\rangle.$$
$$\mathfrak{n}^-_{\Theta_3} = \langle X_{12} + Y_{12}, X_{13} + Y_{13}, X_{14} + Y_{14}\rangle \oplus \langle X_{12} - Y_{12}, X_{13} - Y_{13}, X_{14} - Y_{14}\rangle.$$

We see that each of them decomposes as a sum of two irreducible subspaces V_1 and V_2 which induce non-equivalent representations and such that $\dim V_1 = \dim V_2 = 3$. Lemma 2 implies that any K_Θ-invariant complex structure preserves each of these irreducible components, which is not possible since these are odd dimensional. Therefore $\mathbb{F}_{\Theta_i}$ does not admit K-invariant almost complex structures for $i = 1, 2, 3$.

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