ON CONJUGACY OF CARTAN SUBALGEBRAS IN EXTENDED AFFINE LIE ALGEBRAS

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Abstract. That finite dimensional simple Lie algebras over the complex numbers can be classified by means of purely combinatorial and geometric objects such as Coxeter-Dynkin diagrams and indecomposale irreducible root systems, is arguably one of the most elegant results in mathematics. The definition of the root system is done by fixing a Cartan subalgebra of the given Lie algebra. The remarkable fact is that (up to isomorphism) this construction is independent of the choice of Cartan subalgebra. The modern way of establishing this fact is by showing that all Cartan subalebras are conjugate.

For symmetrizable Kac-Moody Lie algebras, with the appropriate definition of Cartan subalgebra, conjugacy has been established by Peterson and Kac. An immediate consequence of this result is that the root systems and generalized Cartan matrices are invariants of the Kac-Moody Lie algebras. The purpose of this paper is to establish conjugacy of Cartan subalgebras for Extended Affine Lie Algebras; a natural class of Lie algebras that generalizes the finite dimensional simple Lie algebra and affine Kac-Moody Lie algebras.

Introduction

Let $\mathfrak{g}$ be a finite dimensional split simple Lie algebra over a field k of characteristic 0, and let $\mathbf{G}$ be the simply connected Chevalley-Demazure algebraic group associated to $\mathfrak{g}$. Chevalley's theorem (unpublished) asserts that all split Cartan subalgebras of $\mathfrak{g}$ are conjugate under the adjoint action of $\mathbf{G}(k)$ on $\mathfrak{g}$. This is one of the central results of classical Lie theory. One of its immediate consequences is that the corresponding root system is an invariant of the Lie algebra (i.e., it does not depend on the choice of Cartan subalgebra).

We now look at the analogous question in the infinite dimensional set up as it relates to Extended Affine Lie Algebras (EALAs). We assume henceforth that k is algebraically closed. The role of $\mathfrak{g}$ is now played by a pair (E, H) consisting of a Lie algebra E and a "Cartan subalgebra" H. There other Cartan subalgebras, and the question is whether they are conjugate and, if so, under the action of which group.

The first example is that of untwisted affine Kac-Moody Lie algebras. Let $R = k[t^{\pm 1}]$. Then

$$E = \mathfrak{g} \otimes_k R \oplus kc \oplus kd \tag{0.0.1}$$

and

$$H = \mathfrak{h} \otimes 1 \oplus kc \oplus kd \tag{0.0.2}$$

The relevant information is as follows. kc is a central extension (in fact the universal central extension) of the k-Lie algebra $\mathfrak{g} \otimes_k R$. The derivation d of $\mathfrak{g} \otimes_k R$ corresponds to the degree

V. Chernousov was partially supported by the Canada Research Chairs Program and an NSERC research grant.
E. Neher was partially supported by a Discovery grant from NSERC.
A. Pianzola wishes to thank NSERC and CONICET for their continuous support.

derivation td/dt acting on R. Finally $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$. The nature of H is that it is abelian, it acts k-diagonalizably on E, and it is maximal with respect to these properties. These algebras are called MADs (Maximal Abelian Diagonalizable) subalgebras. A celebrated theorem of Peterson and Kac [PK] states that all MADs of E are conjugate (under the action of a group that they construct which is the analogue of the simply connected group in the finite dimensional case). In fact, the conjugacy theorem of Peterson and Kac is not one but there theorems. It also applies if we reply the pair $E, H)$ by

$$\mathfrak{g} \otimes_k R \oplus kc \text{ and } \mathfrak{h} \otimes 1 \oplus kc$$

or

$$\mathfrak{g} \otimes_k R \text{ and } \mathfrak{h} \otimes 1.$$

Similar results hold for the twisted affine Lie algebras. These algebras are of the form

$$E = L \oplus kc \oplus kd.$$

The Lie algebra L is a loop algebra $L = L(\mathfrak{g}, \sigma)$ for some finite order automorphism σ of $\mathfrak{g}$ (see 4.1 below for details). If σ is the identity, we are in the untwisted case . The ring R can be recovered as the centroid of L.

In the EALA set up, the Lie algebras $\mathfrak{g}$ as above are the case of nullity $n = 0$ case, while the affine Lie algebras are the case of nullity $n = 1$. In higher nullity n we have $R = k[t_1^{\pm 1}, \dots, t_\ell^{\pm 1}]$ for some $\ell \leq n$, where again R is the centroid of the centre less core E_{cc} of the given EALA. Most of our work will concentrate in the case when $\ell = n$. In this situation E_{cc} is finitely generated as a module over the centroid R (called the *fgc condition* in EALA theory). We hasten to add that the non-fgc algebras are fully understood and classified (see 2.2 below). The crucial result about the fgc case is that E_{cc} is necessarily a multiloop algebra, hence a twisted form of $\mathfrak{g} \otimes_k R$ for some (unique) $\mathfrak{g}$. This allows methods from Galois cohomology to be used in the study of the algebras under consideration (all of this, with suitable references, will be explained in the main text).

Given (E, H) we obtain a root space decomposition. The "root system" Δ is what is an example of an extended affine root system. It is also possible to attach to Δ an indecomposable finite root system $\dot{\Delta}$ (in the sense of Bourbaki). It need not be reduced. The main question, of course, is whether Δ and $\dot{\Delta}$ are an invariant of E. In other words. If H' is a subalgebra of E for which the pair $E, H')$ is given an EALA structure, are the resulting root systems Δ' and $\dot{\Delta}'$ isomorphic (in the sense of [extended] root systems) to Δ and $\dot{\Delta}$? That this is true follows immediately from the main result of our paper.

0.1. **Theorem.** *Let (E, H) be an extended affine Lie algebra of fgc type. Assume E admits the second structure (E, H') of an extended affine Lie algebra. Then H and H' are conjugate, i.e., there exists an automorphism $\phi \in \mathrm{Aut}_{k-Lie}(E)$ such that $\phi(H) = H'$.*

The main idea of the proof is as follows. Just as as for the affine algebras, an EALA E can be written in the form $E = L \oplus C \oplus D$. Unlike the affine case, starting with L (which is a multiloop algebra given our fgc assumption), one can construct an infinite number of $E's$. The exact nature of all possible C and D, and what the resulting Lie algebra structure is, has been described in works of one of the authors (Neher). For the readers convenience we will recall this construction below. By the main result of [CGP] one knows that conjugacy holds for L. The challenge, which is far from trivial, is to "lift" this conjugacy to E. It worth noting that [PK] proceeds to some extend in the opposite direction. They establish conjugacy "upstairs", i.e. for E, and use this to obtain conjugacy "downstairs", i.e. for L.

It is also worth observing that in the affine case, the most important and useful result is conjugacy upstairs. The same consideration applies to EALAs.

Notation: We suppose throughout that k is a field of characteristic 0. Starting with section §4 we assume that k is algebraically closed.

Comments: *Check that this is indeed the case. For example, that we did not use* [ABFP]

For convenience $\otimes = \otimes_k$.

1. Some general results

Some of the key results needed later to establish our main theorem are true and easier to prove in a more general setting. This is the purpose of this section.

Throughout L will denote a Lie algebra over k.

1.1. **Cohomology.** Let V be an L-module. We denote by $\mathrm{Z}^2(L,V)$ the k-space of 2-cocycles of L with coefficients in V. Its elements consist of alternating maps $\sigma\colon L\times L\to V$ satisfying the cocycle condition ($l_i \in L$)

$$\begin{aligned} l_1\cdot\sigma(l_2,l_3)+l_2\cdot\sigma(l_3,l_1)+l_3\cdot\sigma(l_1,l_2)& \\ =\sigma([l_1,l_2],l_3)+\sigma([l_2,l_3],l_1)+\sigma([l_3,l_1],l_2)& \end{aligned} \tag{1.1.1}$$

Given such a 2-cocycle σ, the vector space $L\oplus V$ becomes a Lie algebra with respect to the product

$$[l_1+v_1,\, l_2+v_2]=[l_1,l_2]_L+\big(l_1\cdot v_2-l_2\cdot v_1+\sigma(l_1,l_2)\big)$$

We will denote this Lie algebra by $L\oplus_\sigma V$. Note that the projection onto the first factor $\mathrm{pr}_L\colon L\oplus_\sigma V\to L$ is an epimorphism of Lie algebras whose kernel is the abelian ideal V. Note that L is not necessarily a subalgebra of $L\oplus_\sigma V$.

A special case of this construction is the situation when V is a trivial L-module. In this case a 2-cocycle will be called a *central* 2*-cocycle*. Note that all terms on the left hand side of (1.1.1) vanish. For a central 2-cocycle, V is a central ideal of $L\oplus_\sigma V$ and $\mathrm{pr}_L\colon L\oplus_\sigma V\to L$ is a central extension.

1.2. **Invariant bilinear forms.** A bilinear form $\beta\colon L\times L\to k$ is *invariant* if $\beta([l_1,l_2],l_3)=\beta(l_1,[l_2,l_3])$ holds for all $l_i\in L$.

Let $\mathfrak{g}$ be a finite-dimensional split simple Lie algebra with Killing form κ. Let $R\in k\text{–}alg$. For any linear form $\varphi\colon R\to k$, i.e., an element of R^*, we obtain an invariant bilinear form $(\cdot|\cdot)$ of the Lie algebra $\mathfrak{g}\otimes_k R$ by $(x\otimes r\mid y\otimes s)=\kappa(x,y)\,\varphi(rs)$. We mention that every invariant bilinear form of $\mathfrak{g}\otimes_k R$ is obtained in this way for a unique $\varphi\in R^*$ (see Cor. 6.2 of [NPPS]).

1.3. **Central 2-cocycles and invariant bilinear forms.** Assume our Lie algebra L comes equipped with an invariant bilinear form $(\cdot|\cdot)$. We denote by $\mathrm{Der}_k(L)$ the Lie algebra of derivations of L and by $\mathrm{SDer}(L)$ the subalgebra of skew derivations, i.e., those derivations d satisfying $\big(d(l)\mid l\big)=0$ for all $l\in L$. Let D be a subalgebra of $\mathrm{SDer}(L)$ and denote by $D^*=\mathrm{Hom}_k(D,k)$ its dual space. It is well-known and easy to check that then $\sigma_D\colon L\times L\to D^*$ defined by

$$\sigma_D\big(l_1,\,l_2\big)\,(d)=\big(d(l_1)\mid l_2\big) \tag{1.3.1}$$

is a central 2-cocycle.

1.4. **A general construction of Lie algebras.** We consider the following data:

(i) Two Lie algebras L and D;

(ii) an action of D on L by derivations of L, written as $d \cdot l$ for $d \in D$, $l \in L$ (thus $[d_1, d_2] \cdot l = d_1 \cdot (d_2 \cdot l) - d_2 \cdot (d_1 \cdot l)$ and $d \cdot [l_1, l_2] = [d \cdot l_1, l_2] + [l_1, d \cdot l_2]$ for $d, d_i \in D$ and $l, l_i \in L$);

(iii) a vector space V which is a D-module and which will also be considered as a trivial L-module;

(iv) a central 2-cocycle $\sigma \colon L \times L \to V$ and a 2-cocycle $\tau \colon D \times D \to V$.

Given these data, we define a product on

$$E = L \oplus V \oplus D$$

by ($v_i \in V$, $l_i \in L$, and $d_i \in D$)

$$\begin{aligned}[l_1 + v_1 + d_1,\, l_2 + v_2 + d_2] = \big([l_1, l_2]_L + d_1 \cdot l_2 - d_2 \cdot l_1\big) \\ + \big(\sigma(l_1, l_2) + d_1 \cdot v_2 - d_2 \cdot v_1 + \tau(d_1, d_2)\big) \\ + [d_1, d_2]_D.\end{aligned} \tag{1.4.1}$$

Here $[.,.]_L$ and $[.,.]_D$ are the Lie algebra products of L and D respectively. To avoid any possible confusion we will sometimes denote the product of E by $[.,.]_E$.

1.5. **Proposition.** *The algebra E defined in (1.4.1) is a Lie algebra.*

We will henceforth denote this Lie algebra (L, σ, τ).

Proof. The product is evidently alternating. For $e_i \in E$ let $J(e_1, e_2, e_3) = \big[[e_1, e_2]\, e_3\big] + \big[[e_2, e_3]\, e_1\big] + \big[[e_3, e_1]\, e_2\big]$ for $e_i \in E$. That $J(E, E, E) = 0$ follows from tri-linearity of J and the following special cases: $J(D, D, D) = 0$ since D is a Lie algebra and τ is a 2-cocycle; $J(D, D, L) = 0$ since L is a D-module; $J(D, D, V) = 0$ since V is a D-module; $J(D, V, V) = 0 = J(D, L, V)$ since all terms vanish by definition (1.4.1); $J(D, L, L) = 0$ since D acts on L by derivations; $J(L \oplus V, L \oplus V, L \oplus V) = 0$ since $L \oplus_\sigma V$ is a Lie algebra by 1.1. $\square$

We will later use this construction for different data. For example, it is the standard construction of an EALA as reviewed in §2.

One of the central themes of this paper is to extend automorphisms from the Lie algebra L to the Lie algebra $E = (L, \sigma, \tau)$. Recall that the elementary automorphism group $\mathrm{EAut}(M)$ of a Lie k-algebra M is by definition the subgroup of $\mathrm{Aut}_k(M)$ generated by the automorphisms $\exp(\mathrm{ad}_M x)$ for $\mathrm{ad}_M x$ a nilpotent derivation. Clearly, any elementary automorphism is $\mathrm{Ctd}_k(M)$-linear, where here and below Ctd_k denotes the centroid of a k-algebra.[1]

1.6. **Proposition.** *Every elementary automorphism f of L lifts to an elementary automorphism $\tilde{f}$ of $E = (L, \sigma, \tau)$ with the following properties:*

[1]We recall that for an arbitrary k-algebra A, $\mathrm{Ctd}_k(A) = \{\chi \in \mathrm{End}_k(A) : \chi(ab) = \chi(a)b = a\chi(b)\, \forall\, a, b \in A\}$. The space A is naturally a left $\mathrm{Ctd}_k(A)$-module via $\chi \cdot a = \chi(a)$. If $\mathrm{Ctd}_k(A)$ is commutative, for example if A is perfect, the above action endows in A an algebra structure over $\mathrm{Ctd}_k(A)$. The reader may refer to [BN] for general facts about centroids.

(i) $\tilde f(L) \subset L \oplus V$; *the* L*-component of* $\tilde f|_L$ *is* f, *i.e.*, $\mathrm{pr}_L \circ \tilde f|_L = f$.

(ii) $\tilde f(V) \subset V$. *In fact* $\tilde f|_V = \mathrm{Id}_V$.

(iii) *For* $d \in D$ *the* D*-component of* $\tilde f(d) \in E$ *is* d, *i.e.*, $\tilde f(d) = d + x_{f,d}$ *for some* $x_{f,d} \in L \oplus V$.

Proof. Let $x \in L$ and denote by $\mathrm{ad}_L\, x$ and $\mathrm{ad}_E\, x$ the corresponding inner derivation of L and E respectively. We let $e = l + v + d \in E$ be an arbitrary element of E with the obvious notation. Then

$$(\mathrm{ad}_E\, x)(e) = \big([x,l]_L - d\cdot x\big) + \sigma(x,l) \in L \oplus V.$$

Putting $e_1 = [x,l] - d \cdot x$ an easy induction shows that

$$(\mathrm{ad}_E\, x)^n(e) = (\mathrm{ad}_L\, x)^{n-1}(e_1) + \sigma\big(x, (\mathrm{ad}_L\, x)^{n-2}(e_1)\big) \in L \oplus V, \quad n \geq 2.$$

In particular, if $\mathrm{ad}_L\, x$ is nilpotent then so is $\mathrm{ad}_E\, x$. Assuming this to be the case, it is immediate from the product formula (1.4.1) that (i)–(iii) hold for $\tilde f = \exp(\mathrm{ad}_E\, x)$. □

2. Review: Lie tori and EALAs

2.1. **Lie tori.** In this paper the term "root system" means a finite, not necessarily reduced root system Δ in the usual sense, except that we will assume $0 \in \Delta$, as for example in [AABGP]. We denote by $\Delta_{\mathrm{ind}} = \{0\} \cup \{\alpha \in \Delta : \frac{1}{2}\alpha \notin \Delta\}$ the subsystem of indivisible roots and by $\mathcal{Q}(\Delta) = \mathrm{span}_{\mathbb{Z}}(\Delta)$ the root lattice of Δ. To avoid some degeneracies we will always assume that $\Delta \neq \{0\}$.

Let Δ be a finite irreducible root system, and let Λ be an abelian group. A *Lie torus of type* (Δ, Λ) is a Lie algebra L satisfying the following conditions (LT1) – (LT4).

(LT1) (a) L is graded by $\mathcal{Q}(\Delta) \oplus \Lambda$. We write this grading as $L = \bigoplus_{\alpha \in \mathcal{Q}(\Delta), \lambda \in \Lambda} L_\alpha^\lambda$ and thus have $[L_\alpha^\lambda, L_\beta^\mu] \subset L_{\alpha+\beta}^{\lambda+\mu}$. It is convenient to define

$$L_\alpha = \textstyle\bigoplus_{\lambda \in \Lambda} L_\alpha^\lambda \quad \text{and} \quad L^\lambda = \bigoplus_{\alpha \in \mathcal{Q}(\Delta)} L_\alpha^\lambda.$$

(b) We further assume that $\mathrm{supp}_{\mathcal{Q}(\Delta)} L = \{\alpha \in \mathcal{Q}(\Delta); L_\alpha \neq 0\} = \Delta$, so that $L = \bigoplus_{\alpha \in \Delta} L_\alpha$.

(LT2) (a) If $L_\alpha^\lambda \neq 0$ and $\alpha \neq 0$, then there exist $e_\alpha^\lambda \in L_\alpha^\lambda$ and $f_\alpha^\lambda \in L_{-\alpha}^{-\lambda}$ such that

$$L_\alpha^\lambda = k e_\alpha^\lambda, \quad L_{-\alpha}^{-\lambda} = k f_\alpha^\lambda,$$

and

$$[[e_\alpha^\lambda, f_\alpha^\lambda], x_\beta] = \langle \beta, \alpha^\vee \rangle x_\beta$$

for all $\beta \in \Delta$ and $x_\beta \in L_\beta$.[2]

(b) $L_\alpha^0 \neq 0$ for all $0 \neq \alpha \in \Delta_{\mathrm{ind}}$.

(LT3) As a Lie algebra, L is generated by $\bigcup_{0 \neq \alpha \in \Delta} L_\alpha$.

(LT4) As an abelian group, Λ is generated by $\mathrm{supp}_\Lambda L = \{\lambda \in \Lambda : L^\lambda \neq 0\}$.

[2]Here an elsewhere $\alpha^\vee$ denotes the coroot corresponding to α in the sense of [Bbk].

We define the *nullity* of a Lie torus L of type (Δ, Λ) as the rank of Λ and the *root-grading type* as the type of Δ. We will say that L is a *Lie torus* (without qualifiers) if L is a Lie torus of type (Δ, Λ) for some pair (Δ, Λ). A Lie torus is called *centreless* if its centre $\mathcal{Z}(L) = \{0\}$. If L is an arbitrary Lie torus, its centre $\mathcal{Z}(L)$ is contained in L_0 from which it easily follows that $L/\mathcal{Z}(L)$ is in a natural way a centreless Lie torus of the same type as L and nullity (see [Yo3, Lemma 1.4]).

An obvious example of a Lie torus of type $(\Delta, \mathbb{Z}^n)$ is the Lie k-algebra $\mathfrak{g} \otimes R$ where $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra of type Δ and $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is the Laurent polynomial ring in n-variables with coefficients in k equipped with the natural $\mathbb{Z}^n$-grading. Another important example, studied in [BGK], is $\mathfrak{sl}_l(k_q)$ for k_q a quantum torus.

Lie tori have been classified, see [Al] for a recent survey of the many papers involved in this classification. Some more background on Lie tori is contained in the papers [ABFP, Ne3, Ne4].

2.2. **Some known properties of centreless Lie tori.** We review the properties of Lie tori used in our present work. This is not a comprehensive survey. The reader can find more information in [ABFP, Ne3, Ne4]. We assume that L is a centreless Lie torus of type (Δ, Λ) and nullity n.

For e_α^λ and f_α^λ as in (LT2) we put $h_\alpha^\lambda = [e_\alpha^\lambda, f_\alpha^\lambda] \in L_0^0$ and observe that $(e_\alpha^\lambda, h_\alpha^\lambda, f_\alpha^\lambda)$ is an $\mathfrak{sl}_2$-triple. Then

$$\mathfrak{h} = \operatorname{span}_k\{h_\alpha^\lambda\} = L_0^0 \tag{2.2.1}$$

is a toral[3] subalgebra of L whose root spaces are the L_α, $\alpha \in \Delta$.

Up to scalars, L has a unique nondegenerate symmetric bilinear form $(\cdot|\cdot)$ which is Λ-graded in the sense that $(L^\lambda \mid L^\mu) = 0$ if $\lambda + \mu \neq 0$, [NPPS, Yo3]. Since the subspaces L_α are the root spaces of the toral subalgebra $\mathfrak{h}$ we also know $(L_\alpha \mid L_\tau) = 0$ if $\alpha + \tau \neq 0$.

The centroid $\operatorname{Ctd}_k(L)$ of L is isomorphic to the group ring $k[\Xi]$ for a subgroup Ξ of Λ, the so-called *central grading group.*[4] Hence $\operatorname{Ctd}_k(L)$ is a Laurent polynomial ring in ν variables, $0 \le \nu \le n$, ([Ne1, 7], [BN, Prop. 3.13]). (All possibilities for ν do in fact occur.) We can thus write $\operatorname{Ctd}_k(L) = \bigoplus_{\xi \in \Xi} k\chi^\xi$, where the χ^ξ satisfy the multiplication rule $\chi^\xi \chi^\delta = \chi^{\xi+\delta}$ and act on L as endomorphisms of Λ-degree ξ.

L is a prime Lie algebra, whence $\operatorname{Ctd}_k(L)$ acts without torsion on L ([Al, Prop. 4.1], [Ne1, 7]). As a $\operatorname{Ctd}_k(L)$-module, L is free. If L is fgc, namely finitely generated as a module over its centroid, then L is a multiloop algebra [ABFP].

If L is not fgc, equivalently $\nu < n$, one knows ([Ne1, Th. 7]) that L has root-grading type A. Lie tori with this root-grading type are classified in [BGK, BGKN, Yo1]. It follows from this classification together with [NY, 4.9] that $L \simeq \mathfrak{sl}_l(k_q)$ for k_q a quantum torus in n variables and $q = (q_{ij})$ an $n \times n$ quantum matrix with at least one q_{ij} not a root of unity.

[3]A subalgebra T of a Lie algebra L is toral, sometimes also called ad-diagonalizable, if $L = \bigoplus_{\alpha \in T^*} L_\alpha(T)$ for $L_\alpha(T) = \{l \in L : [t, l] = \alpha(t)l \text{ for all } t \in T\}$. In this case $\{\operatorname{ad} t : t \in T\}$ is a commuting family of ad-diagonalizable endomorphism. The converse holds whenever T is finite-dimensional.

[4]In [Ne3] the central grading group is denoted by Γ. We will reserve this notation for the Galois group of an extension S/R which is prominently used later in our work.

Any $\theta \in \operatorname{Hom}_{\mathbb{Z}}(\Lambda, k)$ induces a so-called *degree derivation* ∂_θ of L defined by $\partial_\theta(l^\lambda) = \theta(\lambda) l^\lambda$ for $l^\lambda \in L^\lambda$. We put $\mathcal{D} = \{\partial_\theta : \theta \in \operatorname{Hom}_{\mathbb{Z}}(\Lambda, k)\}$ and note that $\theta \mapsto \partial_\theta$ is a vector space isomorphism from $\operatorname{Hom}_{\mathbb{Z}}(\Lambda, k)$ to $\mathcal{D}$, whence $\mathcal{D} \simeq k^n$. We define $\mathrm{ev}_\lambda \in \mathcal{D}^*$ by $\mathrm{ev}_\lambda(\partial_\theta) = \theta(\lambda)$. One knows ([Ne1, 8]) that $\mathcal{D}$ induces the Λ-grading of L in the sense that $L^\lambda = \{l \in L : \partial_\theta(l) = \mathrm{ev}_\lambda(\partial_\theta) l$ for all $\theta \in \operatorname{Hom}_{\mathbb{Z}}(\Lambda, k)\}$ holds for all $\lambda \in \Lambda$.

If $\chi \in \operatorname{Ctd}_k(L)$ then $\chi d \in \operatorname{Der}_k(L)$ for any derivation $d \in \operatorname{Der}_k(L)$. We call

$$\operatorname{CDer}_k(L) := \operatorname{Ctd}_k(L)\mathcal{D} = \textstyle\bigoplus_{\gamma \in \Xi} \chi^\xi \mathcal{D}$$

the *centroidal derivations* of L. Since

$$[\chi^\xi \partial_\theta,\, \chi^\delta \partial_\psi] = \chi^{\xi+\delta}(\theta(\delta)\partial_\psi - \psi(\xi)\partial_\theta) \tag{2.2.2}$$

it follows that $\operatorname{CDer}(L)$ is a Ξ-graded subalgebra of $\operatorname{Der}_k(L)$, a generalized Witt algebra. Note that $\mathcal{D}$ is a toral subalgebra of $\operatorname{CDer}_k(L)$ whose root spaces are the $\chi^\xi \mathcal{D} = \{d \in \operatorname{CDer}(L) : [t, d] = \mathrm{ev}_\xi(t) d$ for all $t \in \mathcal{D}\}$. One also knows ([Ne1, 9]) that

$$\operatorname{Der}_k(L) = \operatorname{IDer}(L) \rtimes \operatorname{CDer}_k(L) \quad \text{(semidirect product)}. \tag{2.2.3}$$

For the construction of EALAs, the Ξ-graded subalgebra $\operatorname{SCDer}_k(L)$ of *skew-centroidal derivations* is important:

$$\begin{aligned}
\operatorname{SCDer}_k(L) &= \{d \in \operatorname{CDer}_k(L) : (d \cdot l \mid l) = 0 \text{ for all } l \in L\} \\
&= \textstyle\bigoplus_{\xi \in \Xi} \operatorname{SCDer}_k(L)^\xi, \\
\operatorname{SCDer}_k(L)^\xi &= \chi^\xi \{\partial_\theta : \theta(\xi) = 0\}.
\end{aligned}$$

Note $\operatorname{SCDer}_k(L)^0 = \mathcal{D}$ and $[\operatorname{SCDer}_k(L))^\xi, \operatorname{SCDer}_k(L)^{-\xi}] = 0$, whence

$$\operatorname{SCDer}_k(L) = \mathcal{D} \ltimes \left(\textstyle\bigoplus_{\xi \neq 0} \operatorname{SCDer}(L)^\xi \right) \quad \text{(semidirect product).}$$[5]

2.3. **Extended affine Lie algebras (EALAs).** An *extended affine Lie algebra* or EALA for short, is a triple $\big(E, H, (\cdot|\cdot)\big)$ (but see Remark 2.4) consisting of a Lie algebra E over k, a subalgebra H of E and a nondegenerate symmetric invariant bilinear form $(\cdot|\cdot)$ satisfying the axioms (EA1) – (EA5) below.

(EA1) H is a nontrivial finite-dimensional toral and self-centralizing subalgebra of E.

Thus $E = \bigoplus_{\alpha \in H^*} E_\alpha$ for $E_\alpha = \{e \in E : [h, e] = \alpha(h)e$ for all $h \in H\}$ and $E_0 = H$. We denote by $\Psi = \{\alpha \in H^* : E_\alpha \neq 0\}$ the set of roots of (E, H) – note that $0 \in \Psi$! Because the restriction of $(\cdot|\cdot)$ to $H \times H$ is nondegenerate, one can in the usual way transfer this bilinear form to H^* and then introduce anisotropic roots $\Psi^{\mathrm{an}} = \{\alpha \in \Psi : (\alpha \mid \alpha) \neq 0\}$ and isotropic (= null) roots $\Psi^0 = \{\alpha \in \Psi : (\alpha \mid \alpha) = 0\}$. The *core of* $\big(E, H, (\cdot|\cdot)\big)$ is by definition the subalgebra generated by $\bigcup_{\alpha \in \Psi^{\mathrm{an}}} E_\alpha$. It will be henceforth denoted by E_c.

(EA2) For every $\alpha \in \Psi^{\mathrm{an}}$ and $x_\alpha \in E_\alpha$, the operator $\operatorname{ad} x_\alpha$ is locally nilpotent on E.

(EA3) Ψ^{an} is connected in the sense that for any decomposition $\Psi^{\mathrm{an}} = \Psi_1 \cup \Psi_2$ with $\Psi_1 \neq \emptyset$ and $\Psi_2 \neq \emptyset$ we have $(\Psi_1 \mid \Psi_2) \neq 0$.

(EA4) The centralizer of the core E_c of E is contained in E_c, i.e., $\{e \in E : [e, E_c] = 0\} \subset E_c$.

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[5]The left-hand side depends a priori on the choice of invariant bilinear form on L, while the right- hand side does not. This is as it should be given that the non-degenerate invariant bilinear form is unique up to non-zero scalar.

(EA5) The subgroup $\Lambda = \mathrm{span}_{\mathbb{Z}}(\Psi^0) \subset H^*$ generated by Ψ^0 in $(H^*, +)$ is a free abelian group of finite rank.

The rank of Λ is called the *nullity* of $\big(E, H, (\cdot|\cdot)\big)$. Some references for EALAs are [AABGP, BGK, BGKN, Ne2, Ne3, Ne4]. It is immediate that any finite-dimensional split simple Lie algebra is an EALA of nullity 0. The converse is also true, [Ne4, Prop. 5.3.24]. It is also known that any affine Kac-Moody algebra is an EALA – in fact, by [ABGP], the affine Kac-Moody algebras are precisely the EALAs of nullity 1. The core E_c of an EALA is in fact an ideal.

2.4. **Remark.** In [Ne2, Ne3, Ne4] an EALA is defined as a pair (E, H) consisting of a Lie algebra E and a subalgebra $H \subset E$ satisfying the axioms (EA1) – (EA5) of 2.3 as well as

(EA0) E has an invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$.

As we will see in Corollary 3.3 below the choice of the invariant bilinear form is not important. To be precise, the sets of isotropic and anisotropic roots, which a priori depend on the form $(\cdot|\cdot)$, are actually independent of the choice of $(\cdot|\cdot)$. In other words, two EALAs of the form $\big(E, H, (\cdot|\cdot)\big)$ and $\big(E, H, (\cdot|\cdot)'\big)$ have the same Ψ (this is obvious), $\Psi^{\rm an}$ and Ψ^0, and hence also the same core E_c and centreless core $E_{cc} = E_c/Z(E_c)$. The role of $(\cdot|\cdot)$ is to show that Ψ is an extended affine root system (EARS) [AABGP][6] and to pair the dimensions between the homogeneous spaces C^λ and $D^{-\lambda}$. In fact, as indicated in [Ne3, §6], it is natural to consider more general EALA structure in which the existence of an invariant form is replaced by the requirement that the set of roots of (E, H) has a specific structure without changing much the structure of EALAs.

2.5. **Isomorphisms of EALAs.** An *isomorphism* between EALAs $\big(E, H, (\cdot|\cdot)\big)$ and $\big(E', H', (\cdot|\cdot)'\big)$ is a Lie algebra isomorphism $f\colon E \to E'$ that maps H onto H'. Any such map induces an isomorphism between the corresponding EARS.

We point out that no assumption is made about the compatibility of the bilinear forms with the given Lie algebra isomorphism $f\colon E \to E'$. In particular, f is not assumed to be an isometry up to scalar as in [AF]. There is a good reason for not making this assumption. While the form is unique on the core E_c up to a scalar, there are many ways to extend it from E_c to an invariant form on E without changing the algebra structure. This can already be seen at the example of an affine Kac-Moody Lie algebra E with the standard choice of H for which there exists an infinite number of invariant bilinear forms $(\cdot|\cdot)$ on E which are not scalar multiple of each other and such that $\big(E, H, (\cdot|\cdot)\big)$ is an EALA. The isometry up to scalar condition will render all these EALAs non-isomorphic. Removing this condition yields the equivalence (up to Lie algebra isomorphism) between the affine Kac-Moody Lie algebras and EALAs of nullity one (see above).

2.6. **Roots.** The set Ψ of roots of an EALA E has special properties: It is a so-called extended affine root system in the sense of [AABGP, Ch. I]. We will not need the precise definition of an extended affine root system or the more general affine reflection system in this paper and therefore refer the interested reader to [AABGP] or the surveys [Ne3, §2, §3] and [Ne4, §5.3]. But we need to recall the structure of Ψ as an affine reflection system: There exists an irreducible root system $\Delta \subset H^*$, an embedding $\Delta_{\rm ind} \subset \Psi$ and a family $(\Lambda_\alpha : \alpha \in \Delta)$ of subsets $\Lambda_\alpha \subset \Lambda$ such that

$$\mathrm{span}_k(\Psi) = \mathrm{span}_k(\Delta) \oplus \mathrm{span}_k(\Lambda) \quad \text{and} \quad \Psi = \textstyle\bigcup_{\alpha \in \Delta} (\alpha + \Lambda_\alpha). \tag{2.6.1}$$

[6] EARS can be defined without invariant forms [LN, Prop. 5.4, §5.3]

Using this (non-unique) decomposition of Ψ, we write any $\psi \in \Psi$ as $\psi = \alpha + \lambda$ with $\alpha \in \Delta$ and $\lambda \in \Lambda_\alpha \subset \Lambda$ and define $(E_c)_\psi^\lambda = E_c \cap E_\alpha$. Then $E_c = \bigoplus_{\alpha\in\Delta,\lambda\in\Lambda}(E_c)_\alpha^\lambda$ is a Lie torus of type (Δ, Λ). Hence $E_{cc} = E_c/\mathcal{Z}(E_c)$ is a centreless Lie torus, called the *centreless core of* E_c.

2.7. **Construction of EALAs.** To construct an EALA one reverses the process described in 2.6. We will use data (L, σ_D, τ) described below. Some background material can be found in [Ne3, §6] and [Ne4, §5.5]:

- L is a centreless Lie torus of type (Δ, Λ). We fix a Λ-graded invariant nondegenerate symmetric bilinear form $(\cdot|\cdot)$ and let Ξ be the central grading group of L.
- $D = \bigoplus_{\xi\in\Xi} D^\xi$ is a graded subalgebra of $\mathrm{SCDer}_k(L)$ such that the evaluation map $\mathrm{ev}_{D^0} : \Lambda \to D^{0\,*}$, $\lambda \to \mathrm{ev}_\lambda|_{D^0}$ is injective. We denote by $C = D^{\mathrm{gr}*}$ the graded dual of D and by σ_D the central 2-cocycle defined in (1.3.1).
- $\tau : D \times D \to C$ is an *affine cocycle* defined to be a 2-cocycle satisfying for all $d, d_i \in D$
$$\tau(D^0, D) = 0, \quad \text{and} \quad \tau(d_1, d_2)(d_3) = \tau(d_2, d_3)(d_1)$$

It is important to point out that there do exist non-trivial affine cocycles, see [BGK, Rem. 3.71].

The data (L, σ_D, τ) as above satisfy all the axioms of our general construction 1.4 and hence, by 1.5, is a Lie algebra with respect to the product (1.4.1).[7] We will denote this Lie algebra by E. By construction we have the decomposition

$$E = L \oplus C \oplus D. \tag{2.7.1}$$

Note that E has the toral subalgebra

$$H = \mathfrak{h} \oplus C^0 \oplus D^0$$

for $\mathfrak{h}$ as in 2.2. The symmetric bilinear form $(\cdot|\cdot)$ on E, defined by

$$\big(l_1 + c_1 + d_1 \mid l_2 + c_2 + d_2\big) = (l_1 \mid l_2)_L + c_1(d_2) + c_2(d_1),$$

is nondegenerate and invariant. Here $(\cdot|\cdot)_L$ is of course our fixed chosen invariant bilinear form of the Lie torus L. We have now indicated part of the following result.

2.8. **Theorem** ([Ne2, Th. 6])**.** (a) *The triple* $\big(E, H, (\cdot|\cdot)\big)$ *constructed above is an extended affine Lie algebra,*[8] *denoted* $\mathrm{EA}(L, D, \tau)$. *Its core is* $L \oplus D^{\mathrm{gr}\,*}$ *and its centreless core is* L.

(b) *Conversely, let* $\big(E, H, (\cdot|\cdot)\big)$ *be an extended affine Lie algebra, and let* $L = E_c/Z(E_c)$ *be its centreless core. Then there exists a subalgebra* $D \subset \mathrm{SCDer}_F(L)$ *and an affine cocycle* τ *satisfying the conditions in* 2.7 *such that* $\big(E, H, (\cdot|\cdot)\big) \simeq \mathrm{EA}(L, (\cdot|\cdot)_L, D, \tau)$ *for some* Λ*-graded invariant nondegenerate bilinear form* $(\cdot|\cdot)_L$ *on* L.

2.9. **Remark.** As mentioned in 2.2, invariant Λ-graded bilinear forms on L are unique up to a scalar. Changing the form on L by the scalar $s \in k$, will result in multiplying the central cocycle $L \times L \to C$ by s. Including for a moment the bilinear form $(\cdot|\cdot)$ on L in the notation, the map $\mathrm{Id}_L \oplus s\,\mathrm{Id}_C \oplus \mathrm{Id}_D$ is an isomorphism from $\mathrm{EA}(L, (\cdot|\cdot)_L, D, \tau)$ to $\mathrm{EA}(L, s(\cdot|\cdot)_L, D, s\tau)$.

[7] Strictly speaking we should write $\mathrm{EA}(L, D, (\cdot|\cdot)_L, \tau)$. The effect that different choice of forms has on the resulting EALA is explained in Remark 2.9.

[8] See Remark 2.4 above.

3. Invariance of the core

In this section $\big(E, H, (\cdot|\cdot)\big)$ is an EALA whose centreless core $E_{cc} = E_c/Z(E_c)$ is an arbitrary Lie torus L, hence not necessarily fgc. We decompose E in the form

$$E = L \oplus C \oplus D$$

as described in the previous section. We have a canonical map $\bar{\ }\colon E_c \to E_c/Z(E_c) = L$.

We start by proving a result of independent interest on the structure of ideals of the Lie algebra E.

3.1. **Proposition.** *Let I be an ideal of the Lie algebra E. Then either $I \subset C = Z(E_c)$ or $E_c \subset I$.*

Since L is centreless, the centre of E_c is C. We note that it is immediate that $C \lhd E$.

Proof. We assume that $I \not\subset C$ and set $I_c = I \cap E_c$ and $I_{cc} = \overline{I_c}$. We will proceed in several steps using without further comments the notation introduced in §2.

(I) $I_{cc} \neq \{0\}$: Let $e = x + c + d \in I$ where $x \in L, c \in C$ and $d \in D$. For any $l \in L$ we then get $[e, l]_E = (\operatorname{ad}_L x + d)(l) \oplus \sigma_D(x, l) \in I$, whence $(\operatorname{ad}_L x + d)(l) \in I_{cc}$. If for all $e \in I$ the corresponding derivation $\operatorname{ad}_L x + d = 0$ it follows that $x = 0 = d$ since L is centreless. But then $I \subset C$ which we excluded. Therefore some $e \in I$ has $\operatorname{ad}_L x + d \neq 0$, hence $0 \neq (\operatorname{ad}_L x + d)(l) \in I_{cc}$ for some $l \in L$.

(II) $d \cdot x \in I_{cc}$ for all $d \in D$ and $x \in I_{cc}$: There exists $c \in C$ such that $x + c \in I_c$. Hence $[d, x + c]_E = d \cdot x + d \cdot c \in I_c$ since I_c is an ideal of E. Therefore $d \cdot x \in I_{cc}$.

(III) $I_{cc} = L$: Since the Λ-grading of L is induced by the action of $D^0 \subset D$ on L, it follows from (II) that I_{cc} is a Λ-graded ideal. By [Yo2, Lemma 4.4], L is a Λ-graded simple. Hence $I_{cc} = L$.

(IV) $E_c \subset I$: Let $c \in C$ be arbitrary. Since E_c is perfect, there exist $l_i, l_i' \in L$ such that $c = \sum_i [l_i, l_i']_E$. By (III) there exist $c_i \in C$ such that $l_i + c_i \in I_c$. Then $[l_i, l_i']_E = [l_i + c_i, l_i']_E \in I_c$ implies $c \in I_c$ which together with (III) forces $E_c \subset I$. □

3.2. **Corollary.** *Let $\big(E, H, (\cdot|\cdot)\big)$ and $\big(E, H', (\cdot|\cdot)'\big)$ be two extended affine Lie algebra structures on E with cores E_c and E_c' respectively. Then $E_c = E_c'$.*

Proof. Since E_c' is an ideal of E, Proposition 3.1 says that either $E_c' \subset Z(E_c)$ or $E_c \subset E_c'$. In the first case E_c' is abelian, a contradiction to the assumption that anisotropic roots exist. Hence $E_c \subset E_c'$. By symmetry, also $E_c' \subset E_c$. □

3.3. **Corollary.** *Let $(E, H, (\cdot|\cdot))$ and $(E, H, (\cdot|\cdot)')$ be two EALAs. We distinguish the notation of 2.3 for $(E, H, (\cdot|\cdot)')$ by $'$.*

(a) $\Psi = \Psi'$, $\Psi^0 = {\Psi'}^0$, $\Psi^{\mathrm{an}} = {\Psi'}^{\,\mathrm{an}}$.

(b) *There exists $0 \neq a \in k$ such that $(\cdot|\cdot)|_{E_c' \times E_c'} = a(\cdot|\cdot)|_{E_c \times E_c}$.*

Proof. (a) The equality $\Psi = \Psi'$ is obvious since Ψ is the set of roots of H. By Corollary 3.2, we have $E_c = {E'}_c$. The algebra E_c is a Lie torus whose root-grading by a finite irreducible root system Δ is induced by $H_c = H \cap E_c$. Let $\pi\colon H^* \to H_c^*$ be the canonical restriction map.

The structure of the root spaces of E, see for example [Ne3, 6.9], shows that $\Psi^0 = \pi^{-1}(\{0\})$ whence $\Psi^0 = \Psi'^0$.

(b) Because E_c is perfect, the centre of E_c equals the radical of $(\cdot|\cdot)|_{E_c\times E_c}$. Indeed. Let $z \in E_c$. Then, using that $(\cdot|\cdot)$ is nondegenerate and invariant and that E_c is perfect we have $z \in Z(E_c) \iff 0 = ([z, E_c] \mid E) = (z \mid [E_c, E]) = (z \mid E_c) \iff z$ lies in the radical of the restriction of $(\cdot|\cdot)$ to E_c. Now (b) follows from the fact that invariant bilinear forms on E_{cc} are unique up to a scalar. □

As a consequence, when no explicit use of the form is being made, we will denote EALAs as couples (E, H).

As an application of Corollary 3.2 we can now prove

3.4. **Proposition.** *The core E_c of an EALA (E, H) is stable under automorphisms of the algebra E, i.e., $f(E_c) = E_c$ for any $f \in \mathrm{Aut}_k(E)$.*

Proof. Let $f \in \mathrm{Aut}_k(E)$. Denote $H' = f(H)$. Let $(\cdot|\cdot)'$ be the bilinear form on E given by

$$(x \,|\, y)' = \big(f^{-1}(x) \,|\, f^{-1}(y)\big).$$

Clearly, $\big(E, H', (\cdot|\cdot)'\big)$ is another EALA-structure on the Lie algebra E. Therefore, by Corollary 3.2, we have that the core E'_c of $\big(E, H', (\cdot|\cdot)'\big)$ is equal to E_c. It remains to show that $E'_c = f(E_c)$.

Let $\alpha \in \Psi$ be a root with respect to H. There exists a unique element t_α in H such that $(t_\alpha \,|\, h) = \alpha(h)$ for all $h \in H$. Recall that α is anisotropic if $(t_\alpha \,|\, t_\alpha) \neq 0$ and that E_c is generated (as an ideal) by $\cup_{\alpha\in\Psi^{\mathrm{an}}} E_\alpha$. Let Ψ' be the set of roots of (E, H'). The mapping ${}^t f^{-1}_{|H} : H^* \to H'^*$ satisfies ${}^t f^{-1}_{|H}(\Psi) = \Psi'$. Notice that $f(t_\alpha) = t_{({}^t f)^{-1}(\alpha)}$. We next have $(t_{({}^t f)^{-1}(\alpha)} \,|\, t_{({}^t f)^{-1}(\alpha)})' = (f(t_\alpha) \,|\, f(t_\alpha))' = (t_\alpha \,|\, t_\alpha)$. Therefore, ${}^t f^{-1}(\Psi^{\mathrm{an}}) = (\Psi')^{\mathrm{an}}$, $f(E_\alpha) = E'_{{}^t f^{-1}}(\alpha)$, and this implies $f(E_c) = E'_c = E_c$. □

By Proposition 3.4 we have a well-defined restriction map

$$\mathrm{res}_c \colon \mathrm{Aut}_k(E) \longrightarrow \mathrm{Aut}_k(E_c).$$

Since L is centreless, the centre of E_c is C. It is left invariant by any automorphism of E_c. Hence $\bar{\ } \colon E_c \to L$ induces a natural group homomorphism

$$\overline{\mathrm{res}} \colon \mathrm{Aut}_k(E_c) \to \mathrm{Aut}_k(L).$$

Composing the two group homomorphisms yields

$$\overline{\mathrm{res}}_c := \overline{\mathrm{res}} \circ \mathrm{res}_c \colon \mathrm{Aut}_k(E) \to \mathrm{Aut}_k(L) \tag{3.4.1}$$

We can easily determine the kernel of $\overline{\mathrm{res}}_c$. For its description we recall that a k-linear map $\varphi \colon D \to C$ is called a *derivation* if $\varphi([d_1, d_2]) = d_1 \cdot \varphi(d_2) - d_2 \cdot \varphi(d_1)$ holds for all $d_i \in D$. We denote by $\mathrm{Der}_k(D, C)$ the k-vector space of derivations from D to C.

3.5. **Proposition.** (a) $\overline{\mathrm{res}}$ *is injective.*

(b) *The kernel of* $\overline{\mathrm{res}}_c$ *consists of the maps* f *of the form*

$$f(l + c + d) = l + \big(c + \varphi(d)\big) + d, \qquad \varphi \in \mathrm{Der}_k(D, C). \tag{3.5.1}$$

In particular, $\mathrm{Ker}(\overline{\mathrm{res}}_c)$ *is a vector group isomorphic to* $\mathrm{Der}_k(D, C)$.

Proof. (a) is immediate from the fact that $L \oplus C = [L, L]_E$. It implies that $\operatorname{Ker}(\overline{\operatorname{res}}_c) = \operatorname{Ker}(\operatorname{res}_c)$. Let $f \in \operatorname{Ker}(\operatorname{res}_c)$. Then there exist linear maps $f_{CD} \in \operatorname{Hom}_k(D, C)$, $f_{LD} \in \operatorname{Hom}_k(D, L)$ and $f_D \in \operatorname{End}_k(D)$ such that $f(d) = f_{LD}(d) + f_{CD}(d) + f_D(d)$ holds for all $d \in D$. For $l \in L$ we then get $d \cdot l = f([d, l]) = [f_{LD}(d) + f_{CD}(d) + f_D(d), l] = \big(\operatorname{ad}_L f_{LD}(d) + f_D(d)\big)(l)$, i.e., $d = \operatorname{ad}_L f_{LD}(d) + f_D(d)$. Since $D \cap \operatorname{IDer} L = \{0\}$ it follows that $f_{LD} = 0$ and $f_D = \operatorname{Id}_D$. One then sees that f_{CD} is a derivation by applying f to a product $[d_1, d_2]_E$. That conversely any map of the form (3.5.1) is an automorphism is a straightforward verification. □

Our next goal is to study in detail the image of $\overline{\operatorname{res}}_c$. From Proposition 1.6 we know

$$\operatorname{EAut}(L) \subset \overline{\operatorname{res}}_c\big(\operatorname{Aut}_k(E)\big).$$

For the Conjugacy Theorem 7.6 it is necessary to know that a bigger group of automorphisms of L lies in the image of $\overline{\operatorname{res}}_c$. We will do this in Theorem 6.1. Its proof requires some preparations to which the next two sections are devoted.

4. Fgc EALAs as subalgebras of untwisted EALAs

We remind the reader that from now on k is assumed to be algebraically closed. In this section we will describe how to embed an fgc EALA into an untwisted EALA. Here, we say that an EALA E is *fgc* if its centreless core is so, and we say that E is *untwisted* if its centreless core E_{cc}, as a Lie torus, is of the form $E_{cc} = \mathfrak{g} \otimes R$ for some finite-dimensional simple Lie algebra $\mathfrak{g}$ over k and Laurent polynomial ring R in finitely many variables.

4.1. **Multiloop algebras.** In order to realize an fgc EALA as a subalgebra of an untwisted EALA, we need some preparation starting with a review of fgc Lie tori which by [ABFP] are multiloop algebras $L = L(\mathfrak{g}, \boldsymbol{\sigma})$. They are constructed as follows: $\mathfrak{g}$ is a simple finite-dimensional Lie algebra and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ is a family of commuting finite order automorphisms. We will denote the order of σ_i by m_i. We fix once and for all a compatible set $(\zeta_\ell)_{\ell \in \mathbb{N}}$ of primitive ℓ-th roots of unity, i.e. $\zeta_{n\ell}^n = \zeta_\ell$ for $n \in \mathrm{N}$. The second ingredient are two rings,

$$R = k[t^{\pm 1}, \dots, t_n^{\pm 1}] \quad \text{and} \quad S = k[t_1^{\pm \frac{1}{m_1}}, \dots, t_n^{\pm \frac{1}{m_n}}].$$

For convenience we set $z_i = t_i^{\frac{1}{m_i}}$. Thus $z_i^{m_i} = t_i$ and $S = k[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$.

Let $\Lambda = \mathbb{Z}^n$. For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Lambda$ let

$$z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n} := t_1^{\frac{\lambda_1}{m_1}} \cdots t_n^{\frac{\lambda_n}{m_n}}.$$

The k-algebra S has a natural Λ-grading by declaring that z^λ is of degree λ. Then R is a graded subalgebra of S whose homogeneous components have degree belonging to the subgroup

$$\Xi = m_1\mathbb{Z} \oplus \cdots \oplus m_n\mathbb{Z} \subset \Lambda$$

Note that $\Xi \simeq \mathbb{Z}^n$.

We set $\overline{\Lambda} = \Lambda/\Xi$ and let $\bar{\ } : \Lambda \to \overline{\Lambda}$ denote the canonical map. After the natural identification of Ξ with $\mathbb{Z}^n$, this is nothing but the canonical map $\bar{\ } : \mathbb{Z}^n \to \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_n\mathbb{Z}$.

The automorphisms σ_i can be simultaneously diagonalized. For $\bar{\lambda} = (\overline{\lambda_1}, \cdots, \overline{\lambda_n}) \in \overline{\Lambda}$ we set

$$\mathfrak{g}^{\bar{\lambda}} = \{x \in \mathfrak{g} : \sigma_i(x) = \zeta_{m_i}^{\lambda_i} x,\ 1 \le i \le n\}$$

then $\mathfrak{g} = \bigoplus_{\bar{\lambda} \in \overline{\Lambda}} \mathfrak{g}^{\bar{\lambda}}$.

Note that $\mathfrak{g} \otimes S$ is a centreless Λ-graded Lie algebra with homogeneous subspaces $(\mathfrak{g} \otimes S)^{\lambda} = \mathfrak{g} \otimes S^{\lambda}$. By definition, the multiloop algebra $L(\mathfrak{g}, \boldsymbol{\sigma})$ is the graded subalgebra of $\mathfrak{g} \otimes S$ given by

$$L = L(\mathfrak{g}, \boldsymbol{\sigma}) = \bigoplus_{\lambda \in \overline{\Lambda}} \mathfrak{g}^{\bar{\lambda}} \otimes z^{\lambda} \subset \mathfrak{g} \otimes S. \tag{4.1.1}$$

Note that the Λ-grading of L is given by $L^{\lambda} = L \cap (\mathfrak{g} \otimes S)^{\lambda} = \mathfrak{g}^{\bar{\lambda}} \otimes z^{\lambda}$. The grading group of L is

$$\Lambda_L := \operatorname{span}\{\lambda \in \Lambda : L^{\lambda} \neq 0\} = \operatorname{span}\{\lambda \in \Lambda : \mathfrak{g}^{\bar{\lambda}} \neq 0\} \subset \Lambda.$$

We shall later see that in the cases we are interested in, namely those related to the realization of Lie tori and EALAs, we always have $\Lambda_L = \Lambda$.

4.2. **The EALA construction with $L(\mathfrak{g}, \boldsymbol{\sigma})$ as centreless core.** From now on we consider an EALA E whose centreless core is fgc. By [ABFP, Prop. 3.2.5 and Th. 3.1] one then knows that E_{cc} is a multiloop algebra $L(\mathfrak{g}, \boldsymbol{\sigma})$ with $\mathfrak{g}$ simple and $\boldsymbol{\sigma}$ as above. The (admittedly delicate) choice of $\boldsymbol{\sigma}$ is such that the Λ-grading of $L(\mathfrak{g}, \boldsymbol{\sigma})$ yields the Λ-grading of the Lie torus E_{cc}. With such a choice $\mathfrak{g}^0$ is simple.

By [BN, GP1] the ring R is canonically isomorphic to the centroid $\operatorname{Ctd}_k(L)$ of the Lie algebra $L = L(\mathfrak{g}, \boldsymbol{\sigma})$. More precisely, for $r \in R$ let $\chi_r \in \operatorname{End}(L)$ be the homothety $l \mapsto rl$. Then the centroid $\operatorname{Ctd}_k(L)$ of L is $\{\chi_r : r \in R\}$ and the map $r \mapsto \chi_r$ is a k-algebra isomorphism $R \to \operatorname{Ctd}_k(L)$. We will henceforth identify these two rings without further mention and view L naturally as an R-Lie algebra.

Let $\varepsilon \in S^*$ be the linear form defined by $\varepsilon(z^{\lambda}) = \delta_{\lambda,\mathbf{0}}$. We will also view ε as a symmetric bilinear form on S defined by $\varepsilon(s_1, s_2) = \varepsilon(s_1 s_2)$ for $s_i \in S$. We denote by κ the Killing form of $\mathfrak{g}$ and define a bilinear form $(\cdot|\cdot)_S$ on $\mathfrak{g} \otimes S$ by

$$(x_1 \otimes s_1 \mid x_2 \otimes s_2)_S = \kappa(x_1, x_2)\, \varepsilon(s_1 s_2),$$

i.e., $(\cdot|\cdot)_S = \kappa \otimes \varepsilon$. The bilinear form $(\cdot|\cdot)_S$ is invariant, nondegenerate and symmetric. By [NPPS, Cor. 7.4], the restriction $(\cdot|\cdot)_L$ of $(\cdot|\cdot)_S$ to the subalgebra $L(\mathfrak{g}, \boldsymbol{\sigma})$ has the same properties and is up to a scalar the only such bilinear form.

Since S is Λ-graded, every $\theta \in \operatorname{Hom}_{\mathbb{Z}}(\Lambda, k)$ gives rise to a derivation ∂_θ of S, defined by $\partial_\theta(z^{\lambda}) = \theta(\lambda) z^{\lambda}$ for $\lambda \in \Lambda$. We get a subalgebra $\mathcal{D}_S = \{\partial_\theta : \theta \in \operatorname{Hom}_{\mathbb{Z}}(\Lambda, k)\}$ of degree 0 derivations of S. The map $\theta \mapsto \partial_\theta$ is a vector space isomorphism. It is well-known that $\operatorname{Der}_k(S) = S\mathcal{D}_S$. It follows that $\operatorname{Der}_k(S)$ is a Λ-graded Lie algebra with homogeneous subspace $(\operatorname{Der}_k(S))^{\lambda} = S^{\lambda}\mathcal{D}$. The analogous facts hold for the Ξ-graded algebra R, i.e., putting $\mathcal{D}_R = \{\partial_\xi : \xi \in \operatorname{Hom}_{\mathbb{Z}}(\Xi, k)\}$ the Lie algebra $\operatorname{Der}_k(R) = R\mathcal{D}_R$ is Ξ-graded with $\operatorname{Der}_k(R)^{\xi} = R^{\xi}\mathcal{D}$. But we can identify $\mathcal{D}_S$ with $\mathcal{D}_R$ and then denote $\mathcal{D} = \mathcal{D}_S = \mathcal{D}_R$ since the restriction map $\operatorname{Hom}_{\mathbb{Z}}(\Lambda, k) \to \operatorname{Hom}_{\mathbb{Z}}(\Xi, k)$ is an isomorphism of vector spaces (this because $\Lambda/\Xi = \Gamma$ is a finite group and k is torsion-free). Hence $\operatorname{Der}_k(R) = R\mathcal{D} \subset \operatorname{Der}_k(S) = S\mathcal{D}$. Observe that the embedding $\operatorname{Der}_k(R) \subset \operatorname{Der}_k(S)$ preserves the degrees of the derivations.[9]

[9]Since S is an étale covering of R, in fact even Galois, every k-linear derivation $\delta \in \operatorname{Der}_k(R)$ uniquely extends to a derivation $\hat{\delta}$ of S. Under our inclusion $\operatorname{Der}_k(R) \subset \operatorname{Der}_k(S)$ we have $\delta = \hat{\delta}$.

One easily verifies that $z^\lambda \partial_\theta$ is skew-symmetric with respect to the bilinear form ε of S if and only if $\theta(\lambda) = 0$. The analogous fact holds for R:

$$\begin{aligned}\mathrm{SDer}_k(R) &= \{\delta \in \mathrm{Der}_k(R) : \delta \text{ is skew-symmetric}\}\\ &= \textstyle\bigoplus_{\xi\in\Xi} z^\xi\{\partial_\theta : \theta \in \mathrm{Hom}_{\mathbb{Z}}(\Xi, k), \theta(\xi) = 0\} \subset \mathrm{SDer}_k(S).\end{aligned}$$

We now consider derivations of $\mathfrak{g} \otimes S$ and of L. It is well-known that the map $\delta \mapsto \mathrm{Id}_{\mathfrak{g}} \otimes \delta$ identifies $\mathrm{Der}_k(S)$ with the subalgebra $\mathrm{CDer}_k(\mathfrak{g} \otimes S)$ of centroidal derivations of $\mathfrak{g} \otimes S$; it maps $\mathrm{SDer}_k(S)$ onto $\mathrm{SCDer}_k(\mathfrak{g} \otimes S)$. Analogously, $\mathrm{Der}_k(R) \to \mathrm{CDer}(L)$, $\delta \mapsto (\mathrm{Id}_{\mathfrak{g}} \otimes \delta)|_L$ is an isomorphism of Lie algebras [Pi].[10] One can check that under this isomorphism $\mathrm{SDer}_k(R)$ is mapped onto $\mathrm{SCDer}_k(L)$. The embedding $\mathrm{SDer}_k(R) \subset \mathrm{SDer}_k(S)$ of above then gives rise to an embedding

$$\mathrm{SCDer}_k(L) \subset \mathrm{SCDer}_k(\mathfrak{g} \otimes S). \tag{4.2.1}$$

To construct an EALA E with $E_{cc} = L$ we follow 2.7 and take a graded subalgebra $D \subset \mathrm{SCDer}_k(L) \simeq \mathrm{SDer}_k(R)$ such that the evaluation map $\mathrm{ev}\colon \Lambda \to D^{0^*}$ is injective. This then provides us with the central cocycle $\sigma_D \colon L \times L \to C = D^{\mathrm{gr}*}$. Using Theorem 2.8 it follows that $\mathrm{EA}(L, D, \tau)$ is an EALA with centreless core L for any affine cocycle $\tau \colon D \times D \to C$ and, conversely, any EALA E with $E_{cc} \simeq L$ is isomorphic to $\mathrm{EA}(L, D, \tau)$ for appropriate choices of D and τ.

4.3. **Example (untwisted EALA).** Let $\sigma_i = \mathrm{Id}$ for all i. Then $S = R$, $L = \mathfrak{g} \otimes S = \mathfrak{g} \otimes R$. Using the invariant bilinear form $(\cdot|\cdot)_S$ on $\mathfrak{g} \otimes S$ described above we observe that for any $D \subset \mathrm{SCDer}(L)$ as above and affine cocycle τ we have an EALA $\mathrm{EA}(\mathfrak{g} \otimes S, D, \tau)$. Any EALA isomorphic to such an EALA will be called *untwisted*.

4.4. **Remark.** Note that if we replace $(\cdot|\cdot)_S$ by $s(\cdot|\cdot)_S$ for some $s \in k^\times$, then, as explained in Remark 2.9, the resulting EALA is $\mathrm{EA}(\mathfrak{g} \otimes S, D, s\tau)$, which is again an untwisted EALA.

By taking into account that the invariant bilinear form $(\cdot|\cdot)_L$ on $L = L(\mathfrak{g}, \boldsymbol{\sigma})$ is by assumption the restriction of $(\cdot|\cdot)_S$ to L, the following lemma is immediate from the above.

4.5. **Lemma.** *Let $E = \mathrm{EA}(L, D, \tau) = L \oplus C \oplus D$ be an EALA with centreless core $L = L(\mathfrak{g}, \boldsymbol{\sigma})$ as in (4.1.1). Assume, without loss of generality, that the invariant bilinear form $(\cdot|\cdot)_E$ of E is such that its restriction to L is the form $(\cdot|\cdot)_L$ above. By means of (4.2.1) view D as a subalgebra of $\mathrm{SCDer}(\mathfrak{g} \otimes S)$. Then*

$$E_S = \mathrm{EA}(\mathfrak{g} \otimes S, D, \tau) = \mathfrak{g} \otimes S \oplus C \oplus D \tag{4.5.1}$$

is an untwisted EALA containing E as a subalgebra.

4.6. **Remark.** That there is no loss of generality on the choice of $(\cdot|\cdot)_E$ follows from Remark 4.4. Indeed, scaling a given form to produce $(\cdot|\cdot)_L$ when restricted to L will result in replacing $\mathrm{EA}(\mathfrak{g} \otimes S, D, \tau)$ by $\mathrm{EA}(\mathfrak{g} \otimes S, D, s\tau)$. The relevant conclusion that E is a subalgebra of an untwisted EALA remains valid.

The following lemma will be useful later.

[10] Note that in the expression $\mathrm{Id}_{\mathfrak{g}} \otimes \delta$ the element $\delta \in \mathrm{Der}_k(R)$ is viewed as an element of $\mathrm{Der}_k(S)$ under the inclusion $\mathrm{Der}_k(R) \subset \mathrm{Der}_k(S)$ described above.

4.7. **Lemma.** *Let $E = L \oplus C \oplus D$ be an EALA with centreless core an fgc Lie torus.*

(a) *Let $g \in \mathrm{Aut}_k(L)$. Then the endomorphism f_g of E defined by*

$$f_g(l \oplus c \oplus d) = g(l) \oplus c \oplus d \tag{4.7.1}$$

is an automorphism of E if and only if $g \circ d = d \circ g$ holds for all $d \in D$.

(b) *The map $g \mapsto f_g$ is an isomorphism between the groups*

$$\mathrm{Aut}_D(L) = \{g \in \mathrm{Aut}_k(L) : g \circ d = d \circ g \text{ for all } d \in D\}$$

and $\{f \in \mathrm{Aut}_k(E) : f(L) = L, f|_{C \oplus D} = \mathrm{Id}\}$. In particular

$$\{g \in \mathrm{Aut}_R(L) : g(L^\lambda) = L^\lambda \text{ for all } \lambda \in \Lambda\} \subset \mathrm{Aut}_D(L). \tag{4.7.2}$$

Proof. (a) It is immediate from (4.7.1) and the multiplication rules (1.4.1) that f_g is an automorphism of E if and only if

(i) $g \circ d = d \circ g$ holds for all $d \in D$ and

(ii) $\sigma\big(g(l_1), g(l_2)\big) = \sigma(l_1, l_2)$ holds for all $l_i \in L$.

To show that the second condition is implied by the first, recall that σ is defined by (1.3.1), whence (ii) is equivalent to $\big((d \circ g)(l_1) \mid g(l_2)\big) = \big(d(l_1) \mid l_2\big)$. Because of (i) this holds as soon as g is orthogonal with respect to $(\cdot|\cdot)$. But this is exactly what [NPPS, Cor. 7.4] says.

The first part of (b) is immediate. Any automorphism stabilizing the homogeneous spaces L^λ commutes with $\mathcal{D}$ viewed as a subset of $\mathrm{SCDer}(L)$. If it is also R-linear it commutes with all of $\mathrm{SCDer}(L)$ and so in particular with the subalgebra $D \subset \mathrm{SCDer}(L)$. $\square$

5. Lifting automorphisms in the untwisted case

In this section we assume that E is an extended affine Lie algebra whose centreless core E_{cc} is untwisted in the sense that $E_{cc} = L = \mathfrak{g} \otimes R$. In other words $L = L(\mathfrak{g}, \mathrm{Id})$. In particular $R = S$ and $t_i = z_i$.

5.1. **Notation.** We let $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ be the adjoint and simply connected algebraic k–groups corresponding to $\mathfrak{g}$. Recall that we have a central isogeny

$$1 \to \boldsymbol{\mu} \to \widetilde{\mathbf{G}} \to \mathbf{G} \to 1 \tag{5.1.1}$$

where $\boldsymbol{\mu}$ is either $\boldsymbol{\mu}_m$ or $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$.

The algebraic k–group of automorphisms of $\mathfrak{g}$ will be denoted by $\mathbf{Aut}(\mathfrak{g})$. For any (associative commutative unital) k-algebra K by definition $\mathbf{Aut}(\mathfrak{g})(K)$ is the (abstract) group $\mathrm{Aut}_K(\mathfrak{g} \otimes K)$ of automorphisms of the K–Lie algebra $\mathfrak{g} \otimes K$.

Recall that we have a split exact sequence of k–groups (see [SGA3] Exp. XXIV Théorème 1.3 and Proposition 7.3.1)

$$1 \to \mathbf{G} \to \mathbf{Aut}(\mathfrak{g}) \to \mathbf{Out}(\mathfrak{g}) \to 1 \tag{5.1.2}$$

where $\mathbf{Out}(\mathfrak{g})$ is the finite constant k–group $\mathrm{Out}(\mathfrak{g})$ corresponding to the group of symmetries of the Dynkin diagram of $\mathfrak{g}$.[11]

[11] The group $\mathbf{Out}(\mathfrak{g})$ is denoted by $\mathbf{Aut}\big(\mathrm{Dyn}(\mathfrak{g})\big)$ in [SGA3].

There is no canonical splitting of the above exact sequence. A splitting is obtained (see [SGA3]) once we fix a base of the root system of a Killing couple of $\widetilde{\mathbf{G}}$ or $\mathbf{G}$. Accordingly, we henceforth fix a maximal (split) torus $\widetilde{\mathbf{T}} \subset \widetilde{\mathbf{G}}$. Let $\Sigma = \Sigma(\widetilde{\mathbf{G}}, \widetilde{\mathbf{T}})$ be the root system of $\widetilde{\mathbf{G}}$ relative to $\widetilde{\mathbf{T}}$. We fix a Borel subgroup $\widetilde{\mathbf{T}} \subset \widetilde{\mathbf{B}} \subset \widetilde{\mathbf{G}}$. It determines a system of simple roots $\{\alpha_1, \dots, \alpha_n\}$. Fix a Chevalley basis $\{H_{\alpha_1}, \dots H_{\alpha_n},\ X_\alpha,\ \alpha \in \Sigma\}$ of $\mathfrak{g}$ corresponding to the pair $(\widetilde{\mathbf{T}}, \widetilde{\mathbf{B}})$. The Killing couple $(\widetilde{\mathbf{B}}, \widetilde{\mathbf{T}})$ induces a Killing couple $(\mathbf{B}, \mathbf{T})$ of $\mathbf{G}$.

In what follows we need to consider the R-groups obtained by the base the change R/k of all of the algebraic k–groups described above. Note that $\mathbf{Aut}(\mathfrak{g})_R = \mathbf{Aut}(\mathfrak{g} \otimes R)$. Since no confusion will arise we will omit the use of the subindex R (so that for example (5.1.1) and (5.1.2) should now be thought as an exact sequence of group schemes over R).

5.2. **Theorem.** *The group $\mathrm{Aut}_R(\mathfrak{g} \otimes R)$ is in the image of the map $\overline{\mathrm{res}}_c$ of (3.4.1), i.e., every R-linear automorphism of $\mathfrak{g} \otimes R$ can be lifted to an automorphism of E.*

Proof. By (5.1.2) we have

$$\mathrm{Aut}_R(\mathfrak{g} \otimes R) = \mathbf{G}(R) \rtimes \mathrm{Out}(\mathfrak{g}). \tag{5.2.1}$$

We will proceed in 3 steps:

(1) Lifting of automorphisms in the image of $\widetilde{\mathbf{G}}(R)$ in $\mathbf{G}(R)$.

(2) Lifting of automorphisms in $\mathbf{G}(R)$.

(3) Lifting of the elements of $\mathrm{Out}(\mathfrak{g})$.

To make our proof more accessible we start by recalling the main ingredients of the construction of E, see 1.4 and 2.7.

(a) Up to a scalar in k, the Lie algebra $\mathfrak{g} \otimes R$ has a unique nondegenerate invariant bilinear form $(\cdot|\cdot)_R$, namely $(x \otimes r \mid x' \otimes r')_R = \kappa(x, x')\, \varepsilon(rr')$ where κ is the Killing form of $\mathfrak{g}$, $x, x' \in \mathfrak{g}$ and $\varepsilon \in R^*$ is given by $\varepsilon(\sum_{\lambda \in \Lambda} a_\lambda t^\lambda) = a_{\mathbf{0}}$. Recall that $t^\lambda = t_1^{\lambda_1} \cdots t_n^{\lambda_n}$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda = \mathbb{Z}^n$.

(b) The Lie algebra D is a Λ-graded Lie algebra of skew-centroidal derivations of R acting on $\mathfrak{g} \otimes R$ by $\mathrm{Id} \otimes d$ for $d \in D$. Every homogeneous $d \in D$, say of degree λ, can be uniquely written as $d = t^\lambda \partial_\theta$ for some additive map $\theta \colon \Lambda \to k$, where $\partial_\theta(t^\mu) = \theta(\mu) t^\mu$ for $\mu \in \Lambda$.

(c) The Lie algebra E is constructed using the general construction 1.4 with $L = \mathfrak{g} \otimes R$, D as above, $V = C = D^{\mathrm{gr}*}$ with the canonical D-action on L and C, the central 2-cocycle of (1.3.1) using the bilinear form $(\cdot|\cdot)_R$ of (a) above, and some 2-cocycle $\tau \colon D \times D \to C$.

In our proofs of steps 1 and 2 we will embed E as a subalgebra of a Lie algebra $\widetilde{E}$ and use the following general result.

5.3. **Lemma.** *Assume that R is a subring of a commutative associative ring $\widetilde{R}$. We put $\widetilde{L} = \mathfrak{g} \otimes \widetilde{R}$.*

(a) *Assume that*

(i) *the action of D on R extends to an action of D on $\widetilde{R}$ by derivations.*

(ii) *$\widetilde{\sigma} \colon \widetilde{L} \times \widetilde{L} \to C$ is a central 2-cocycle such that $\widetilde{\sigma}|_{L \times L} = \sigma$.*

Then D acts on $\widetilde{L}$ by $d \cdot (x \otimes s) = x \otimes d(s)$ for $d \in D$, $x \in \mathfrak{g}$ and $s \in \widetilde{R}$. The data $(\widetilde{L}, \widetilde{\sigma}, \tau)$ satisfy the conditions of the construction 1.4, hence define a Lie algebra $\widetilde{E} = \widetilde{L} \oplus C \oplus D$. It contains E as a subalgebra.

(b) *Let $\tilde{f} \in \mathrm{Aut}(\widetilde{E})$ satisfy $\tilde{f}(L \oplus C) = L \oplus C$. Then $\tilde{f}(E) = E$.*

Proof. The easy proof of (a) will be left to the reader. In (b) it remains to show that $\tilde{f}(D) \subset L \oplus C \oplus D$. We fix $d \in D$. We then know $\tilde{f}(d) = \tilde{l} + \tilde{c} + \tilde{d}$ for appropriate $\tilde{l} \in \widetilde{L}$, $\tilde{c} \in C$ and $\tilde{d} \in D$. We claim that $\tilde{l} \in L$. For arbitrary $l \in L$ we have $d \cdot l = [d, l]_E = [d, l]_{\widetilde{E}}$ where $[.,.]_E$ and $[.,.]_{\widetilde{E}}$ are the products of E and $\widetilde{E}$ respectively. Hence $\tilde{f}\big(d \cdot l\big) = [\tilde{f}(d),\, \tilde{f}(l)]_{\widetilde{E}} = [\tilde{l} + \tilde{c} + \tilde{d},\, \tilde{f}(l)]_{\widetilde{E}}$. Denoting by $(\cdot)_{\widetilde{L}}$ the $\widetilde{L}$-component of elements of $\widetilde{E}$, it follows that

$$\tilde{f}\big(d(l)\big)_{\widetilde{L}} = [\tilde{l},\, \tilde{f}(l)_{\widetilde{L}}]_{\widetilde{E}} + [\tilde{d}, \tilde{f}(l)_{\widetilde{L}}]_{\widetilde{E}}.$$

By assumption for all $x \in L$, $\tilde{f}(x)_{\widetilde{L}} \in L$. It follows that the last term in the displayed equation and the left hand side lie in L. Since C is the centre of $\widetilde{L} \oplus C$ we know $\tilde{f}(C) = C$ whence $(\mathrm{pr}_L \circ \tilde{f})(L) = L$ for $\mathrm{pr}_L \colon L \oplus C \to L$ the canonical projection. The displayed equation above therefore implies $[\tilde{l}, L]_{\widetilde{L}} \subset L$.

We will prove that this in turn forces $\tilde{l} \in L$. Indeed, let $\{r_i : i \in I\}$ be a k-basis of R and extend it to a k-basis of $\widetilde{R}$, say by $\{s_j : j \in J\}$. Thus $\tilde{l} = \sum_i x_i \otimes r_i + \sum_j y_j \otimes s_j$ for suitable $x_i,\, y_j \in \mathfrak{g}$. For every $z \in \mathfrak{g}$ we then have $[\tilde{l}, z \otimes 1] = \sum_i [x_i, z] \otimes r_i + \sum_j [y_j, z] \otimes s_j \in \mathfrak{g} \otimes R$. Hence $[y_j, z] = 0$ for all $j \in J$. Since this holds for all $z \in \mathfrak{g}$, we get $y_j = 0$ for all $j \in J$ proving $\tilde{l} \in \mathfrak{g} \otimes R$. $\square$

After these preliminaries we can now start the proof of Theorem 5.2 proper. In what follows we view R as a subring of the iterated Laurent series field $F = k((t_1))((t_2)) \cdots ((t_n))$.[12]

Step 1. Lifting of automorphisms of $\mathbf{G}(R)$ coming from $\widetilde{\mathbf{G}}(R)$.

We will follow the strategy suggested by Lemma 5.3 and construct a Lie algebra $\widetilde{E} = (\mathfrak{g} \otimes F) \oplus C \oplus D$ containing $E = (\mathfrak{g} \otimes R) \oplus C \oplus D$ as subalgebra, and then show that if $g \in \widetilde{\mathbf{G}}(R) \subset \widetilde{\mathbf{G}}(F)$, then $\mathrm{Ad}\, g \in \mathrm{Aut}_F(\mathfrak{g} \otimes F)$ can be lifted to an automorphism of $\widetilde{E}$ that stabilizes E and whose restriction to L is precisely $\mathrm{Ad}\, g \in \mathrm{Aut}_R(\mathfrak{g} \otimes R)$.

The following lemma implies that the conditions of Lemma 5.3(a) are satisfied.

5.4. **Lemma.** (a) *The linear form $\varepsilon \in R^*$ extends to a linear form $\widetilde{\varepsilon}$ of F.*

(b) *The bilinear form $(\cdot|\cdot)^{\sim}$ defined by $(x \otimes f \mid x' \otimes f')^{\sim} = \kappa(x, x')\, \widetilde{\varepsilon}(f f')$ for $x, x' \in \mathfrak{g}$, $f, f' \in F$, is an invariant symmetric bilinear form extending the bilinear form $(\cdot|\cdot)$ of $\mathfrak{g} \otimes R$.*

(c) *Every derivation $d \in D$ extends to a derivation $\tilde{d}$ of F such that*

(i) *$\tilde{d}$ is skew symmetric with respect to the bilinear form $(\cdot|\cdot)^{\sim}$,*

(ii) *$d \mapsto \tilde{d}$ is an embedding of D into $\mathrm{Der}_k(F)$.*

[12] The field F is more natural to use than the function field $K = k(t_1, \cdots, t_n)$. The extensions of forms and derivations of R are easier to see on F than K.. There is also a much more important reason: The absolute Galois group of F coincides with the algebraic fundamental group of R. This fact is essential in [GP2].

(iii) *Every* $d \in D$ *acts on* $\widetilde{L} = \mathfrak{g} \otimes F$ *by the derivation* $\mathrm{Id} \otimes \tilde{d}$ *which is skew-symmetric with respect to the bilinear form* $\widetilde{(\cdot|\cdot)}$.

(d) *Let* $\sigma_D \colon \widetilde{L} \times \widetilde{L} \to D^*$ *be the central* 2*-cocycle of* (1.3.1) *with respect to the action of* D *on* $\widetilde{L}$ *defined in* (c). *Let* $\mathrm{pr} \colon D^* \to C$ *be any projection of* D^* *onto* C *whose restriction to* $C \subset D^*$ *is the identity map. Then* $\widetilde{\sigma} = \mathrm{pr} \circ \sigma_D \colon \widetilde{L} \times \widetilde{L} \to C = D^{\mathrm{gr}*}$ *is a central* 2*-cocycle such that* $\widetilde{\sigma}|_{L \times L}$ *is the central* 2*-cocycle appearing in the construction of* E.

Proof. An *arbitrary* k–derivation of R extends to a k–derivation of F. To see this use the fact that $\mathrm{Der}_k(R)$ is a free R–module admitting the degree derivations $\partial_i = t_i \partial / \partial t_i$ as a basis. It is thus sufficient to show that the ∂_i extend to k–derivations of F, but this is easy to see. The rest of the proof is straightforward and will be left to the reader. □

Comments: *A. An outline of why* ∂_i *extends is given in lv.*

We can now apply Lemma 5.3(a) and get a Lie algebra $\widetilde{E} = \widetilde{L} \oplus C \oplus D$, with $\widetilde{L} = \mathfrak{g} \otimes F$, containing $E = L \oplus C \oplus D$ as a subalgebra.

Since $\mathrm{ad}\, X_\alpha$, $\alpha \in \Sigma$, is a nilpotent derivation, $\exp(\mathrm{ad}\, f X_\alpha)$ is an elementary automorphism of $\mathfrak{g} \otimes F$ for all $f \in F$. It is well-known that, since F is a field, the group $\widetilde{\mathbf{G}}(F)$ is generated by root subgroups $U_\alpha = \{x_\alpha(f) \mid f \in F\}$, $\alpha \in \Sigma$ and that $\mathrm{Ad}\, x_\alpha(f) = \exp(\mathrm{ad}\, f X_\alpha)$. By Proposition 1.6, $\mathrm{Ad}\, x_\alpha(f)$ lifts to an automorphism of $\widetilde{E}$ which maps $\mathfrak{g} \otimes F$ to $(\mathfrak{g} \otimes F) \oplus C$ and such that its $(\mathfrak{g} \otimes F)$-component is $\mathrm{Ad}\, x_\alpha(f)$. Consequently, for any $g \in \widetilde{\mathbf{G}}(F)$ there is an automorphism $\tilde{f}_g \in \mathrm{Aut}_k(\widetilde{E})$ such that $(\mathrm{pr}_{\mathfrak{g} \otimes F_n} \circ \tilde{f}_g)|(\mathfrak{g} \otimes F) = \mathrm{Ad}\, g \in \mathrm{Aut}_F(\mathfrak{g} \otimes F)$. Moreover, again by Proposition 1.6, $\tilde{f}_g(C) = C$, whence $\tilde{f}_g(L \oplus C) = L \oplus C$ whenever $g \in \widetilde{\mathbf{G}}(R)$. Therefore, by Lemma 5.3, we get $\tilde{f}_g(E) = E$. This finishes the proof of *Step 1.*

Step 2. Lifting automorphisms from $\mathbf{G}(R)$.

We begin with a preliminary simple observation.

5.5. **Lemma.** *There exist an integer* $m > 0$ *such that the algebra* $\widetilde{R} = k[t_1^{\pm \frac{1}{m}}, \dots, t_n^{\pm \frac{1}{m}}]$ *has the following property: All the elements of* $\mathbf{G}(R)$*, when viewed as elements of* $\mathbf{G}(\widetilde{R})$*, belong to the image of* $\widetilde{\mathbf{G}}(\widetilde{R})$ *in* $\mathbf{G}(\widetilde{R})$.

Proof. Recall that $H^1(R, \boldsymbol{\mu}_m) \simeq R^\times / (R^\times)^m$. Let m be the order of $\boldsymbol{\mu}(k)$ (if $\boldsymbol{\mu} = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ we can take $m = 2$ instead of $m = 4$.) Consider the exact sequence

$$\widetilde{\mathbf{G}}(R) \to \mathbf{G}(R) \to H^1(R, \boldsymbol{\mu})$$

resulting from (5.1.1). Let $g \in \mathbf{G}(R)$ and consider its image $[g] \in H^1(R, \boldsymbol{\mu})$. Then g is in the image of $\widetilde{\mathbf{G}}(R)$ if and only if $[g] = 1$. It is clear that the image of $[g]$ in $H^1(\widetilde{R}, \boldsymbol{\mu})$ is trivial. The lemma follows. □

Let $\widetilde{R}$ be as in the previous lemma, and let $\widetilde{L} = \mathfrak{g} \otimes \widetilde{R}$. By Lemma 5.3(a) we have a Lie algebra $\widetilde{E} = \widetilde{L} \oplus C \oplus D$ containing $E = L \oplus C \oplus D$ as a subalgebra.

Let $g \in \mathbf{G}(R) \subset \mathbf{G}(\widetilde{R})$. To avoid any possible confusion when g is viewed as an element of $\mathbf{G}(\widetilde{R})$ we denote it by $\widetilde{g}$. By *Step 1* there is a lifting $\tilde{f}_g \in \mathrm{Aut}_k(\widetilde{E})$ of $\mathrm{Ad}\, \widetilde{g} \in \mathrm{Aut}_{\widetilde{R}}(\mathfrak{g} \otimes \widetilde{R})$. To establish this we used that $\tilde{f}_g(\widetilde{L} \oplus C) = \widetilde{L} \oplus C$. But since $g \in \mathbf{G}(R)$ and $\tilde{f}_g$ lifts $\mathrm{Ad}\, \widetilde{g}$ we conclude that $\tilde{f}_g(L \oplus C) = L \oplus C$. We can thus apply Lemma 5.3(b) and conclude

$\tilde{f}(E) = E$. Hence $\tilde{f}|_E$ is the desired lift of $\operatorname{Ad} g \in \operatorname{Aut}_R(\mathfrak{g} \otimes R)$. This completes the proof of *Step 2.*

Step 3. Lifting automorphisms from $\operatorname{Out}(\mathfrak{g})$

Let g be a diagram automorphism of $\mathfrak{g}$, or more generally any automorphism of $\mathfrak{g}$. We identify g with $g \otimes \operatorname{Id}_R$ and note that g is an R-linear automorphism of $\mathfrak{g} \otimes R$ preserving the Λ-grading. Hence Lemma 4.7(c) shows that g lifts to an automorphism of E. This completes the proof of Theorem 5.2. □

6. Lifting automorphisms in the fgc case

In this section we will consider an EALA E whose centreless core L is an fgc Lie torus. If $R = k[t^{\pm 1}, \dots, t_n^{\pm 1}]$ is the centroid of L (see the second paragraph of 4.2), we will show that any R-linear automorphism of L lifts to an automorphism of E. Although our method of proof is inspired by general Galois descent considerations, we will give a self-contained presentation (with some hints for the expert readers regarding the Galois formalism) .

Throughout we will use the notation and definitions of §4. Thus $L = L(\mathfrak{g}, \boldsymbol{\sigma})$ is a multiloop Lie torus with $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ consisting of commuting automorphisms $\sigma_i \in \operatorname{Aut}_k(\mathfrak{g})$ of order m_i. The crucial point here is that the subalgebras $L \subset \mathfrak{g} \otimes S$ and $E \subset E_S$ are the fixed point subalgebras under actions of $\Gamma = \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_n\mathbb{Z}$ on $\mathfrak{g} \otimes S$ and E_S respectively. In this section we will write the group operation of Γ as multiplication.

Indeed, let γ_i be the image of $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n$ in Γ. Then γ_i can be viewed as an automorphism of S via $\gamma_i \cdot z^\lambda = \zeta_{m_i}^{\lambda_i} z^\lambda$ for $\lambda \in \Lambda = \mathbb{Z}^n$. This defines in a natural way an action of Γ as automorphisms of S. Clearly $R = S^\Gamma$. The group Γ also acts on $\mathfrak{g}$ by letting γ_i act on $\mathfrak{g}$ via σ_i^{-1}. The two actions of Γ combine to the tensor product action of Γ on $\mathfrak{g} \otimes S$. Note that Γ acts on $\mathfrak{g} \otimes S$ as automorphisms. The subalgebra $L \subset \mathfrak{g} \otimes S$ is the fixed point subalgebra under this action.[13]

By construction every $\gamma \in \Gamma$ acts on $\mathfrak{g} \otimes S$ by an R-linear automorphism preserving the Λ-grading of $\mathfrak{g} \otimes S$. Identifying (with any risk of confusion) $\gamma \in \Gamma$ with this automorphism, the inclusion (4.7.2) applied to $E_S = \mathfrak{g} \otimes S \oplus C \oplus D$ says that γ extends to an automorphism $f_\gamma \in \operatorname{Aut}_k(E_S)$ given by (4.7.1). Moreover, the group homomorphism $\gamma \mapsto f_\gamma$ defines an action of Γ on E_S by automorphisms. By construction, E is the fixed point subalgebra of E_S under this action. To summarize,

$$L = (\mathfrak{g} \otimes S)^\Gamma \quad \text{and} \quad E = (E_S)^\Gamma.$$

The action of Γ on $\mathfrak{g} \otimes S$ gives rise to an action of Γ on the automorphism group $\operatorname{Aut}_k(\mathfrak{g} \otimes S)$ by conjugation: $\gamma \cdot g = \gamma \circ g \circ \gamma^{-1}$ for $g \in \operatorname{Aut}_k(\mathfrak{g} \otimes S)$ and $\gamma \in \Gamma$. Similarly, Γ acts on $\operatorname{Aut}_k(E_S)$ by conjugation. The first part of the following theorem shows that these two actions are compatible with the restriction map $\overline{\operatorname{res}}_c$ of (3.4.1).

6.1. **Theorem.** (a) *The restriction map* $\overline{\operatorname{res}}_c\colon \operatorname{Aut}_k(E_S) \to \operatorname{Aut}_k(\mathfrak{g} \otimes S)$ *is* Γ*-equivariant. Its kernel is fixed pointwise under the action of* Γ*.*

[13]In fact, S/R is a Galois extension with Galois group Γ. The action of Γ on $\mathfrak{g} \otimes S$ is the twisted action of Γ given by the loop cocycle $\eta(\boldsymbol{\sigma})$ mapping $\gamma_i \in \Gamma$ to $\sigma_i^{-1} \otimes \operatorname{Id}_S \in \operatorname{Aut}_S(\mathfrak{g} \otimes S)$.

(b) *The canonical map*

$$\mathrm{Aut}_k(E_S)^\Gamma \to \mathrm{Im}(\overline{\mathrm{res}}_c)^\Gamma$$

induced by $\overline{\mathrm{res}}_c$ *is surjective.*

(c) *Every R-linear automorphism g of L lifts to an automorphism f_g of E, i.e.,* $\overline{\mathrm{res}}_c(f_g) = g$.

Proof. (a) Let $\gamma \in \Gamma$ and view γ as an automorphism of $\mathfrak{g} \otimes S$. By construction $\overline{\mathrm{res}}_c(f_\gamma) = \gamma$. Since $\overline{\mathrm{res}}_c$ is a group homomorphism, for any $f \in \mathrm{Aut}_k(E_S)$ we get $\overline{\mathrm{res}}_c(\gamma \cdot f) = \overline{\mathrm{res}}_c(f_\gamma \circ f \circ f_\gamma^{-1}) = \gamma \circ \overline{\mathrm{res}}_c(f) \circ \gamma^{-1} = \gamma \cdot \overline{\mathrm{res}}_c(f)$. We have determined the kernel of $\overline{\mathrm{res}}_c$ in Proposition 3.5. The description in loc. cit. together with the definition of the lift f_γ in (4.7.1) implies the last statement of (a).

(b) By [Se, I§5.5, Prop. 38], the exact sequence of Γ-modules $1 \to \mathrm{Ker}(\overline{\mathrm{res}}_c) \to \mathrm{Aut}_k(E_S) \to \mathrm{Im}(\overline{\mathrm{res}}_c) \to 1$ gives rise to the long exact cohomology sequence

$$1 \to \mathrm{Ker}(\overline{\mathrm{res}}_c) \to \mathrm{Aut}_k(E_S)^\Gamma \to \mathrm{Im}(\overline{\mathrm{res}}_c)^\Gamma \to H^1\big(\Gamma, \mathrm{Ker}(\overline{\mathrm{res}}_c)\big) \to \cdots$$

of pointed sets. Since $\mathrm{Ker}(\overline{\mathrm{res}}_c)$ is a torsion-free abelian group and Γ is finite, we have $H^1\big(\Gamma, \mathrm{Ker}(\overline{\mathrm{res}}_c)\big) = 1$. Now (b) follows.

(c) Every automorphism $g \in \mathrm{Aut}_k(\mathfrak{g} \otimes S)^\Gamma$ leaves $(\mathfrak{g} \otimes S)^\Gamma = L$ invariant and in this way gives rise to an automorphism $\rho_L(g) \in \mathrm{Aut}_k(L)$. Similarly we have a group homomorphism $\rho_E \colon \mathrm{Aut}_k(E_S)^\Gamma \to \mathrm{Aut}_k(E)$. Since $\overline{\mathrm{res}}_c \colon \mathrm{Aut}_k(E_S) \to \mathrm{Aut}_k(\mathfrak{g} \otimes S)$ is Γ-equivariant, it preserves the Γ-fixed points. We thus get the following commutative diagram where $\overline{\mathrm{res}}_{c,E} \colon \mathrm{Aut}_k(E) \to \mathrm{Aut}_k(L)$ is the map (3.4.1):

(6.1.1)
$$\begin{tikzcd}
\mathrm{Aut}_k(E_S)^\Gamma \arrow[r, "\overline{\mathrm{res}}_c"] \arrow[d, "\rho_E"'] & \mathrm{Aut}_k(\mathfrak{g} \otimes S)^\Gamma \arrow[d, "\rho_L"] \\
\mathrm{Aut}_k(E) \arrow[r, "\overline{\mathrm{res}}_{c,E}"] & \mathrm{Aut}_k(L)
\end{tikzcd}$$

We will prove (c) by restricting the diagram (6.1.1) to subgroups. Observe that ρ_L maps $\mathrm{Aut}_S(\mathfrak{g} \otimes S)^\Gamma$ to $\mathrm{Aut}_R(L)$. In fact, we claim

$$\rho_L \colon \mathrm{Aut}_S(\mathfrak{g} \otimes S)^\Gamma \to \mathrm{Aut}_R(L) \quad \text{is an isomorphism.}$$

This can be proven as a particular case of a general Galois descent result of affine group schemes. That said, due to the concrete nature of the algebras involved it is easy to give a direct proof (which we now do). The Lie algebra L is an S/R-form of $\mathfrak{g} \otimes S$. Indeed, the S-linear Lie algebra homomorphism

$$\theta \colon L \otimes_R S \to \mathfrak{g} \otimes_k S, \quad \textstyle\sum_i x_i \otimes s_i \otimes s \mapsto \sum_i x_i \otimes s_i s$$

where $\sum_i x_i \otimes s_i \in L$, $s \in S$, is an isomorphism. This can be checked directly [ABP, Lem. 5.7], or derived from the fact that L is given by the Galois descent described in the last footnote. It follows that $L \subset \mathfrak{g} \otimes S$ is a spanning set of the S-module $\mathfrak{g} \otimes S$, which implies that ρ_L is injective. For the proof of surjectivity, we associate to $g \in \mathrm{Aut}_R(L)$ the automorphisms $g \otimes \mathrm{Id}_S \in \mathrm{Aut}_S(L \otimes S)$ and $\tilde{g} = \theta \circ (g \otimes \mathrm{Id}_S) \circ \theta^{-1} \in \mathrm{Aut}_S(\mathfrak{g} \otimes S)$. We contend that $\tilde{g} \in \mathrm{Aut}_S(\mathfrak{g} \otimes S)^\Gamma$, i.e., $\gamma \circ \tilde{g} \circ \gamma^{-1} = \tilde{g}$ holds for all $\gamma \in \Gamma$. Since both sides are S-linear, it suffices to prove this equality by applying both sides to $l \in L$. Since $\theta(l \otimes 1) = l$ we get $(\theta \circ (g \otimes \mathrm{Id}) \circ \theta^{-1})(l) = \big(\theta \circ (g \otimes \mathrm{Id})\big)(l \otimes 1) = \theta^{-1}(g(l) \otimes 1) = g(l)$ and since γ fixes $L \subset \mathfrak{g} \otimes S$ pointwise the invariance of $\tilde{g}$ follows. It is immediate that $\rho_L(\tilde{g}) = g$.

By Theorem 5.2, every S-linear automorphism of $\mathfrak{g} \otimes S$ lifts to an automorphism of E_S, in other words $\mathrm{Aut}_S(\mathfrak{g} \otimes S) \subset \mathrm{Im}(\overline{\mathrm{res}}_c)$. Using (b) this implies that the canonical map $\overline{\mathrm{res}}_c^{-1}\big(\mathrm{Aut}_S(\mathfrak{g} \otimes S)^\Gamma\big) \to \mathrm{Aut}_S(\mathfrak{g} \otimes S)^\Gamma$ is surjective. By restricting the diagram (6.1.1) we now get the commutative diagram

$$\begin{tikzcd}
\overline{\mathrm{res}}_c^{-1}\big(\mathrm{Aut}_S(\mathfrak{g} \otimes S)^\Gamma\big) \arrow[r, two heads] \arrow[d, "\rho_E"'] & \mathrm{Aut}_S(\mathfrak{g} \otimes S)^\Gamma \arrow[d, "\rho_L", "\simeq"'] \\
\overline{\mathrm{res}}_{c,E}^{-1}\big(\mathrm{Aut}_R(L)\big) \arrow[r] & \mathrm{Aut}_R(L)
\end{tikzcd}$$

which implies that the bottom horizontal map is surjective and thus finishes the proof. $\square$

7. The conjugacy theorem

In this section we will prove the main result of our paper: Theorem 0.1 asserting the conjugacy of Cartan subalgebras of a Lie algebra E which give rise to fgc EALA structures on a Lie algebra E (Theorem 7.6). Assume therefore that H and H' are subalgebras of E such that (E, H) and (E, H') are fgc EALAs.[14] The strategy of our proof is as follows:

(a) Show that the canonical images H_{cc} and H'_{cc} of H and H' respectively in the centreless core E_{cc} are conjugate by an automorphism that can be lifted to E.

This allows us to assume $H_{cc} = H'_{cc}$. Then we prove that

(b) Two Cartan subalgebras H and H' of E with $H_{cc} = H'_{cc}$ are conjugate in $\mathrm{Aut}_k(E)$.

It turns out that part (b) can be proven for all EALAs, not only for fgc EALAs. In view of later applications we therefore start with part (b,) which is the theorem below.

7.1. **Theorem.** *Let (E, H) and (E, H') be two EALA structures on the Lie algebra E. We put $H_c = H \cap E_c$, $H_{cc} = \overline{H_c} \subset E_{cc}$ and use $'$ to denote the analogous data for (E, H') keeping in mind that $E_c = E'_c$ by Corollary 3.2. Assume $H_{cc} = H'_{cc}$. Then.*

(a) $H_c = H'_c$.

(b) *There exists $f \in \mathrm{Ker}(\overline{\mathrm{res}}_c) \subset \mathrm{Aut}_k(E)$ such that $f(H) = H'$.*

Proof. (a) Let $x \in H_c$. Since $H_{cc} = H'_{cc}$ there exists $y \in H'_c$ such that $\overline{x} = \overline{y} \in E_{cc}$. Then $c = x - y \in C = Z(E_c)$, so that the elements x and y commute. Being elements of H_c and H'_c, both $\mathrm{ad}_E\, x$ and $\mathrm{ad}_E\, y$ are k-diagonalizable endomorphisms of E. It follows that $\mathrm{ad}_E\, c$ is also k-diagonalizable.

We now note that it follows from $[C, D]_E \subset C$ and $[C, E_c]_E = 0$ that any eigenvector of $\mathrm{ad}_E\, c$ with a nonzero D-component necessarily commutes with c. Therefore $c \in Z(E) \subset H'_c$. Thus $x = y + c \in H'_c$, and therefore $H_c \subset H'_c$. Thus $H'_c = H_c$ by symmetry finishing the proof of (a).

Since the proof of (b) is much more involved, we have divided it into a series of lemmas (Lemma 7.2 – Lemma 7.5). The reader will find the proof of (b) after the proof of Lemma 7.5.

[14] We have seen that the core, in particular the fgc assumption, is independent of the chosen invariant bilinear form.

Because $H'_c = H_c = H_{cc} \oplus C^0$ we have decompositions $H = H_{cc} \oplus C^0 \oplus D^0$ and $H' = H_{cc} \oplus C^0 \oplus D'^0$ for a (non-unique) subspace $D'^0 \subset E$. Our immediate goal is restrict the possibilities for D'^0.

Comments: *(E, June 2014) New proof not using fgc.*

7.2. **Lemma.** $D'^0 \subset H_{cc} \oplus C \oplus D^0$

Proof. Let $d'^0 \in D'^0$, say $d'^0 = l' + c + d$ with obvious notation. Since $[d'^0, h]_E = 0$ for $h \in H'_{cc} = H_{cc}$ we get $0 = [l' + c + d, h]_E = \big([l', h]_L + d(h)\big) + \sigma(l', h) = [l', h]_L$ because $\mathrm{CDer}(L)^0(H_{cc}) = 0$ and therefore $d(h) = \sigma(l', h) = 0$. Thus $L' \in C_L(H_{cc}) = L_0$.

We have two Lie tori structures on L, the second one is denoted by L'; the L'-structure has a Λ'-grading $L' = \oplus_{\lambda' \in \Lambda'} L^{\lambda'}$, induced by D'^0. Similarly, $L = \oplus_{\lambda \in \Lambda} L^\lambda$ is induced by D^0. Since $H_{cc} = H'_{cc}$ the identity map of L is an isotopy (see [Al, Theorem 7.2]). Thus

$$L^\lambda_\alpha = {L'}^{\phi_\Lambda(\lambda)+\phi_s(\alpha)}_{\phi_r(\alpha)}.$$

The nature of the maps ϕ is given in *loc. cit.* All that is relevant to us is the fact that for all λ, α there exist appropriate α', λ' such that $L^\lambda_\alpha = {L'}^{\lambda'}_{\alpha'}$. Since D'^0 induces the Λ-grading of L, we have for $l^\lambda \in L^\lambda$ that

$$kl^\lambda \ni [d'^0, l^\lambda]_E = [l' + c + d, l^\lambda]_E = \big([l', l^\lambda]_L + d(l^\lambda)\big) + \sigma(l', l^\lambda).$$

Thus $0 = \sigma(l', l^\lambda)(\tilde d) = (\tilde d(l') \mid l^\lambda)$ for all $\tilde d \in D$ and all l^λ. By the nondegeneracy of $(\cdot|\cdot)$ on L we get $\tilde d(l') = 0$ for all $\tilde d \in D$. As $D^0 \subset D$ induces the Λ-grading of L this forces $l' \in L^0$, whence $l' \in L^0_0 = H_{cc}$. But then $[l', l^\lambda]_L \in kl^\lambda$ so that the equation above implies $d(l^\lambda) \in kl^\lambda$. We can write $d = \sum_{\gamma\in\Gamma} r^\gamma d^{0\gamma}$ for some $r^\gamma \in R^\gamma$ and $d^{0\gamma} \in D^0$. Since $r^\gamma d^{0\gamma}(l^\lambda) \in L^{\lambda+\gamma}$ we get $r^\gamma d^{0\gamma}(l^\lambda) = 0$ for all $\gamma \neq 0$. But R acts without torsion on L, so $r^\gamma = 0$ or $d^{0\gamma} = 0$ for $\gamma \neq 0$, and $d \in D^0$ follows. □

We keep the above notation and set $C^{\neq\mu} = \oplus_{\lambda\neq\mu} C^\lambda$.

7.3. **Lemma.** *There exists a subspace $V \subset H'$ such that*

(a) $H' = H_c \oplus V$, $V \subset C^{\neq 0} \oplus D^0$, *and*

(b) V *is the graph of some linear map* $\xi \in \mathrm{Hom}(D^0, C^{\neq 0})$.

Proof. (a) By the already proven part (a) of Theorem 7.1 we have $H' = H'_c \oplus D'^0 = H_c \oplus D'^0$ and by Lemma 7.2, $D'^0 \subset H_{cc} \oplus C \oplus D^0$. We decompose

$$H_{cc} \oplus C \oplus D^0 = (H_{cc} \oplus C^0) \oplus (C^{\neq 0} \oplus D^0). \tag{7.3.1}$$

Let $p : H_{cc} \oplus C \oplus D^0 \to C^{\neq 0} \oplus D^0$ be the projection with kernel $H_{cc} \oplus C^0$ and put $V = p(D'^0)$. Since $D'^0 \cap (H_{cc} \oplus C^0) \subset D'^0 \cap E_c = 0$, we see that $p|_{D'^0} \colon D'^0 \to V$ is a vector space isomorphism. Note also that $V \subset H'$. Indeed, every $v \in V$ is of the form $v = p(d'^0)$ for some $d'^0 \in D'^0$, whence $d'^0 = h + c^0 + v$ for unique $c^0 \in C^0, h \in H_{cc}$. Since $h, c_0 \in H'$ it follows that $v = d'^0 - c^0 - h \in H'$. Moreover the inclusion $V \subset C^{\neq 0} \oplus D^0$ implies $V \cap (H_{cc} \oplus C^0) = 0$ by (7.3.1). By a dimension argument we now get $H' = H_c \oplus V$.

(b) The multiplication rule (2.2.2) together with the fact that the Λ-grading of D is induced by D^0 shows $[D, D] = \bigoplus_{\lambda\neq 0} D^\lambda$. Hence, using (1.4.1) and the perfectness of E_c, we have $E = [E, E] \oplus D^0$ and then $D^0 \simeq E/[E, E] \simeq D'^0$. In particular, $\dim(V) = \dim(D'^0) = \dim(D^0)$. Note also that $V \cap C^{\neq 0} = \{0\}$. Indeed, let $v = p(d'^0)$ for some $d'^0 = h_{cc} + c^0 +$

$c^{\neq 0} + d^0$ (obvious notation). Then $p(d'^0) = c^{\neq 0} + d^0 \in C^{\neq 0}$ forces $d^0 = 0$, whence $d'^0 \in E_c$. But then $d'^0 = 0$ because $E_c \cap D'^0 = \{0\}$. Therefore $v = p(d'^0) = 0$. It now follows that the projection $p_1 \colon C^{\neq 0} \oplus D^0$ with kernel $C^{\neq 0}$ is injective on V. By reasons of dimensions $p_1|_V \colon V \to D^0$ is a vector space isomorphism. Its inverse is the map ξ whose graph is V. $\square$

7.4. **Lemma.** *The weights of the toral subalgebra V of $C \oplus D$ are the linear forms $\mathrm{ev}'_\mu \in V^*$ for $\mu \in \mathrm{supp}(C) = \mathrm{supp}(D) \subset \Lambda$, defined by*

$$\mathrm{ev}'_\mu(\xi(d^0) \oplus d^0) = \mathrm{ev}_\mu(d^0)$$

for $d^0 \in D^0$ and ξ as in Lemma 7.3. There exists a unique linear map $\psi_\mu \colon D^\mu \to C^{\neq \mu}$ such that the ev'_μ-weight space of $C \oplus D$ is given by

$$(C \oplus D)_{\mathrm{ev}'_\mu} = C^\mu \oplus \{\psi_\mu(d^\mu) + d^\mu : d^\mu \in D^\mu\}. \tag{7.4.1}$$

We have $\psi_0 = \xi$.

Proof. Since $V \subset H' \cap (C \oplus D)$ the space V is indeed a toral subalgebra of $C \oplus D$. We write the elements of V in the form $\xi(d^0) + d^0$. Since $\tau(D^0, D) = 0$ we then have the following multiplication rule for the action of V on $C \oplus D$:

$$[\xi(d^0) + d^0,\, c + d]_E = (d^0 \cdot c - d \cdot \xi(d^0)) + [d^0, d]_D. \tag{7.4.2}$$

It follows that $C^\mu \subset (C \oplus D)_{\mathrm{ev}'_\mu}$. Moreover, for any eigenvector $c + d$ of $\mathrm{ad}\, V$ with $d \neq 0$ the D-component d is an eigenvector of the toral subalgebra D^0 of D, whence $d \in D^\mu$ for some $\mu \in \mathrm{supp}\, D$ and thus $c + d \in (C \oplus D)_{\mathrm{ev}'_\mu}$. It now remains to establish the existence of the map ψ_μ in the description (7.4.1) of the ev'_μ-weight space. By (7.4.2) we have $c + d \in (C \oplus D)_{\mathrm{ev}'_\mu}$ with $d \neq 0$ if and only if $d = d^\mu$ and

$$\begin{aligned} \mathrm{ev}_\mu(d^0)\,(c + d^\mu) &= \mathrm{ev}'_\mu(\xi(d^0) + d^0)\,(c + d^\mu) = [\xi(d^0) + d^0,\, c + d^\mu]_E \\ &= \big(d^0 \cdot c - d^\mu \cdot \xi(d^0)\big) + \mathrm{ev}_\mu(d^0) d^\mu \end{aligned}$$

holds for all $d^0 \in D^0$. Thus $\mathrm{ev}_\mu(d^0)c = d^0 \cdot c - d^\mu \cdot \xi(d^0)$. Writing c in the form $c = \sum_{\lambda \in \Lambda} c^\lambda$ with $c^\lambda \in C^\lambda$ and comparing homogeneous components we get $\mathrm{ev}_\mu(d^0)c^\lambda = \mathrm{ev}_\lambda(d^0)c^\lambda - (d^\mu \cdot \xi(d^0))^\lambda$ for every $\lambda \in \Lambda$, whence

$$(d^\mu \cdot \xi(d^0))^\lambda = \mathrm{ev}_{\lambda - \mu}(d^0)c^\lambda. \tag{7.4.3}$$

Since $C^\mu \subset (C \oplus D)_{\mathrm{ev}'_\mu}$ we can assume $c^\mu = 0$. But for $\lambda \neq \mu$ there exists $d^0 \in D^0$ such that $\mathrm{ev}_{\lambda - \mu}(d^0) \neq 0$ and then (7.4.3) uniquely determines c^λ. Thus $c + d = \psi_\mu(d^\mu) + d^\mu$ for a unique $\psi_\mu(d^\mu) \in C^{\neq \mu}$. That ψ_μ is linear now follows from uniqueness.

Finally, the weight space decomposition of $C \oplus D$ together with $C^0 \oplus V \subset (C \oplus D)_{\mathrm{ev}'_0}$ forces this inclusion to be an equality. But then $\psi_0 = \xi$ follows from Lemma 7.3(b) and the uniqueness of ψ_0. $\square$

7.5. **Lemma.** *Let $\psi \colon D \to C$ be the unique linear map satisfying $\psi|_{D^\mu} = \psi_\mu$ with ψ_μ as in Lemma 7.4. Then ψ is a derivation, i.e., for $d^\lambda \in D^\lambda$ and $d^\mu \in D^\mu$ we have*

$$\psi_{\lambda + \mu}([d^\lambda, d^\mu]_D) = d^\lambda \cdot \psi_\mu(d^\mu) - d^\mu \cdot \psi_\lambda(d^\lambda). \tag{7.5.1}$$

Proof. The multiplication in $C \oplus D$ yields

$$[\psi_\lambda(d^\lambda) + d^\lambda,\, \psi_\mu(d^\mu) + d^\mu]_{C \oplus D} = \big(\tau(d^\lambda, d^\mu) + d^\lambda \cdot \psi_\mu(d^\mu) - d^\mu \cdot \psi_\lambda(d^\lambda)\big) + [d^\lambda, d^\mu]_D.$$

Since $\tau(d^\lambda, d^\mu) \in C^{\lambda + \mu}$ the $C^{\neq(\lambda + \mu)}$-component of this element is

$$[\psi_\lambda(d^\lambda) + d^\lambda,\, \psi_\mu(d^\mu) + d^\mu]_{C^{\neq(\lambda+\mu)}} = d^\lambda \cdot \psi_\mu(d^\mu) - d^\mu \cdot \psi_\lambda(d^\lambda) \tag{7.5.2}$$

But because $\psi_\lambda(d^\lambda) + d^\lambda \in (C \oplus D)_{\mathrm{ev}'_\lambda}$ and $\psi_\mu(d^\mu) + d^\mu \in (C \oplus D)_{\mathrm{ev}'_\mu}$ we also know

$$[\psi_\lambda(d^\lambda) + d^\lambda,\, \psi_\mu(d^\mu) + d^\mu]_{C\oplus D} \in (C \oplus D)_{\mathrm{ev}'_{\lambda+\mu}}.$$

By (7.4.1) there are therefore two cases to be considered, $[d^\lambda, d^\mu] \neq 0$ and $[d^\lambda, d^\mu] = 0$.

Case $[d^\lambda, d^\mu]_D \neq 0$: In this case

$$[\psi_\lambda(d^\lambda) + d^\lambda,\, \psi_\mu(d^\mu) + d^\mu]_{C\oplus D} = \psi_{\lambda+\mu}([d^\lambda, d^\mu]_D) + [d^\lambda, d^\mu]_D$$

with $C^{\neq(\lambda+\mu)}$-component equal to $\psi_{\lambda+\mu}([d^\lambda, d^\mu]_D)$ so that (7.5.1) follows by comparison with (7.5.2).

Case $[d^\lambda, d^\mu]_D = 0$: In this case (7.5.1) becomes

$$d^\lambda \cdot \psi_\mu(d^\mu) = d^\mu \cdot \psi_\lambda(d^\lambda)$$

with both sides being contained in $C^{\neq(\lambda+\mu)}$. We prove this equality by comparing the C^ρ-component of both sides for some $\rho \neq \lambda + \mu$. By (7.4.3)

$$\begin{aligned} &\mathrm{ev}_{(\rho-\lambda)-\mu}(d^0)\, \psi(d^\mu)^{\rho-\lambda} = d^\mu \cdot (\xi(d^0)^{(\rho-\lambda)-\mu}) \quad \text{and} \\ &\mathrm{ev}_{(\rho-\mu)-\lambda}(d^0)\, \psi(d^\lambda)^{\rho-\mu} = d^\lambda \cdot (\xi(d^0)^{(\rho-\mu)-\lambda}) \end{aligned}$$

Hence, choosing $d^0 \in D^0$ such that $\mathrm{ev}_{\rho-\lambda-\mu}(d^0) \neq 0$, setting $e = \mathrm{ev}_{\rho-\lambda-\mu}(d^0)^{-1}$ and using $[d^\lambda, d^\mu]_D = 0$ we have

$$\begin{aligned} d^\lambda \cdot \psi(d^\mu)^{\rho-\lambda} &= d^\lambda \cdot (e\, d^\mu \cdot \xi(d^0)^{\rho-\lambda-\mu}) = e\, d^\mu \cdot (d^\lambda \cdot \xi(d^0)^{\rho-\lambda-\mu}) \\ &= e\, d^\mu \cdot (\mathrm{ev}_{\rho-\lambda-\mu}(d^0) \psi(d^\lambda)^{\rho-\mu}) = d^\mu \cdot \psi(d^\lambda)^{\rho-\lambda}, \end{aligned}$$

This finishes the proof of (7.5.1). □

End of the proof of Theorem 7.1(b): It follows from Lemma 7.5 that the map f defined by (3.5.1) lies in $\mathrm{Ker}(\overline{\mathrm{res}}_c)$. This map fixes $L \oplus C$ pointwise and maps D^0 to $(\psi + \mathrm{Id})(D^0) = V$. Thus $f(H) = H'$ in view of Lemma 7.3. □

We can now prove the main result of this paper: Conjugacy of Cartan subalgebras of a Lie algebra E which give rise to fgc EALA structures on E.

7.6. **Theorem.** *Let (E, H) be an EALA whose centreless core E_{cc} is fgc, and let (E, H') be a second EALA structure. Then there exists an automorphism f of the* Lie algebra E *such that $f(H) = H'$.*

Proof. Using the notation of Theorem 7.1, we know that (E_{cc}, H_{cc}) and (E_{cc}, H'_{cc}) are fgc Lie tori. Both subalgebras H_{cc} and H'_{cc} are MADs of $L = E_{cc}$ ([Al, Cor. 5.5]). We can now apply the Conjugacy Theorem of [CGP, ???].

Comments: *(E) Need precise reference*

Hence there exists $g \in \mathrm{Aut}_R(L)$ such that $g(H'_{cc}) = H_{cc}$.[15] According to Theorem 6.1(c), $g \in \mathrm{Aut}_R(L) \subset \mathrm{Aut}_k(L)$ can be lifted to an automorphism, say f, of E. So replacing

[15]Even though it is not needed for this work, we remind the reader that g can be chosen in the image of a natural map $\widetilde{\mathfrak{G}}(R) \to \mathrm{Aut}_R(L)$ where $\widetilde{\mathfrak{G}}$ is a simple simply connected group scheme over R with Lie algebra L.

the second structure (E, H') by $(E, f(H'))$ we may assume without loss of generality that $H_{cc} = H'_{cc}$.[16] An application of Theorem 7.1 now finishes the proof. $\square$

7.7. **Remark.** We point out that conjugacy does not hold for all maximal ad-diagonalizable subalgebras of an EALA $E, H)$, see [**?**]

Comments: *(E, June 2014) Reference to Volodya's paper. Remarks still to be worked out depending on the paper. I did not check the counter example.*

Comments: *(E, June 2014) We should add a remark on the paper [CGPY] on conjugacy in the affine Kac-Moody case.*

Comments: *References need to be checked and updated. Do we actually use all of them?*

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